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Definition. Let $(S,\cdot)$ be a semigroup.

(1) S satisfies the *Strong Følner Condition* (SFC) if and only if
$\big(\forall H\in\mathcal{P}_f(S)\big)\big(\forall\epsilon>0\big)\big(\exists K\in\mathcal{P}_f(S)\big)\big(\forall s\in H\big)\big(|K\setminus sK|<\epsilon\cdot|K|\big)$.

(2) If S satisfies the SFC and $A\subseteq S$, the *Følner density of* A is defined by
$d(A)=\sup\{\alpha\in[0,1]:\big(\forall H\in\mathcal{P}_f(S)\big)(\forall\epsilon>0)\big(\exists K\in\mathcal{P}_f(S)\big)$
$\big((\forall s\in H)(|K\setminus sK|<\epsilon\cdot|K|)\text{ and }|A\cap K|\geq\alpha\cdot|K|\big)\}$.

Note that for $H\in\mathcal{P}_f(S)$, $\epsilon>0$, $K\in\mathcal{P}_f(S)$, and $s\in H$, the statements $|K\setminus sK|<\epsilon\cdot|K|$ and $|K\cap sK|>(1-\epsilon)\cdot|K|$ are equivalent. Note also that to show that $d(A)=1$, it suffices to let $\alpha\in(0,1)$ be given, let $H\in\mathcal{P}_f(S)$ and $\epsilon>0$ be given, and show that there exists $K\in\mathcal{P}_f(S)$ such that

(1) for each $s\in H$, $|K\setminus sK|<\epsilon\cdot|K|$ and

(2) $|A\cap K|\geq\alpha\cdot|K|$.

Lemma 1. *Let $(S,\cdot)$ be a semigroup satisfying SFC. Assume that there exists $b\in\mathbb{N}$ such that for each $x\in S$, ρ_x is at most b-to-1. If A is a thick subset of S, then $d(A)=1$. In particular this holds if S is cancellative.*

Proof. This is [H, Lemma 3.9]. (This is a four line elementary proof, so it might be good to include it.) □

Definition. Let $(S,+)$ be a commutve semigroup. Define an equivalence relation $\sim$ on S by, for $a,b\in S$, $a\sim b$ if and only if there exists $x\in S$ such that $a+x=b+x$. For $x\in S$ let $[a]$ denote the $\sim$ equivalence class of a and let $T=S/{\sim}=\{[a]:a\in S\}$ with the operation $[a]+[b]=[a+b]$. Let h be the projection from S onto T so that, for $x\in S$, $h(x)=[x]$.

It is an easy exercise to verify that the operation on T is well defined and that T is a cancellative semigroup. Recall that the projection, h, is a homomorphism.

Theorem. *Let $(S,+)$ be a commutative semigroup and let A be a thick subset of S. Then $d(A)=1$.*

Proof. By [HS, Theorem 4.48] a subset of S is thick if and only if its closure contains a left ideal of βS. We may pick a left ideal L of S such that $L\subseteq\overline{A}$. Since every left ideal contains a minimal left ideal, we may assume that L is minimal. Pick $p\in L$ and let $B = \{x \in A : -x + A \in p\}$. We claim that $L \subseteq \overline{B}$. To see this, let $q \in L$. Then $q + p \in L \subseteq \overline{A}$ so $\{x \in S : -x + A \in p\} \in q$ and $A \in q$ so $B \in q$.

Let $g : T \to S$ be a choice function for the $\sim$ equivalence classes, requiring that $g(t) \in B$ if $t \in h[B]$. Note that $h \circ g$ is the identity on T.

Since $L \subseteq \overline{B}$ we have that B is thick in S. We claim that $h[B]$ is thick in T. To see this, let $F \in \mathcal{P}_f(T)$. Pick $x \in S$ such that $g[F] + x \subseteq B$. We claim that $F + h(x) \subseteq h[B]$. Let $t \in F$. Then $g(t) + x \in B$ so $t + h(x) = h\big(g(t)\big) + h(x) \in h[B]$.

Since $h[B]$ is thick in T and T is cancellative, we have by Lemma 1 that $d(h[B]) = 1$. To see that $d(A) = 1$, let $0 < \alpha < 1$. We shall show that for every $H \in \mathcal{P}_f(S)$ and every $\epsilon > 0$, there exists $G \in \mathcal{P}_f(S)$ such that

(1) for every $s \in H$, $|(s + G) \cap G| > (1 - \epsilon) \cdot |G|$ and

(2) $|G \cap A| > \alpha \cdot |G|$.

Let $H \in \mathcal{P}_f(S)$ and $\epsilon > 0$ be given. We may presume that $\epsilon \leq 1 - \alpha$. Using the fact that $d(h[B]) = 1$ and $h[H] \in \mathcal{P}_f(T)$, pick $F \in \mathcal{P}_f(T)$ such that

(3) for every $s \in H$, $|(h(s) + F) \cap F| > (1 - \epsilon) \cdot |F|$ and

(4) $|F \cap h[B]| > (1 - \epsilon) \cdot |F|$.

For each $s \in H$ and each $t \in F \cap (h(s) + F)$, pick $r(s,t) \in F$ such that $t = h(s) + r(s,t)$ and let $R_{s,t} = \big\{y \in S : y + g(t) = y + s + g\big(r(s,t)\big)\big\}$. Since $h\big(g(t)\big) = t$ and $h\big(s + g\big(r(s,t)\big)\big) = h(s) + r(s,t)$, $R_{s,t} \neq \emptyset$ so by [HS, Corollary 4.18], $\overline{R(s,t)}$ is an ideal of βS and thus contains $K(\beta S)$ so $R_{s,t} \in p$.

If $t \in F \cap h[B]$, then pick $x \in B$ such that $t = h(x)$. Since we required $g(t)$ to be a member of B when $t \in h[B]$, we have $-g(t) + A \in p$.

Pick $u \in \bigcap\{R_{s,t} : s \in H \text{ and } t \in F \cap (h(s) + F)\} \cap \bigcap\{-g(t) + A : t \in F \cap h[B]\}$. Let $G = u + g[F]$. We shall show that G satisfies statements (1) and (2) above.

We first claim that $|G| = |F|$. For this we show that h is injective on G. Then $h[G] = h(u) + h\big[g[F]\big] = h(u) + F$ and since T is cancellative, $|h[G]| = |F|$ so that $|G| = |F|$. So assume that $x, y \in G$ and $h(x) = h(y)$. Pick z and w in F such that $x = u + g(z)$ and $y = u + g(w)$. Then $h(u) + z = h(x) = h(y) = h(u) + w$ so $z = w$ by cancellation in T.

Next we establish statement (1). Equivalently, since $|G| = |F|$, we show that for every $s \in H$, $|(s + u + g[F]) \cap (u + g[F])| > (1 - \epsilon) \cdot |F|$. For this, we let $s \in H$ be given, define $\Psi : F \cap (h(s) + F) \to (s + u + g[F]) \cap (u + g[F])$ by $\Psi(t) = u + g(t)$, and show that Ψ is a bijection.

To see that $\Psi[F\cap(h(s)+F)]\subseteq(s+u+g[F])\cap(u+g[F])$, let $t\in F\cap(h(s)+F)$. Then $u\in R_{s,t}$ so $u+g(t)=u+s+g\big(r(s,t)\big)$ so $\Psi(t)\in(s+u+g[F])\cap(u+g[F])$.

To see that Ψ is onto $(s+u+g[F])\cap(u+g[F])$, let $x\in(s+u+g[F])\cap(u+g[F])$ and pick $t,r\in F$ such that $x=s+u+g(t)=u+g(r)$. Then $h(s)+h(u)+t=h(u)+r$ so $r=h(s)+t\in F\cap(h(s)+F)$ and $\Psi(r)=u+g(r)=x$.

To see that Ψ is one-to-one, assume that $t,x\in F\cap(h(s)+F)$ and $\Psi(t)=\Psi(x)$. Then $u+g(t)=u+g(x)$ so $h(u)+t=h(u)+x$ so $t=x$.

Finally we establish statement (2). Let $F_1=F\cap h[B]$ and $F_2=F\setminus h[B]$. If $t\in F_1$, then $u\in -g(t)+A$ so $(u+g[F_1])\cap A=u+g[F_1]$

By statement (4), $|F_1|>(1-\epsilon)\cdot|F|$ so $|F|=|F_1|+|F_2|>(1-\epsilon)\cdot|F|+|F_2|$ so

$$(*)\qquad |F_2|<\epsilon\cdot|F|\,.$$

Also $|u+g[F]|=|(u+g[F_1])\cup(u+g[F_2])|\leq|u+g[F_1]|+|u+g[F_2]|$ so

$$(**)\qquad |u+g[F_1]|\geq|u+g[F]|-|u+g[F_2]|\,.$$

Consequently

$$\begin{aligned}|G\cap A|&=|\big((u+g[F_1])\cap A\big)\cup\big((u+g[F_2])\cap A\big)|\\&=|(u+g[F_1])\cup\big((u+g[F_2])\cap A\big)|\\&\geq|u+g[F_1]|\\&\geq|u+g[F]|-|u+g[F_2]|\text{ by }(**)\\&\geq|u+g[F]|-|g[F_2]|=|u+g[F]|-|F_2|\\&>|u+g[F]|-\epsilon\cdot|F|\text{ by }(*)\\&=|G|-\epsilon\cdot|G|\geq\alpha\cdot|G|\,.\end{aligned}$$

□

References

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