Proof that $d = d^* = d_t$ and $\Delta^*(S)$ is an ideal

1 The proof that $d = d^* = d_t$

We are grateful to Daniel Glasscock for pointing out that our proof that $d = d^*$ could be easily modified to prove that $d = d^* = d_t$.

Definition 1.1. Let $(S, \cdot)$ be a semigroup which satisfies SFC and let $A \subseteq S$. The *Følner density of A* is defined by
$d(A) = \sup\{\alpha \in [0,1] : \big(\forall H \in \mathcal{P}_f(S)\big)(\forall \epsilon > 0)\big(\exists K \in \mathcal{P}_f(S)\big)$
$\big((\forall s \in H)(|K \setminus sK| < \epsilon \cdot |K|) \text{ and } |A \cap K| \geq \alpha \cdot |K|\big)\}$.

Definition 1.2. Let $(S, \cdot)$ be a left amenable semigroup and let $A \subseteq S$. The *Banach density of A* is defined by $d^*(A) = \sup\{\lambda(\chi_A) : \lambda$ is a left invariant mean on $S\}$.

Definition 1.3. Let $(S, \cdot)$ be a semigroup and let $A \subseteq S$. Then $d_t(A) = \sup\{\alpha \in [0,1] : \big(\forall F \in \mathcal{P}_f(S)\big)(\exists s \in S)(|F \cap As^{-1}| \geq \alpha \cdot |F|)\}$.

Theorem 1.4. *Let $(S, \cdot)$ be a semigroup satisfying SFC, let $A \subseteq S$, and let $\delta = d(A)$. There is a Følner net $\langle F_\alpha \rangle_{\alpha \in D}$ such that the net $\left\langle \frac{|F_\alpha \cap A|}{|F_\alpha|} \right\rangle_{\alpha \in D}$ converges to δ.*

If ν is any cluster point of the net $\langle \mu_{F_\alpha} \rangle_{\alpha \in D}$ in $\times_{f \in l_\infty(S)}[-||f||_\infty, ||f||_\infty]$ Then $\nu \in LIM_0(S)$ and $\nu(\chi_A) = \delta$. In particular $d(A) \leq d^*(A)$.

Proof. Since $d(A) = \delta$, it is a routine exercise to show that
$\big(\forall H \in \mathcal{P}_f(S)\big)(\forall \epsilon > 0)\big(\exists K \in \mathcal{P}_f(S)\big)\big((\forall s \in H)(|K \setminus sK| < \epsilon \cdot |K|) \text{ and }$
$(\delta - \epsilon) \cdot |K| < |A \cap K| < (\delta + \epsilon) \cdot |K|\big)$.

Let $D = \mathcal{P}_f(S) \times \mathbb{N}$ and direct D by $(H, n) \leq (K, m)$ if and only if $H \subseteq K$ and $n \leq m$. For $\alpha = (H, n) \in D$, pick $F_\alpha \in \mathcal{P}_f(S)$ such that
$(\forall s \in H)(|F_\alpha \setminus sF_\alpha| < \frac{1}{n} \cdot |F_\alpha|$ and $(\delta - \frac{1}{n}) \cdot |F_\alpha| < |F_\alpha \cap A| < (\delta + \frac{1}{n}) \cdot |F_\alpha|$.

Let ν be a cluster point of the net $\langle \mu_{F_\alpha} \rangle_{\alpha \in D}$. Since $\langle \mu_{F_\alpha}(\chi_A) \rangle_{\alpha \in D}$ converges to δ, we have that $\nu(\chi_A) = \delta$. □

The proof of the next lemma is based on the proof of [1, Theorem 3.2].

Lemma 1.5. *Let $(S,\cdot)$ be a left amenable semigroup, let $A \subseteq S$, let μ be a left invariant mean on S such that $\mu(\chi_A) > 0$. Assume that $F \in \mathcal{P}_f(S)$ and $0 < \eta < \delta \leq \mu(\chi_A)$ and let $R = \{s \in S : |F \cap As^{-1}| \geq \eta \cdot |F|\}$. Then $\mu(\chi_R) \geq \dfrac{\delta - \eta}{1 - \eta}$. In particular, $R \neq \emptyset$.*

Proof. Note that for $x \in S$, $\chi_{x^{-1}A} = \chi_A \circ \lambda_x$ so $\mu(\chi_{x^{-1}A}) = \mu(\chi_A \circ \lambda_x) = \mu(\chi_A)$. Define $g : S \to [0,1]$ by $g(t) = \dfrac{|F \cap At^{-1}|}{|F|}$. Then for $t \in S$, $g(t) = \frac{1}{|F|}\sum_{x \in F} \chi_{At^{-1}}(x) = \frac{1}{|F|}\sum_{x\in F} \chi_{x^{-1}A}(t)$ so $g = \frac{1}{|F|}\sum_{x\in F}\chi_{x^{-1}A}$. Thus $\mu(g) = \frac{1}{|F|}\sum_{x\in F}\mu(\chi_{x^{-1}A}) = \frac{1}{|F|}\sum_{x\in F}\mu(\chi_A) = \mu(\chi_A)$.

Since μ is additive, $\mu(\chi_A) = \mu(g) \leq \mu(g\chi_R) + \mu(g\chi_{S\setminus R})$. Since $g\chi_R \leq \chi_R$, $\mu(g\chi_R) \leq \mu(\chi_R)$. For $t \in S \setminus R$, $|F \cap At^{-1}| < \eta \cdot |F|$ so $g(t) < \eta$. Also, $\mu(\chi_{S\setminus R}) = 1 - \mu(\chi_R)$ so $\mu(\chi_A) \leq \mu(g\chi_R) + \mu(g\chi_{S\setminus R}) \leq \mu(\chi_R) + \eta\mu(\chi_{S\setminus R}) = \mu(\chi_R) + \eta\big(1 - \mu(\chi_R)\big)$. Therefore $\mu(\chi_A) - \eta \leq \mu(\chi_R)\cdot(1-\eta)$ so $\mu(\chi_R) \geq \dfrac{\mu(\chi_A) - \eta}{1-\eta} \geq \dfrac{\delta - \eta}{1-\eta}$. □

Lemma 1.6. *Let $(S,\cdot)$ be a left amenable semigroup and let $A \subseteq S$. Then $d^*(A) \leq d_t(A)$.*

Proof. Suppose $d_t(A) < d^*(A)$, pick η such that $d_t(A) < \eta < d^*(A)$, and pick $\mu \in LIM(S)$ such that $\delta = \mu(\chi_A) > \eta$. Since $d_t(A) < \eta$ pick $F \in \mathcal{P}_f(S)$ such that for all $s \in S$, $|F \cap As^{-1}| < \eta \cdot |F|$. But then the set R in Lemma 1.5 is empty, a contradiction. □

Lemma 1.7. *Let $(S,\cdot)$ be a semigroup satisfying SFC and define a relation $\sim$ on S by, for $a, b \in S$, $a \sim b$ if and only if there exists $x \in S$ such that $ax = bx$. Then $\sim$ is an equivalence relation on S and the quotient $T = S/{\sim}$ is a cancellative semigroup which satisfies SFC.*

Proof. This is [4, Lemma 3.2]. Its proof was based on the proofs of [2, Lemma 2 and Remark 3] and [5, Theorem 2.2]. □

It is well known and easy to prove that if S is a left amenable semigroup, then the intersection of finitely many right ideals of S is nonempty.

Lemma 1.8. *Let S be a left amenable semigroup, let $h : S \to S/{\sim}$ be the projection in Lemma 1.7, let $n \in \mathbb{N}$, and let $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ be elements of S with the property that $h(a_i) = h(b_i)$ for every $i \in \{1, 2, \ldots, n\}$. There is a right ideal R of S such that $a_i u = b_i u$ for every $u \in R$ and every $i \in \{1, 2, \ldots, n\}$.*

Proof. For each $i \in \{1, 2, \ldots, n\}$, pick $x_i \in S$ such that $a_i x_i = b_i x_i$ and let $R_i = \{u \in S : a_i u = b_i u\}$. Then each R_i is a right ideal of S. Let $R = \bigcap_{i=1}^n R_i$. □

Theorem 1.9. *Let $(S, \cdot)$ be a semigroup which satisfies SFC. Then, for every subset A of S, $d(A) = d^*(A) = d_t(A)$.*

Proof. By Theorem 1.4 and Lemma 1.6 we have that $d(A) \leq d^*(A) \leq d_t(A)$ so it suffices to show that $d_t(A) \leq d(A)$. To see this we will show that if $\eta < d_t(A)$, then $d(A) \geq \eta$. So let $\eta < d_t(A)$ be given.

To see that $d(A) \geq \eta$, let $H \in \mathcal{P}_f(S)$ and $\epsilon > 0$ be given and let $\mathcal{F} = \{K \in \mathcal{P}_f(S) : (\forall s \in H)(|K \setminus sK| < \epsilon \cdot |K|)\}$. We shall show that there exists $K \in \mathcal{F}$ such that $|K \cap A| \geq \eta \cdot |K|$.

Let $g : T \to S$ be a choice function for $\sim$ and note that $h\big(g(t)\big) = t$ for every $t \in T$. Observe that g is injective and h is injective on $g[T]$. We claim that for each $s \in S$, ρ_s is injective on $g[T]$. To see this, let $a, b \in T$ and assume that $g(a)s = g(b)s$. Then $g(a) \sim g(b)$ so $a = h\big(g(a)\big) = h\big(g(b)\big) = b$.

Now $h[H] \in \mathcal{P}_f(T)$ and by Lemma 1.7, T satisfies SFC, so pick $F \in \mathcal{P}_f(T)$ such that for all $s \in H$, $|F \setminus h(s)F| < \epsilon \cdot |F|$. Now $y \in F \cap h(s)F$ if and only if $y \in F$ and $y = h(s)y'$ for some $y' \in F$. The equation $y = h(s)y'$ implies that $h\big(sg(y')\big) = h(s)h\big(g(y')\big) = h(s)y' = y = h\big(g(y)\big)$ so, by Lemma 1.8, there is a right ideal R of S such that $g(y)u = sg(y')u$ for every $y, y' \in F$ and every $s \in H$ for which $y = h(s)y'$.

For each $u \in R$, let $G_u = g[F]u$ and note that $|G_u| = |g[F]u| = |g[F]| = |F|$, because ρ_u is injective on $g[F]$ and g is injective. We claim that for each $u \in R$, $G_u \in \mathcal{F}$. To see this, let $u \in R$ and let $s \in H$. It suffices to show that $G_u \setminus sG_u \subseteq \{g(y)u : y \in F \setminus h(s)F\}$, for then $|G_u \setminus sG_u| \leq |F \setminus h(s)F| < \epsilon \cdot |F| = \epsilon \cdot |G_u|$. So let $x \in G_u \setminus sG_u$ and pick $y \in F$ such that $x = g(y)u$. If we had $y \in h(s)F$, there would be some $y' \in F$ such that $y = h(s)y'$ so $x = g(y)u = sg(y')u$ and thus $x \in sG_u$.

Choose $u \in R$. Since $d_t(A) > \eta$, we may pick $x \in S$ such that $|G_u \cap Ax^{-1}| \geq \eta \cdot |G_u|$. If $z \in G_u \cap Ax^{-1}$, then $zx \in G_u x \cap A$. So $(G_u \cap Ax^{-1})x \subseteq G_u x \cap A$. Now ρ_x is injective on G_u, because ρ_{ux} is injective on $g[F]$. So $|G_u x \cap A| \geq |G_u \cap Ax^{-1}|$ and therefore $|G_u x \cap A| \geq \eta \cdot |G_u| = \eta \cdot |G_u x|$. Since $G_u x = G_{ux} \in \mathcal{F}$ we are done. $\square$

2 The proof that $\Delta^*(S)$ is an ideal

Lemma 2.1. *Let S be a left amenable semigroup. A mean μ on S is left invariant if and only if $\mu(s^{-1}A) = \mu(A)$ for every $s \in S$ and every $A \subseteq S$.*

Proof. Observe that $\chi_{s^{-1}A} = \chi_A \circ \lambda_s$ for every $s \in S$ and every $A \subseteq S$. So, if $\mu \in LIM(S)$, $\mu(s^{-1}A) = \mu(A)$ for every $s \in S$ and every $A \subseteq S$.

To prove the converse, assume that μ is a mean on S with the property that $\mu(s^{-1}A) = \mu(A)$ for every $s \in S$ and every $A \subseteq S$.

For $s \in S$, define $\tau_s : l_\infty(S)^* \to l_\infty(S)^*$ by, for $\eta \in l_\infty(S)^*$ and $f \in l_\infty(S)$, $\tau_s(\eta)(f) = \eta(f \circ \lambda_s)$. Note that τ_s is a linear map. We claim that τ_s is continuous for the norm topology on $l_\infty(S)^*$. By [3, Theorem B.10] it suffices to show that τ_s is bounded. So let $\eta \in l_\infty(S)^*$. We claim that $||\tau_s(\eta)|| \leq ||\eta||$, for which it suffices to let $f \in l_\infty(S)$ and note that $||f \circ \lambda_s|| \leq ||f||$ so that $||\tau_s(\eta)(f)|| = ||\eta(f \circ \lambda_s)|| \leq ||\eta(f)||$.

Let E denote the linear subspace of $l_\infty(S)$ generated by the functions χ_A, where A denotes a subset of S. We claim that E is uniformly dense in $l_\infty(S)$. This is easy to prove directly by an elementary argument. It also follows from the Stone-Weierstrass Theorem, because $\{\widetilde{f} : f \in E\}$ is a subalgebra of $C(\beta S)$ which separates points and contains the constant function $\chi_{\beta S}$.

Given $s \in S$ and $A \subseteq S$, we have noted that $\chi_{s^{-1}A} = \chi_A \circ \lambda_s$ so that $\tau_s(\mu)(\chi_A) = \mu(\chi_A \circ \lambda_s) = \mu(\chi_{s^{-1}A}) = \mu(\chi_A)$. This implies that $\tau_s(\mu)(f) = \mu(f)$ for every $f \in E$. Since E is dense in $l_\infty(S)$ and τ_s is continuous, this implies that $\mu(f \circ \lambda_s) = \mu(f)$ for every $f \in l_\infty(S)$ so that μ is left invariant. □

Definition 2.2. Let $(S, \cdot)$ be a semigroup satisfying SFC. Then
$\Delta(S) = \{p \in \beta S : (\forall A \in p)(d(A) > 0)\}$.

Definition 2.3. Let $(S, \cdot)$ be a left amenable semigroup. Then
$\Delta^*(S) = \{p \in \beta S : (\forall A \in p)(d^*(A) > 0)\}$.

Lemma 2.4. *Let $(S, \cdot)$ be a left amenable semigroup. Then $\Delta^*(S)$ is a left ideal of βS.*

Proof. Let $p \in \Delta^*(S)$, let $q \in \beta S$, and let $A \in qp$. Then $\{s \in S : s^{-1}A \in p\} \in q$ so pick $s \in S$ such that $s^{-1}A \in p$. Then $d^*(s^{-1}A) > 0$ so pick $\eta \in LIM(S)$ such that $\eta(\chi_{s^{-1}A}) > 0$. Since $\chi_A \circ \lambda_s = \chi_{s^{-1}A}$, $\eta(\chi_A) = \eta(\chi_{s^{-1}A})$ so $d^*(A) > 0$. □

Theorem 2.5. *Let $(S,\cdot)S$ be a left amenable semigroup. Then $\Delta^*(S)$ is a two sided ideal of βS.*

Proof. Let $p \in \Delta^*(S)$, let $q \in \beta S$, and let $A \in pq$. We shall show that $d^*(A) > 0$. This will establish that $pq \in \Delta^*(S)$ and hence that $\Delta^*(S)$ is a right ideal of $\beta(S)$.

Let $P = \{s \in S : s^{-1}A \in q\} = \{s \in S : sq \in \overline{A}\}$. Then $P \in p$ so $d^*(P) > 0$ and we can choose $\mu \in LIM(S)$ for which $\mu(P) > 0$. Note that, as usual, if $s \in S$ and $B \subseteq S$, by $s^{-1}B$ we mean $\{t \in S : st \in B\}$. On the other hand, $s^{-1}\overline{B} = \{r \in \beta S : sr \in \overline{B}\}$.

We define ν on $\mathcal{P}(S)$ by $\nu(B) = \mu(S \cap \rho_q^{-1}[\,\overline{B}\,])$ for every $B \subseteq S$. Clearly, ν extends to a positive, bounded, linear functional on $l_\infty(S)$. The fact that $\nu(S) = 1$ implies that $\|\nu\| = 1$, because χ_S is the maximum element of the unit ball of $l_\infty(S)$. So ν is a mean on S. We shall show that ν is left invariant, for which it suffices by Lemma 2.1 to show that $\nu(s^{-1}B) = \nu(B)$ for every $s \in S$ and every $B \subseteq S$.

To see this, let $s \in S$ and $B \subseteq S$ be given. We claim that $\overline{s^{-1}B} = s^{-1}\overline{B}$. Indeed, let $r \in \beta S$.

$$\begin{aligned} r \in \overline{s^{-1}B} &\Leftrightarrow s^{-1}B \in r \\ &\Leftrightarrow B \in sr \\ &\Leftrightarrow sr \in \overline{B} \\ &\Leftrightarrow r \in s^{-1}\overline{B} \end{aligned}$$

so if $t \in S$, then

$$\begin{aligned} t \in S \cap \rho_q^{-1}[\,\overline{s^{-1}B}\,] &\Leftrightarrow t \in \rho_q^{-1}[\,\overline{s^{-1}B}\,] \\ &\Leftrightarrow t \in \rho_q^{-1}[s^{-1}\overline{B}\,] \\ &\Leftrightarrow tq \in s^{-1}\overline{B} \\ &\Leftrightarrow stq \in \overline{B} \\ &\Leftrightarrow st \in \rho_q^{-1}[\,\overline{B}\,] \\ &\Leftrightarrow t \in s^{-1}(S \cap \rho_q^{-1}[\,\overline{B}\,])\,. \end{aligned}$$

Therefore $\nu(s^{-1}B) = \mu(S \cap \rho_q^{-1}[\,\overline{s^{-1}B}\,]) = \mu\big(s^{-1}(S \cap \rho_q^{-1}[\,\overline{B}\,])\big) =$ $\mu(S \cap \rho_q^{-1}[\,\overline{B}\,]) = \nu(B)$ as required.

Now $P = S \cap \rho_q^{-1}[\,\overline{A}\,]$ so $\nu(A) = \mu(S \cap \rho_q^{-1}[\,\overline{A}\,]) = \mu(P) > 0$. □

Corollary 2.6. *Let $(S,\cdot)S$ be a semigroup that satisfies SFC. Then $\Delta(S)$ is a two sided ideal of βS.*

Proof. Theorems 1.9 and 2.5. □

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