

Preservation of size in Cartesian products

Seminar Version – This is a work in progress.

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Abstract

There are several notions of size for subsets of semigroups that make sense in any semigroup. A number of them such as *central* have strong combinatorial consequences. In this paper we investigate 40 of these notions with respect to the following four questions involving Cartesian products. (1) If A is large (in one of these senses) in S and B is large in T , must $A \times B$ be large in $S \times T$? (2) If $A \times B$ is large in $S \times T$, must A be large in S and B be large in T ? (3) If for each $\alpha \in I$, A_α is large in S_α , must $\times_{\alpha \in I} A_\alpha$ be large in $\times_{\alpha \in I} S_\alpha$? (4) If $\times_{\alpha \in I} A_\alpha$ is large in $\times_{\alpha \in I} S_\alpha$ must each A_α be large in S_α ? Another question which is behind most of the affirmative answers to questions 2 and 4 is this: If $h : S \rightarrow T$ is a surjective homomorphism and A is large in S , must $h[A]$ be large in T ?

1 Introduction

If R is a property which may or may not be possessed by subsets of a semigroup S , we say that R is a *notion of largeness for S* if and only if S is an R set, $\emptyset$ is not an R set, and whenever $A \subseteq B \subseteq S$ and A is an R set, so is B . If R is a notion of largeness for S , then another notion of largeness R^* for S is defined by: a subset A of S is an R^* set if and only if A has nonempty intersection with every R subset of S ; equivalently a subset A of S is an R^* set if and only if $S \setminus A$ is not an R subset of S . Note that a subset A of S is an $(R^*)^*$ set if and only if A is an R set.

We will introduce twenty defined notions of largeness for subsets of a semigroup S . If R is any one of our defined notions, then R and R^* are distinct, so our twenty defined notions become a total of forty.

It can get more complicated. All but one of our notions are one sided notions. (The exception is P .) So for example A is thick in S if and only if for every finite nonempty subset F of S , there exists $x \in S$ such that $Fx \subseteq A$. One can consider separately the version replacing Fx by xF . And in [4], the first

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author of this paper did consider the right and left versions separately. But in this paper we are concerned with Cartesian products, so the answers to our questions will be the same for the right and left versions. Here we exclusively use what were called the right versions in [4].

Many of the notions have combinatorial definitions and of these several have simple algebraic characterizations in terms of the algebraic structure of $(\beta S, \cdot)$ where S has the discrete topology. Others have definitions in terms of that algebraic structure. The reader is referred to [7, Part I] for an elementary derivation of that algebraic structure.

We will introduce our twenty basic notions in groups.

Given a set X we write $\mathcal{P}_f(X)$ for the set of finite nonempty subsets of X . Given a semigroup S , $X \subseteq S$, and $x \in S$ we let $x^{-1}X = \{y \in S : xy \in X\}$ and $Xx^{-1} = \{y \in S : yx \in X\}$. We write ${}^{\mathbb{N}}S$ for the set of sequences in S . Given $m \in \mathbb{N}$ we let $\mathcal{J}_m = \{(t(1), t(2), \dots, t(m)) \in \mathbb{N}^m : t(1) < t(2) < \dots < t(m)\}$.

Given a sequence $\langle x_n \rangle_{n=1}^{\infty}$ in a semigroup $(S, \cdot)$, we let $FP(\langle x_n \rangle_{n=1}^{\infty}) = \{\prod_{t \in F} x_t : F \in \mathcal{P}_f(\mathbb{N})\}$ where $\prod_{t \in F} x_t$ is computed in increasing order of indices. For $r \in \mathbb{N}$, $FP(\langle x_n \rangle_{n=1}^r)$ is defined analogously.

Definition 1.1. Let $(S, \cdot)$ be a semigroup and let $A \subseteq S$.

- (1) A is a *Q set* in S if and only if there exists a sequence $\langle x_n \rangle_{n=1}^{\infty}$ in S such that whenever $m < n$ in $\mathbb{N}$, $x_n \in x_m A$.
- (2) A is an *IP set* in S if and only if there exists a sequence $\langle x_n \rangle_{n=1}^{\infty}$ in S such that $FP(\langle x_n \rangle_{n=1}^{\infty}) \subseteq A$.
- (3) For $r \in \mathbb{N} \setminus \{1\}$, A is an *IP_r set* in S if and only if there exists a sequence $\langle x_n \rangle_{n=1}^r$ in S such that $FP(\langle x_n \rangle_{n=1}^r) \subseteq A$.
- (4) For $r \in \mathbb{N} \setminus \{1\}$, A is a *strong IP_r set* in S (*SIP_r*) if and only if whenever A is finitely colored, there exists a sequence $\langle x_n \rangle_{n=1}^r$ in S such that $FP(\langle x_n \rangle_{n=1}^r)$ is monochromatic.
- (5) A is an *IP_{<ω} set* if and only if for each $r \in \mathbb{N} \setminus \{1\}$ A is an *IP_r set*.

By [7, Theorem 5.12] A is an *IP set* if and only if A is a member of an idempotent in $(\beta S, \cdot)$.

The *Q* in *Q set* comes from *quotient*; if $(S, \cdot)$ is a group then A is a *Q set* if and only if there is a sequence $\langle x_n \rangle_{n=1}^{\infty}$ in S such that $\{\frac{x_n}{x_m} : m < n \text{ in } \mathbb{N}\} \subseteq A$. The *IP* notation is due to Furstenberg and is variously thought to represent *idempotent* or *infinite dimensional parallelepiped*.

Definition 1.2. Let $(S, \cdot)$ be a semigroup and let $A \subseteq S$.

- (1) A is *thick* in S if and only if for each $F \in \mathcal{P}_f(S)$ there exists $x \in S$ such that $Fx \subseteq A$.
- (2) A is *weakly thick* in S (*WT*) if and only if there exists $s \in S$ such that $s^{-1}A$ is thick.

- (3) A is *piecewise syndetic in S (PS)* if and only if there exists $H \in \mathcal{P}_f(S)$ such that $\bigcup_{t \in H} t^{-1}A$ is thick.
- (4) A is *strongly piecewise syndetic in S (SPS)* if and only if there exists $H \in \mathcal{P}_f(S)$ such that $\bigcup_{t \in H} At^{-1}$ is thick.

The notion thick* is usually called *syndetic* and we shall do so in the rest of this paper. A simple characterization of syndetic is that A is syndetic in S if and only if there exists $G \in \mathcal{P}_f(S)$ such that $S = \bigcup_{t \in G} t^{-1}A$.

We write $K(\beta S)$ for the smallest two sided ideal of $(\beta S, \cdot)$. By [7, Theorem 4.40] A is piecewise syndetic if and only if $\overline{A} \cap K(\beta S) \neq \emptyset$.

Definition 1.3. Let $(S, \cdot)$ be a semigroup and let $A \subseteq S$.

- (1) A is *central in S* if and only if A is a member of an idempotent in $K(\beta S)$.
- (2) A is *strongly central in S (SC)* if and only if for every minimal left ideal L of βS , A is a member of an idempotent in L .
- (3) A is *quasi central in S (QC)* if and only if A is a member of an idempotent in $\text{cl}K(\beta S)$.

In our opinion *central* is the most significant of the notions of largeness. It originated in [2] where it was defined for subsets of $\mathbb{N}$. Central sets are guaranteed to have substantial combinatorial properties. See, for example [7, Theorem 15.24] for properties that any central subset of $\mathbb{N}$ must hold. Most of these properties are consequences of the Central Sets Theorem. The original Central sets Theorem [2, Proposition 8.21] applied to subsets of $\mathbb{N}$. The following, much more general, version is from [7].

Theorem 1.4. Let $(S, \cdot)$ be a semigroup and let A be a central subset of S . There exist

$$m : \mathcal{P}_f(\mathbb{N}S) \rightarrow \mathbb{N}, \alpha \in \times_{F \in \mathcal{P}_f(\mathbb{N}S)} S^{m(F)+1}, \text{ and } \tau \in \times_{F \in \mathcal{P}_f(\mathbb{N}S)} \mathcal{J}_{m(F)},$$

such that

- (1) if $F, G \in \mathcal{P}_f(\mathbb{N}S)$ and $F \subsetneq G$, then $\tau(F)(m(F)) < \tau(G)(1)$ and
- (2) whenever $n \in \mathbb{N}$, $G_1, G_2, \dots, G_n \in \mathcal{P}_f(\mathbb{N}S)$, $G_1 \subsetneq G_2 \subsetneq \dots \subsetneq G_n$, and for each $i \in \{1, 2, \dots, n\}$, $f_i \in G_i$, one has

$$\prod_{i=1}^n \left(\left(\prod_{j=1}^{m(G_i)} \alpha(G_i)(j) \cdot f_i(\tau(G_i)(j)) \right) \cdot \alpha(G_i)(m(G_i) + 1) \right) \in A.$$

Proof. [7, Corollary 14.14.10]. □

Definition 1.5. Let $(S, \cdot)$ be a semigroup and let $A \subseteq S$.

- (1) A is a *J set in S* if and only if for every $F \in \mathcal{P}_f(\mathbb{N}S)$ there exist $m \in \mathbb{N}$, $a \in S^{m+1}$, and $t \in \mathcal{J}_m$ such that for each $f \in F$,

$$a(1)f(t(1))a(2) \cdots a(m)f(t(m))a(m+1) \in A$$

- (2) A is a P set in S if and only if for each $k \in \mathbb{N}$, there exist $m \in \mathbb{N}$, $a \in S^{m+1}$, and $d \in S$ such that $\{a(1)d^t a(2) \cdots a(m)d^t a(m+1) : t \in \{1, 2, \dots, k\}\} \subseteq A$.
- (3) A is a C set in S if and only if A satisfies the conclusion of Theorem 1.4.

The name P comes from “progression”. It was called WP in [4]. We do not consider in this paper the notion that was called P in [4]. The names J and C have no particular significance.

We let $J(S) = \{p \in \beta S : \text{every member of } p \text{ is a } J \text{ set in } S\}$. Then $J(S)$ is a two sided ideal of βS . (This is [7, Theorem 14.14.4] except that we neglected to show that $J(S) \neq \emptyset$ so Theorem 14.14.4 should have been placed after Lemma 14.14.6.) By [7, Theorem 14.15.1] A is a C set if and only if A is a member of an idempotent in $J(S)$.

Our last five notions involve three versions of density of subsets of S . We refer the reader to [3] for the definitions of left invariant mean, left amenable, and the Strong Følner Condition (SFC).

Definition 1.6. Let $(S, \cdot)$ be a semigroup and let $A \subseteq S$.

- (1) The *translation density* of A is defined by $d_t(A) = \sup\{\alpha \in [0, 1] : \text{for every } F \in \mathcal{P}_f(S) \text{ there exists } s \in S \text{ such that } |F \cap As^{-1}| \geq \alpha|F|\}$.
- (2) If S is left amenable, then the *Banach density* of A is defined by $d^*(A) = \sup\{\lambda(\chi_A) : \lambda \text{ is a left invariant mean on } S\}$, where χ_A is the characteristic function of A .
- (3) If S satisfies SFC, then the Følner density of A is defined by $d(A) = \sup\{\alpha \in [0, 1] : \text{for every } H \in \mathcal{P}_f(S) \text{ and every } \epsilon > 0, \text{ there exists } K \in \mathcal{P}_f(S) \text{ such that (i) for every } s \in H, |K \setminus sK| < \epsilon \cdot |K| \text{ and (ii) } |A \cap K| \geq \alpha \cdot |K|\}$.
- (4) If S is left amenable, then $\Delta^*(S) = \{p \in \beta S : \text{for all } A \in p, d^*(A) > 0\}$.
- (5) If S satisfies SFC, then $\Delta(S) = \{p \in \beta S : \text{for all } A \in p, d(A) > 0\}$.

If a semigroup satisfies SFC, then it is left amenable. By [3, Theorem 3.15] if S satisfies SFC and $A \subseteq S$, then $d(A) = d^*(A) = d_t(A)$. By [3, Theorem 2.8] if S is left amenable, then $\Delta^*(S)$ is a two sided ideal of βS so if S satisfies SFC, then $\Delta(S)$ is a two sided ideal of βS . If S is the free semigroup generated by $\{a, b\}$ and X is the set of words whose leftmost letter is a , then $d_t(X) = d_t(S \setminus X) = 0$ so there is no ultrafilter all of whose members have positive translation density.

Definition 1.7. Let $(S, \cdot)$ be a semigroup and let $A \subseteq S$.

- (1) A is a B_t set in S if and only if $d_t(A) > 0$.
- (2) A is a B_a set in S if and only if S is left amenable and $d^*(A) > 0$.
- (3) A is a B_f set in S if and only if S satisfies SFC and $d(A) > 0$.

- (4) A is a D_a set in S if and only if S is left amenable and A is a member of an idempotent in $\Delta^*(S)$.
- (5) A is a D_f set in S if and only if S satisfies SFC and A is a member of an idempotent in $\Delta(S)$.

If S is left amenable, then d^* is subadditive by definition so by [7, Theorem 3.11] a subset A of S is a B_a set if and only if $\Delta^*(S) \cap \overline{A} \neq \emptyset$. If S satisfies SFC, then d is subadditive by [4, Lemma 3.4] so by [7, Theorem 3.11] a subset A of S is a B_f set if and only if $\Delta(S) \cap \overline{A} \neq \emptyset$.

If S is a left amenable semigroup, then B_a and D_a are notions of largeness for S ; if S is not left amenable, then B_a and D_a are not notions of largeness for S since S is neither a B_a set nor a D_a set. If we applied the usual definition, that is if we said that a subset A of S is B_a^* if and only if it has nonempty intersection with every B_a subset of S , we would have that every subset of S is B_a^* .

Definition 1.8. Let S be a subsemigroup and let $A \subseteq S$.

- (1) A is a B_a^* set in S if and only if S is left amenable and A has nonempty intersection with every B_a set in S .
- (2) A is a D_a^* set in S if and only if S is left amenable and A has nonempty intersection with every D_a set in S .
- (3) A is a B_f^* set in S if and only if S satisfies SFC and A has nonempty intersection with every B_f set in S .
- (4) A is a D_f^* set in S if and only if S satisfies SFC and A has nonempty intersection with every D_f set in S .

We omit the very easy proof of the following lemma.

Lemma 1.9. Let S be a semigroup and let $A \subseteq S$.

(I) Assume that S is left amenable. The following statements are equivalent.

- (1) A is a B_a^* set in S .
- (2) $S \setminus A$ is not a B_a set in S .
- (3) $\Delta^*(S) \subseteq \overline{A}$.

(II) Assume that S is left amenable. The following statements are equivalent.

- (1) A is a D_a^* set in S .
- (2) $S \setminus A$ is not a D_a set in S .
- (3) Every idempotent in $\Delta^*(S)$ is a member of $\overline{A}$.

(III) Assume that S satisfies SFC. The following statements are equivalent.

- (1) A is a B_f^* set in S .

(2) $S \setminus A$ is not a B_f set in S .

(3) $\Delta(S) \subseteq \overline{A}$.

(IV) Assume that S satisfies SFC. The following statements are equivalent.

(1) A is a D_f^* set in S .

(2) $S \setminus A$ is not a D_f set in S .

(3) Every idempotent in $\Delta(S)$ is a member of $\overline{A}$.

We conclude this introduction with diagrams showing the implications that are known to hold among these notions. All of the notions defined above appear as right versions of notions in [4] except P , WT , $IP_{<\omega}$, IP_r , SIP_r , B_t , B_a , B_f , D_a , D_f , and their duals. And P is WP in [4]. Verification of the implications among the notions in [4] are given in Section 4 of that paper.

It is trivial that thick implies WT and the verification that WT implies central is [6, Lemma 3.3] so WT^* implies syndetic and central* implies WT^* .

The implications involving $IP_{<\omega}$, IP_r , SIP_r , $IP_{<\omega}^*$, IP_r^* , and SIP_r^* , are all immediate except the fact that $IP_{<\omega}$ implies SIP_r and SIP_r^* implies $IP_{<\omega}^*$ which follows from the fact that, given r and k there exists $m \in \mathbb{N}$ such that whenever an IP_m set is colored with k colors, there is a monochromatic IP_r set.

It remains to verify the implications involving B_a , B_f , B_t , D_f , D_a , B_a^* , B_f^* , B_t^* , D_f^* , and D_a^* . The fact that B_a implies B_t is [3, Corollary 1.18]. If S is not left amenable, then no subset of S is a B_a^* set so the fact that B_t^* implies B_a^* requires the assumption that S is left amenable. For the same reason, the facts that B_a^* implies B_f^* and D_a^* implies D_f^* require the assumption that S satisfies SFC, while the facts that B_f implies B_a and D_f implies D_a are immediate. If S is not left amenable, then there are no D_a sets in S so the fact that IP^* implies D_a^* requires the assumption that S is left amenable while the fact that D_a implies IP is immediate. If A is a B_f^* set in S , then S satisfies SFC so $\Delta(S)$ is a compact ideal of βS and thus $clK(\beta S) \subseteq \Delta(S)$ so B_f^* implies PS^* and D_f^* implies QC^* . If S does not satisfy SFC, then there are no B_f sets in S so the facts that PS implies B_f and QC implies D_f requires the assumption that S satisfies SFC.



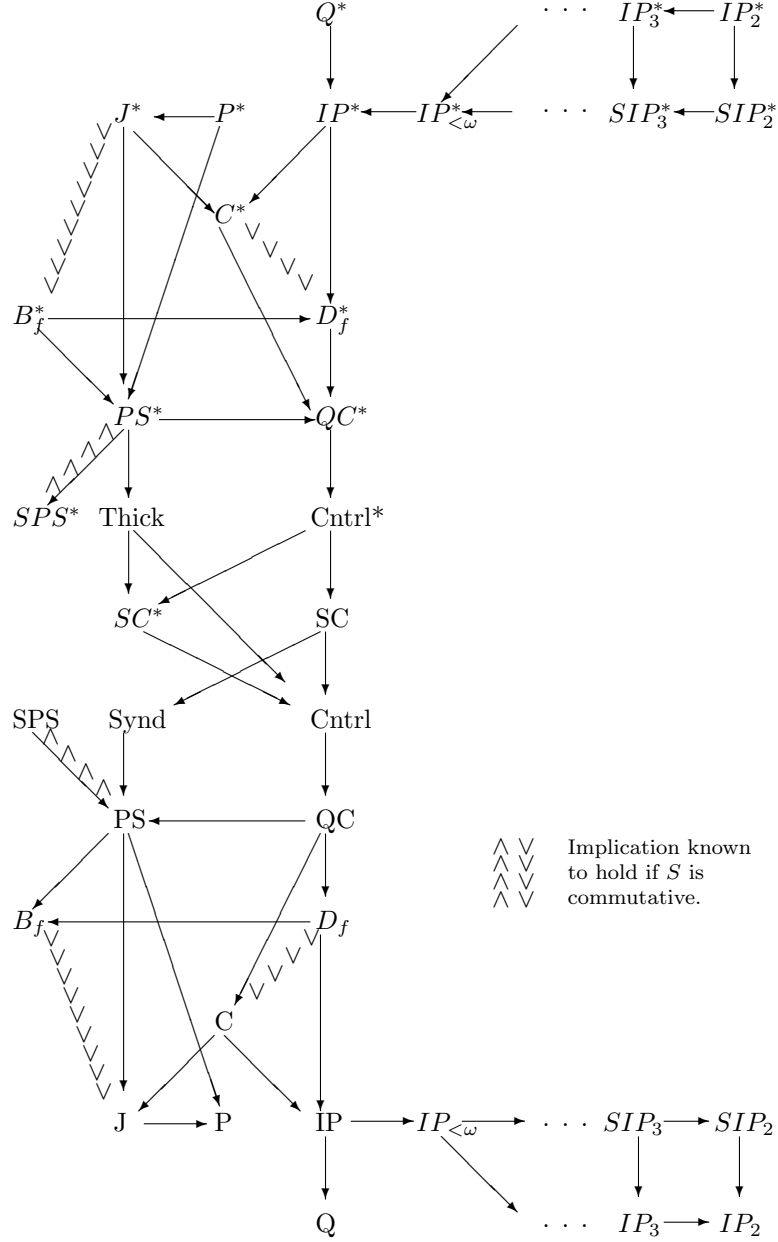


Figure 2: Implications for S satisfying SFC

As we have noted if S satisfies SFC, then for all $A \subseteq S$, $d(A) = d^*(A) = d_t(A)$ and $\Delta(S) = \Delta^*(S)$ so B_f , B_a , and B_t coincide and D_f and D_a coincide. Also, if S satisfies SFC, then S is left amenable so by [5, Theorem 5.11] any weakly thick set in S is thick. This explains the differences between Figure 1 and Figure 2.

In the remaining sections of this paper we address the questions posed in the abstract.

2 Preservation under surjective homomorphisms

In this section we show that all except possibly three of the notions we are considering are preserved under passage to homomorphic images.

Definition 2.1. Let S be a semigroup and let R be a property which may or may not be possessed by subsets of S . We say that R is *preserved under surjective homomorphisms* if and only if whenever S and T are semigroups, $h : S \rightarrow T$ is a surjective homomorphism, and A is an R set in S , one has $h[A]$ is an R set in T .

Lemma 2.2. Let S be a semigroup and let R be a notion of largeness for S . Then R^* is preserved by surjective homomorphisms if and only if whenever S and T are semigroups, $h : S \rightarrow T$ is a surjective homomorphism, and A is an R set in T , one has $h^{-1}[A]$ is an R set in S .

Proof. For the necessity, assume that R^* is preserved by surjective homomorphisms, let S and T be semigroups, let $h : S \rightarrow T$ be a surjective homomorphism, and let A be an R set in T . Suppose that $h^{-1}[A]$ is not an R set in S . Let $B = S \setminus h^{-1}[A]$. Then $S \setminus B$ is not an R set in S so B is an R^* set in S so $h[B]$ is an R^* set in T so $T \setminus h[B]$ is not an R set in T . This is a contradiction because $A \subseteq T \setminus h[B]$.

For the sufficiency let S and T be semigroups, let $h : S \rightarrow T$ be a surjective homomorphism, and let A be an R^* set in S . Suppose that $h[A]$ is not an R^* set in T . Then $T \setminus h[A]$ is an R set in T so $h^{-1}[T \setminus h[A]]$ is an R set in S . But $h^{-1}[T \setminus h[A]] \cap A = \emptyset$, a contradiction. $\square$

Definition 2.3. Let h be a homomorphism from the semigroup S onto the semigroup T . Choose $\varphi : T \rightarrow S$ such that $h(\varphi(x)) = x$ for every $x \in T$.

We see first that all of the notions with simple elementary definitions are preserved by surjective homomorphisms as are most of their dual notions. (The exceptions are Q^* and SIP_r^* which we are unable to determine whether they are preserved by surjective homomorphisms.)

Theorem 2.4. Each of Q , IP , IP_r and SIP_r for $r > 1$, $IP_{<\omega}$, *thick*, WT , PS , SPS , J , P , IP^* , IP_r^* for $r > 1$, $IP_{<\omega}^*$, *thick** = *syndetic*, WT^* , PS^* , SPS^* , J^* , and P^* is preserved by surjective homomorphisms.

Proof. Assume that $h : S \rightarrow T$ is a surjective homomorphism and that $A \subseteq S$. For Q , IP , and IP_r , if $\langle x_n \rangle_{n=1}^\infty$ or $\langle x_n \rangle_{n=1}^r$ is as required in S for A to satisfy the definition, then $\langle h(x_n) \rangle_{n=1}^\infty$ or $\langle h(x_n) \rangle_{n=1}^r$ is as required in T for $h[A]$ to satisfy the definition. And $IP_{<\omega}$ is preserved because each IP_r is preserved.

To verify SIP_r , Assume that A is SIP_r , let $m \in \mathbb{N}$ and assume that $h[A] \subseteq \bigcup_{i=1}^m J_i$. Then $A \subseteq \bigcup_{i=1}^m h^{-1}[J_i]$ so pick $i \in \{1, 2, \dots, m\}$ and $\langle x_n \rangle_{n=1}^r$ in S such that $FP(\langle x_n \rangle_{n=1}^r) \subseteq h^{-1}[J_i]$. so $FP(\langle h(x_n) \rangle_{n=1}^r) \subseteq J_i$.

The verifications for thick, WT , PS , SPS , J and P are similarly routine. Two of the more complicated of these are SPS and J ; we will give the details for those. These proofs illustrate the general technique: a finite subset F of S corresponds to the finite subset $h[F]$ of T ; the finite subset G of T corresponds to the finite subset $\varphi[G]$ of S .

Let A be SPS in S . Then

$$(\exists H \in \mathcal{P}_f(S))(\forall F \in \mathcal{P}_f(S))(\exists x \in S)(\forall z \in F)(\exists t \in H)(zxt \in A).$$

Pick $H \in \mathcal{P}_f(S)$ as guaranteed for A . To see that $h[A]$ is SPS in T , let $H' = h[H]$ and let $F' \in \mathcal{P}_f(T)$. Let $F = \varphi[F']$ and pick $x \in S$ such that $(\forall z \in F)(\exists t \in H)(zxt \in A)$. Let $x' = h(x)$. We claim that $(\forall z' \in F')(\exists t' \in H')(z'x't' \in h[A])$ so let $z' \in F'$ and let $z = \varphi(z')$. Pick $t \in H$ such that $zxt \in A$. Then $z'x't' = h(zxt) \in h[A]$.

Now assume that A is a J set in S . To see that $h[A]$ is a J set in T , Let $F' \in \mathcal{P}_f(\mathbb{N}T)$. For $f \in F'$, let $g_f = \varphi \circ f$ and let $F = \{g_f : f \in F'\}$. Since A is a J set in S pick $m \in \mathbb{N}$, $a \in S^{m+1}$, and $t \in \mathcal{J}_m$ such that for each $f \in F'$,

$$a(1)g_f(t(1))a(2) \cdots a(m)g_f(t(m))a(m+1) \in A.$$

Define $b \in T^{m+1}$ by, for $i \in \{1, 2, \dots, m+1\}$, $b(i) = h(a(i))$. Then for each $f \in F'$,

$$h(a(1))h(g_f(t(1)))h(a(2)) \cdots h(a(m))h(g_f(t(m)))h((a(m+1))) \in h[A].$$

Now given $i \in \{1, 2, \dots, m+1\}$, $h(g_f(t(i))) = h(\varphi(f(t(i)))) = f(t(i))$ so we have

$$b(1)f(t(1))a(2) \cdots a(m)f(t(m))a(m+1) \in h[A].$$

We have yet to verify the assertions about the dual notions. We shall use Lemma 2.2 so let $h : S \rightarrow T$ be a surjective homomorphism and let $A \subseteq T$.

If $\langle x_n \rangle_{n=1}^\infty$ is a sequence in T and $FP(\langle x_n \rangle_{n=1}^\infty) \subseteq A$, then $FP(\langle \varphi(x_n) \rangle_{n=1}^\infty) \subseteq h^{-1}[A]$ so IP^* is preserved, as is IP_r^* for $r \in \mathbb{N} \setminus \{1\}$ and consequently so is $IP_{<\omega}^*$.

To verify for syndetic=thick*, assume that A is thick in T and let $F \in \mathcal{P}_f(S)$. Then $h[F] \in \mathcal{P}_f(T)$ so pick $x \in T$ such that $h[F]x \subseteq A$. Then $F\varphi(x) \subseteq h^{-1}[A]$.

The general technique described earlier works for WT^* , PS^* , SPS^* , J^* , and P^* . We conclude this proof by verifying the general technique for P^* .

Assume that A is a P set in T . To see that $h^{-1}[A]$ is a P set in S , let $k \in \mathbb{N}$. Pick $m \in \mathbb{N}$, $a \in T^{m+1}$, and $d \in T$ such that $\{a(1)d^k a(2) \cdots a(m)d^k a(m+1) :$

$t \in \{1, 2, \dots, k\} \subseteq A$. Define $b \in S^{m+1}$ and $c \in S$ by $b(i) = \varphi(a(i))$ for $i \in \{1, 2, \dots, m+1\}$ and $c = \varphi(d)$. Then, given $t \in \{1, 2, \dots, k\}$,

$$\begin{aligned} b(1)c^t b(2) \cdots b(m)c^t b(m+1) &= \\ \varphi(a(1))(\varphi(d))^t \varphi(a(2)) \cdots \varphi(a(m))(\varphi(d))^t \varphi(a(m+1)) &\in h^{-1}[A]. \end{aligned}$$

Note that we did not claim that $c^t = \varphi(d^t)$, which may not be true since φ need not be a homomorphism. $\square$

We are unable to prove that Q^* and SIP_r^* are preserved under surjective homomorphisms. In trying to use the general technique for Q^* , we would let A be a Q set in T and pick a sequence $\langle x_n \rangle_{n=1}^\infty$ in T such that whenever $m < n$, $x_n \in x_m \cdot A$. Then one would want $\varphi(x_n) \in \varphi(x_m) \cdot h^{-1}[A]$. But if $x_n = x_m \cdot a$ it does not follow that $\varphi(x_n) = \varphi(x_m) \cdot \varphi(a)$.

We shall use the next lemma when dealing with the notions of central, strongly central, quasi central, and C .

Lemma 2.5. *Let $h : S \rightarrow T$ be a surjective homomorphism and let $\tilde{h} : \beta S \rightarrow \beta T$ be the continuous extension of h . Then $\tilde{h}$ is a surjective homomorphism and $\tilde{h}[K(\beta S)] = K(\beta T)$.*

Proof. By [7, Corollary 4.22], $\tilde{h}$ is a homomorphism and by elementary topology $\tilde{h}[\beta S] = \beta T$. By [7, Exercise 1.7.3], $\tilde{h}[K(\beta S)] = K(\beta T)$. $\square$

Theorem 2.6. *Each of central, SC , QC , C , central * , SC^* , QC^* , and C^* is preserved by surjective homomorphisms.*

Proof. Let $h : S \rightarrow T$ be a surjective homomorphism and let $A \subseteq S$. If A is central in S , pick an idempotent $p \in K(\beta S)$ such that $A \in p$. Then $h[A] \in \tilde{h}(p)$. By Lemma 2.5, $\tilde{h}(p) \in K(\beta T)$ so $h[A]$ is central in T .

By Lemma 2.5 and elementary topology $\tilde{h}[clK(\beta S)] = clK(\beta T)$ so if A is QC in S , then $h[A]$ is QC in T .

Now assume that A is SC in S . To see that $h[A]$ is SC in T , let L be a minimal left ideal of βT . By [7, Exercise 1.7.3(1)] pick a minimal left ideal L' of βS such that $\tilde{h}[L'] = L$. Pick an idempotent $p \in L'$ such that $A \in p$. Then $\tilde{h}(p)$ is an idempotent in L and $h[A] \in \tilde{h}(p)$.

Now assume that A is a C set in S and pick an idempotent $p \in J(S)$ such that $A \in p$. Then $h[A] \in \tilde{h}(p)$ so it suffices to show that $\tilde{h}(p) \in J(T)$. So let $B \in \tilde{h}(p)$ so that $h^{-1}[B] \in p$. Then $h^{-1}[B]$ is a J set in S so by Theorem 2.4 $h[h^{-1}[B]]$ is a J set in T and $h[h^{-1}[B]] \subseteq B$.

We now establish the dual versions using Lemma 2.2 so assume that $h : S \rightarrow T$ is a surjective homomorphism and let $A \subseteq T$.

Assume that A is central in T and pick an idempotent $p \in K(\beta T)$ such that $A \in p$. By Lemma 2.5 $\tilde{h}^{-1}[\{p\}] \cap K(\beta S) \neq \emptyset$ so there is a minimal left ideal L of S such that $\tilde{h}^{-1}[\{p\}] \cap L \neq \emptyset$. Then $\tilde{h}^{-1}[\{p\}] \cap L$ is a compact subsemigroup of βS so pick an idempotent $q \in \tilde{h}^{-1}[\{p\}] \cap L$. Then $A \in p$ so $h^{-1}[A] \in q$ and therefore $h^{-1}[A]$ is central.

The proof for QC is slightly shorter. Assume that A is QC in T and pick an idempotent $p \in \mathcal{c}K(\beta T)$ such that $A \in p$. Then $\tilde{h}^{-1}[\{p\}] \cap \mathcal{c}K(\beta S) \neq \emptyset$ so $\tilde{h}^{-1}[\{p\}] \cap \mathcal{c}K(\beta S)$ is a compact subsemigroup of βS so pick an idempotent $q \in \tilde{h}^{-1}[\{p\}] \cap \mathcal{c}K(\beta S)$. Then $A \in p$ so $h^{-1}[A] \in q$ and therefore $h^{-1}[A]$ is QC .

Assume that A is SC in T . To see that $h^{-1}[A]$ is SC in S , let L be a minimal left ideal of βS . By [7, Exercise 1.7.3] $\tilde{h}[L]$ is a minimal left ideal of βT so pick an idempotent $p \in \tilde{h}[L]$ such that $A \in p$. Then $\tilde{h}^{-1}[\{p\}] \cap L \neq \emptyset$ so $\tilde{h}^{-1}[\{p\}] \cap L$ is a compact subsemigroup of βS so pick an idempotent $q \in \tilde{h}^{-1}[\{p\}] \cap L$. Since $A \in p$, $h^{-1}[A] \in q$ so $h^{-1}[A]$ is SC .

Finally, assume that A is a C set in T . Pick an idempotent $p \in J(T)$ such that $A \in p$. By Theorem 2.4, J^* is preserved by surjective homomorphisms and given $B \in p$, B is a J set in T so by Lemma 2.2, $h^{-1}[B]$ is a J set in S so by [7, Theorem 14.14.7], $J(S) \cap \overline{h^{-1}[B]} \neq \emptyset$. So $\{J(S) \cap \overline{h^{-1}[B]} : B \in p\}$ has the finite intersection property so $J(S) \cap \bigcap_{B \in p} \overline{h^{-1}[B]} \neq \emptyset$. Now $\bigcap_{B \in p} \overline{h^{-1}[B]} \subseteq \tilde{h}^{-1}[\{p\}]$. (In fact, equality holds.) So $J(S) \cap \tilde{h}^{-1}[\{p\}]$ is a compact subsemigroup of βS so we may pick an idempotent $q \in J(S) \cap \tilde{h}^{-1}[\{p\}]$. Since $A \in p$, $h^{-1}[A] \in q$. $\square$

It remains for us to deal with the density related notions.

Theorem 2.7. *Let $h : S \rightarrow T$ be a surjective homomorphism and let $A \subseteq S$. Then $d_t(h[A]) \geq d_t(A)$. Consequently B_t is preserved by surjective homomorphisms.*

Proof. Let $\alpha = d_t(A)$. If $\alpha = 0$, the conclusion is trivial so assume we have $\alpha > \delta > 0$. We show that for every $G \in \mathcal{P}_f(T)$ there exists $t \in T$ such that $|G \cap h[A]t^{-1}| \geq \delta \cdot |G|$. Let $G \in \mathcal{P}_f(T)$ and let $F = \varphi[G]$. Pick $s \in S$ such that $|F \cap As^{-1}| \geq \delta \cdot |F| = \delta \cdot |G|$. We claim that $h[F \cap As^{-1}] \subseteq G \cap h[A]h(s)^{-1}$. That will suffice since h is injective on F . Let $x \in h[F \cap As^{-1}]$ and pick $y \in F \cap As^{-1}$ such that $x = h(y)$. Then $ys \in A$ so $h(y)h(s) \in h[A]$ and thus $x \in G \cap h[A]h(s)^{-1}$ as required. $\square$

We do not know whether B_t^* is preserved by surjective homomorphisms.

If S is not left amenable (so also does not satisfy SFC), then $B_a, B_f, D_a, D_f, B_a^*, B_f^*, D_a^*$, and D_f^* are not notions of largeness for subsets of S , so we need to phrase results about these notions differently.

If S is a left amenable semigroup, it is a fact, attributed to Day [1] by Klawe in [8] that any homomorphic image of S is also left amenable. (That fact also follows from [3, Lemma 3.2].)

In [8], Klawe noted that it was an open problem whether SFC was preserved by surjective homomorphisms. In [11, Proposition 4.8], Yang showed that Klawe's example, [8, Counterexample 3.5], which she showed satisfies SFC has a homomorphic image which does not. So SFC is not preserved by surjective homomorphisms.

Theorem 2.8. *Let $h : S \rightarrow T$ be a surjective homomorphism and assume that S is left amenable.*

- (1) If A is a B_a set in S , then $h[A]$ is a B_a set in T .
- (2) If A is a D_a set in S , then $h[A]$ is a D_a set in T .
- (3) If A is a B_a set in T , then $h^{-1}[A]$ is a B_a set in S .
- (4) If A is a D_a set in T , then $h^{-1}[A]$ is a D_a set in S .
- (5) If A is a B_a^* set in S , then $h[A]$ is a B_a^* set in T .
- (6) If A is a D_a^* set in S , then $h[A]$ is a D_a^* set in T .

Proof. (1) By [3, Theorem 3.5], $d^*(h[A]) \geq d^*(A)$.

(2) Pick an idempotent $p \in \Delta^*(S)$ such that $A \in p$. Then $h[A] \in \tilde{h}(p)$ and by [3, Theorem 3.6], $\tilde{h}(p) \in \Delta^*(T)$.

(3) By [3, Theorem 3.4], $d^*(A) = d^*(h^{-1}[A])$.

(4) Pick an idempotent $p \in \Delta^*(T)$ such that $A \in p$. By [3, Theorem 3.6], $\tilde{h}[\Delta^*(S)] = \Delta^*(T)$ so $\Delta^*(S) \cap \tilde{h}^{-1}[\{p\}]$ is a compact subsemigroup of βS so there is an idempotent $q \in \Delta^*(S) \cap \tilde{h}^{-1}[\{p\}]$ and $h^{-1}[A] \in q$.

(5) Let A be a B_a^* set in S . To see that $h[A]$ is a B_a^* set in T let C be a B_a set in T . By (3), $h^{-1}[C]$ is a B_a set in S so $A \cap h^{-1}[C] \neq \emptyset$ and consequently $h[A] \cap C \neq \emptyset$.

(6) Let A be a D_a^* set in S . To see that $h[A]$ is a D_a^* set in T let C be a D_a set in T . By (4), $h^{-1}[C]$ is a D_a set in S so $A \cap h^{-1}[C] \neq \emptyset$ and consequently $h[A] \cap C \neq \emptyset$. $\square$

Theorem 2.9. *Let $h : S \rightarrow T$ be a surjective homomorphism and assume that S and T satisfy SFC.*

- (1) If A is a B_f set in S , then $h[A]$ is a B_f set in T .
- (2) If A is a D_f set in S , then $h[A]$ is a D_f set in T .
- (3) If A is a B_f set in T , then $h^{-1}[A]$ is a B_f set in S .
- (4) If A is a D_f set in T , then $h^{-1}[A]$ is a D_f set in S .
- (5) If A is a B_f^* set in S , then $h[A]$ is a B_f^* set in T .
- (6) If A is a D_f^* set in S , then $h[A]$ is a D_f^* set in T .

Proof. By [3, Theorem 3.15], d^* and d agree on subsets of S and on subsets of T so $\Delta(S) = \Delta^*(S)$ and $\Delta(T) = \Delta^*(T)$ so Theorem 2.8 applies. $\square$

Question 2.10. *Are any or all of Q^* , B_t^* , and SIP_r^* for $r > 1$ preserved by surjective homomorphisms?*

3 Answers to Question 1

For all but four of the 40 notions that we have been considering we get affirmative answers to Question 1; that is, if A is large in S and B is large in T , must $A \times B$ be large in $S \times T$? For three of these four, namely B_t , P , and SIP_r for $r > 1$, we have not been able to determine the answer. (We are counting SIP_r and IP_r for $r > 1$ as one notion each.) For one notion, we know the answer is “no”. We present that proof first.

Theorem 3.1. *Let $n, m \in \mathbb{N} \setminus \{1\}$. Then $n\mathbb{N} \times m\mathbb{N}$ is not IP_{n+m-2}^* in $\mathbb{N} \times \mathbb{N}$ under addition.*

Proof. For $t \in \{1, 2, \dots, n-1\}$ let $x_t = (1, m)$ and for $t \in \{n, n+1, \dots, n+m-2\}$, let $x_t = (n, 1)$. We claim that $FS(\langle x_t \rangle_{t=1}^{n+m-2}) \cap (n\mathbb{N} \times m\mathbb{N}) = \emptyset$. To see this, let $\emptyset \neq F \subseteq \{1, 2, \dots, n+m-2\}$. Let $k = |F \cap \{1, 2, \dots, n-1\}|$ and let $l = |F \cap \{n, n+1, \dots, n+m-2\}|$ so $0 \leq k < n$, $0 \leq l < m$, and $k+l \geq 1$. Then $\sum_{t \in F} x_t = (k+ln, km+l) \notin (n\mathbb{N} \times m\mathbb{N})$. $\square$

Corollary 3.2. *Let $r \in \mathbb{N} \setminus \{1\}$. Then $r\mathbb{N}$ is IP_r^* in $(\mathbb{N}, +)$ but $r\mathbb{N} \times r\mathbb{N}$ is not IP_r^* in $\mathbb{N} \times \mathbb{N}$ under addition.*

Proof. To see that $r\mathbb{N}$ is an IP_r^* set, let $\langle x_t \rangle_{t=1}^r$ be a sequence in $\mathbb{N}$. For each $t \in \{1, 2, \dots, r\}$ pick $s_t \in \{0, 1, \dots, r-1\}$ such that $\sum_{i=1}^t x_i \equiv s_t \pmod{r}$. If any $s_t = 0$, we are done so assume that each $s_t \in \{1, 2, \dots, r-1\}$. Pick $j < t$ in $\{1, 2, \dots, r\}$ such that $s_j = s_t$. Then $\sum_{i=j+1}^t x_i \in r\mathbb{N}$.

By Theorem 3.1, $r\mathbb{N} \times r\mathbb{N}$ is not IP_{2r-2}^* . Since $2r-2 \geq r$, $r\mathbb{N} \times r\mathbb{N}$ is not IP_r^* . $\square$

We need the following lemma to deal with notions whose definitions or characterizations involve the smallest ideal.

Lemma 3.3. *Let I be a nonempty set, for $\alpha \in I$ let S_α be a semigroup, and let $\tilde{\iota} : \beta(\times_{\alpha \in I} S_\alpha) \rightarrow \times_{\alpha \in I} \beta S_\alpha$ be the continuous extension of the identity on $\times_{\alpha \in I} S_\alpha$. Then $\tilde{\iota}$ is a surjective homomorphism,*

$$\tilde{\iota}[K(\beta(\times_{\alpha \in I} S_\alpha))] = K[\times_{\alpha \in I} \beta S_\alpha] = \times_{\alpha \in I} K(\beta S_\alpha)$$

and

$$\tilde{\iota}[clK(\beta(\times_{\alpha \in I} S_\alpha))] = clK[\times_{\alpha \in I} \beta S_\alpha] = \times_{\alpha \in I} clK(\beta S_\alpha)$$

Proof. By [7, Corollary 4.22] $\tilde{\iota}$ is a homomorphism and by elementary topology $\tilde{\iota}[\beta(\times_{\alpha \in I} S_\alpha)] = \times_{\alpha \in I} \beta S_\alpha$. By [7, Exercise 1.7.3], $\tilde{\iota}[K(\beta(\times_{\alpha \in I} S_\alpha))] = K[\times_{\alpha \in I} \beta S_\alpha]$. By [7, Theorem 2.23], $K[\times_{\alpha \in I} \beta S_\alpha] = \times_{\alpha \in I} K(\beta S_\alpha)$.

The second displayed equalities follow by elementary topology. $\square$

We will also need the fact that P is a partition regular notion.

Lemma 3.4. *Let A be a P set in a semigroup S and let $A = A_1 \cup A_2$. Then either A_1 or A_2 is a P set.*

Proof. Suppose not and pick $k \in \mathbb{N}$ such that for $i \in \{1, 2\}$ there do not exist $m \in \mathbb{N}$, $a \in S^{m+1}$, and $d \in S$ such that

$$\{a(1)d^t a(2) \cdots a(m)d^t a(m+1) : t \in \{1, 2, \dots, k\}\} \subseteq A_i.$$

Pick by van der Waerden's Theorem some $l \in \mathbb{N}$ such that whenever $\{1, 2, \dots, l\} = B_1 \cup B_2$, there are $i \in \{1, 2\}$ and $f, g \in \{1, 2, \dots, l\}$ such that $\{f, f+g, \dots, f+kg\} \subseteq B_i$.

Pick $m \in \mathbb{N}$, $a \in S^{m+1}$, and $c \in S$ such that

$$\{a(1)c^t a(2) \cdots a(m)c^t a(m+1) : t \in \{1, 2, \dots, l\}\} \subseteq A.$$

For $i \in \{1, 2\}$ let $B_i = \{t \in \{1, 2, \dots, l\} : a(1)c^t a(2) \cdots a(m)c^t a(m+1) \in A_i\}$. Pick $i \in \{1, 2\}$ and $f, g \in \{1, 2, \dots, l\}$ such that $\{f, f+g, \dots, f+kg\} \subseteq B_i$. Let $d = c^g$, for $j \in \{1, 2, \dots, m\}$ let $b(j) = a(j)c^f$, and let $b(m+1) = a(m+1)$. Then $\{b(1)d^t b(2) \cdots b(m)d^t b(m+1) : t \in \{1, 2, \dots, k\}\} \subseteq A_i$, a contradiction. $\square$

We will need a characterization of Q^* .

Lemma 3.5. *Let A be a subset of a semigroup S . Then A is a Q^* set if and only if, for every sequence $\langle x_n \rangle_{n=1}^\infty$ in S , there is an infinite subset B of $\mathbb{N}$ such that whenever $m < n$ in B , $x_n \notin x_m(S \setminus A)$.*

Proof. Necessity. Assume that $S \setminus A$ is not a Q set and let $\langle x_n \rangle_{n=1}^\infty$ be a sequence in S . Choose by Ramsey's Theorem an infinite subset B of $\mathbb{N}$ such that either

- (1) for all $m < n$ in B , $x_n \in x_m(S \setminus A)$ or
- (2) for all $m < n$ in B , $x_n \notin x_m(S \setminus A)$.

If (1) held, the sequence $\langle x_n \rangle_{n \in B}$ would establish that $S \setminus A$ is a Q set.

Sufficiency. Assume that for every sequence $\langle x_n \rangle_{n=1}^\infty$ in S , there is an infinite subset B of $\mathbb{N}$ such that whenever $m < n$ in B , $x_n \notin x_m(S \setminus A)$. Suppose that $S \setminus A$ is a Q set and choose a sequence $\langle x_n \rangle_{n=1}^\infty$ in S such that for all $m < n$ in $\mathbb{N}$, $x_n \in x_m(S \setminus A)$. No set B can be chosen. $\square$

As in Theorem 2.4 we shall see that for most of the notions with simple elementary definitions, Question 1 has an affirmative answer arrived at in a reasonably simple fashion.

Theorem 3.6. *Each of Q , IP , IP_r for $r > 1$, $IP_{<\omega}$, thick, WT , J , PS , SPS , SIP_r^* for $r > 1$, IP^* , $IP_{<\omega}^*$, syndetic, WT^* , SPS^* , PS^* , J^* , P^* , and Q^* has an affirmative answer to Question 1.*

Proof. The proofs for Q , IP , IP_r for $r > 1$, $IP_{<\omega}$, thick, and WT are very easy, and we omit them.

The proof for J is [6, Theorem 2.11].

For PS , assume that A is PS in S and B is PS in T . Pick $G \in \mathcal{P}_f(S)$ and $H \in \mathcal{P}_f(T)$ such that $\bigcup_{s \in G} s^{-1}A$ is thick in S and $\bigcup_{t \in H} t^{-1}B$ is thick in

T . Then $(\bigcup_{s \in G} s^{-1}A) \times (\bigcup_{t \in H} t^{-1}B)$ is thick in $S \times T$ (using the fact that the answer is affirmative for thick) and

$$(\bigcup_{s \in G} s^{-1}A) \times (\bigcup_{t \in H} t^{-1}B) \subseteq \bigcup_{(s,t) \in G \times H} (s,t)^{-1}(A \times B).$$

For SPS proceed as above replacing $s^{-1}A$ and $t^{-1}B$ by As^{-1} and Bt^{-1} .

For SIP_r^* , assume that $r > 1$, A is SIP_r^* in S , and B is SIP_r^* in T . Since $S \setminus A$ is not SIP_r , pick $m \in \mathbb{N}$ and $J_1, J_2, \dots, J_m$ such that $S \setminus A = \bigcup_{i=1}^m J_i$ and no J_i is IP_r . Since $T \setminus B$ is not SIP_r , pick $k \in \mathbb{N}$ and $L_1, L_2, \dots, L_k$ such that $T \setminus B = \bigcup_{i=1}^k L_i$ and no K_i is IP_r . Note that if $i \in \{1, 2, \dots, m\}$, then $J_i \times T$ is not an IP_r set and if $i \in \{1, 2, \dots, k\}$, then $S \times L_i$ is not an IP_r set. Thus $(S \times T) \setminus (A \times B) = \bigcup_{i=1}^m (J_i \times T) \cup \bigcup_{i=1}^k (S \times L_i)$ so $(S \times T) \setminus (A \times B)$ is not an SIP_r set so $A \times B$ is an SIP_r^* set in $S \times T$.

For IP^* we use the fact from [7, Theorem 16.6] that a subset A of a semigroup S is an IP^* set if and only if A is a member of every idempotent in βS . Assume A is IP^* in S , B is IP^* in T , and let p be an idempotent in $\beta(S \times T)$. By Lemma 3.3, $\tilde{\iota}$ is a homomorphism onto $\beta S \times \beta T$ so there exist an idempotent $q \in \beta S$ and an idempotent $r \in \beta T$ such that $\tilde{\iota}(p) = (q, r)$. Then $A \in q$ and $B \in r$ so $\bar{A} \times \bar{B}$ is a neighborhood of (q, r) so there exists $C \in p$ such that $\tilde{\iota}[C] \subseteq \bar{A} \times \bar{B}$. Then $C \subseteq A \times B$ so $A \times B \in p$.

For $IP_{<\omega}^*$ assume that A and B are $IP_{<\omega}^*$ so that $S \setminus A$ is not $IP_{<\omega}$ and $T \setminus B$ is not $IP_{<\omega}$. Pick n and m in $\mathbb{N}$ such that $S \setminus A$ is not IP_n and $T \setminus B$ is not IP_m . Let $r = \max\{n, m\}$. Then $S \setminus A$ is not SIP_r and $T \setminus B$ is not SIP_r so A and B are SIP_r^* so $A \times B$ is SIP_r^* and therefore $IP_{<\omega}^*$.

For syndetic, we use the characterization that A is syndetic in S if and only if there exists $G \in \mathcal{P}_f(S)$ such that $S = \bigcup_{s \in G} s^{-1}A$. Given $G \in \mathcal{P}_f(S)$ and $H \in \mathcal{P}_f(T)$ such that $S = \bigcup_{s \in G} s^{-1}A$ and $T = \bigcup_{t \in H} t^{-1}B$, then $S \times T = (\bigcup_{s \in G} s^{-1}A) \times (\bigcup_{t \in H} t^{-1}B) \subseteq \bigcup_{(s,t) \in G \times H} (s,t)^{-1}(A \times B)$.

For WT^* assume that A is WT^* in S and B is WT^* in T . Then

$$\begin{aligned} &(\forall s \in S)(\exists F \in \mathcal{P}_f(S))(\forall x \in S)(\exists y \in F)(syx \in A) \text{ and} \\ &(\forall t \in T)(\exists G \in \mathcal{P}_f(T))(\forall z \in T)(\exists u \in G)(tuz \in B). \end{aligned}$$

To see that $A \times B$ is WT^* in $S \times T$, let $(s, t) \in S \times T$. Pick $F \in \mathcal{P}_f(S)$ such that for all $x \in S$ there exists $y \in F$ such that $syx \in A$ and pick $G \in \mathcal{P}_f(T)$ such that for all $z \in T$ there exists $u \in G$ such that $tuz \in B$. Let $H = F \times G$. To see that for all $(x, z) \in S \times T$, there exists $(y, u) \in H$ such that $(s, t)(y, u)(x, z) \in A \times B$, let $(x, z) \in S \times T$. Pick $y \in F$ and $u \in G$ such that $syx \in A$ and $tuz \in B$.

For SPS^* assume that A is SPS^* in S and B is SPS^* in T . We show that $(S \times T) \setminus (A \times B)$ is not SPS in $S \times T$. So let $H \in \mathcal{P}_f(S \times T)$. We will show that $\bigcup_{(s,t) \in H} ((S \times T) \setminus (A \times B))(s, t)^{-1}$ is not thick in $S \times T$. Since $S \setminus A$ is not SPS in S and $T \setminus B$ is not SPS in T , $\bigcup_{s \in \pi_1[H]} (S \setminus A)s^{-1}$ is not thick in S and $\bigcup_{t \in \pi_2[H]} (T \setminus B)t^{-1}$ is not thick in T so pick $F \in \mathcal{P}_f(S)$ and $G \in \mathcal{P}_f(T)$ such that for all $x \in S$ and all $y \in T$,

$$Fx \setminus \bigcup_{s \in \pi_1[H]} (S \setminus A)s^{-1} \neq \emptyset \text{ and } Gy \setminus \bigcup_{t \in \pi_2[H]} (T \setminus B)t^{-1} \neq \emptyset.$$

We claim that for all $(x, y) \in S \times T$, $(F \times G)(x, y) \setminus \bigcup_{(s, t) \in H} ((S \times T) \setminus (A \times B)) \neq \emptyset$. To see this let $(x, y) \in S \times T$. Pick $z \in F$ such that for all $s \in \pi_1[H]$, $zxs \in A$ and pick $w \in G$ such that for all $t \in \pi_2[H]$, $wyt \in B$. Then for all $(s, t) \in H$, $(z, w)(x, y)(s, t) \in (A \times B)$.

For PS^* one can use a nearly identical proof as for SPS^* , but it is simpler to use the fact, already noted, that A is PS in S if and only if $\overline{A} \cap K(\beta S) \neq \emptyset$ so that A is PS^* in S if and only if $K(\beta S) \subseteq \overline{A}$. So assume that A is PS^* in S and B is PS^* in T . We show that $K(\beta(S \times T)) \subseteq \overline{A \times B}$. So let $p \in K(\beta(S \times T))$. By Lemma 3.3 $\tilde{\iota}(p) \in K(\beta S) \times K(\beta T) \subseteq \overline{A} \times \overline{B}$. Pick $C \in p$ such that $\tilde{\iota}[C] \subseteq \overline{A} \times \overline{B}$. Then $C \subseteq A \times B$ so $A \times B \in p$.

For J^* , assume that A is J^* in S and B is J^* in T . Suppose that $A \times B$ is not J^* in $S \times T$ so that $(S \times T) \setminus (A \times B)$ is a J set in $S \times T$. Since $(S \times T) \setminus (A \times B) \subseteq ((S \setminus A) \times T) \cup (S \times (T \setminus B))$ and by [7, Theorem 14.14.6], J sets are partition regular, we may presume that $((S \setminus A) \times T)$ is a J set in $S \times T$. $(S \setminus A)$ is not a J set in S so pick $F \in \mathcal{P}_f(\mathbb{N}S)$ such that for all $m \in \mathbb{N}$, all $a \in S^{m+1}$, and all $t \in \mathcal{J}_m$, there exists $f \in F$ such that

$$a(1)f(t(1))a(2) \cdots a(m)f(t(m))a(m+1) \in A.$$

Pick $x \in T$ and for $f \in F$ define $h_f \in \mathbb{N}(S \times T)$ by $h_f(n) = (f(n), x)$. Then $\{h_f : f \in F\} \in \mathcal{P}_f(\mathbb{N}(S \times T))$ so pick $m \in \mathbb{N}$, $b \in (S \times T)^{m+1}$, and $t \in \mathcal{J}_m$ such that for every $f \in F$,

$$b(1)h_f(t(1))b(2) \cdots b(m)h_f(t(m))b(m+1) \in (S \setminus A) \times T.$$

Define $a \in S^{m+1}$ by, for $j \in \{1, 2, \dots, m+1\}$, $a(j) = \pi_1(b(j))$. Pick $f \in F$ such that $a(1)f(t(1))a(2) \cdots a(m)f(t(m))a(m+1) \in A$. But

$$a(1)f(t(1))a(2) \cdots a(m)f(t(m))a(m+1)$$

is the first coordinate of $b(1)h_f(t(1))b(2) \cdots b(m)h_f(t(m))b(m+1)$ which is in $S \setminus A$, a contradiction.

The proof for P^* is very similar to the proof for J^* , using the fact from Lemma 3.4 that the property P is partition regular.

For Q^* assume that A is Q^* in S and B is Q^* in T . We use Lemma 3.5. Let $\langle (x_n, y_n) \rangle_{n=1}^\infty$ be a sequence in $S \times T$. Since A is Q^* in S , pick an infinite subset C of $\mathbb{N}$ such that for all $m < n$ in C , $x_n \notin x_m(S \setminus A)$. Then $\langle y_n \rangle_{n \in C}$ is a sequence in T and B is a Q^* subset of T , so pick an infinite subset D of C such that for all $m < n$ in D , $y_n \notin y_m(T \setminus B)$. Then for all $m < n$ in D , $(x_n, y_n) \notin (x_m, y_m)((S \times T) \setminus (A \times B))$. $\square$

For our next group of properties we need the following theorem. We note that by [8, Proposition 3.4 and Proposition 4.4] $S \times T$ is left amenable if S and T are, and $S \times T$ satisfies SFC if S and T do.

Theorem 3.7. *Let S and T be semigroups and let $\tilde{\iota} : \beta(S \times T) \rightarrow \beta S \times \beta T$ be the continuous extension of the identity on $S \times T$.*

(1) $\tilde{\iota}[J(S \times T)] = J(S) \times J(T)$.

(2) if S and T are left amenable, then $\tilde{\iota}[\Delta^*(S \times T)] = \Delta^*(S) \times \Delta^*(T)$.

(3) if S and T satisfy SFC, then $\tilde{\iota}[\Delta(S \times T)] = \Delta(S) \times \Delta(T)$.

Proof. (1) To see that $J(S) \times J(T) \subseteq \tilde{\iota}[J(S \times T)]$, let $(p, q) \in J(S) \times J(T)$. We claim that $\{\overline{A \times B} \cap J(S \times T) : A \in p \text{ and } B \in q\}$ has the finite intersection property for which it suffices to let $A \in p$ and $B \in q$ and show that $\overline{A \times B} \cap J(S \times T) \neq \emptyset$. So let $A \in p$ and $B \in q$. By Theorem 3.6 $A \times B$ is a J set so by [7, Theorem 14.14.7], $\overline{A \times B} \cap J(S \times T) \neq \emptyset$. Pick $r \in \bigcap \{\overline{A \times B} \cap J(S \times T) : A \in p \text{ and } B \in q\}$. We claim that $\tilde{\iota}(r) = (p, q)$. Suppose $\tilde{\iota}(r) = (u, v)$ and, say, $u \neq p$. Pick $A \in p \setminus u$ and let $B = T$. Pick $C \in r$ such that $\tilde{\iota}[C] \subseteq S \setminus A \times B$. Then $C \subseteq (S \setminus A) \times B$ so $(S \setminus A) \times B \in r$ while $(A \times B) \in r$, a contradiction.

For the other inclusion, let $r \in J(S \times T)$ and let $\tilde{\iota}(r) = (p, q)$. To see that $p \in J(S)$ let $A \in p$ and let $B = T$. Pick $C \in r$ such that $\tilde{\iota}[C] \subseteq \overline{A \times B}$. Then $C \subseteq A \times B$ so $\pi_1[C] \subseteq A$. Since $r \in J(S \times T)$, C is a J set in $S \times T$ so by Theorem 2.4 $\pi_1[C]$ is a J set in S so $p \in J(S)$.

(2) To see that $\Delta^*(S) \times \Delta^*(T) \subseteq \tilde{\iota}[\Delta^*(S \times T)]$, let $(p, q) \in \Delta^*(S) \times \Delta^*(T)$.

Let $\mathcal{R} = \{C \subseteq S \times T : d^*(C) > 0\}$. Then $\mathcal{R}$ is partition regular. Let $\mathcal{A} = \{A \times B : A \in p \text{ and } B \in q\}$. By [3, Theorem 6.2], $\mathcal{A} \subseteq \mathcal{R}$. By [7, Theorem 3.11] pick $s \in \beta(S \times T)$ such that $\mathcal{A} \subseteq s \subseteq \mathcal{R}$. Then $s \in \Delta^*(S \times T)$. Let $(u, v) = \tilde{\iota}(s)$. We claim that $(p, q) = (u, v)$. To see that $u = p$, suppose instead that $u \neq p$ and pick $A \in p \setminus u$. Then $(u, v) \in \overline{S \setminus A \times T}$ so pick $C \in s$ such that $\tilde{\iota}[C] \subseteq \overline{S \setminus A \times T}$, so $C \cap (A \times T) = \emptyset$, a contradiction. Similarly, $v = q$.

To see that $\tilde{\iota}[\Delta^*(S \times T)] \subseteq \Delta^*(S) \times \Delta^*(T)$, let $p \in \Delta^*(S \times T)$ and let $\tilde{\iota}(p) = (q, r)$. To see that $q \in \Delta^*(S)$ and $r \in \Delta^*(T)$, let $A \in q$ and $B \in r$. Then $\tilde{\iota}(p) \in \overline{A \times B}$ so pick $C \in p$ such that $\tilde{\iota}[C] \subseteq \overline{A \times B}$. Then $C \subseteq A \times B$ so $d^*(A \times B) > 0$ and by [3, Theorem 6.2] $d^*(A \times B) = d^*(A)d^*(B)$ so $d^*(A) > 0$ and $d^*(B) > 0$.

(3) By [3, Theorem 3.15], since S , T , and $S \times T$ satisfy SFC, $\Delta(S) = \Delta^*(S)$, $\Delta(T) = \Delta^*(T)$, and $\Delta(S \times T) = \Delta^*(S \times T)$, so this follows from (2). $\square$

Theorem 3.8. *Each of C , B_a , B_f , D_a , D_f , B_t^* , C^* , B_a^* , D_a^* , B_f^* , and D_f^* has an affirmative answer for Question 1.*

Proof. The result for C is [6, Theorem 2.16].

For B_a , assume that A is a B_a set in S and B is a B_a set in T . Then S and T are left amenable and by [3, Theorem 6.2] $d^*(A \times B) = d^*(A)d^*(B)$ so $A \times B$ is a B_a set in $S \times T$.

For B_f , assume that A is a B_f set in S and B is a B_f set in T . Then S and T satisfy SFC and by [3, Theorem 3.15], d and d^* agree on subsets of S , T , and $S \times T$ so the conclusion follows from the result for B_a .

For D_a assume that A is D_a in S and B is D_a in T . Then S and T are left amenable. Pick an idempotent $p \in \overline{A} \cap \Delta^*(S)$ and pick an idempotent $q \in \overline{B} \cap \Delta^*(T)$. By Theorem 3.7(2), $(p, q) \in \tilde{\iota}[\Delta^*(S \times T)]$. Since $\tilde{\iota}$ is a homomorphism, $\Delta^*(S \times T) \cap \tilde{\iota}^{-1}[\{(p, q)\}]$ is a compact semigroup so pick an idempotent $r \in \Delta^*(S \times T)$ such that $\tilde{\iota}(r) = (p, q)$. Then $A \times B \in r$.

The proof for D_f is similar using Theorem 3.7(3).

For B_t^* , assume that A is B_t^* in S and B is B_t^* in T so that $S \setminus A$ is not B_t in S and $T \setminus B$ is not B_t in T . Suppose that $A \times B$ is not B_t^* in $S \times T$, so that $(S \times T) \setminus (A \times B)$ is B_t in $S \times T$. Let $\alpha = d_t((S \times T) \setminus (A \times B))$. Since $d_t(S \setminus A) = 0$, pick $F \in \mathcal{P}_f(S)$ such that

$$(\forall s \in S)(|F \cap (S \setminus A)s^{-1}| < \frac{\alpha}{3} \cdot |F|)$$

and pick $G \in \mathcal{P}_f(T)$ such that

$$(\forall t \in T)(|G \cap (T \setminus B)t^{-1}| < \frac{\alpha}{3} \cdot |G|).$$

Pick $(s, t) \in S \times T$ such that

$$|(F \times G) \cap ((S \times T) \setminus (A \times B))(s, t)^{-1}| \geq \frac{2\alpha}{3} \cdot |F \times G|.$$

Now $(F \times G) \cap ((S \times T) \setminus (A \times B))(s, t)^{-1} \subseteq ((F \cap (S \setminus A)s^{-1}) \times G) \cup (F \times (G \cap (T \setminus B)t^{-1}))$ so

$$|(F \times G) \cap ((S \times T) \setminus (A \times B))(s, t)^{-1}| \leq \frac{\alpha}{3} \cdot |F| \cdot |G| + \frac{\alpha}{3} \cdot |F| \cdot |G| = \frac{2\alpha}{3} |F \times G|,$$

a contradiction.

For C^* assume that A is a C^* set in S and B is a C^* set in T . Then A is a member of every idempotent in $J(S)$. (Since $S \setminus A$ is not a C set, if p is an idempotent in $J(S)$, then $p \notin S \setminus A$.) And B is a member of every idempotent in $J(T)$. We show that $A \times B$ is a member of every idempotent in $J(S \times T)$. So let p be an idempotent in $J(S \times T)$. Let $\tilde{\iota}(p) = (q, r)$. By Theorem 3.7(1), (q, r) is an idempotent in $J(S) \times J(T)$ so $A \in q$ and $B \in r$ so $A \times B \in p$.

For B_a^* , assume that A is a B_a^* set in S and B is a B_a^* set in T . Then S and T are left amenable. By Lemma 1.9, $d^*(S \setminus A) = d^*(T \setminus B) = 0$. We claim that $d^*((S \times T) \setminus (A \times B)) = 0$. Suppose instead that $d^*((S \times T) \setminus (A \times B)) > 0$. Then $(S \times T) \setminus (A \times B) \subseteq ((S \setminus A) \times T) \cup (S \times (T \setminus B))$ so without loss of generality $d^*((S \setminus A) \times T) > 0$ and $d^*((S \setminus A) \times T) = d^*(S \setminus A) \cdot d^*(T) = d^*(S \setminus A)$, a contradiction.

For D_a^* , assume that A is a D_a^* set in S and B is a D_a^* set in T . Then by Lemma 1.9, A is a member of every idempotent in $\Delta^*(S)$ and B is a member of every idempotent in $\Delta^*(T)$. If p is an idempotent in $\Delta^*(S \times T)$, then by Theorem 3.7(2), $\tilde{\iota}(p)$ is an idempotent in $\Delta^*(S) \times \Delta^*(T)$ so $A \times B \in p$.

Since semigroups that satisfy SFC are left amenable, the conclusions for B_f^* and D_f^* follow from the conclusions for B_a^* and D_a^* . $\square$

To deal with SC we need the next lemma.

Lemma 3.9. *Let S and T be semigroups, let $\tilde{\iota} : \beta(S \times T) \rightarrow \beta S \times \beta T$ be the continuous extension of the identity on $S \times T$, and let M be a minimal left ideal of $\beta(S \times T)$. Let $L_1 = \pi_1 \circ \tilde{\iota}[M]$ and let $L_2 = \pi_2 \circ \tilde{\iota}[M]$. Then L_1 is a minimal left ideal of βS , L_2 is a minimal left ideal of βT , $\tilde{\iota}[M]$ is a minimal left ideal of $\beta S \times \beta T$, and $\tilde{\iota}[M] = L_1 \times L_2$.*

Proof. That L_1 is a minimal left ideal of βS , L_2 is a minimal left ideal of βT , and $\tilde{\iota}[M]$ is a minimal left ideal of $\beta S \times \beta T$ follows from [7, Exercise 1.7.3]. We claim that $L_1 \times L_2$ is a minimal left ideal of $\beta S \times \beta T$, for which it suffices by [7, Lemma 1.52(a)] to note that if $(x, y) \in L_1 \times L_2$, then $(L_1 \times L_2)(x, y) = L_1 \times L_2$. Since $L_1 \times L_2$ and $\tilde{\iota}[M]$ are minimal left ideals of $\beta S \times \beta T$ with nonempty intersection, they are equal. $\square$

Theorem 3.10. *Each of central, QC, SC, central*, QC*, and SC* has an affirmative answer for Question 1.*

Proof. The proofs for central, QC, central* and QC* are easy consequences of Lemma 3.3 .

For SC assume that A is SC in S and B is SC in T . Let M be a minimal left ideal of $\beta(S \times T)$, let $L_1 = \pi_1 \circ \tilde{\iota}[M]$ and let $L_2 = \pi_2 \circ \tilde{\iota}[M]$. By Lemma 3.9 L_1 is a minimal left ideal of βS , L_2 is a minimal left ideal of βT , and $\tilde{\iota}[M] = L_1 \times L_2$. Pick an idempotent $p \in L_1 \cap \overline{A}$ and an idempotent $q \in L_2 \cap \overline{B}$. Then $\tilde{\iota}^{-1}[\{(p, q)\}] \cap M$ is a subsemigroup of M so pick an idempotent $r \in \tilde{\iota}^{-1}[\{(p, q)\}] \cap M$. Then $A \times B \in r$.

For SC* assume that A is SC* in S and B is SC* in T . Pick a minimal left ideal L of βS and a minimal left ideal K of βT such that A is a member of every idempotent in L and B is a member of every idempotent in K . Then $\tilde{\iota}^{-1}[L \times K]$ is a left ideal of $\beta(S \times T)$. Pick a minnimal left ideal M of $\beta(S \times T)$ such that $M \subseteq \tilde{\iota}^{-1}[L \times K]$. If r is an idempotent in M , then $\tilde{\iota}(r) \in L \times K$ so $A \times B \in r$. $\square$

Question 3.11. *For any or all of B_t , P , or SIP_r for $r > 1$, is the answer to Question 1 “yes”?*

4 Answers to Question 2 and Question 4

Because projections from Cartesian products are surjective homomorphisms it is immediate that if R is a property which is preserved by surjective homomorphisms, then the answer to Question 4, and therefore to Question 2, is affirmative.

Theorem 4.1. *Let R be any of the notions Q , IP , IP_r and SIP_r for $r > 1$, $IP_{<\omega}$, thick, WT, PS, SPS, J , P , central, SC, QC, C , B_t , B_a , D_a , B_f , D_f , IP^* , IP_r^* for $r > 1$, $IP_{<\omega}^*$, thick* = syndetic, WT*, PS*, SPS*, J^* , P^* , central*, SC*, QC*, C^* , B_a^* , D_a^* , B_f^* , and D_f^* . Assume that I is a set and for each $\alpha \in I$ S_α is a semigroup and $A_\alpha \subseteq S_\alpha$. If $\times_{\alpha \in I} A_\alpha$ is an R set in $\times_{\alpha \in I} S_\alpha$ and $\delta \in I$, then A_δ is an R set in S_δ .*

Proof. Assume that $\times_{\alpha \in I} A_\alpha$ is an R set in $\times_{\alpha \in I} S_\alpha$ and $\delta \in I$.

If R is any of B_a , D_a , B_a^* , or D_a^* , then $\times_{\alpha \in I} S_\alpha$ is left amenable. Since $\times_{\alpha \in I} S_\alpha \approx (\times_{\alpha \in I \setminus \{\delta\}} S_\alpha) \times S_\delta$ we have by [8, Proposition 3.8] that S_δ is left amenable so the conclusion follows from Theorem 2.8 (1), (2), (5), or (6).

If R is any of B_f , D_f , B_f^* , or D_f^* , then $\times_{\alpha \in I} S_\alpha$ satisfies SFC. Since $\times_{\alpha \in I} S_\alpha \approx (\times_{\alpha \in I \setminus \{\delta\}} S_\alpha) \times S_\delta$ we have by [8, Proposition 4.7] that S_δ satisfies SFC so the conclusion follows from Theorem 2.9 (1), (2), (5), or (6).

If R is any of the other listed notions, then the conclusion follows from Theorem 2.4, Theorem 2.6, or Theorem 2.7. $\square$

The next lemma says that if R is a notion of largeness, then an affirmative answer to Question 1 for R implies an affirmative answer to Question 4 for R^* .

Theorem 4.2. *Let R be a notion of largeness and assume that whenever S and T are semigroups, A is an R set in S , and B is an R set in T , then $A \times B$ is an R set in $S \times T$. Then whenever I is a set, for each $\alpha \in I$, S_α is a semigroup and $A_\alpha \subseteq S_\alpha$, $\times_{\alpha \in I} A_\alpha$ is an R^* set in $\times_{\alpha \in I} S_\alpha$, and $\delta \in I$, then A_δ is an R^* set in S_δ .*

Proof. We use the contrapositive of the assumption. That is, if $A \times B$ is not an R set in $S \times T$, then either A is not an R set in S or B is not an R set in T .

Assume that $\times_{\alpha \in I} A_\alpha$ is an R^* set in $\times_{\alpha \in I} S_\alpha$ and let $\delta \in I$. Then $\times_{\alpha \in I} S_\alpha \setminus \times_{\alpha \in I} A_\alpha$ is not an R set in $\times_{\alpha \in I} S_\alpha$.

For $\alpha \in I$, let $T_\alpha = \begin{cases} S_\alpha & \text{if } \alpha \neq \delta \\ S_\delta \setminus A_\delta & \text{if } \alpha = \delta. \end{cases}$

Then $\times_{\alpha \in I} T_\alpha \subseteq \times_{\alpha \in I} S_\alpha \setminus \times_{\alpha \in I} A_\alpha$ so $\times_{\alpha \in I} T_\alpha$ is not an R set in $\times_{\alpha \in I} S_\alpha$. Now $\times_{\alpha \in I} T_\alpha \approx (\times_{\alpha \in I \setminus \{\delta\}} S_\alpha) \times (S_\delta \setminus A_\delta)$. Since $\times_{\alpha \in I \setminus \{\delta\}} S_\alpha$ is an R set in itself, we must have that $S_\delta \setminus A_\delta$ is not a R set in S_δ so that A_δ is an R^* set in S_δ . $\square$

Corollary 4.3. *Assume that I is a set and for each $\alpha \in I$ S_α is a semigroup and $A_\alpha \subseteq S_\alpha$. If $\times_{\alpha \in I} A_\alpha$ is a Q^* set in $\times_{\alpha \in I} S_\alpha$ and $\delta \in I$, then A_δ is a Q^* set in S_δ .*

Proof. By Theorem 3.6, whenever S and T are semigroups, A is a Q set in S , and B is a Q set in T , then $A \times B$ is a Q set in $S \times T$. So Theorem 4.2 applies. $\square$

We do not have any example of a property with an affirmative answer to Question 2 and a negative answer to Question 4.

Question 4.4. *If $\times_{\alpha \in I} A_\alpha$ is B_t^* , respectively SIP_r^* for $r > 1$, in $\times_{\alpha \in I} S_\alpha$ and $\delta \in I$, must A_δ be B_t^* , respectively SIP_r^* , in S_δ ? If not, what about the corresponding question for $A \times B$ in $S \times T$?*

5 Answers to Question 3

We have negative answers to Question 3 for 26 of our 40 notions. That is we produce I , S_α , and A_α for $\alpha \in I$ such that each A_α is large in S_α and $\times_{\alpha \in I} A_\alpha$ is not large in $\times_{\alpha \in I} S_\alpha$. For all of our examples $I = \mathbb{N}$ and for 21 of them $S_n = \mathbb{N}$ for each $n \in \mathbb{N}$.

We need the following result from [6].

Lemma 5.1. *Let I be a set, for each $\alpha \in I$, let S_α be a semigroup, let $A_\alpha \subseteq S_\alpha$, and let $J = \{\alpha \in I : A_\alpha \text{ is not weakly thick in } S_\alpha\}$. If J is infinite, then $\times_{\alpha \in I} A_\alpha$ is not piecewise syndetic in $\times_{\alpha \in I} S_\alpha$.*

Proof. [6, Lemma 3.6]. $\square$

Theorem 5.2. *Let R be a property which may be possessed by subsets of a semigroup. If $2\mathbb{N}$ is an R set in $(\mathbb{N}, +)$ and every R set in $\times_{n \in \mathbb{N}} \mathbb{N}$ is piecewise syndetic, then $\times_{n \in \mathbb{N}} 2\mathbb{N}$ is not an R set in $\times_{n \in \mathbb{N}} \mathbb{N}$. In particular, the answer to Question 3 is negative for each of IP_r^* and SIP_r^* for $r > 1$, $IP_{<\omega}^*$, Q^* , IP^* , C^* , D_a^* , D_f^* , QC^* , central*, SC^* , SC , SPS , syndetic, WT^* , central, QC , and PS .*

Proof. Note that $2\mathbb{N}$ is not weakly thick in $\mathbb{N}$. From Figure 1, one can see that each of the listed notions implies PS and so by Lemma 5.1 $\times_{n \in \mathbb{N}} 2\mathbb{N}$ does not satisfy any of the listed properties in $\times_{n \in \mathbb{N}} \mathbb{N}$. To see that $2\mathbb{N}$ satisfies each of the listed properties it suffices, again using Figure 1, to note that $2\mathbb{N}$ is IP_2^* (which is trivial), $2\mathbb{N}$ is SPS , and $2\mathbb{N}$ is a Q^* set in $\mathbb{N}$. To see the latter, suppose that $\mathbb{N} \setminus 2\mathbb{N}$ is a Q set. Pick a sequence $\langle x_n \rangle_{n=1}^\infty$ in $\mathbb{N}$ such that for all $m < n$, $x_n \in x_m + (\mathbb{N} \setminus 2\mathbb{N})$. Choosing $m \equiv n \pmod{2}$ we get a contradiction. $\square$

Theorem 5.3. *Let R be a property which may be possessed by subsets of a semigroup. If $n\mathbb{N}$ is an R set in $(\mathbb{N}, +)$ for each $n \in \mathbb{N}$ and every R set in $\times_{n \in \mathbb{N}} \mathbb{N}$ is a J set, then $\times_{n \in \mathbb{N}} n\mathbb{N}$ is not an R set in $\times_{n \in \mathbb{N}} \mathbb{N}$. In particular, the answer to Question 3 is negative for C and J .*

Proof. By [7, Lemma 5.19.1], every $n\mathbb{N}$ is an IP^* set in $(\mathbb{N}, +)$, hence a C set and a J set. We need to show that $\times_{n \in \mathbb{N}} n\mathbb{N}$ is not a J set in $\times_{n \in \mathbb{N}} \mathbb{N}$.

Let $S = \times_{n \in \mathbb{N}} \mathbb{N}$ and let $A = \times_{n \in \mathbb{N}} n\mathbb{N}$. Define $\vec{f}$ and $\vec{g}$ in ${}^{\mathbb{N}}S$ by, for $t, n \in \mathbb{N}$, $f(t)_n = n$ and $g(t)_n = 1$. Suppose that we have $\vec{a} \in S$ and $H \in \mathcal{P}_f(\mathbb{N})$ such that $\vec{a} + \sum_{t \in H} \vec{f}(t) \in A$ and $\vec{a} + \sum_{t \in H} \vec{g}(t) \in A$. Let $k = |H|$ and pick $n > k$. Then $a_n + \sum_{t \in H} f(t)_n \in n\mathbb{N}$ and $a_n + \sum_{t \in H} g(t)_n \in n\mathbb{N}$ so $a_n + kn \in n\mathbb{N}$ and $a_n + k \in n\mathbb{N}$ so n divides $kn - k$ which is impossible. $\square$

We need the following result of M. Day.

Theorem 5.4. *Let G be the free group on the letters a and b , let $G_0 = G$ and for $n \in \omega$, let $G_{n+1} = [G, G_n]$ which is the group generated by*

$$\{fgf^{-1}g^{-1} : f \in G \text{ and } g \in G_n\}.$$

For each $n \in \mathbb{N}$, G/G_n is left amenable and $\times_{n \in \mathbb{N}} G/G_n$ is not left amenable.

Proof. One can piece together this proof from the results in [1, Section 4], especially page 517 between (F'') and (K) except for the assertion that $\bigcap_{n \in \mathbb{N}} G_n = \{1\}$ which is stated but no hint for a proof is given. That fact can be found in [10, 6.1.10] where it is attributed to W. Magnus. $\square$

We note that by the same methods, one can show that if $S = \bigoplus_{n \in \mathbb{N}} G/G_n$, then S is left amenable and $\times_{n \in \mathbb{N}} S$ is not left amenable.

Theorem 5.5. *Each of B_f , B_a , D_f , and D_a has a negative answer for Question 3.*

Proof. Let G be the free group on the letters a and b , for $n \in \mathbb{N}$, let G_n be as in Theorem 5.4 and let $S_n = G/G_n$. By [9, Corollary 3.6] any left amenable and left cancellative semigroup satisfies SFC so each S_n satisfies SFC. So for each n , $\Delta^*(S_n) = \Delta(S_n)$ and S_n is a B_f set, a B_a set, a D_f set, and a D_a set in S_n . Since $\times_{n \in \mathbb{N}} S_n$ is not left amenable, so does not satisfy SFC, it is none of B_f , B_a , D_f , or D_a . $\square$

Theorem 5.6. *Each of P and P^* has a negative answer for Question 3.*

Proof. For P^* , for each $n \in \mathbb{N}$, let $A_n = \mathbb{N} \setminus \{2^n + 1, 2^n + 2, \dots, 2^n + n\}$. Notice that in $(\mathbb{N}, +)$ a set is a P set if and only if it contains arbitrarily long arithmetic progressions. So given n , A_n has nonempty intersection with every arithmetic progression longer than n so each A_n is a P^* set in $\mathbb{N}$.

Now let $A = \times_{n \in \mathbb{N}} A_n$ and let $S = \times_{n \in \mathbb{N}} \mathbb{N}$. We show that A is not a P^* set by showing that $S \setminus A$ is a P set. So let $k \in \mathbb{N}$ be given. Define $\vec{a} \in S$ by

$$a_n = \begin{cases} 2^k & \text{if } n = k; \\ 1 & \text{otherwise} \end{cases} \quad \text{and define } \vec{d} \in S \text{ by } d_n = 1. \text{ Then}$$

$$\{a_k + td_k : t \in \{1, 2, \dots, k\}\} \subseteq \mathbb{N} \setminus A_k \text{ so } \{\vec{a} + t\vec{d} : t \in \{1, 2, \dots, k\}\} \subseteq S \setminus A.$$

For P , let G be the free semigroup on the letters a and b . For $n \in \mathbb{N}$ let $A_n = \{(ab^t)^n a : t \in \mathbb{N}\}$, let $S = \times_{n \in \mathbb{N}} G$ and let $A = \times_{n \in \mathbb{N}} A_n$. Trivially each A_n is a P set in G . We show that A is not a P set in S .

Suppose A is a P set in S . Pick $m \in \mathbb{N}$, $\vec{w} \in S^{m+1}$, and $\vec{e} \in S$ such that

$$\begin{aligned} \vec{x} &= \vec{w}(1)\vec{e}\vec{w}(2) \cdots \vec{w}(m)\vec{e}\vec{w}(m+1) \in A \text{ and} \\ \vec{y} &= \vec{w}(1)\vec{e}^2\vec{w}(2) \cdots \vec{w}(m)\vec{e}^2\vec{w}(m+1) \in A. \end{aligned}$$

Let $n = m + 1$. Then

$$\begin{aligned} x_n &= w(1)_n e_n w(2)_n \cdots w(m)_n e_n w(m+1)_n = ((ab^s)^n a \text{ and} \\ y_n &= w(1)_n e_n^2 w(2)_n \cdots w(m)_n e_n^2 w(m+1)_n = ((ab^t)^n a \end{aligned}$$

for some $s < t$ in $\mathbb{N}$. Since the number of occurrences of a in x_n and y_n are the same (namely $n + 1$), there are no occurrences of a in e_n . So for some $j \in \{1, 2, \dots, m + 1\}$ there are two occurrences of A in $w(j)_n$. But in x_n consecutive occurrences of a are separated by s occurrences of a while in y_n consecutive occurrences of a are separated by t occurrences of a . This is a contradiction. $\square$

The only notions for which we can currently establish a positive answer for Question 3 are ones for which the proofs are routine.

Theorem 5.7. *Question 3 has a positive answer for each of thick, WT, Q, IP, IP_r for $r > 1$, and $IP_{<\omega}$.*

Proof. All of these are routine. We illustrate by providing details for WT . Let I be a set, for $\alpha \in I$ let S_α be a semigroup and let A_α be a WT set in S_α . Let $S = \times_{\alpha \in I} S_\alpha$ and let $A = \times_{\alpha \in I} A_\alpha$. We define $\vec{s} \in S$ by choosing for each $\alpha \in I$, $s_\alpha \in S_\alpha$ such that $s_\alpha^{-1}A_\alpha$ is thick in S_α . Given $F \in \mathcal{P}_f(S)$ and $\alpha \in I$, pick $x_\alpha \in S_\alpha$ such that $\pi_\alpha[F]x_\alpha \subseteq s_\alpha^{-1}A_\alpha$ so that $s_\alpha\pi_\alpha[F]x_\alpha \subseteq A_\alpha$. Then $\vec{s}F\vec{x} \subseteq A$ so $F\vec{x} \subseteq \vec{s}^{-1}A$. $\square$

We have a large number of unanswered questions.

Question 5.8. *For any or all of B_t , SIP_r for $r > 1$, SPS^* , PS^* , B_f^* , B_a^* , B_t^* , or J^* is the answer to Question 3 “yes”?*

References

- [1] M. Day, *Amenable semigroups*, Illinois J. Math. **1** (1957), 509-544.
- [2] H. Furstenberg, *Recurrence in ergodic theory and combinatorial number theory*, Princeton University Press, Princeton, 1981.
- [3] D. Glasscock, N. Hindman, and D. Strauss, *Følner, Banach, and translation density are equal and other new results about density in left amenable semigroups*, Semigroup Forum **111** (2025), 121-143.
- [4] N. Hindman, *Notions of size in a semigroup – an update from a historical perspective*, Semigroup Forum **100** (2020), 52-76.
- [5] N. Hindman and D. Strauss, *Density and invariant means in left amenable semigroups*, Topology and its Applications **156** (2009), 2614-2628.
- [6] N. Hindman and D. Strauss, *Cartesian products of sets satisfying the Central Sets Theorem*, Topology Proceedings **35** (2010), 203-223.
- [7] N. Hindman and D. Strauss, *Algebra in the Stone-Čech compactification: theory and applications*, Second revised and extended edition. Walter de Gruyter & Co., Berlin, 2012.
- [8] M. Klawe, *Semidirect product of semigroups in relation to amenability, cancellation properties, and strong Følner conditions*, Pacific J. Math. **73** (1977), 91-106.
- [9] I. Namioka, *Følner’s conditions for amenable semi-groups*, Math. Scand **15** (1964), 18-28.
- [10] D. Robinson, *A course in the theory of groups*, Second edition, Springer-Verlag, New York, 1996.
- [11] Z. Yang, *Følner numbers and Følner type conditions for amenable semigroups*, Illinois J. Math. **31** (1987), 496-517.