24 April 2026

File `quotient4-24`

Continuation of file `quotient` of 16 April.

The first theorem is [D2, Theorem 6]. I will mostly use his notation.

Theorem 1. *Let H be a group and let G be a normal subgroup of H. If G and H/G are left amenable, then so is H.*

Proof. Let $\alpha \in LIM(G)$ and let $\beta \in LIM(H/G)$. Given $h \in H$ and $x \in l_\infty(H)$ define $x_h \in l_\infty(G)$ by, for $g \in G$, $x_h(g) = x(hg)$ (so x_h is the restriction of $x \circ \lambda_h$ to G.

Define $\widehat{x} : H \to \mathbb{R}$ by $\widehat{x}(h) = \alpha(x_h)$. We claim that if $h, h' \in H$ and $hG = h'G$, then $\widehat{x}(h) = \widehat{x}(h')$.

So let $h, h' \in H$ and assume that $hG = h'G$. Then $h = h'g'$ for some $g' \in G$. Given $g \in G$,

$$x_h(g) = x(hg) = x(h'g'g) = x_{h'}\big(\lambda_{g'}(g)\big) = (x_{h'} \circ \lambda_{g'})(g)\,,$$

so viewing $\lambda_{g'}$ as a function from G to G, we have $x_h = x_{h'} \circ \lambda_{g'}$. Since $\alpha \in LIM(G)$, we have

$$\widehat{x}(h') = \alpha(x_{h'}) = \alpha(x_{h'} \circ \lambda_{g'}) = \alpha(x_h) = \widehat{x}(h)$$

so the claim is established.

Now define $\overline{x} : H/G \to \mathbb{R}$ by $\overline{x}(hG) = \widehat{x}(h)$. By the claim, $\overline{x}$ is well defined. Now define $\gamma : l_\infty(H) \to \mathbb{R}$ by $\gamma(x) = \beta(\,\overline{x}\,)$.

We show now that $\gamma \in LIM(H)$. That is

(1) if $x \in l_\infty(H)$ and $(\forall h \in H)(x(h) \geq 0)$, then $\gamma(x) \geq 0$,

(2) $||\gamma||_\infty = 1$, and

(3) $(\forall v \in H)\big(\forall x \in l_\infty(H)\big)\big(\gamma(x \circ \lambda_v) = \gamma(x)\big)$.

For (1), Let $x \in l_\infty(H)$ and assume that for all $h \in H$, $x(h) \geq 0$. Then for all $h \in H$ and all $g \in G$, $x_h(g) \geq 0$ so for all $h \in H$, $\widehat{x}(h) = \alpha(x_h) \geq 0$. Then for all $h \in H$, $\overline{x}(hG) \geq 0$ so $\gamma(x) = \beta(\,\overline{x}\,) \geq 0$.

Given that (1) holds, to verify (2) it suffices that if $x \in l_\infty(H)$ is the function constantly equal to 1, then $\gamma(x) = 1$. Assume that $x \in l_\infty(H)$ is the function constantly equal to 1. Then for all $h \in H$, x_h is constantly 1 so that $\alpha(x_h) = 1$. And then $\overline{x}$ is constantly 1 so $\gamma(x) = \beta(\,\overline{x}\,) = 1$.

To verify (3), let $v \in H$, let $x \in l_\infty(H)$ and let $y = x \circ \lambda_v$. It suffices to show that $\overline{y} = \overline{x} \circ \lambda_{vG}$ for then

$$\gamma(y) = \beta(\,\overline{y}\,) = \beta(\,\overline{x} \circ \lambda_{vG}) = \beta(\,\overline{x}\,) = \gamma(x)\,.$$

To verify that $\overline{y} = \overline{x} \circ \lambda_{vG}$, let $h \in H$. Then $(\,\overline{x} \circ \lambda_{vG})(hG) = \overline{x}(vGhG) = \overline{x}(vhG) = \widehat{x}(vh) = \alpha(x_{vh}) = \alpha(x \circ \lambda_{vh}|_G) = \alpha(x \circ \lambda_v \circ \lambda_h|_G) = \alpha(y \circ \lambda_h|_G) = \alpha(y_h) = \widehat{y}(h) = \overline{y}(hG)$. □

Definition. Let G be the free group on the generators $\{a, b\}$. Let $G_0 = G$ and for $n \in \omega$, let $G_{n+1} = [G, G_n]$ which is the group generated by $\{fgf^{-1}g^{-1} : f \in G \text{ and } g \in G_n\}$.

The next theorem surprises me. It is actual equality, not just isomorphism. I presume that Day knew this, which is why he had such a brief proof for the assertion that G/G_n is amenable.

Theorem 2. *Let H be a normal subgroup of G and let K be a normal subgroup of H which is also normal in G. Then $(G/K)/(H/K) = G/H$.*

Proof. Let $X \in (G/K)/(H/K)$. Pick $a \in G$ such that $X = aK \cdot (H/K)$. We show that $X = aH$.

$X = aK \cdot \{cK : c \in H\} = \{aKcK : c \in H\} = \{acK : c \in H\}$ since K is normal in H. Then $X = \{acK : c \in H\} = aHK = aH$. □

Theorem 3. *For each $n \in \mathbb{N}$, G/G_n is amenable.*

Proof. G/G_1 is G_0/G_1 which is abelian. Now assume that $n \in \mathbb{N}$ and G/G_n is amenable.

By Theorem 2, $(G/G_{n+1})/(G_n/G_{n+1}) = G/G_n$. By assumption G/G_n is amenable and G_n/G_{n+1} is abelian so by Theorem 1, G/G_{n+1} is amenable. □

Day [D3] says that the next theorem "can be gotten from von Neumann [vN]". A proof is included in [D2]. The proof given here is from a web page of Terry Tau. (He has several.)

Theorem 4. *The free group on 2 generators is not left amenable.*

Proof. Let G be the free group generated by $\{a, b\}$. We will show that G does not satisfy the Følner Condition. So suppose that it does. Let $H = \{a, b, a^{-1}, b^{-1}\}$ and let $\epsilon = 1/2$. Pick $K \in \mathcal{P}_f(G)$ such that for all $s \in H$, $|sK \setminus K| < \epsilon|K|$.

For $s \in H$ let $E_s = \{w \in G : \text{the leftmost letter of } w \text{ is } s\}$.

Let $s \in H$ be given. Then

$$s(K \setminus E_{s^{-1}}) \subseteq sK \cap E_s \subseteq (K \cap E_s) \cup (sK \setminus K)\,,$$

so $|K| - |K \cap E_{s^{-1}}| = |K \setminus E_{s^{-1}}| = |s(K \setminus E_{s^{-1}})| < |K \cap E_s| + \epsilon|K|$.

We then have

$$\begin{array}{cl} |K| - |K \cap E_{a^{-1}}| & < |K \cap E_a| + \epsilon|K|\,, \\ |K| - |K \cap E_{b^{-1}}| & < |K \cap E_b| + \epsilon|K|\,, \\ |K| - |K \cap E_a| & < |K \cap E_{a^{-1}}| + \epsilon|K|, \text{ and} \\ |K| - |K \cap E_b| & < |K \cap E_{b^{-1}}| + \epsilon|K|\,. \end{array}$$

Therefore $4|K| - |K| < |K| + 4\epsilon|K|$ so $1/2 < \epsilon$, a contradiction. □

I was surprised to see how easy the proof of the following theorem is.

The result was announced in [D1]. It was proved in [F], and that is the proof used here.

Theorem 5. *Let G be an amenable group with identity e and let H be a not necessarily normal subgroup of G. Then H is amenable.*

Proof. Pick $\alpha \in LIM(G)$.

The right cosets of H partition G so we may pick $A \subseteq G$ such that for each $x \in G$, $|A \cap Hx| = 1$. We may assume that $A \cap H = \{e\}$. (We don't use this assumption, but it has the pleasing consequence that $\widehat{f}$ defined below is an extension of f.) Define $\varphi : G \to G$ by $\varphi_x \in A \cap Hx$. Note that given $x \in G$, $x\varphi_x^{-1} \in H$.

For $f \in l_\infty(H)$ define $\widehat{f} \in l_\infty(G)$ by for $x \in G$, $\widehat{f}(x) = f(x\varphi_x^{-1})$ and define $\beta(f) = \alpha(\widehat{f}\,)$.

We claim that $\beta \in LIM(H)$. As in the proof of Theorem 1 we easily verify that

(1) if $f \in l_\infty(H)$ and $(\forall x \in H)(f(h) \geq 0)$, then $\beta(f) \geq 0$ and

(2) $||\beta||_\infty = 1$.

To see that β is left invariant, let $f \in l_\infty(H)$ and let $a \in H$. In the rest of this proof we will write λ_a for multiplication on the left by a in H and λ_a^G for multiplication on the left by a in G. We will show that $\beta(f \circ \lambda_a) = \beta(f)$. Note that for any $x \in G$, $\varphi_{ax} = \varphi_x$.

Let $g = f \circ \lambda_a$. We need to show that $\beta(g) = \beta(f)$.

Given $x \in G$,

$$\widehat{g}(x) = g(x\varphi_x^{-1}) = (f \circ \lambda_a)(x\varphi_x^{-1}) = f(ax\varphi_x^{-1}) = f(ax\varphi_{ax}^{-1}) = \widehat{f}(ax) = \widehat{f} \circ \lambda_a^G(x)\,.$$

Consequently $\beta(f \circ \lambda_a) = \beta(g) = \alpha(\,\widehat{g}\,) = \alpha(\,\widehat{f} \circ \lambda_a^G) = \alpha(\,\widehat{f}\,) = \beta(f)$. □

Theorem 6. *Let $(S,\cdot)$ and $(T,\cdot)$ be semigroups.*

(a) If S and T satisfy FC so does $S\times T$.

(b) If S and T satisfy SFC so does $S\times T$.

Proof. (a) Let $H\in\mathcal{P}_f(S\times T)$ and let $\epsilon>0$. We need to show that there is some $K\in\mathcal{P}_f(S\times T)$ such that

$$(\forall (s,t)\in H)(|(s,t)K\setminus K|<\epsilon|K|)\,.$$

Let $H_1=\pi_1[H]$ and $H_2=\pi_2[H]$. Pick $K_1\in\mathcal{P}_f(S)$ and $K_2\in\mathcal{P}_f(T)$ such that $(\forall s\in H_1)(|sK_1\setminus K_1|<(\epsilon/2)|K_1|)$ and $(\forall t\in H_2)(|tK_2\setminus K_2|<(\epsilon/2)|K_2|)$. Let $K=K_1\times K_2$.

Let $(s,t)\in H$. Then $(s,t)K\setminus K\subseteq\big((sK_1\setminus K_1)\times tK_2\big)\cup\big(sK_1\times(tK_2\setminus K_2)\big)$ so

$$\begin{array}{rl}|(s,t)K\setminus K| & \leq|(sK_1\setminus K_1)\times tK_2|+|sK_1\times(tK_2\setminus K_2)|\\ & =|sK_1\setminus K_1|\cdot|tK_2|+|sK_1|\cdot|tK_2\setminus K_2|\\ & <(\epsilon/2)|K_1|\cdot|K_2|+(\epsilon/2)|K_1|\cdot|K_2|=\epsilon|K|\,.\end{array}$$

(b) Let $H\in\mathcal{P}_f(S\times T)$ and let $\epsilon>0$. We need to show that there is some $K\in\mathcal{P}_f(S\times T)$ such that

$$(\forall (s,t)\in H)(|K\setminus(s,t)K|<\epsilon|K|)\,.$$

Let $H_1=\pi_1[H]$ and $H_2=\pi_2[H]$. Pick $K_1\in\mathcal{P}_f(S)$ and $K_2\in\mathcal{P}_f(T)$ such that $(\forall s\in H_1)(|K_1\setminus sK_1|<(\epsilon/2)|K_1|)$ and $(\forall t\in H_2)(|K_2\setminus tK_2|<(\epsilon/2)|K_2|)$. Let $K=K_1\times K_2$.

Let $(s,t)\in H$. Then $K\setminus(s,t)K\subseteq\big((K_1\setminus sK_1)\times K_2\big)\cup\big(K_1\times(K_2\setminus tK_2)\big)$ so

$$\begin{array}{rl}|K\setminus K| & \leq|(K_1\setminus sK_1)\times K_2|+|K_1\times(K_2\setminus tK_2)|\\ & =|K_1\setminus sK_1|\cdot|K_2|+|K_1|\cdot|K_2\setminus tK_2|\\ & <(\epsilon/2)|K_1|\cdot|K_2|+(\epsilon/2)|K_1|\cdot|K_2|=\epsilon|K|\,.\end{array}$$

□

Theorem 7. *Let I be a set and for $i\in I$, let S_i be a semigroup with a two sided identity 1_i which satisfies SFC. Recall that $\bigoplus_{i\in I}S_i=\{x\in\times_{i\in I}S_i:\{i\in I:x_i\neq 1_i\}$ is finite$\}$. Let $S=\bigoplus_{i\in I}S_i$. Then S satisfies SFC.*

Proof. For $x\in S$ let $\mathrm{supp}(x)=\{i\in I:x_i\neq 1_i\}$. To verify that S satisfies SFC, let $H\in\mathcal{P}_f(S)$ and let $\epsilon>0$. Let $J=\bigcup_{s\in H}\mathrm{supp}(s)$. Let $T=\times_{i\in J}S_i$. By Theorem 6(b), T satisfies SFC. Define $\varphi:T\to S$ by $\varphi(x)_i=\begin{cases}x_i & \text{if } i\in J\\ 1_i & \text{otherwise.}\end{cases}$

Then φ is injective. For $s \in H$ let $s' \in T$ be the restriction of s to J. Pick $K' \in \mathcal{P}_f(T)$ such that for each $s \in H$, $|K' \setminus s'K'| < \epsilon|K'|$. Let $K = \varphi[K']$. Then for $s \in H$,

$$|K \setminus sK| = |K' \setminus s'K'| < \epsilon|K'| = \epsilon|K|\,.$$

□

With a nearly identical proof, Theorem 7 holds if SFC is replaced by FC. In [D3, (F'') p. 517] the corresponding assertion of amenable (not just left amenable) semigroups is made.

Definition. $S = \bigoplus_{n=1}^{\infty} G/G_n$.

We proved all parts of the assertion that $\times_{n=1}^{\infty}(G/G_n)$ is not amenable except the assertion that $\bigcap_{n=1}^{\infty} G_n = \{1\}$. This assertion is made in [D3, p. 517] just before (K). We need that assertion for the next proof as well.

Modulo that gap, we have an example of countably many distinct amenable groups whose Cartesian product is not amenable. We set out to show that S is a single amenable group such that $\times_{i=1}^{\infty} S$ is not amenable.

Theorem 8. *S is an amenable group and $\times_{i=1}^{\infty} S$ is not amenable.*

Proof. By Theorem 3 each G/G_n is an amenable group hence satisfies SFC so by Theorem 7, S satisfies SFC so is an amenable group.

We have that $\times_{i=1}^{\infty} S = \times_{i=1}^{\infty} \bigoplus_{n=1}^{\infty}(G/G_n) = \big\{\big\langle\langle H_{i,n}\rangle_{n=1}^{\infty}\big\rangle_{i=1}^{\infty} :$
$(\forall i \in \mathbb{N})(\forall n \in \mathbb{N})(H_{i,n} \in G/G_n)$ and $(\forall i \in \mathbb{N})(\{n \in \mathbb{N} : H_{i,n} \neq G_n\}$ is finite$\big\}$.

Define $\psi : G \to \times_{i=1}^{\infty} S$ by $\psi(g) = \big\langle\langle H_{i,n}\rangle_{n=1}^{\infty}\big\rangle_{i=1}^{\infty}$ where for $i, n \in \mathbb{N}$, $H_{i,n} = \begin{cases} gG_n & \text{if } n = i \\ G_n & \text{otherwise.} \end{cases}$ Then ψ is a homomorphism. To see that ψ is injective assume that $\psi(g) = \psi(h)$ and suppose that $g \neq h$. Then $gh^{-1} \neq 1$ so pick $i \in \mathbb{N}$ such that $gh^{-1} \notin G_i$. Then $\psi(g)_i = \langle H_{i,n}\rangle_{n=1}^{\infty}$ where $H_{i,n} = \begin{cases} gG_n & \text{if } n = i \\ G_n & \text{otherwise} \end{cases}$ and $\psi(h)_i = \langle K_{i,n}\rangle_{n=1}^{\infty}$ where $K_{i,n} = \begin{cases} hG_n & \text{if } n = i \\ G_n & \text{otherwise} \end{cases}$ so $gG_i = hG_i$, a contradictiion.

Thus $\times_{i=1}^{\infty} S$ contains a copy of G so by Theorems 4 and 5, $\times_{i=1}^{\infty} S$ is not amenable.

□

References

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