Combinatorially rich sets in arbitrary semigroups

Neil Hindman, Hedie Hosseini, Dona Strauss and M. A. Tootkaboni

Abstract. *Combinatorially Rich* sets were introduced by Bergelson and Glasscock [1] for commutative semigroups and shown to have several properties justifying their name. We extend the definition to arbitrary semigroups and establish the relationships of combinatorially rich sets to other notions of largeness in semigroups.

Contents

1. Introduction

There are many notions of largeness in a semigroup $(S,\cdot)$ that are related to the algebraic structure of the Stone-Čech compactification, βS, of the discrete set S. See the survey [4] for information about many of these notions.

In [1] Bergelsoon and Glasscock introduced a new such notion for a commutative semigroup $(S,+)$. They used matrix notation. Given an $r\times k$ matrix M we denote by $m_{i,j}$ the element in row i and column j of M.

Definition 1.1. Let $(S,+)$ be a commutative semigroup and let $A\subseteq S$. Then A is a *combinatorially rich set* (denoted CR-set) if and only if for each $k\in\mathbb{N}$, there exists $r\in\mathbb{N}$ such that whenever M is an $r\times k$ matrix with entries from S, there exist $a\in S$ and nonempty $H\subseteq\{1,2,\ldots,r\}$ such that for each $j\in\{1,2,\ldots,k\}$, $a+\sum_{t\in H} m_{t,j}\in A$.

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As they noted, the notion of CR-set is intimately related to the notion of J-set. (J-sets are key to characterizing C-sets, which are sets satisfying the conclusion of the Central Sets Theorem, a very strong combinatorial result.)

We write ${}^{A}B$ for the set of functions from A to B. In particular, ${}^{\mathbb{N}}S$ is the set of sequences in S. And we write $\mathcal{P}_f(X)$ for the set of finite nonempty subsets of X. Given a set X and a cardinal κ, $[X]^\kappa = \{F \subseteq X : |F| = \kappa\}$ and $[X]^{\leq\kappa} = \{F \subseteq X : |F| \leq \kappa\}$

Given $n \in \mathbb{N}$, we will occasionally write $[n]$ for $\{1, 2, \ldots, n\}$. When we write $\prod_{i=k}^{n} x_i$, we mean the product in increasing order of indices.

Definition 1.2. Let $(S, +)$ be a commutative semigroup and let $A \subseteq S$. A is a *J-set* if and only if whenever $F \in \mathcal{P}_f({}^{\mathbb{N}}S)$, there exist $a \in S$ and $H \in \mathcal{P}_f(\mathbb{N})$ such that for each $f \in F$, $a + \sum_{t\in H} f(t) \in A$.

The relationship between J-sets and CR-sets becomes clearer when we rephrase the definition of CR-sets.

Lemma 1.3. *Let $(S, +)$ be a commutative semigroup and let $A \subseteq S$. The following statements are equivalent.*

(a) *A is a CR-set.*
(b) *For each $k \in \mathbb{N}$ there exists $r \in \mathbb{N}$ such that whenever $\emptyset \neq F \subseteq \{f : f : \{1, 2, \ldots, r\} \to S\}$ and $|F| \leq k$, there exist $a \in S$ and $H \in \mathcal{P}_f(\{1, 2, \ldots, r\})$ such that for all $f \in F$, $a + \sum_{t\in H} f(t) \in A$.*
(c) *For each $k \in \mathbb{N}$ there exists $r \in \mathbb{N}$ such that whenever $F \in \mathcal{P}_f({}^{\mathbb{N}}S)$ with $|F| \leq k$, there exist $a \in S$ and $H \in \mathcal{P}_f(\{1, 2, \ldots, r\})$ such that for all $f \in F$, $a + \sum_{t\in H} f(t) \in A$.*

Proof. That (a) and (b) are equivalent is immediate. (One might have $|F| < k$ since two columns of M might be equal.)

To see that (b) implies (c), assume that $k \in \mathbb{N}$ and r has been chosen satisfying (b). Let $F \in \mathcal{P}_f({}^{\mathbb{N}}S)$ with $|F| \leq k$. For $f \in F$, let $g_f : \{1, 2, \ldots, r\} \to S$ be the restriction of f to $\{1, 2, \ldots, r\}$. Pick $a \in S$ and $H \in \mathcal{P}_f(\{1, 2, \ldots, r\})$ such that for all $f \in F$, $a + \sum_{t\in H} g_f(t) \in A$. Then for each $f \in F$, $a + \sum_{t\in H} f(t) \in A$.

To see that (c) implies (b), let $k \in \mathbb{N}$ and pick $r \in \mathbb{N}$ as guaranteed by (c). Let $\emptyset \neq F \subseteq \{f : f : \{1, 2, \ldots, r\} \to S\}$ such that $|F| \leq k$. Pick $z \in S$. Given $f \in F$, define $g_f \in {}^{\mathbb{N}}S$ by $g_f(t) = f(t)$ if $t \leq r$ and $g_f(t) = z$ if $t > r$. Pick $a \in S$ and $H \in \mathcal{P}_f(\{1, 2, \ldots, r\})$ such that for each $f \in F$, $a + \sum_{t\in H} g_f(t) \in A$. Then for each $f \in F$, $a + \sum_{t\in H} f(t) \in A$. □

We introduce now a finer gradation.

Definition 1.4. Let $(S, +)$ be a commutative semigroup, let $k \in \mathbb{N}$, and let $A \subseteq S$. Then A is a *k-CR-set* if and only if there exists $r \in \mathbb{N}$ such that whenever $F \in \mathcal{P}_f({}^{\mathbb{N}}S)$ with $|F| \leq k$, there exist $a \in S$ and $H \in \mathcal{P}_f(\{1, 2, \ldots, r\})$ such that for all $f \in F$, $a + \sum_{t\in H} f(t) \in A$.

Note that a set is a CR-set if and only if for each $k \in \mathbb{N}$, it is a k-CR-set.

Given a discrete semigroup $(S, \cdot)$, the Stone-Čech compactification, βS of S is the set of ultrafilters on S. We identify a point $x \in S$ with the principal ulrafilter $\{A \subseteq S : x \in A\}$. The topology on βS has a basis consisting of the open and closed subsets $\{\overline{A} : A \subseteq S\}$. The operation on S extends to βS making $(\beta S, \cdot)$ a right topological semigroup with S contained in the topological center of βS. That is, for each $p \in \beta S$, the function $\rho_p : \beta S \to \beta S$ defined by $\rho_p(q) = qp$ is continuous and for each $x \in S$, the function $\lambda_x : \beta S \to \beta S$ defined by $\lambda_x(q) = xq$ is continuous. For basic information about the algebraic structure of βS see [7, Part I].

In Section 2 we extend the definition of CR-sets to arbitrary semigroups and obtain results about the algebraic properties. Section 3 deals with the relation with other notions of largeness. In Section 4 we address the question of whether the Cartesian products of two CR-sets must be a CR-set.

2. The notion of combinatorially rich sets in arbitrary semigroups

The definition of CR-sets in an arbitrary semigroup is based on the corresponding definition for J-sets.

Definition 2.1. Let $(S, \cdot)$ be a semigroup and let $A \subseteq S$. Then A is a J-set if and only if for each $F \in \mathcal{P}_f({}^{\mathbb{N}}S)$, there exist $m \in \mathbb{N}$, $\vec{a} \in S^{m+1}$, and $t(1) < t(2) < \ldots < t(m)$ in $\mathbb{N}$ such that for each $f \in F$, $a(1)f\big(t(1)\big)a(2)\cdots a(m)f\big(t(m)\big)a(m+1) \in A$.

Definition 2.2. Let $(S, \cdot)$ be a semigroup and let $A \subseteq S$.

(1) Then A is a *combinatorially rich set* (a CR-set) if and only if for each $k \in \mathbb{N}$ there exists $r \in \mathbb{N}$ such that for each $F \in \mathcal{P}_f({}^{\mathbb{N}}S)$ with $|F| \leq k$, there exist $m \in \mathbb{N}$, $\vec{a} \in S^{m+1}$, and $t(1) < t(2) < \ldots < t(m) \leq r$ in $\mathbb{N}$ such that for each $f \in F$, $a(1)f\big(t(1)\big)a(2)\cdots a(m)f\big(t(m)\big)a(m+1) \in A$.

(2) Given $k \in \mathbb{N}$, A is a k-CR-set if and only if there exists $r \in \mathbb{N}$ such that for each $F \in \mathcal{P}_f({}^{\mathbb{N}}S)$ with $|F| \leq k$, there exist $m \in \mathbb{N}$, $\vec{a} \in S^{m+1}$, and $t(1) < t(2) < \ldots < t(m) \leq r$ in $\mathbb{N}$ such that for each $f \in F$, $a(1)f\big(t(1)\big)a(2)\cdots a(m)f\big(t(m)\big)a(m+1) \in A$.

We note that the definitions of CR-sets agree in commutative semigroups.

Lemma 2.3. *Let $(S, \cdot)$ be a commutative semgroup and let $A \subseteq S$. Then A is a CR-set according to Definition 1.1 if and only if A is a CR-set according to Definition 2.2.*

Proof. For the sufficiency, let $k \in \mathbb{N}$ and assume that we have $r \in \mathbb{N}$ such that for each $F \in \mathcal{P}_f({}^{\mathbb{N}}S)$ with $|F| \leq k$, there exist $m \in \mathbb{N}$, $\vec{a} \in S^{m+1}$, and $t(1) < t(2) < \ldots < t(m) \leq r$ in $\mathbb{N}$ such that for each $f \in F$, $a(1)f\big(t(1)\big)a(2)\cdots a(m)f\big(t(m)\big)a(m+1) \in A$. Given $F \in \mathcal{P}_f({}^{\mathbb{N}}S)$ with

$|F| \leq k$, pick $m \in \mathbb{N}$, $\vec{a} \in S^{m+1}$, and $t(1) < t(2) < \ldots < t(m) \leq r$ as guaranteed, let $a = a(1)a(2)\cdots a(m+1)$ and $H = \{t(1), t(2), \ldots, t(m)\}$. Then a and H are as required by Lemma 1.3(c).

For the necessity, let $k \in \mathbb{N}$ and assume we have $r \in \mathbb{N}$ such that whenever $\emptyset \neq G \subseteq \{f : f : \{1, 2, \ldots, r\} \to S\}$ and $|G| \leq k$, there exist $a \in S$ and $H \in \mathcal{P}_f(\{1, 2, \ldots, r\})$ such that for all $f \in G$, $a \cdot \prod_{t \in H} f(t) \in A$, as is guaranteed by Lemma 1.3(b). Let $F \in \mathcal{P}_f({}^{\mathbb{N}}S)$ with $|F| \leq k$ be given. Then whenever $\emptyset \neq G \subseteq \{f : f : \{1, 2, \ldots, r\} \to S\}$ and $|G| \leq k$, there exist $a \in S$ and $H \in \mathcal{P}_f(\{1, 2, \ldots, r\})$ such that for all $f \in G$, $a \cdot \prod_{t \in H} f(t) \in A$. Pick $c \in S$ and for $f \in F$ define $g_f : \{1, 2, \ldots, r\} \to S$ by for $j \in \{1, 2, \ldots, r\}$, $g_f(j) = f(j)c$. Let $G = \{g_f : f \in F\}$. Then $|G| \leq k$ so pick $b \in S$ and $H \in \mathcal{P}_f(\{1, 2, \ldots, r\})$ such that for all $f \in F$, $b \cdot \prod_{t \in H} g_f(t) \in A$. Let $m = |H|$, let $\langle t(1), t(2), \ldots, t(m) \rangle$ enumerate H in increasing order, let $a(1) = b$, and for $j \in \{2, 3, \ldots, m+1\}$ let $a(j) = c$. Then for each $f \in F$, $a(1)f\big(t(1)\big)a(2)\cdots a(m)f\big(t(m)\big)a(m+1) \in A$. $\square$

We now show that the notion of combinatorially rich is partition regular. The proof is based on the proof of [7, Lemma 14.14.5], and includes a section from that proof verbatim.

Theorem 2.4. *Let $(S, \cdot)$ be a semigroup and let A_1 and A_2 be subsets of S. If $A_1 \cup A_2$ is a CR-set in S, then either A_1 or A_2 is a CR-set in S.*

Proof. Assume that $A_1 \cup A_2$ is a CR-set in S and neither A_1 nor A_2 is a CR-set in S.

For $i \in \{1, 2\}$ pick $k(i) \in \mathbb{N}$ such that for every $r \in \mathbb{N}$ there exists $F \in \mathcal{P}_f({}^{\mathbb{N}}S)$ such that $|F| \leq k(i)$ and

$$(\forall m \in \mathbb{N})(\forall \vec{a} \in S^{m+1})(\forall t(1) < t(2) < \ldots < t(m) \leq r)$$
$$(\exists f \in F)\big(a(1)f\big(t(1)\big)a(2)\cdots a(m)f\big(t(m)\big)a(m+1)\big) \notin A_i)\,.$$

Let $k = k(1) + k(2)$ and pick by [7, Lemma 14.8.1] some $n \in \mathbb{N}$ such that whenever the length n words over $\{1, 2, \ldots, k\}$ are 2-colored, there is a variable word $w(v)$ beginning and ending with a constant and having no adjacent occurrences of v such that $\big\{w(l) : l \in \{1, 2, \ldots, k\}\big\}$ is monochromatic.

Let W be the set of length n words over $\{1, 2, \ldots, k\}$ and let $\alpha = k^n = |W|$. Pick $\beta \in \mathbb{N}$ such that for every $G \in \mathcal{P}_f({}^{\mathbb{N}}S)$ with $|G| \leq \alpha$

$$(\exists m \in \mathbb{N})(\exists \vec{a} \in S^{m+1})(\exists t(1) < t(2) < \ldots < t(m) \leq \beta)$$
$$(\forall f \in G)\big(a(1)f\big(t(1)\big)a(2)\cdots a(m)f\big(t(m)\big)a(m+1)\big) \in A_1 \cup A_2)\,.$$

For $i \in \{1, 2\}$, $n\beta + n$ is not big enough to serve as r for $k(i)$ so pick $F_i \in [{}^{\mathbb{N}}S]^{\leq k(i)}$ such that

$$(\dagger) \quad (\forall m \in \mathbb{N})(\forall \vec{a} \in S^{m+1})(\forall t(1) < t(2) < \ldots < t(m) \leq n\beta + n)$$
$$(\exists f \in F_i)\big(a(1)f\big(t(1)\big)a(2)\cdots a(m)f\big(t(m)\big)a(m+1)\big) \notin A_i)\,.$$

Write $F_1 = \{f_1, f_2, \ldots, f_{k(1)}\}$ and $F_2 = \{f_{k(1)+1}, f_{k(1)+2}, \ldots, f_{k(1)+k(2)}\}$, with repetition if need be.

For $w = b_1 b_2 \cdots b_n \in W$ (where each $b_i \in \{1, 2, \ldots, k\}$), define $g_w : \mathbb{N} \to S$ by, for $y \in \mathbb{N}$, $g_w(y) = \prod_{i=1}^{n} f_{b_i}(ny + i)$. Let $G = \{g_w : w \in W\}$. Then $G \in [{}^{\mathbb{N}}S]^{\leq \alpha}$ so pick $m \in \mathbb{N}$, $\vec{a} \in S^{m+1}$, and $t(1) < t(2) < \ldots < t(m)$ in $\{1, 2, \ldots, \beta\}$ such that for all $w \in W$,

$$\big(a(1) g_w\big(t(1)\big) a(2) \cdots a(m) g_w\big(t(m)\big) a(m+1)\big) \in A_1 \cup A_2\,.$$

Define $\varphi : W \to \{1, 2\}$ by

$$\varphi(w) = 1 \text{ iff } \big(a(1) g_w\big(t(1)\big) a(2) \cdots a(m) g_w\big(t(m)\big) a(m+1)\big) \in A_1\,.$$

Pick a variable word $w(v)$, beginning and ending with a constant and without successive occurrences of v such that φ is constant on $\big\{w(l) : l \in \{1, 2, \ldots, k\}\big\}$. Assume without loss of generality that $\varphi\big(w(l)\big) = 1$ for all $l \in \{1, 2, \ldots, k\}$. That is, for all $l \in \{1, 2, \ldots, k\}$,

$$\big(a(1) g_{w(l)}\big(t(1)\big) a(2) \cdots a(m) g_{w(l)}\big(t(m)\big) a(m+1)\big) \in A_1\,.$$

Let $w(v) = b_1 b_2 \cdots b_n$ where each $b_i \in \{1, 2, \ldots, k\} \cup \{v\}$, some $b_i = v$, $b_1 \neq v$, $b_n \neq v$, and if $b_i = v$, then $b_{i+1} \neq v$. Let r be the number of occurrences of v in $w(v)$ and pick $L(1), L(2), \ldots, L(r+1)$ and $s(1), s(2), \ldots, s(r)$ such that for each $p \in \{1, 2, \ldots, r\}$, $\max L(p) < s(p) < \min L(p+1)$,

$$\begin{aligned} &\textstyle\bigcup_{p=1}^{r+1} L(p) = \big\{i \in \{1, 2, \ldots, n\} : b_i \in \{1, 2, \ldots, k\}\big\} \text{ and,} \\ &\{s(1), s(2), \ldots, s(r)\} = \big\{i \in \{1, 2, \ldots, n\} : b_i = v\big\}\,. \end{aligned}$$

For example, assume that $w(v) = 12v131v2v1121v32$. Then $r = 4$, $L = (\{1, 2\}, \{4, 5, 6\}, \{8\}, \{10, 11, 12, 13\}, \{15, 16\})$, and $s = (3, 7, 9, 14)$.

We shall show now that, given $y \in \mathbb{N}$, there exist $\vec{c_y} \in S^{r+1}$ and $z_y(1) < z_y(2) < \ldots < z_y(r)$ such that for all $l \in \{1, 2, \ldots, k\}$,

$$g_{w(l)}(y) = c_y(1) f_l\big(z_y(1)\big) c_y(2) \cdots c_y(r) f_l\big(z_y(r)\big) c_y(r+1)$$

and further, for each y, $z_y(r) < z_{y+1}(1)$. So let $y \in \mathbb{N}$ be given. For $p \in \{1, 2, \ldots, r+1\}$, let $c_y(p) = \prod_{i \in L(p)} f_{b_i}(ny + i)$ and for $p \in \{1, 2, \ldots, r\}$, let $z_y(p) = ny + s(p)$. To see that these are as required, first note that $z_y(r) \leq ny + n < z_{y+1}(1)$. Now let $l \in \{1, 2, \ldots, k\}$ be given. Then $w(l) = d_1 d_2 \cdots d_n$ where for $i \in \{1, 2, \ldots, n\}$,

$$d_i = \begin{cases} b_i & \text{if } i \in \bigcup_{p=1}^{r+1} L(p) \\ l & \text{if } i \in \{s(1), s(2), \ldots, s(r)\}\,. \end{cases}$$

Therefore

$$\begin{aligned} g_{w(l)}(y) &= \textstyle\prod_{i=1}^{n} f_{d_i}(ny + i) \\ &= \Big(\textstyle\prod_{p=1}^{r} \big(\prod_{i \in L(p)} f_{b_i}(ny + i)\big) \cdot \\ &\quad f_l(ny + s(p))\Big) \cdot \textstyle\prod_{i \in L(r+1)} f_{b_i}(ny + i) \\ &= \Big(\textstyle\prod_{p=1}^{r} c_y(p) \cdot f_l\big(z_y(p)\big)\Big) \cdot c_y(r+1) \end{aligned}$$

as required.

Let $u = mr$. For $j \in \{1, 2, \ldots, m\}$ and $p \in \{1, 2, \ldots, r\}$, let

$$q\big((j-1)r + p\big) = z_{t(j)}(p)\,.$$

Note that $q(1) < q(2) < \ldots < q(u) = z_{t(m)}(r) = nt(m) + s(r) < nt(m) + n \leq n\beta + r$. For $j \in \{1, 2, \ldots, m\}$ and $p \in \{2, 3, \ldots, r\}$ let $d\big((j-1)r + p\big) = c_{t(j)}(p)$. Let $d(1) = a(1)c_{t(1)}(1)$, let $d(u+1) = c_{t(m)}(r+1)a(m+1)$, and for $j \in \{1, 2, \ldots, m-1\}$, let $d(jr+1) = c_{t(j)}(r+1)a(j+1)c_{t(j+1)}(1)$.

Note that $d \in S^{u+1}$ and for $l \in \{1, 2, \ldots, k\}$

$$\begin{aligned}
&a(1)g_{w(l)}\big((t(1)\big)a(2)\cdots a(m)g_{w(l)}\big(t(m)\big)a(m+1) = \\
&a(1)c_{t(1)}(1)f_l\big(z_{t(1)}(1)\big)c_{t(1)}(2)\cdots c_{t(1)}(r)f_l\big(z_{t(1)}(r)\big)c_{t(1)}(r+1)\cdot \\
&a(2)c_{t(2)}(1)f_l\big(z_{t(2)}(1)\big)c_{t(2)}(2)\cdots c_{t(2)}(r)f_l\big(z_{t(2)}(r)\big)c_{t(2)}(r+1)\cdot \\
&\qquad\qquad\qquad\qquad\vdots \\
&a(m)c_{t(m)}(1)f_l\big(z_{t(m)}(1)\big)c_{t(m)}(2)\cdots \\
&c_{t(m)}(r)f_l\big(z_{t(m)}(r)\big)c_{t(m)}(r+1)a(m+1) \\
&= d(1)f_l\big(q(1)\big)d(2)\cdots d(u)f_l\big(q(u)\big)d(u+1)\,.
\end{aligned}$$

We have $u \in \mathbb{N}$, $d \in S^{u+1}$, and $q(1) < q(2) < \ldots < q(u) \leq n\beta + n$ so by (†), we may pick $l \in \{1, 2, \ldots, k(1)\}$ such that

$$d(1)f_l\big(q(1)\big)d(2)\cdots d(u)f_l\big(q(u)\big)d(u+1) \notin A_1\,.$$

This is a contradiction. □

Definition 2.5. Let $(S, \cdot)$ be a semigroup.

(1) $CR(S) = \{p \in \beta S : (\forall A \in p)(A \text{ is a CR-set})\}$.
(2) For $k \in \mathbb{N}$, k-$CR(S) = \{p \in \beta S : (\forall A \in p)(A \text{ is a } k\text{-CR-set})\}$.

Theorem 2.6. *Let $(S, \cdot)$ be an infinite semigroup. Then $CR(S)$ is a compact two sided ideal of βS and for each $k \in \mathbb{N}$, k-$CR(S)$ is a compact two sided ideal of βS.*

Proof. By Theorem 2.4 and [7, Theorem 3.11], $CR(S) \neq \emptyset$. Since $CR(S) = \bigcap_{k=1}^{\infty} k\text{-}CR(S)$, we have that each k-$CR(S) \neq \emptyset$ and it suffices to show that each k-$CR(S)$ is a two sided ideal of βS, so let $k \in \mathbb{N}$.

Let $p \in k$-$CR(S)$ and let $q \in \beta S$. To see that $qp \in k$-$CR(S)$, let $A \in qp$, and let $B = \{x \in S : x^{-1}A \in p\}$. Then $B \in q$ so $B \neq \emptyset$. Pick $x \in B$. Then $x^{-1}A \in p$ so pick $r \in \mathbb{N}$ such that for each $F \in \mathcal{P}_f(^{\mathbb{N}}S)$ with $|F| \leq k$, there exist $m \in \mathbb{N}$, $\vec{a} \in S^{m+1}$, and $t(1) < t(2) < \ldots < t(m) \leq r$ in $\mathbb{N}$ such that for each $f \in F$, $a(1)f\big(t(1)\big)a(2)\cdots a(m)f\big(t(m)\big)a(m+1) \in x^{-1}A$. Let $F \in \mathcal{P}_f(^{\mathbb{N}}S)$ with $|F| \leq k$ and pick m, $\vec{a}$, and $t(1) < t(2) < \ldots < t(m) \leq r$ as guaranteed for $x^{-1}A$. Define $\vec{b} \in S^{m+1}$ by $b(1) = xa(1)$ and for $j \in \{2, 3, \ldots, m+1$, let $b(j) = a(j)$. Then for each $f \in F$, $b(1)f\big(t(1)\big)b(2)\cdots b(m)f\big(t(m)\big)b(m+1) \in A$.

To see that $pq \in k$-$CR(S)$, let $A \in pq$ and let $B = \{x \in S : x^{-1}A \in q\}$. Then $B \in p$ so pick $r \in \mathbb{N}$ such that for each $F \in \mathcal{P}_f(^{\mathbb{N}}S)$ with $|F| \leq k$, there exist $m \in \mathbb{N}$, $\vec{a} \in S^{m+1}$, and $t(1) < t(2) < \ldots < t(m) \leq r$ in $\mathbb{N}$ such that for each $f \in F$, $a(1)f\big(t(1)\big)a(2)\cdots a(m)f\big(t(m)\big)a(m+1) \in B$. Let $F \in \mathcal{P}_f({}^{\mathbb{N}}S)$ with $|F| \leq k$ and pick m, $\vec{a}$, and $t(1) < t(2) < \ldots < t(m) \leq r$ as guaranteed for B. Then $\bigcap_{f\in F}\big(a(1)f\big(t(1)\big)a(2)\cdots a(m)f\big(t(m)\big)a(m+1)\big)^{-1}A \in q$ so pick $y \in \bigcap_{f\in F}\big(a(1)f\big(t(1)\big)a(2)\cdots a(m)f\big(t(m)\big)a(m+1)\big)^{-1}A$. Define $\vec{b} \in S^{m+1}$ by $b(j) = a(j)$ for $j \in \{1,2,\ldots,m\}$ and $b(m+1) = a(m+1)y$. Then for each $f \in F$, $b(1)f\big(t(1)\big)b(2)\cdots b(m)f\big(t(m)\big)b(m+1) \in A$. □

We shall see in Theorem 3.9 that the notion of k-CR need not be partition regular, so one could not appeal to [7, Theorem 3.11] to conclude that k-$CR(S) \neq \emptyset$.

Corollary 2.7. *Let $(S,\cdot)$ be an infinite semigroup. A subset A of S is a CR-set if and only if $\overline{A} \cap CR(S) \neq \emptyset$.*

Proof. The necessity is trivial and the sufficiency follows from Theorem 2.4 and [7, Theorem 3.11]. □

3. Relations with other notions of largeness

Recall that in an arbitrary infinite semigroup $(S,\cdot)$, $J(S) = \{p \in \beta S : (\forall A \in p)(A \text{ is a J-set}\}$.

The other notions of largeness with which we will be concerned in this section are k-J-sets, piecewise syndetic sets, C-sets, and B-sets.

Definition 3.1. Let $(S,\cdot)$ be an infinite semigroup and let $k \in \mathbb{N}$.

(1) A subset A of S is a k-J-set if and only if whenever $F \in \mathcal{P}_f({}^{\mathbb{N}}S)$ with $|F| \leq k$, there exist $m \in \mathbb{N}$, $\vec{a} \in S^{m+1}$, and $t(1) < t(2) < \ldots < t(m)$ in $\mathbb{N}$ such that for each $f \in F$,
$$a(1)f\big(t(1)\big)a(2)\cdots a(m)f\big(t(m)\big)a(m+1) \in A\,.$$
(2) k-$J(S) = \{p \in \beta(S) : (\forall A \in p)(A \text{ is a } k\text{-J-set})\}$.

C-sets were originally defined as sets satisfying the conclusion of the Central Sets Theorem. Since they do satisfy the conclusion of the Central Sets Theorem, they satisfy very strong Ramsey theoretic conclusions. For example, any C-set in $(\mathbb{N},+)$ contains solutions to any finite partition regular system of homogeneous linear equations with rational coefficients. We use a simpler characterization, established in [7, Theorem 14.27], as the definition of C-sets.

Definition 3.2. Let $(S,\cdot)$ be an infinite semigroup and let $A \subseteq S$.

(1) A is piecewise syndetic if and only if there exists $G \in \mathcal{P}_f(S)$ such that for every $F \in \mathcal{P}_f(S)$ there exists $x \in S$ such that $Fx \subseteq \bigcup_{t\in G} t^{-1}A$.
(2) A is a C-set if and only if there is a downward directed family $\langle C_F\rangle_{F\in I}$ of subsets of A such that

(i) for each $F \in I$ and each $x \in C_F$ there exists $G \in I$ with $C_G \subseteq x^{-1}C_F$ and
(ii) for each $\mathcal{F} \in \mathcal{P}_f(I)$, $\bigcap_{F\in\mathcal{F}} C_F$ is a J-set.

By [7, Theorem 14.15.1], A subset of the semigroup S is a C-set if and only if A is a member of an idempotent in $J(S)$.

Theorem 3.3. *Let $(S,\cdot)$ be an infinite semigroup and let A be a piecewise syndetic subset of S. Then A is a CR-set.*

Proof. By Theorem 2.6, $CR(S)$ is a two sided ideal of βS so $K(\beta S) \subseteq CR(S)$. By [7, Theorem 4.40], $\overline{A} \cap K(\beta S) \neq \emptyset$ so $\overline{A} \cap CR(S) \neq \emptyset$ so Corollary 2.7 applies. $\square$

It follows immediately from the definitions that CR-sets are J-sets. We see now that the converse fails badly in free semigroups. Given $x \in S$ we write $\overline{x}$ for the element of ${}^{\mathbb{N}}S$ which is constantly equal to x.

Theorem 3.4. *Let κ be an infinite cardinal and let S be the free semigroup on the distinct letters $\langle b_\sigma\rangle_{\sigma<\kappa}$. There is a C-set $A \subseteq S$ which is not a 2-CR set.*

Proof. Given $w \in S$ we write $\ell(w)$ for the length of w.

We note that we can choose for each $F \in \mathcal{P}_f({}^{\mathbb{N}}S)$ an infinite subset B_F of $\mathbb{N}$ such that for each f and g in F, either $f(t) = g(t)$ for all $t \in B_F$ or $f(t) \neq g(t)$ for all $t \in B_F$.

Enumerate B_F in order as $\langle n_F(t)\rangle_{t=1}^{\infty}$. For $f \in F$, define $g_{F,f} \in {}^{\mathbb{N}}S$ by $g_{F,f}(t) = f\big(n_F(t)\big)$. Let $\gamma(F) = \{g_{F,f} : f \in F\}$.

For $F \in \mathcal{P}_f({}^{\mathbb{N}}S)$, define $\phi(F) = 1 + \max\{n \in \omega : (\exists f \in F)(b_n \text{ occurs in } g_{F,f}(1)\}$.

For $F \in \mathcal{P}_f({}^{\mathbb{N}}S)$, let $G_F = \{g_{|\{1,2,\ldots,\phi(F)\}} : g \in \gamma(F)\}$. Let $\mathcal{G} = \{G_F : F \in \mathcal{P}_f({}^{\mathbb{N}}S)\}$. We claim that $|\mathcal{G}| \leq \kappa$. (One can in fact show that $|\mathcal{G}| = \kappa$, but we will not need that fact.) To see this let $\mathcal{T} = \bigcup_{n=1}^{\infty}({}^{[n]}S)$. Then $|\mathcal{T}| = \sum_{n=1}^{\infty}\kappa = \kappa$ and each $G_F \in \mathcal{P}_f(\mathcal{T})$ so $\mathcal{G} \subseteq \mathcal{P}_f(\mathcal{T})$ and thus $|\mathcal{G}| \leq \kappa$. Enumerate $\mathcal{G}$ as $\langle H_\sigma\rangle_{\sigma<\kappa}$ with repetition if need be.

Note that if F and K are in $\mathcal{P}_f({}^{\mathbb{N}}S)$ and $G_F = G_K$, then $\phi(F) = \phi(K) = \max\mathrm{domain}(g)$ for any $g \in \gamma(F)$. For $\sigma < \kappa$, let $s_\sigma = \max\mathrm{domain}(g)$ for some $g \in H_\sigma$ (hence for any $g \in H_\sigma$). Note that if $F \in \mathcal{P}_f({}^{\mathbb{N}}S)$ and $G_F = H_\sigma$, then $s_\sigma = \phi(F)$.

Define $\psi : \kappa \to \omega$ by $\sigma = \lambda + \psi(\sigma)$ for some limit ordinal λ. (We are taking 0 to be a limit ordinal.)

Inductively choose an injective function $\eta : \kappa \to \kappa$ such that for each $\sigma < \kappa$, $\psi\big(\eta(\sigma)\big) \in 3\omega$ and

$$\eta(\sigma) > \max\{\tau < \kappa : (\exists g \in H_\sigma)(\exists j \in \{1,2,\ldots,s_\sigma\})(b_\tau \text{ occurs in } g(j)\}\,.$$

For $\sigma<\kappa$, let $D_\sigma=\{b_{\eta(\sigma)}g(1)b_{\eta(\sigma)}g(2)\cdots b_{\eta(\sigma)}g(s_\sigma)b_{\eta(\sigma)}:g\in H_\sigma\}$. For $\tau<\kappa$, let

$$C_\tau=\{\textstyle\prod_{\sigma\in K}x_\sigma:K\in\mathcal{P}_f(\kappa)\,,\,\min K\geq\tau,\text{ and }(\forall\sigma\in K)(x_\sigma\in D_\sigma\}\,.$$

Let $A=C_0$.

We first show that A is not a 2-CR set. Suppose instead that we have $r\in\mathbb{N}$ such that for every $F\in\mathcal{P}_f({}^{\mathbb{N}}S)$ with $|F|\leq 2$, there exist $m\in\mathbb{N}$, $\vec{a}\in S^{m+1}$ and $t(1)<t(2)<\ldots<t(m)\leq r$ such that for all $f\in F$, $(a(1)f\big(t(1)\big)a(2)\cdots a(m)f\big(t(m)\big)a(m+1)\in A)$.

Pick $k>r$ with $k\equiv 1\pmod 3$ and let $F=\{\overline{b_k},\overline{b_{k+1}}\}$. Pick m, $\vec{a}$, and $t(1)<t(2)<\ldots<t(m)\leq r$ as guaranteed for F. Pick K and L in $\mathcal{P}_f(\kappa)$ such that

$$\begin{aligned} w&=a(1)b_ka(2)\cdots a(m)b_ka(m+1)\\ &=\textstyle\prod_{\sigma\in K}(b_{\eta(\sigma)}c_\sigma(1)b_{\eta(\sigma)}c_\sigma(2)\cdots b_{\eta(\sigma)}c_\sigma(s_\sigma)b_{\eta(\sigma)})\text{ and}\\ v&=a(1)b_{k+1}a(2)\cdots a(m)b_{k+1}a(m+1)\\ &=\textstyle\prod_{\sigma\in L}(b_{\eta(\sigma)}d_\sigma(1)b_{\eta(\sigma)}d_\sigma(2)\cdots b_{\eta(\sigma)}d_\sigma(s_\sigma)b_{\eta(\sigma)})\,,\end{aligned}$$

where for $\sigma\in K$, $c_\sigma\in H_\sigma$ and for $\sigma\in L$, $d_\sigma\in H_\sigma$.

We may presume that $\ell(w)=\ell(v)$ is the smallest among all such examples.

Notice that for each $\sigma<\kappa$, $\psi\big(\eta(\sigma)\big)\in 3\omega$ so $b_{\eta(\sigma)}\notin\{b_k,b_{k+1}\}$ so all occurrences of $b_{\eta(\sigma)}$ for $\sigma\in K\cup L$ occur in some $a(j)$ and occur in the same position in w and in v. Let $\sigma=\min K$. Based on the first letter of $a(1)$, we see that also $\sigma=\min L$ so c_σ and d_σ are in H_σ. Assume first that $c_\sigma\neq d_\sigma$, in which case for all $j\in\{1,2,\ldots,s_\sigma\}$, $c_\sigma(j)\neq d_\sigma(j)$. The second occurrence of $b_{\eta(\sigma)}$ in v occurs in $a(j)$ for some $j\in\{1,2,\ldots,m+1\}$ and occurs in some position t. So the second occurrence of b_{n_i} in w also occurs in $a(j)$ at position t so $\ell\big(c_\sigma(1)\big)=\ell\big(d_\sigma(1)\big)$. If $\ell\big(a(1)\big)>\ell\big(c_\sigma(1)\big)$, then $c_\sigma(1)$ and $d_\sigma(1)$ occur in the same position in $a(1)$, which is impossible. So $\ell\big(a(1)\big)\leq\ell\big(c_\sigma(1)\big)$, so b_k occurs in $c_\sigma(1)$ and thus $s_\sigma\geq k+1$.

We have that $b_{\eta(\sigma)}$ does not occur in $c_\sigma(t)$ or $d_\sigma(t)$ for any $t\in\{1,2,\ldots,s_\sigma\}$ so the first $s_\sigma+1$ occurrences of $b_{\eta(\sigma)}$ in w are exactly those shown. Since $s_\sigma\geq k+1>r+1\geq m+1$, there exist $j\in\{1,2,\ldots,m+1\}$ and $t\in\{1,2,\ldots,s_\sigma\}$ such that occurrences number t and number $t+1$ of $b_{\eta(\sigma)}$ in w are both in $a(j)$. But then $c_\sigma(t)$ and $d_\sigma(t)$ occur in the same position in $a(j)$, which is impossible.

Thus we must have that $c_\sigma=d_\sigma$. Since $w\neq v$, we can't have $K=L=\{\sigma\}$. Since $\ell(w)=\ell(v)$ we must have that both K and L properly contain $\{\sigma\}$. If we had $\ell\big(a(1)\big)$ less than the length of

$$b_{\eta(\sigma)}c_\sigma(1)b_{\eta(\sigma)}\cdots b_{\eta(\sigma)}c_\sigma(s_\sigma)b_{\eta(\sigma)}=b_{\eta(\sigma)}d_\sigma(1)b_{\eta(\sigma)}\cdots b_{\eta(\sigma)}d_\sigma(s_\sigma)b_{\eta(\sigma)}\,,$$

then b_k and b_{k+1} would occur in the same location in

$$b_{\eta(\sigma)}c_\sigma(1)b_{\eta(\sigma)}c_\sigma(2)\cdots b_{\eta(\sigma)}c_\sigma(s_\sigma)b_{\eta(\sigma)}\,.$$

So the length of $b_{\eta(\sigma)}c_\sigma(1)b_{\eta(\sigma)}c_\sigma(2)\cdots b_{\eta(\sigma)}c_\sigma(s_\sigma)b_{\eta(\sigma)}$ must be less than or equal to $\ell\big(a(1)\big)$. Equality can't hold since then one would have $b_k = b_{\eta(\tau)}$ where $\tau = \min(K \setminus \{i\})$. So we have a nonempty word $a'(1)$ such that $a(1) = b_{\eta(\sigma)}c_\sigma(1)b_{\eta(\sigma)}c_\sigma(2)\cdots b_{\eta(\sigma)}c_\sigma(s_\sigma)b_{\eta(\sigma)}a'(1)$. For $j \in \{2,3,\ldots,m+1\}$ let $a'(j) = a(j)$. Then we get

$$\begin{aligned} w' &= a'(1)b_k a'(2)\cdots a'(m)b_k a'(m+1) \\ &= \textstyle\prod_{\tau\in K\setminus\{\sigma\}}\big(b_{\eta(\tau)}c_\tau(1)b_{\eta(\tau)}c_\tau(2)\cdots b_{\eta(\tau)}c_\tau(s_\tau)b_{\eta(\tau)}\big) \text{ and} \\ v' &= a'(1)b_{k+1}a'(2)\cdots a'(m)b_{k+1}a'(m+1) \\ &= \textstyle\prod_{\tau\in L\setminus\{\sigma\}}\big(b_{\eta(\tau)}d_\tau(1)b_{\eta(\tau)}d_\tau(2)\cdots b_{\eta(\tau)}d_\tau(s_\tau)b_{\eta(\tau)}\big)\,, \end{aligned}$$

where for $\tau \in K\setminus\{\sigma\}$, $c_\tau \in H_\tau$ and for $\tau \in L\setminus\{\sigma\}$, $d_\tau \in H_\tau$.

This contradicts the minimality of $\ell(w)$.

To show that A is a C-set, we show that $\langle C_\sigma\rangle_{\sigma<\kappa}$ is as required by Definition 3.2(2). That is $\langle C_\sigma\rangle_{\sigma<\kappa}$ is downward directed, for $\mathcal{F} \in \mathcal{P}_f(\kappa)$, $\bigcap_{\sigma\in\mathcal{F}} C_\sigma$ is a J-set, and for each $\sigma < \kappa$ and each $x \in C_\sigma$, there exists $\tau < \kappa$ such that $C_\tau \subseteq x^{-1}C_\sigma$. Since κ is linearly ordered, the first two assertions amount to the assertions that $\langle C_\sigma\rangle_{\sigma<\kappa}$ is decreasing and that each C_σ is a J-set.

It is trivial that $\langle C_\sigma\rangle_{\sigma<\kappa}$ is decreasing. To verify the third assertion, let $\sigma < \kappa$ be given and let $x \in C_\sigma$. Pick $K \in \mathcal{P}_f(\kappa)$ such that $\min K \geq \sigma$ and for $\tau \in K$, there is some $g_\tau \in H_\tau$ such that

$$x = \textstyle\prod_{\tau\in K}\big(b_{\eta(\tau)}g_\tau(1)b_{\eta(\tau)}g_\tau(2)\cdots b_{\eta(\tau)}g(s_\tau)b_{\eta(\tau)}\big)\,.$$

Let $\delta = \max K + 1$. Then $C_\delta \subseteq x^{-1}C_\sigma$.

To complete the proof, we need to show that for each $\sigma < \kappa$, C_σ is a J-set. We claim that it suffices to show that for each $\sigma < \kappa$ and each $F \in \mathcal{P}_f({}^{\mathbb{N}}S)$ there exists $K \in \mathcal{P}_f({}^{\mathbb{N}}S)$ such that $F \subseteq K$ and $G_K = H_\mu$ for some $\mu \geq \sigma$. Assume that we have done this. Let $\sigma < \kappa$ and $F \in \mathcal{P}_f({}^{\mathbb{N}}S)$ be given and pick $K \in \mathcal{P}_f({}^{\mathbb{N}}S)$ and $\mu \geq \sigma$ such that $F \subseteq K$ and $G_K = H_\mu$. Then $D_\mu \subseteq C_\sigma$. Let $m = s_\mu$. For $j \in \{1,2,\ldots,m+1\}$, let $a(j) = b_{\eta(\mu)}$, and for $j \in \{1,2,\ldots,m\}$, let $t(j) = n_K(j)$. Then for $f \in F$,

$$\begin{aligned} &a(1)f\big(t(1)\big)a(2)\cdots a(m)f\big(t(m)\big)a(m+1) \\ &= b_{\eta(\mu)}f\big(n_K(1)\big)b_{\eta(\mu)}\cdots b_{\eta(\mu)}f\big(n_K(m)\big)b_{\eta(\mu)} \\ &= b_{\eta(\mu)}g_{K,f}(1)b_{\eta(\mu)}\cdots b_{\eta(\mu)}g_{K,f}(m)b_{\eta(\mu)} \\ &= b_{\eta(\mu)}g_{K,f}(1)b_{\eta(\mu)}\cdots b_{\eta(\mu)}g_{K,f}(s_\mu)b_{\eta(\mu)} \in D_\mu\,. \end{aligned}$$

We now consider two cases. Suppose first that $\kappa = \omega$. Let $n < \omega$ and let $F \in \mathcal{P}_f({}^{\mathbb{N}}S)$ be given. Pick $m(0) \geq \phi(F)$ and let $K_1 = F \cup \{\overline{b_{m(0)}}\}$. Then $b_{m(0)}$ occurs in $g_{K_1,\overline{b_{m(0)}}}(1)$ so $\phi(K_1) > m(0)$. Having chosen K_q, let $m(q) > \phi(K_q)$ and let $K_{q+1} = F \cup \{\overline{b_{m(q)}}\}$. Then $\phi(K_{q+1}) > \phi(K_q)$. Since $\phi(F) < \phi(K_1) < \phi(K_2) < \ldots$ we have that $G_F, G_{K_1}, G_{K_2}, \ldots$ are all distinct so there is some j such that $G_{K_j} = H_l$ for some $l \geq n$.

Now assume that $\kappa > \omega$. Let $\sigma < \kappa$ and $F \in \mathcal{P}_f({}^{\mathbb{N}}S)$ be given. Let $X = \{\tau < \kappa : (\exists t \in \mathbb{N})(\exists f \in F)\big(b_\tau \text{ occurs in } f(t)\big)\}$. Then X is countable so pick an injective function $\theta : \kappa \to \kappa \setminus X$. For $\tau < \kappa$, let $K_\tau = F \cup \{\overline{b_{\theta(\tau)}}\}$. Then $\overline{b_{\theta(\tau)}}_{|\{1,2,\ldots,\phi(K_\tau)\}}$ occurs in G_{K_τ} and not in any other G_{K_δ} so there must exist $\tau < \kappa$ and $\mu > \sigma$ such that $G_{K_\tau} = H_\mu$. □

Question 3.5. *Do there exist an infinite commutative semigroup $(S, +)$ and a set $A \subseteq S$ such that A is a J-set but not a CR-set?*

Lemma 3.6. *Let $(S, +)$ and $(T, +)$ be infinite commutative semigroups and let $\varphi : S \to T$ be a surjective homomorphism. Let $k \in \mathbb{N}$ and let $A \subseteq T$.*

(1) *A is a k-J-set in T if and only if $\varphi^{-1}[A]$ is a k-J-set in S.*
(2) *A is a k-CR-set in T if and only if $\varphi^{-1}[A]$ is a k-CR-set in S.*

Proof. We do the proof for (2), the other proof being very similar. For the necessity, pick $r \in \mathbb{N}$ such that for every $F \in \mathcal{P}_f({}^{\mathbb{N}}T)$ with $|F| \le k$, there exist $a \in T$ and $H \in \mathcal{P}_f(\{1, 2, \ldots, r\})$ such that for each $f \in F$, $a + \sum_{t \in H} f(t) \in A$.

To see that $\varphi^{-1}[A]$ is a k-CR-set in S, let $F \in \mathcal{P}_f({}^{\mathbb{N}}S)$ with $|F| \le k$. Let $G = \{\varphi \circ f : f \in F\}$. Then $G \in \mathcal{P}_f({}^{\mathbb{N}}T)$ and $|G| \le k$ so pick $a \in T$ and $H \in \mathcal{P}_f(\{1, 2, \ldots, r\})$ such that for each $f \in F$, $a + \sum_{t \in H} (\varphi \circ f)(t) \in A$. Pick $b \in S$ such that $\varphi(b) = a$. Then for each $f \in F$, $\varphi\big(b + \sum_{t \in H} f(t)\big) \in A$ so $b + \sum_{t \in H} f(t) \in \varphi^{-1}[A]$.

For the sufficiency, pick $r \in \mathbb{N}$ such that for every $F \in \mathcal{P}_f({}^{\mathbb{N}}S)$ with $|F| \le k$, there exist $a \in S$ and $H \in \mathcal{P}_f(\{1, 2, \ldots, r\})$ such that for each $f \in F$, $a + \sum_{t \in H} f(t) \in \varphi^{-1}[A]$.

To see that A is a k-CR-set in T, let $F \in \mathcal{P}_f({}^{\mathbb{N}}T)$ with $|F| \le k$. For $f \in F$, pick $f^* \in {}^{\mathbb{N}}S$ such that for $t \in \mathbb{N}$, $\varphi\big(f^*(t)\big) = f(t)$ and let $G = \{f^* : f \in F\}$. Pick $a \in S$ and $H \in \mathcal{P}_f(\{1, 2, \ldots, r\})$ such that for each $f \in F$, $a + \sum_{t \in H} f^*(t) \in \varphi^{-1}[A]$. Then $\varphi(a) \in T$ and for $f \in F$, $\varphi(a) + \sum_{t \in H} \varphi\big(f^*(t)\big) = \varphi(a) + \sum_{t \in H} f(t) \in A$. □

Lemma 3.7. *Let $k \in \mathbb{N}$ and $A \subseteq \mathbb{N}$.*

(1) *A is a k-J-set in $(\mathbb{N}, \cdot)$ if and only if $A \setminus \{1\}$ is a k-J-set in $(\mathbb{N} \setminus \{1\}, \cdot)$.*
(2) *A is a k-CR-set in $(\mathbb{N}, \cdot)$ if and only if $A \setminus \{1\}$ is a k-CR-set in $(\mathbb{N} \setminus \{1\}, \cdot)$.*

Proof. Again we do the proof for (2), the other proof being very similar. Given $f \in {}^{\mathbb{N}}\mathbb{N}$, define $g_f \in {}^{\mathbb{N}}(\mathbb{N} \setminus \{1\})$ by $g_f(n) = 2f(n)$.

For the necessity, assume A is a k-CR-set in $(\mathbb{N}, \cdot)$ and pick $r \in \mathbb{N}$ such that whenever $F \in \mathcal{P}_f({}^{\mathbb{N}}\mathbb{N})$ with $|F| \le k$, there exist $a \in \mathbb{N}$ and $H \in \mathcal{P}_f(\{1, 2, \ldots, r\})$ such that for each $f \in F$, $a \cdot \prod_{t \in H} f(t) \in A$. To see that r works for $\mathbb{N} \setminus \{1\}$, let $F \in \mathcal{P}_f({}^{\mathbb{N}}(\mathbb{N} \setminus \{1\}))$ with $|F| \le k$. Let $G = \{g_f : f \in F\}$. Then $G \in \mathcal{P}_f({}^{\mathbb{N}}\mathbb{N})$ and $|G| \le k$ so pick $a \in \mathbb{N}$ and $H \in \mathcal{P}_f(\{1, 2, \ldots, r\})$ such that for each $f \in F$, $a \cdot \prod_{t \in H} g_f(t) \in A$. Let $m = |H|$ and let $b = 2^m a$. Then $b \in \mathbb{N} \setminus \{1\}$ and for each $f \in F$, $b \cdot \prod_{t \in H} f(t) \in A \setminus \{1\}$.

For the sufficiency, assume $A \setminus \{1\}$ is a k-CR-set in $(\mathbb{N} \setminus \{1\}, \cdot)$ and pick $r \in \mathbb{N}$ such that whenever $F \in \mathcal{P}_f({}^{\mathbb{N}}(\mathbb{N} \setminus \{1\}))$ with $|F| \leq k$, there exist $a \in \mathbb{N} \setminus \{1\}$ and $H \in \mathcal{P}_f(\{1, 2, \ldots, r\})$ such that for each $f \in F$, $a \cdot \prod_{t \in H} f(t) \in A \setminus \{1\}$. To see that r works for $\mathbb{N}$, let $F \in \mathcal{P}_f({}^{\mathbb{N}}\mathbb{N})$ with $|F| \leq k$. Let $G = \{g_f : f \in F\}$. Then $G \in \mathcal{P}_f({}^{\mathbb{N}}(\mathbb{N} \setminus \{1\}))$ and $|G| \leq k$ so pick $a \in \mathbb{N} \setminus \{1\}$ and $H \in \mathcal{P}_f(\{1, 2, \ldots, r\})$ such that for each $f \in F$, $a \cdot \prod_{t \in H} g_f(t) \in A \setminus \{1\}$. Let $m = |H|$ and let $b = 2^m a$. Then for each $f \in F$, $b \cdot \prod_{t \in H} f(t) \in A \setminus \{1\} \subseteq A$. □

Theorem 3.8. *If there exist a countable commutative semigroup $(S, +)$ and a subset A of S which is a J-set and is not a CR-set, then there is a subset B of $\mathbb{N}$ which is a J-set in $(\mathbb{N}, \cdot)$ and is not a CR-set in $(\mathbb{N}, \cdot)$.*

Proof. Enumerate S as $\langle x_n \rangle_{n=1}^{\infty}$ and enumerate the primes as $\langle p_n \rangle_{n=1}^{\infty}$. Define $f : \{p_n : n \in \mathbb{N}\} \to S$ by $f(p_n) = x_n$. Then f extends to a surjective homomorphism $\varphi : \mathbb{N} \setminus \{1\} \to S$ so by Lemma 3.6, $\varphi^{-1}[A]$ is a J-set in $(\mathbb{N} \setminus \{1\}, \cdot)$ and is not a CR-set in $(\mathbb{N} \setminus \{1\}, \cdot)$. By Lemma 3.7, $\varphi^{-1}[A]$ is a J-set in $(\mathbb{N}, \cdot)$ and is not a CR-set in $(\mathbb{N}, \cdot)$. □

Theorem 3.9. *Let S be $(\mathbb{N}, +)$ or $(\mathbb{N}, \cdot)$ and let $k \in \mathbb{N}$. There is a set $A_k \subseteq S$ such that*

(1) *A_k is a k-CR-set;*
(2) *A_k is not a $(k+1)$-CR-set, in fact not a $(k+1)$-J-set;*
(3) *if $k > 1$, then there exist sets B and C such that $A_k = B \cup C$ and neither B nor C is a k-J-set; and*
(4) *if $k > 1$, then $\overline{A_k} \cap k\text{-}J(S) = \emptyset$, so that $\overline{A_k} \cap k\text{-}CR(S) = \emptyset$.*

Proof. By Lemmas 3.6 and 3.7 It suffices to prove the theorem under the assumption that $S = (\mathbb{N}, +)$.

Let $\Gamma = \{(b_1, b_2, \ldots, b_k) \in \mathbb{N}^k : b_1 \leq b_2 \leq \ldots \leq b_k\}$ and enumerate Γ as $\langle \vec{x_n} \rangle_{n=1}^{\infty}$. For each n, let $\vec{x_n} = (b_{n,1}, b_{n,2}, \ldots, b_{n,k})$. Let $\eta_1 = 1$ and for $n \in \mathbb{N}$, let $\eta_{n+1} = 2(\eta_n + b_{n,k})$. For $n \in \mathbb{N}$, let $E_n = \big\{\eta_n + b_{n,i} : i \in \{1, 2, \ldots, k\}\big\}$ and let $A_k = \bigcup_{n=1}^{\infty} E_n$.

(1) To see that A_k is a k-CR-set, let $r = 1$. Let $F \in \mathcal{P}_f({}^{\mathbb{N}}\mathbb{N})$ with $|F| \leq k$. Let $D = \{f(1) : f \in F\}$ and note that $|D| \leq k$ so write D as $\{c_1, c_2, \ldots, c_k\}$ with $c_1 \leq c_2 \leq \ldots \leq c_k$. Pick n such that $(c_1, c_2, \ldots, c_k) = (b_{n,1}, b_{n,2}, \ldots, b_{n,k})$. Let $a = \eta_n$ and let $H = \{1\}$. Then given $f \in H$, $a + \sum_{t \in H} f(t) = \eta_n + f(1) \in E_n \subseteq A_k$.

(2) To see that A_k is not a $(k+1)$-J-set, let $F = \{\overline{k}, \overline{k+1}, \ldots, \overline{2k}\}$. Suppose we have $a \in \mathbb{N}$ and $H \in \mathcal{P}_f(\mathbb{N})$ such that for each $i \in \{0, 1, \ldots, k\}$, $a + \sum_{t \in H} \overline{k+i} \in A_k$. Let $m = |H|$. Then with $i = 0$, $a + \sum_{t \in H} \overline{k+i} = a + mk \in A_k$ and with with $i = k$, $a + \sum_{t \in H} \overline{k+i} = a + 2mk \in A_k$ so there exist $n \leq l$ and $i, j \in \{1, 2, \ldots, k\}$ such that $a + mk = \eta_n + b_{n,i}$ and $a + 2mk = \eta_l + b_{l,j}$.

We claim that $n = l$. Suppose instead that $n < l$. Then

$$\begin{aligned} mk &= \eta_l - (\eta_n + b_{n,i}) + b_{l,j} > \eta_l - (\eta_n + b_{n,i}) \geq \eta_l - (\eta_n + b_{n,k}) \\ &\geq \eta_n + b_{n,k} \geq \eta_n + b_{n,i} = a + mk > mk \,, \end{aligned}$$

a contradiction.

Thus $\{a+mk, a+m(k+1), \ldots, a+2mk\} \subseteq \{\eta_n + b_{n,1}, \eta_n + b_{n,2}, \ldots, \eta_n + b_{n,k}\}$. The left hand side has cardinality $k+1$ while the right hand side has cardinality at most k.

(3) Assume that $k > 1$. Let $B = \bigcup_{n=1}^{\infty} \{\eta_n + b_{n,i} : i \in \{1, 2, \ldots, k-1\}\}$ and let $C = \{\eta_n + b_{n,k} : n \in \mathbb{N}\}$. Let $F = \{\overline{k+1}, \overline{k+2}, \ldots, \overline{2k}\}$. Arguing as in (2), F establishes that B is not a k-J-set and letting $F = \{\overline{k}, \overline{2k}\}$ establishes that C is not a 2-J-set.

(4) Assume that $k > 1$. Pick B and C as guaranteed by (2). Suppose we have $p \in \overline{A_k} \cap k\text{-}J(S)$. Then either B or C is a k-J-set. $\square$

It is an easy exercise to show that for subsets of $\mathbb{N}$, the notions of 1-CR-set, 1-J-set, and infinite are equivalent. So the requirement that $k > 1$ in Theorem 3.9(3) and (4) is needed.

Definition 3.10. Let $(S, \cdot)$ be a semigroup.

(1) S satisfies the *Strong Følner Condition* (SFC) if and only if

$$\big(\forall H \in \mathcal{P}_f(S)\big)(\forall \epsilon > 0)\big(\exists F \in \mathcal{P}_f(S)\big)(\forall s \in H)(|F \setminus sF| < \epsilon \cdot |F|)\,.$$

(2) A *left invariant mean* μ on S is a positive linear functional on the space $l_1(S)$ of bounded real valued functions on S with $||\mu|| = 1$ which is left invariant. That is, for each $g \in l_1(S)$ and each $x \in S$, $\mu(g) = \mu(g \circ \lambda_x)$.

(3) S is left amenable if and only if there exists a left invariant mean on S.

Every semigroup that satisfies SFC is left amenable and every left cancellative and left amenable semigroup satisfies SFC. If S is left amenable, right cancellative, and not left cancellative, then S does not satisfy SFC. See [8, Paragraph 4.22] for details and references for these facts.

We denote the characteristic function of a set A by χ_A.

Definition 3.11. Let $(S, \cdot)$ be a semigroup and let $A \subseteq S$.

(1) If S satisfies SFC, then

$$\begin{aligned} d(A) = \sup\big\{\alpha : \ &\big(\forall H \in \mathcal{P}_f(S)\big)\big(\forall \epsilon > 0\big)\big(\exists K \in \mathcal{P}_f(S)\big)\big(|A \cap K| \geq \alpha \cdot |K| \\ &\text{and } \big(\forall s \in H\big)\big(|K \triangle sK| < \epsilon \cdot |K|\big)\big)\big\}\,. \end{aligned}$$

(2) If S is left amenable, then $d^*(A) = \sup\{\mu(\chi_A) : \mu$ is a left invariant mean on $S\}$.

(3) A is a B-set if and only if S is left amenable and $d^*(A) > 0$.

By [5, Theorems 2.12 and 2.14], if S satisfies SFC, then for each $A \subseteq S$, $d(A) \leq d^*(A)$ and if in addition S is left cancellative, then $d(A) = d^*(A)$. In particular if S satisfies SFC and $d(A) > 0$, then A is a B-set.

We set out to show in Theorem 3.15 that any B-set is a CR-set.

Theorem 3.12 (Density Hales-Jewett)**.** *Let $n \in \mathbb{N}$ and $\eta \in (0,1)$. There exists $k \in \mathbb{N}$ such that whenever $C \subseteq [n]^k$ and $|C| \geq \eta n^k$, there is a length k variable word u over the alphabet $[n]$ such that $\{u(t) : t \in [n]\} \subseteq C$.*

Proof. This is due to Furstenberg and Katznelson in [2]. For a simplified elementary proof see [9] which is an anonymous collaborative effort. □

We will need the following strengthening of the Density Hales-Jewett Theorem.

Corollary 3.13. *Let $n \in \mathbb{N}$ and $\eta \in (0,1)$. There exists $r \in \mathbb{N}$ such that whenever $C \subseteq [n]^r$ and $|C| \geq \eta n^r$, there is a length r variable word w over the alphabet $[n]$ which begins with a constant and has no successive occurrences of the variable such that $\{w(t) : t \in [n]\} \subseteq C$.*

Proof. Pick $k \in \mathbb{N}$ as guaranteed for n and η by Theorem 3.12 and let $r = 2k$. For each $y = y_1 y_2 \cdots y_k \in [n]^k$ define $\varphi_y : [n]^k \to [n]^r$ by, for $z = z_1 z_2 \cdots z_k \in [n]^k$, $\varphi_y(z) = y_1 z_1 y_2 z_2 \cdots y_k z_k$. For $y \in [n]^k$ let $B_y = \varphi_y\big[[n]^k\big]$ Let $C \subseteq [n]^r$ such that $|C| \geq \eta n^r$.

Now $C = \bigcup_{y \in [n]^k} (B_y \cap C)$ and if x and y are distinct members of $[n]^k$, then $B_x \cap B_y = \emptyset$ so $|C| = \sum_{y \in [n]^k} |B_y \cap C|$ so we may pick some $y \in [n]^k$ such that $|B_y \cap C| \geq \eta n^k$. Since φ_y is injective, $|\varphi_y^{-1}[B_y \cap C]| \geq \eta n^k$. Pick a length k variable word $u = u_1 u_2 \cdots u_k$ over the alphabet $[n]$ such that $\{u(t) : t \in [n]\} \subseteq \varphi_y^{-1}[B_y \cap C]$. Let $w = y_1 u_1 y_2 u_2 \cdots y_k u_k$. Then w is a length r variable word which begins with a constant and has no successive occurrences of the variable. And, given $t \in [n]$, $w(t) = \varphi_y\big(u(t)\big) \in C$. □

The proof of the next lemma is adapted from the proof of [1, Theorem 3.2].

Lemma 3.14. *Let $(S,\cdot)$ be a left amenable semigroup, let μ be a left invariant mean on S, let $F \in \mathcal{P}_f(S)$, let $\mathcal{F} : F \to \mathbb{N}$, let $0 < \eta < \delta < 1$, let $A \subseteq S$ such that $\mu(\chi_A) \geq \delta$, and let*

$$R = \{t \in S : \textstyle\sum_{x \in F \cap (At^{-1})} \mathcal{F}(x) \geq \eta \sum_{x \in F} \mathcal{F}(x)\}\,.$$

Then $\mu(\chi_R) \geq \dfrac{\delta - \eta}{1 - \eta}$.

Proof. Define $g : S \to [0,1]$ by $g(t) = \dfrac{\sum_{x \in F \cap (At^{-1})} \mathcal{F}(x)}{\sum_{x \in F} \mathcal{F}(x)}$. Then for $t \in S$,

$$\begin{aligned} g(t) &= \frac{1}{\sum_{x \in F} \mathcal{F}(x)} \textstyle\sum_{x \in F} \mathcal{F}(x) \cdot \chi_{At^{-1}}(x) \\ &= \frac{1}{\sum_{x \in F} \mathcal{F}(x)} \textstyle\sum_{x \in F} \mathcal{F}(x) \cdot \chi_{(x^{-1}A)}(t)\,, \end{aligned}$$

so

$$\begin{aligned}\mu(g) &= \frac{1}{\sum_{x\in F}\mathcal{F}(x)}\cdot\big(\textstyle\sum_{x\in F}\mathcal{F}(x)\cdot\mu(\chi_{x^{-1}A})\big)\\ &= \frac{1}{\sum_{x\in F}\mathcal{F}(x)}\cdot\big(\textstyle\sum_{x\in F}\mathcal{F}(x)\cdot\mu(\chi_{A})\big)\end{aligned}$$

since μ is invariant. Therefore $\mu(g) = \dfrac{\mu(\chi_A)}{\sum_{x\in F}\mathcal{F}(x)}\cdot\sum_{x\in F}\mathcal{F}(x) = \mu(\chi_A)$.

Since μ is additive, $\mu(\chi_A) = \mu(g) \leq \mu(g\chi_R) + \mu(g\chi_{S\setminus R})$. Since $g\chi_R \leq \chi_R$, $\mu(g\chi_R) \leq \mu(\chi_R)$. For $t \in S\setminus R$, $\sum_{x\in F\cap(At^{-1})}\mathcal{F}(x) < \eta\sum_{x\in F}\mathcal{F}(x)$ so $g(t) = \dfrac{\sum_{x\in F\cap At^{-1}}\mathcal{F}(x)}{\sum_{x\in F}\mathcal{F}(x)} < \eta$ and $\mu(\chi_{S\setminus R}) = 1-\mu(\chi_R)$ so $\mu(\chi_A) \leq \mu(\chi_R) + \eta\big(1-\mu(\chi_R)\big)$. Therefore $\mu(\chi_A)-\eta \leq \mu(\chi_R)\cdot(1-\eta)$ so $\mu(\chi_R) \geq \dfrac{\delta-\eta}{1-\eta}$. $\square$

In the event that S is commutative, the next theorem is [1, Theorem 3.3]. Our proof is a modification of the proof of that result.

Theorem 3.15. *Let $(S,\cdot)$ be a semigroup and let $A \subseteq S$ be a B-set. Then A is a CR-set.*

Proof. Pick a left invariant mean μ on S such that $\mu(\chi_A) > 0$ and pick η and δ such that $0 < \eta < \delta \leq \mu(\chi_A)$. Let $n \in \mathbb{N}$. We will show that A is an n-CR-set. Pick $r \in \mathbb{N}$ as guaranteed by Corollary 3.13 for n and η. Let $G \in \mathcal{P}_f({}^{\mathbb{N}}S)$ with $|G| \leq n$. Write $G = \{f_1, f_2, \ldots, f_n\}$ with repetition if $|G| < n$.

Define $\pi : [n]^r \to S$ by, for $w = w_1w_2\cdots w_r \in [n]^r$,

$$\pi(w) = f_{w_1}(1)f_{w_2}(2)\cdots f_{w_r}(r)\,.$$

Let $F = \pi\big[[n]^r\big]$ and define $\mathcal{F}; F \to \mathbb{N}$ by $\mathcal{F}(x) = |\{w \in [n]^r : \pi(w) = x\}|$. Let $R = \{s \in S : \sum_{x\in F\cap(As^{-1})}\mathcal{F}(x) \geq \eta\sum_{x\in F}\mathcal{F}(x)\}$. By Lemma 3.14, $\mu(\chi_R) > 0$ so in particular, $R \neq \emptyset$. Pick $s \in R$.

Then $\sum_{x\in F\cap(As^{-1})}\mathcal{F}(x) \geq \eta\sum_{x\in F}\mathcal{F}(x)$. Note that $\sum_{x\in F}\mathcal{F}(x) = [n]^r$ so

$$\begin{aligned}\eta[n]^r &\leq \textstyle\sum_{x\in F\cap(As^{-1})}\mathcal{F}(x)\\ &= |\{w \in [n]^r : \pi(w) \in As^{-1}\}| = |\{w \in [n]^r : \pi(w)s \in A\}|\,.\end{aligned}$$

Pick a variable word $w = w_1w_2\cdots w_r$ over the alphabet $[n]$ which begins with a constant and has no successive occurrences of the variable such that $\{w(u) : u \in [n]\} \subseteq \{w \in [n]^r : \pi(w)s \in A\}$. Let v be the variable and let $m = \{i \in \{1,2,\ldots,r\} : w_i = v\}$. Let $t(1) < t(2) < \ldots < t(m) \leq r$ enumerate $\{i \in \{1,2,\ldots,r\} : w_i = v\}$.

Let $a(1) = \prod_{i=1}^{t(1)-1} f_{w_i}(i)$. For $j \in \{2,3,\ldots,m\}$, if any, let $a(j) = \prod_{i=t(j-1)+1}^{t(j)-1} f_{w_i}(i)$. If $t(m) = r$, let $a(m+1) = s$. If $t(m) < r$, let $a(m+1) = \big(\prod_{i=t(m)+1}^{r} f_{w_i}(i)\big)s$. Then for $u \in [n]$,

$$\pi\big(w(u)\big)s = a(1)f_u\big(t(1)\big)a(2)\cdots a(m)f_u\big(t(m)\big)a(m+1) \in A\,.$$

□

4. Cartesian products

In [6, Theorem 2.11] it was shown that if S and T are arbitrary semigroups, A is a J-set in S and B is a J-set in T, then $A \times B$ is a J-set in $S \times T$. We would like to know whether the corresponding result holds for CR-sets.

Theorem 4.1. *Let $(S, \cdot)$ be an infinite semigroup, let $k \in \mathbb{N}$, and let A be a $2k$-CR-set in S. Then $A \times A$ is a k-CR-set in $S \times S$. In particular, if A is a CR-set in S, then $A \times A$ is a CR-set in $S \times S$.*

Proof. Let $r \in \mathbb{N}$ be as guaranteed for the fact that A is a $2k$-CR-set in S. We will show that r is as required to show that $A \times A$ is a k-CR-set in $S \times S$ Let $F \in \mathcal{P}_f({}^{\mathbb{N}}(S \times S))$ with $|F| \leq k$. Let $G = \{\pi_1 \circ f : f \in F\} \cup \{\pi_2 \circ f : f \in F\}$. Pick $m \in \mathbb{N}$, $\vec{a} \in S^{m+1}$ and $t(1) < t(2) < \ldots < t(m) \leq r$ such that for all $g \in G$, $a(1)g\big(t(1)\big)a(2) \cdots a(m)g\big(t(m)\big)a(m+1) \in A$. Define $\vec{b} \in (S \times S)^{m+1}$ by $b(i) = \big(a(i), a(i)\big)$. Then for $f \in F$,

$$b(1)f\big(t(1)\big)b(2) \cdots b(m)f\big(t(m)\big)b(m+1) \in A \times A\,.$$

□

Question 4.2. *Do there exist infinite semigroups S and T, a CR-set $A \subseteq S$, and a CR-set $B \subseteq T$ such that $A \times B$ is not a CR-set in $S \times T$?*

For countable commutative semigroups we shall show in Theorem 4.4 that the answer for CR-sets in different semigroups is the same as the answer for CR-sets in $(\mathbb{N}, \cdot)$.

Lemma 4.3. *Let C and D be subsets of $\mathbb{N}$ and let $k \in \mathbb{N}$. Then $C \times D$ is a k-CR-set in $(\mathbb{N} \times \mathbb{N}, \cdot)$ if and only if $(C \setminus \{1\}) \times (D \setminus \{1\})$ is a k-CR-set in $\big((\mathbb{N} \setminus \{1\}) \times (\mathbb{N} \setminus \{1\}, \cdot)\big)$.*

Proof. The proof is very similar to the proof of Lemma 3.7(2). Given $f \in {}^{\mathbb{N}}(\mathbb{N} \times \mathbb{N})$ define $g_f \in {}^{\mathbb{N}}\big((\mathbb{N} \setminus \{1\}) \times (\mathbb{N} \setminus \{1\})\big)$ by $g_f(n) = (2, 2) \cdot f(n)$. In the necessity proof, given $F \in \mathcal{P}_f({}^{\mathbb{N}}\big((\mathbb{N} \setminus \{1\}) \times (\mathbb{N} \setminus \{1\})\big))$ with $|F| \leq k$, let $G = \{g_f : f \in F\}$ and use the fact that $G \in \mathcal{P}_f({}^{\mathbb{N}}(\mathbb{N} \times \mathbb{N}))$.

In the sufficiency proof given $F \in \mathcal{P}_f({}^{\mathbb{N}}(\mathbb{N} \times \mathbb{N}))$ with $|F| \leq k$, let $G = \{g_f : f \in F\}$ and use the fact that $G \in \mathcal{P}_f({}^{\mathbb{N}}\big((\mathbb{N} \setminus \{1\}) \times (\mathbb{N} \setminus \{1\})\big))$. □

Theorem 4.4. *If there exist countable commutative semigroups S and T, a CR-set $A \subseteq S$, and a CR-set $B \subseteq T$ such that $A \times B$ is not a CR-set in $S \times T$, then there are CR-sets C and D in $(\mathbb{N}, \cdot)$ such that $C \times D$ is not a CR-set in $(\mathbb{N} \times \mathbb{N}, \cdot)$.*

Proof. Assume such S, T, A, and B exist. Enumerate S as $\langle x_n \rangle_{n=1}^{\infty}$, enumerate T as $\langle y_n \rangle_{n=1}^{\infty}$, and enumerate the primes as $\langle p_n \rangle_{n=1}^{\infty}$. Define

$f : \{p_n : n \in \mathbb{N}\} \to S$ and $g : \{p_n : n \in \mathbb{N}\} \to T$ by $f(p_n) = x_n$ and $g(p_n) = y_n$. Then f extends to a homomorphism $\varphi : (\mathbb{N} \setminus \{1\}) \to S$ and g extends to a homomorphism $\psi : (\mathbb{N} \setminus \{1\}) \to T$. Let $C = \varphi^{-1}[A]$ and $D = \psi^{-1}[B]$. By Lemma 3.6, C and D are CR-sets in $(\mathbb{N} \setminus \{1\}, \cdot)$ so by Lemma 3.7(2), C and D are CR-sets in $(\mathbb{N}, \cdot)$.

Define $\tau : (\mathbb{N} \setminus \{1\}) \times (\mathbb{N} \setminus \{1\}) \to S \times T$ by $\tau(a, b) = \big(\varphi(a), \psi(b)\big)$. Then by Lemma 3.6, $C \times D = \tau^{-1}[A \times B]$ is not a CR-set in $(\mathbb{N} \setminus \{1\}) \times (\mathbb{N} \setminus \{1\})$. By Lemma 4.3, $C \times D$ is not a CR-set in $\mathbb{N} \times \mathbb{N}$. $\square$

References

[1] Bergelson, V.; Glasscock, D. On the interplay between additive and multiplicative largeness and its combinatorial applications. *J. Combin. Theory Ser. A* **172** (2020), 105203. MR4050733, Zbl 1433.05313.

[2] Furstenberg, H.; Katznelson, Y. A density version of the Hales-Jewett theorem. *J. Anal. Math.* **57** (1991), 64-119. MR1191743 (94f:28020), Zbl 0770.05097.

[3] Hindman, N. Small sets satisfying the central sets theorem. Combinatorial number theory, 57-63, *Walter de Gruyter, Berlin,* 2009. MR2521953 (2011d:05382), Zbl 1173.05045.

[4] Hindman, N. Notions of size in a semigroup: an update from a historical perspective. *Semigroup Forum* **100** (2020), no. 1, 52-76. MR4061568, Zbl 1479.54050.

[5] Hindman, N.; Strauss, D. Density and invariant means in left cancellative left amenable semigroups. *Topology Appl.* **156** (2009), no. 16, 2614-2628. MR2561213 (2011b:43003), Zbl 1189.22003.

[6] Hindman, N.; Strauss, D. Cartesian products of sets satisfying the central sets theorem. *Topology Proc.* **35** (2010), 203-223. MR2534785 (2011a:54008), Zbl 1213.54020.

[7] Hindman, N.; Strauss, D. Algebra in the Stone-Čech compactification: theory and applications. Second revised and extended edition. De Gruyter Textbook. *Walter de Gruyter & Co., Berlin* 2012. xviii+591 pp. ISBN: 978-3-11-025623-9. MR2893605, Zbl 1241.22001, doi:10.1515/9783110258356.

[8] Paterson, A. Amenability. Mathematical Surveys and Monographs, 29. *American Mathematical Society, Providence, RI* 1988. xx+452 pp. ISBN: 0-8218-1529-6. MR0961261 (90e:43001), Zbl 0648.43001.

[9] Polymath, D. A new proof of the density Hales-Jewett theorem. *Ann. of Math.* (2) **175** (2012), no. 3, 1283-1327. MR2912706, Zbl 1267.11010.

Department of Mathematics, Howard University, Washington, DC 20059, USA.
`nhindman@aol.com`

Department of mathematics, Faculty of Basic Science, Shahed University, Tehran, Iran
`hoseinihedie1@gmail.com`

University of Hull, Hull HU6 7RX, UK
`d.strauss@emeritus.hull.ac.uk`

Department of Pure Mathematics, Faculty of Mathematical Sciences, University of Guilan, Rasht, Iran
`tootkaboni.akbari@gmail.com`