Følner, Banach, and translation density are equal and other new results about density in left amenable semigroups

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Abstract

In any semigroup S satisfying the *Strong Følner Condition*, there are three natural notions of density for a subset A of S: Følner density $d(A)$, Banach density $d^*(A)$, and translation density $d_t(A)$ introduced in [2]. If S is commutative or left cancellative, it is known that these three notions coincide. We shall show that these notions coincide for every semigroup S which satisfies the Strong Følner Condition. Using this fact, we solve a problem that has been open for decades, showing that the set of ultrafilters every member of which has positive Følner density is a two sided ideal of βS. We also show that, if S is a left amenable semigroup, then the set of ultrafilters every member of which has positive Banach density is a two sided ideal of βS. We investigate the density properties of subsets of S in the case in which the minimal left ideals of the Stone-Čech compactification βS are singletons. This occurs in many familiar examples, including all semilattices and all semigroups which have a right zero. We show that this is equivalent to the statement that S satisfies the strong Følner condition and that, for every subset A of S, $d(A) \in \{0, 1\}$. We also examine the relation between the density properties of two semigroups when one is a quotient of the other. The Følner density of a subset of S is always determined by some Følner net in S. We show that an arbitrary Følner net in S determines the density of all of the subsets of S. And we prove that, if S and T are left amenable semigroups, then $d^*(A \times B) = d^*(A)d^*(B)$ for every subset A of S and every subset B of T.

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1 Three notions of density in semigroups

We begin by introducing some of the notions that we are concerned with here. Given a set X, we let $\mathcal{P}_f(X)$ be the set of finite nonempty subsets of X. If $(S, \cdot)$ is a semigroup and $x \in S$, we denote left and right multiplication by x by $\lambda_x : S \to S$ and $\rho_x : S \to S$, respectively. For $A \subseteq S$, we define $x^{-1}A = \lambda_x^{-1}[A] = \{y \in S : xy \in A\}$ and $Ax^{-1} = \rho_x^{-1}[A] = \{y \in S : yx \in A\}$.

In the following subsections, we treat three notions of density in semigroups: Banach density d^*, Følner density d, and translation density d_t.

1.1 Banach density

Let $(S, \cdot)$ be a semigroup. Let $l_\infty(S)$ be the set of bounded real valued functions on S with the supremum norm, denoted by $\| \; \|_\infty$. Let $l_\infty(S)^*$ be the set of continuous real valued linear functionals on $l_\infty(S)$ with the dual norm $||\mu|| = \sup\{\mu(f) : ||f||_\infty \leq 1\}$. A *mean* on S is an element of $l_\infty(S)^*$ such that $||\mu|| = 1$ and $\mu \geq 0$, that is, whenever $g \in l_\infty(S)$ and for all $s \in S$, $g(s) \geq 0$, one has that $\mu(g) \geq 0$. A *left invariant mean* on S is a mean μ such that for all $s \in S$ and all $g \in l_\infty(S)$, $\mu(g \circ \lambda_s) = \mu(g)$. The semigroup S is defined to be *left amenable* if and only if there exists a left invariant mean on S.

We denote by $MN(S)$ and $LIM(S)$ the set of means and left invariant means on S, respectively. If μ is a mean on S and $A \subseteq S$, we may use $\mu(A)$ to denote $\mu(\chi_A)$, where χ_A is the characteristic function of A.

The weak* topology of $l_\infty(S)^*$ is defined by stating that a net $\langle \mu_\alpha \rangle_{\alpha \in D}$ converges to a limit μ in this space if and only if the net $\langle \mu_\alpha(f) \rangle_{\alpha \in D}$ converges to $\mu(f)$ in $\mathbb{R}$ for every $f \in l_\infty(S)$. The weak* topology is the restriction to $l_\infty(S)^*$ of the product topology on $\times_{g \in l_\infty(S)} \mathbb{R}$. By the Alaoglu Theorem [6, Theorem B25], the closed unit ball of $l_\infty(S)^*$ is compact in the weak* topology.

The following notion of density is defined in any left amenable semigroup. Following [2], we call it *Banach density*.

Definition 1.1. Let $(S, \cdot)$ be a left amenable semigroup, and let $A \subseteq S$. The *Banach density of A* is defined by $d^*(A) = \sup\{\lambda(\chi_A) : \lambda$ is a left invariant mean on $S\}$.

In the remainder of this subsection, we collect some preliminary results that will be key to relating Banach density to other notions of density later on.

Lemma 1.2. *Let $(S, \cdot)$ be a left amenable semigroup, let λ be a left invariant mean on S, and let R be a right ideal of S. Then $\lambda(\chi_R) = 1$.*

Proof. Pick $a \in R$. Then $\chi_R \circ \lambda_a = \chi_S$ so $\lambda(\chi_R) = \lambda(\chi_R \circ \lambda_a) = \lambda(\chi_S) = 1$. □

Lemma 1.3. *Let $(S, \cdot)$ be a semigroup, and let $\mu \in l_\infty(S)^*$. If $\mu \geq 0$, then $\mu \in MN(S)$ if and only if $\mu(S) = 1$.*

Proof. Since χ_S is the maximum element of the unit ball of $l_\infty(S)$, for any $\mu \geq 0$ in $l_\infty(S)^*$, $||\mu|| = \mu(S)$. Thus $||\mu|| = 1$ if and only if $\mu(S) = 1$. □

In the following lemmas, we make use of the linear subspace E of $l_\infty(S)$ generated by the functions of the form χ_A, where A denotes a subset of S. Note that this is precisely the subset of $l_\infty(S)$ consisting of those functions with finite range.

Lemma 1.4. *Let $(S,\cdot)$ be a semigroup, and let E denote the linear subspace of $l_\infty(S)$ generated by the functions of the form χ_A, where A denotes a subset of S. Then E is uniformly dense in $l_\infty(S)$.*

Proof. This is easy to prove directly by an elementary argument. It also follows from the Stone-Weierstrass Theorem, because $\{\widetilde{f} : f \in E\}$ is a subalgebra of $C(\beta S)$ which separates points and contains the constant function $\chi_{\beta S}$. (Here $\widetilde{f} : \beta S \to \mathbb{R}$ is the continuous extension of f.) □

Lemma 1.5. *Let $(S,\cdot)$ be a semigroup, and let $\mu : \mathcal{P}(S) \to \mathbb{R}$ be a non-negative, finitely additive set function such that $\mu(S) = 1$. Then μ extends uniquely to a mean on S.*

Proof. Denote by E the subset of $l_\infty(S)$ consisting of those functions with finite range. By Lemma 1.4, the set E is uniformly dense in $l_\infty(S)$.

We define a function $\nu : E \to \mathbb{R}$ as follows: for $f \in E$,

$$\nu(f) = \sum_{x \in \mathrm{Range}(f)} x\mu\big(f^{-1}[\{x\}]\big).$$

Note that ν extends μ in the sense that for all $A \subseteq S$, $\nu(\chi_A) = \mu(A)$. By the finite additivity of μ, it is easy to see that for all $n \in \mathbb{N}$, all $c_1, \ldots, c_n \in \mathbb{R}$, and all pairwise disjoint $A_1, \ldots, A_n \subseteq S$, $\nu\left(\sum_{i=1}^n c_i\chi_{A_i}\right) = \sum_{i=1}^n c_i\mu(A_i)$. Now we note that

(*) if $f \in E$, $a > 0$, and $\mathrm{Range}(f) \subseteq [-a, a]$, then $|\nu(f)| \leq a$.

To see this, let such f and a be given. Note that since $\{f^{-1}[\{x\}] : x \in \mathrm{Range}(f)\}$ is a set of pairwise disjoint sets and $\mu(S) = 1$, $\sum_{x\in\mathrm{Range}(f)} \mu\big(f^{-1}[\{x\}]\big) \leq 1$. Then $|\nu(f)| \leq \sum_{x\in\mathrm{Range}(f)} |x|\mu\big(f^{-1}[\{x\}]\big) \leq a\sum_{x\in\mathrm{Range}(f)} \mu\big(f^{-1}[\{x\}]\big) \leq a$, as required.

We claim that for all $f, g \in E$ and $c \in \mathbb{R}$, $\nu(cf) = c\nu(f)$ and $\nu(f+g) = \nu(f)+\nu(g)$. Indeed, the first is immediate. To see the second, suppose $\mathrm{Range}(f) = \{c_1, \ldots, c_n\}$ and $\mathrm{Range}(g) = \{d_1, \ldots, d_m\}$. Let $A_{i,j} = \{s \in S \mid f(s) = c_i \text{ and } g(s) = d_j\}$. Note that $f = \sum_{i=1}^n \sum_{j=1}^m c_i\chi_{A_{i,j}}$, $g = \sum_{i=1}^n \sum_{j=1}^m d_j\chi_{A_{i,j}}$, and that the $A_{i,j}$'s are pairwise disjoint. It follows that

$$\sum_{i=1}^n \sum_{j=1}^m (c_i + d_j)\mu(A_{i,j}) = \sum_{i=1}^n \sum_{j=1}^m c_i\mu(A_{i,j}) + \sum_{i=1}^n \sum_{j=1}^m d_j\mu(A_{i,j}).$$

The left hand side is $\nu(f + g)$ and the right hand side is $\nu(f) + \nu(g)$, whence $\nu(f + g) = \nu(f) + \nu(g)$.

We claim that $\nu : E \to \mathbb{R}$ is uniformly continuous. Indeed, suppose $f, g \in E$ are such that $\|f - g\|_\infty < \epsilon$. By the definition of $|| \cdot ||_\infty$, the range of

$f - g$ is contained in the interval $[-\epsilon, \epsilon]$. Since $\mu(S) = 1$, we see from (*) that $|\nu(f) - \nu(g)| = |\nu(f - g)| < \epsilon$.

Because ν is uniformly continuous and E is dense, the function ν extends uniquely to a uniformly continuous function $\nu : l_\infty(S) \to \mathbb{R}$. Clearly $\nu(\chi_S) = 1$ and $\nu \geq 0$. For $c \in \mathbb{R}$, the functions $f \mapsto \nu(cf)$ and $f \mapsto c\nu(f)$ are continuous and agree on E, hence are equal. Similarly, the functions $(f, g) \mapsto \nu(f + g)$ and $(f, g) \mapsto \nu(f) + \nu(g)$ are continuous and agree on E, hence are equal. Therefore, ν is linear, and hence is a mean on S.

It is easy to see that any mean extending μ must be equal to ν when restricted to the set E. It follows by the work above, then, that ν is the unique mean on S extending μ. □

We remind the reader that, if $s \in S$ and $A \subseteq S$, $s^{-1}A$ denotes $\{t \in S : st \in A\}$.

Lemma 1.6. *Let $(S, \cdot)$ be a semigroup, and let $\mu \in MN(S)$. Then $\mu \in LIM(S)$ if and only if $\mu(s^{-1}A) = \mu(A)$ for every $s \in S$ and every $A \subseteq S$.*

Proof. Observe that $\chi_{s^{-1}A} = \chi_A \circ \lambda_s$ for every $s \in S$ and every $A \subseteq S$. So, if $\mu \in LIM(S)$, $\mu(s^{-1}A) = \mu(A)$ for every $s \in S$ and every $A \subseteq S$.

To prove the converse, assume that μ is a mean on S with the property that $\mu(s^{-1}A) = \mu(A)$ for every $s \in S$ and every $A \subseteq S$.

For $s \in S$, define $\tau_s : l_\infty(S)^* \to l_\infty(S)^*$ by, for $f \in l_\infty(S)$, $\tau_s(\eta)(f) = \eta(f \circ \lambda_s)$. Note that τ_s is a linear map. We claim that τ_s is continuous for the norm topology on $l_\infty(S)^*$. By [6, Theorem B.10], it suffices to show that τ_s is bounded. So let $\eta \in l_\infty(S)^*$. We claim that $||\tau_s(\eta)|| \leq ||\eta||$, for which it suffices to let $f \in l_\infty(S)$ and note that $||f \circ \lambda_s||_\infty \leq ||f||_\infty$, whereby $|\tau_s(\eta)(f)| = |\eta(f \circ \lambda_s)| \leq |\eta(f)|$.

Let E denote the linear subspace of $l_\infty(S)$ generated by the functions χ_A, where A denotes a subset of S, which is uniformly dense in $l_\infty(S)$ by Lemma 1.4. Given $s \in S$ and $A \subseteq S$, we have noted that $\chi_{s^{-1}A} = \chi_A \circ \lambda_s$, so that $\tau_s(\mu)(\chi_A) = \mu(\chi_A \circ \lambda_s) = \mu(\chi_{s^{-1}A}) = \mu(\chi_A)$. This implies that $\tau_s(\mu)(f) = \mu(f)$ for every $f \in E$. Since E is dense in $l_\infty(S)$ and τ_s is continuous, this implies that $\mu(f \circ \lambda_s) = \mu(f)$ for every $f \in l_\infty(S)$, so that $\mu \in LIM(S)$. □

1.2 Følner density

The following conditions provide a way to understand left amenability in terms of sets and their images under left multiplication.

Definition 1.7. Let $(S, \cdot)$ be a semigroup.

(a) The semigroup S satisfies the *Følner Condition* (FC) if and only if $\big(\forall H \in \mathcal{P}_f(S)\big)(\forall \epsilon > 0)\big(\exists K \in \mathcal{P}_f(S)\big)(\forall s \in H)(|sK \setminus K| < \epsilon \cdot |K|)$.

(b) The semigroup S satisfies the *Strong Følner Condition* (SFC) if and only if $\big(\forall H \in \mathcal{P}_f(S)\big)(\forall \epsilon > 0)\big(\exists K \in \mathcal{P}_f(S)\big)(\forall s \in H)(|K \setminus sK| < \epsilon \cdot |K|)$.

The Strong Følner Condition leads to a natural notion of density.

Definition 1.8. Let $(S,\cdot)$ be a semigroup which satisfies SFC, and let $A \subseteq S$. The *Følner density of A* is defined by
$d(A) = \sup\{\alpha \in [0,1] : \big(\forall H \in \mathcal{P}_f(S)\big)(\forall \epsilon > 0)\big(\exists K \in \mathcal{P}_f(S)\big)$
$\big((\forall s \in H)(|K \setminus sK| < \epsilon \cdot |K|) \text{ and } |A \cap K| \geq \alpha \cdot |K|\big)\}$.

The reader is referred to [16, Section 4.22] for a readable discussion of the relationship among FC, SFC, and left amenability and relevant references. In particular, any left amenable semigroup satisfies FC and any semigroup satisfying SFC is left amenable. By [1, Theorem 4], any commutative semigroup satisfies SFC and by (the left-right switch of) [15, Corollary 3.6], any left amenable and left cancellative semigroup satisfies SFC.

Definition 1.9. Let $(S,\cdot)$ be a semigroup. A *Følner net* in $\mathcal{P}_f(S)$ is a net $\langle F_\alpha \rangle_{\alpha\in D}$ such that for each $s \in S$, $\lim_{\alpha \in D} \dfrac{|F_\alpha \setminus sF_\alpha|}{|F_\alpha|} = 0$.

Of course, a *Følner sequence* is a Følner net in which the relevant directed set is the set $\mathbb{N}$ of positive integers, with its usual order. It is immediate that if there exists a Følner net in $\mathcal{P}_f(S)$, then S satisfies SFC. It is a consequence of Theorem 1.13 below that the converse holds.

Definition 1.10. Let $(S,\cdot)$ be a semigroup.

(1) For $F \in \mathcal{P}_f(S)$, $\mu_F \in l_\infty(S)^*$ is defined by, for $g \in l_\infty(S)$, $\mu_F(g) = \frac{1}{|F|}\sum_{t\in F} g(t)$.

(2) $LIM_0(S) = \{\nu :$ there exists a Følner net $\langle F_\alpha\rangle_{\alpha\in D}$ in $\mathcal{P}_f(S)$ such that ν is a cluster point of the net $\langle \mu_{F_\alpha}\rangle_{\alpha\in D}$ in the weak* topology of $l_\infty(S)^*\}$.

As the notation suggests, elements of $LIM_0(S)$ are, in fact, left invariant means. This is recorded in the following lemma.

Lemma 1.11. *Let $(S,\cdot)$ satisfy SFC. Then $LIM_0(S) \subseteq LIM(S)$.*

Proof. This is shown in [12, Lemma 2.2]. □

Theorem 1.12. *Let $(S,\cdot)$ be a semigroup satisfying SFC. Then $LIM(S)$ is convex, and $LIM(S)$ and $LIM_0(S)$ are both compact in the weak* topology of $l_\infty(S)^*$.*

Proof. It is easy to see that $LIM(S)$ is convex and weak* compact. We shall show that $LIM_0(S)$ is weak* compact.

For every $H \in \mathcal{P}_f(S)$ and every $\epsilon > 0$, let

$$\mathcal{F}_{H,\epsilon} = \{F \in \mathcal{P}_f(S) : (\forall s \in H)(|F \setminus sF| < \epsilon \cdot |F|)\},$$

and let $\mu_{H,\epsilon} = \{\mu_F : F \in \mathcal{F}_{H,\epsilon}\}$. We shall show that

$$LIM_0(S) = \bigcap\{c\ell\mu_{H,\epsilon} : H \in \mathcal{P}_f(S) \text{ and } \epsilon > 0\}.$$

This will suffice since each μ_F is a mean on S. First, let $\nu \in LIM_0(S)$ and pick a Følner net $\langle F_\alpha \rangle_{\alpha \in D}$ in $\mathcal{P}_f(S)$ such that ν is a cluster point of $\langle \mu_{F_\alpha} \rangle_{\alpha \in D}$ in $l_\infty(S)^*$. Let $H \in \mathcal{P}_f(S)$ and $\epsilon > 0$. To see that $\nu \in c\ell\mu_{H,\epsilon}$, let U be a neighborhood of ν in $l_\infty(S)^*$. Pick $\gamma \in D$ such that for all $s \in H$ and all $\alpha \geq \gamma$ in D, $|F_\alpha \setminus sF_\alpha| < \epsilon \cdot |F_\alpha|$ and pick $\alpha \geq \gamma$ such that $\mu_{F_\alpha} \in U$. Then $\mu_{F_\alpha} \in U \cap \mu_{H,\epsilon}$.

Now let $\nu \in \bigcap\{c\ell\mu_{H,\epsilon} : H \in \mathcal{P}_f(S) \text{ and } \epsilon > 0\}$. Let $\mathcal{U}$ be the set of open neighborhoods of ν in $l_\infty(S)^*$. Let $D = \mathcal{U} \times \mathcal{P}_f(S) \times (0,1)$ and direct D by $(U, H, \epsilon) \leq (V, K, \delta)$ provided $V \subseteq U$, $H \subseteq K$, and $\delta \leq \epsilon$. For $\alpha = (U, H, \epsilon) \in D$, pick $F_\alpha \in \mathcal{F}_{H,\epsilon}$ such that $\mu_{F_\alpha} \in U \cap \mu_{H,\epsilon}$. Then $\langle F_\alpha \rangle_{\alpha \in D}$ is a Følner net in $\mathcal{P}_f(S)$ and $\langle \mu_{F_\alpha} \rangle_{\alpha \in D}$ converges to ν. □

Theorem 1.13. *Let $(S, \cdot)$ be a semigroup satisfying SFC, let $A \subseteq S$, and let $\delta = d(A)$. There is a Følner net $\langle F_\alpha \rangle_{\alpha \in D}$ such that the net $\left\langle \dfrac{|F_\alpha \cap A|}{|F_\alpha|} \right\rangle_{\alpha \in D}$ converges to δ.*

If ν is any cluster point of the net $\langle \mu_{F_\alpha} \rangle_{\alpha \in D}$ in $\times_{f \in l_\infty(S)}[-||f||_\infty, ||f||_\infty]$, then $\nu \in LIM_0(S)$ and $\nu(\chi_A) = \delta$. In particular $d(A) \leq d^*(A)$.

Proof. Since $d(A) = \delta$, it is a routine exercise to show that
$\big(\forall H \in \mathcal{P}_f(S)\big)(\forall \epsilon > 0)\big(\exists K \in \mathcal{P}_f(S)\big)\big((\forall s \in H)(|K \setminus sK| < \epsilon \cdot |K|)$ and
$(\delta - \epsilon) \cdot |K| < |A \cap K| < (\delta + \epsilon) \cdot |K|\big)$.

Let $D = \mathcal{P}_f(S) \times \mathbb{N}$, and direct D by $(H, n) \leq (K, m)$ if and only if $H \subseteq K$ and $n \leq m$. For $\alpha = (H, n) \in D$, pick $F_\alpha \in \mathcal{P}_f(S)$ such that
$(\forall s \in H)(|F_\alpha \setminus sF_\alpha| < \frac{1}{n} \cdot |F_\alpha|$ and $(\delta - \frac{1}{n}) \cdot |F_\alpha| < |F_\alpha \cap A| < (\delta + \frac{1}{n}) \cdot |F_\alpha|$.

Let ν be a cluster point of the net $\langle \mu_{F_\alpha} \rangle_{\alpha \in D}$. Since $\langle \mu_{F_\alpha}(\chi_A) \rangle_{\alpha \in D}$ converges to δ, we have that $\nu(\chi_A) = \delta$. □

Theorem 1.14. *Let $(S, \cdot)$ be a semigroup satisfying SFC, and let $A \subseteq S$. Then $d(A) = \max\{\nu(\chi_A) : \nu \in LIM_0(S)\}$.*

Proof. By Theorem 1.13, it suffices to show that if $\nu \in LIM_0(S)$, then $\nu(\chi_A) \leq d(A)$. So let $\nu \in LIM_0(S)$ and suppose that $\nu(\chi_A) = \delta > \gamma = d(A)$. Pick a Følner net $\langle F_a \rangle_{a \in D}$ in $\mathcal{P}_f(S)$ such that ν is a cluster point of the net $\langle \mu_{F_a} \rangle_{a \in D}$ in $\times_{f \in l_\infty(S)}[-||f||_\infty, ||f||_\infty]$.

Let $\beta = \delta - \gamma$. We shall show that $d(A) \geq \delta - \frac{\beta}{2}$, a contradiction. So let $H \in \mathcal{P}_f(S)$ and $\epsilon > 0$ be given. Since $\langle F_a \rangle_{a \in D}$ is a Følner net, pick $a \in D$ such that for every $b \in D$ with $b \geq a$ and every $s \in H$, $\dfrac{|F_b \setminus sF_b|}{|F_b|} < \epsilon$. Let $g = \chi_A$ and let $U = \pi_g^{-1}[(\delta - \frac{\beta}{2}, \delta + \frac{\beta}{2})]$. Then U is a neighborhood of ν, so pick $b \in D$ such that $b \geq a$ and $\mu_{F_b} \in U$. Then $\dfrac{|A \cap F_b|}{|F_b|} = \mu_{F_b}(g) > \delta - \dfrac{\beta}{2}$. □

Question 1.15. *Let $(S, \cdot)$ be a semigroup satisfying SFC. Is $LIM(S)$ the weak* closed convex hull of $LIM_0(S)$?*

We know that the answer is affirmative if S is left cancellative. This follows from the Krein-Milman Theorem and [7, Corollary 2.13].

1.3 Translation density

We will consider a third notion of density, denoted by d_t, which is defined in any semigroup. (The t stands for *translate* which seems appropriate if the operation is written additively (so $As^{-1} = \{x \in S : xs \in A\}$ in the definition becomes $A - s = \{x \in S : x + s \in A\}$).) This notion of density was defined first in [2, Theorem 3.2] but appeared and was considered in relation to the upper Banach density in various other places: in [3, Lemma 9.6] for $(\mathbb{R}^d, +)$, in [5, Corollary 9.2] for $(\mathbb{Z}, +)$, and in a pre-publication version of [13] (as Theorem G) for cancellative semigroups satisfying SFC.

Definition 1.16. Let $(S, \cdot)$ be a semigroup and let $A \subseteq S$. Then $d_t(A) = \sup\{\alpha \in [0,1] : \big(\forall F \in \mathcal{P}_f(S)\big)(\exists s \in S)(|F \cap As^{-1}| \geq \alpha \cdot |F|)\}$.

The proof of the next lemma is based on the proof of [2, Theorem 3.2]. This lemma plays an important role in our paper.

Lemma 1.17. *Let $(S, \cdot)$ be a left amenable semigroup, let $A \subseteq S$, let μ be a left invariant mean on S such that $\mu(\chi_A) > 0$. Assume that $F \in \mathcal{P}_f(S)$ and $0 < \eta < \delta \leq \mu(\chi_A)$ and let $R = \{s \in S : |F \cap As^{-1}| \geq \eta \cdot |F|\}$. Then $\mu(\chi_R) \geq \dfrac{\delta - \eta}{1 - \eta}$. In particular, $R \neq \emptyset$.*

Proof. Note that for $x \in S$, $\chi_{x^{-1}A} = \chi_A \circ \lambda_x$ so $\mu(\chi_{x^{-1}A}) = \mu(\chi_A \circ \lambda_x) = \mu(\chi_A)$. Define $g : S \to [0,1]$ by $g(t) = \dfrac{|F \cap At^{-1}|}{|F|}$. Then for $t \in S$, $g(t) = \frac{1}{|F|}\sum_{x \in F} \chi_{At^{-1}}(x) = \frac{1}{|F|}\sum_{x \in F} \chi_{x^{-1}A}(t)$ so $g = \frac{1}{|F|}\sum_{x \in F} \chi_{x^{-1}A}$. Thus $\mu(g) = \frac{1}{|F|}\sum_{x \in F} \mu(\chi_{x^{-1}A}) = \frac{1}{|F|}\sum_{x \in F} \mu(\chi_A) = \mu(\chi_A)$.

Since μ is additive, $\mu(\chi_A) = \mu(g) \leq \mu(g\chi_R) + \mu(g\chi_{S \setminus R})$. Since $g\chi_R \leq \chi_R$, $\mu(g\chi_R) \leq \mu(\chi_R)$. For $t \in S \setminus R$, $|F \cap At^{-1}| < \eta \cdot |F|$ so $g(t) < \eta$. Also, $\mu(\chi_{S \setminus R}) = 1 - \mu(\chi_R)$ so $\mu(\chi_A) \leq \mu(g\chi_R) + \mu(g\chi_{S \setminus R}) \leq \mu(\chi_R) + \eta\mu(\chi_{S \setminus R}) = \mu(\chi_R) + \eta\big(1 - \mu(\chi_R)\big)$. Therefore $\mu(\chi_A) - \eta \leq \mu(\chi_R) \cdot (1 - \eta)$ so $\mu(\chi_R) \geq \dfrac{\mu(\chi_A) - \eta}{1 - \eta} \geq \dfrac{\delta - \eta}{1 - \eta}$. □

Corollary 1.18. *Let $(S, \cdot)$ be a left amenable semigroup. For every subset A of S, $d_t(A) \geq d^*(A)$.*

Proof. Suppose $d_t(A) < d^*(A)$, pick η such that $d_t(A) < \eta < d^*(A)$, and pick $\mu \in LIM(S)$ such that $\delta = \mu(\chi_A) > \eta$. Since $d_t(A) < \eta$ pick $F \in \mathcal{P}_f(S)$ such that for all $s \in S$, $|F \cap As^{-1}| < \eta \cdot |F|$. But then the set R in Lemma 1.17 is empty, a contradiction. □

Combining the conclusions of Theorem 1.13 and Corollary 1.18, we have shown that in any semigroup $(S, \cdot)$ satisfying SFC, for all $A \subseteq S$,

$$d(A) \leq d^*(A) \leq d_t(A).$$

Bergelson and Glasscock [2, Theorem 3.5] showed that these three quantities are equal if S satisfies SFC and is either left cancellative or commutative. We shall show in Theorem 3.15 in Section 3 that the same conclusion holds for any semigroup satisfying SFC.

It is easy to see that in any left amenable semigroup, d^* is both left invariant and subadditive. Therefore, as a consequence of Theorem 3.15, if S satisfies SFC, then both d and d_t are left invariant and subadditive. While d_t is defined on the subsets of every semigroup, there are semigroups in which d_t is neither subadditive nor left invariant. For example, let S be the free semigroup on two generators a and b, let A be the set of elements of S whose first letter is a, and let B be the set of elements of S whose first letter is b. Then $d_t(A) = d_t(B) = 0$, because $\{a\} \cap Bs^{-1} = \emptyset$ for every $s \in S$. However, $d_t(A \cup B) = 1$. Furthermore, $d_t(a^{-1}A) = 1$, because $a^{-1}A = S$.

Question 1.19. *Let $(S, \cdot)$ be a semigroup.*

(1) If S is left amenable and $A \subseteq S$, must $d_t(A) = d^*(A)$?

(2) Is $d_t(x^{-1}A) \geq d_t(A)$ for every $A \subseteq S$ and every $x \in S$?

If S is any semigroup, $A \subseteq S$, and x is a left cancelable element of S, then it is easy to show that $d_t(x^{-1}A) \geq d_t(A)$. We do not know of any example for which $d_t(x^{-1}A) < d_t(A)$.

2 The ideal of ultrafilters whose members have positive density

Given a discrete semigroup $(S, \cdot)$, the Stone-Čech compactification, βS of S, is the set of ultrafilters on S. We identify a point $x \in S$ with the principal ulrafilter $\{A \subseteq S : x \in A\}$. The topology on βS has a basis consisting of the open and closed subsets $\{\overline{A} : A \subseteq S\}$, where $\overline{A} = \{p \in \beta S : A \in p\}$. The notation is justified by the fact that $\overline{A} = c\ell_{\beta S}(A)$.

If X is a compact Hausdorff space and $f : S \to X$, we denote by $\widetilde{f}$ the continuous extension of f taking βS to X. The operation on S extends to βS, making $(\beta S, \cdot)$ a right topological semigroup with S contained in the topological center of βS. That is, for each $p \in \beta S$, the function $\rho_p : \beta S \to \beta S$ defined by $\rho_p(q) = qp$ is continuous and for each $x \in S$, the function $\lambda_x : \beta S \to \beta S$ defined by $\lambda_x(q) = xq$ is continuous. As does any compact Hausdorff right topological semigroup, βS has a smallest two sided ideal, denoted $K(\beta S)$, which is the union of all minimal left ideals and is also the union of all minimal right ideals. Any two minimal left ideals are isomorphic, any two minimal right ideals are isomorphic, and the intersection of any minimal right ideal and any minimal left ideal is a group. For basic information about the algebraic structure of βS see [8, Part I].

Definition 2.1. Let $(S, \cdot)$ be a semigroup satisfying SFC. Then
$\Delta(S) = \{p \in \beta S : (\forall A \in p)(d(A) > 0)\}$.

Definition 2.2. Let $(S, \cdot)$ be a left amenable semigroup. Then
$\Delta^*(S) = \{p \in \beta S : (\forall A \in p)(d^*(A) > 0)\}$.

For any semigroup $(S, \cdot)$ that satisfies SFC, $\Delta(S)$ is a left ideal of βS. It has been an open question for decades whether $\Delta(S)$ is a right ideal of βS. We will show at the conclusion of this section that for any left amenable semigroup S, $\Delta^*(S)$ is an ideal of βS. When we have shown that $d = d^*$ for semigroups satisfying SFC, the following lemma will be an immediate consequence. But we need it earlier, and it follows quickly from a recent result.

Lemma 2.3. *If $(S, \cdot)$ is a semigroup satisfying SFC, $c\ell K(\beta S) \subseteq \Delta(S)$.*

Proof. Let $p \in c\ell K(\beta S)$ and let $A \in p$. By [8, Corollary 4.41], A is piecewise syndetic, so by [12, Corollary 3.5(2)], $d(A) > 0$. □

It is the first point in Lemma 2.5 that justifies the notation "Aq^{-1}" in the following definition.

Definition 2.4. Let $(S, \cdot)$ be a semigroup, $A \subseteq S$, and $q \in \beta S$. We define

$$Aq^{-1} = \big\{s \in S : s^{-1}A \in q\big\}.$$

Lemma 2.5. *Let $(S, \cdot)$ be a semigroup. For all $A, B \subseteq S$, $t \in S$, and $p, q \in \beta S$,*

1. *$A \in pq$ if and only if $Aq^{-1} \in p$.*
2. *$t^{-1}(Aq^{-1}) = (t^{-1}A)q^{-1}$.*
3. *$(A \cup B)q^{-1} = Aq^{-1} \cup Bq^{-1}$.*
4. *$(A \cap B)q^{-1} = Aq^{-1} \cap Bq^{-1}$. Therefore, if the sets A and B are disjoint, then the sets Aq^{-1} and Bq^{-1} are disjoint.*

Proof. These are all routine computations using the fact from [8, Theorem 4.12] that for any p and $q \in \beta S$ and any $A \subseteq S$, $A \in pq$ if and only if $\{s \in S : s^{-1} \in q\} \in p$. □

Lemma 2.6. *Let $(S, \cdot)$ be a semigroup, $q \in \beta S$, and $\mu \in MN(S)$. Define $\nu : \mathcal{P}(S) \to \mathbb{R}$ by putting $\nu(A) = \mu(Aq^{-1})$ for every $A \subseteq S$. Then the set function ν extends uniquely to a mean on S. If μ is left invariant, then ν extends uniquely to a left invariant mean on S.*

Proof. The set function ν is real valued and non-negative. It follows from Lemma 2.5 and the finite additivity of μ that ν is finitely additive. Moreover, since $Sq^{-1} = S$, we have that $\nu(S) = 1$. It follows from Lemma 1.5 that ν extends uniquely to a mean on S. Abusing notation slightly, we denote this mean by ν.

We wish to show that if μ is left invariant, then ν is left invariant. Let $A \subseteq S$ and $s \in S$. We see by Lemma 2.5 that

$$\nu(s^{-1}A) = \mu\big((s^{-1}A)q^{-1}\big) = \mu\big(s^{-1}(Aq^{-1})\big) = \mu(Aq^{-1}) = \nu(A).$$

It follows by Lemma 1.6 that ν is left invariant. □

We present now one of the major results of this paper, namely that whenever S is a left amenable semigroup, $\Delta^*(S)$ is an ideal of βS. Part of that conclusion is very easy.

Lemma 2.7. *Let $(S,\cdot)$ be a left amenable semigroup. Then $\Delta^*(S)$ is a left ideal of βS.*

Proof. Let $p \in \Delta^*(S)$, let $q \in \beta S$, and let $A \in qp$. Then $\{s \in S : s^{-1}A \in p\} \in q$ so pick $s \in S$ such that $s^{-1}A \in p$. Then $d^*(s^{-1}A) > 0$ so pick $\eta \in LIM(S)$ such that $\eta(\chi_{s^{-1}A}) > 0$. Since $\chi_A \circ \lambda_s = \chi_{s^{-1}A}$, $\eta(\chi_A) = \eta(\chi_{s^{-1}A})$ so $d^*(A) > 0$. □

Theorem 2.8. *Let $(S,\cdot)$ be a left amenable semigroup. Then $\Delta^*(S)$ is a two sided ideal of βS.*

Proof. Let $p \in \Delta^*(S)$, let $q \in \beta S$, and let $A \in pq$. We shall show that $d^*(A) > 0$. This will establish that $pq \in \Delta^*(S)$ and hence that $\Delta^*(S)$ is a right ideal of $\beta(S)$. It will follow then from Lemma 2.7 that $\Delta^*(S)$ is a two-sided ideal of $\beta(S)$.

Let $P = Aq^{-1}$. Then $P \in p$ so $d^*(P) > 0$, and we can choose $\mu \in LIM(S)$ for which $\mu(P) > 0$.

We define ν on $\mathcal{P}(S)$ by $\nu(B) = \mu(Bq^{-1})$ for every $B \subseteq S$. By Lemma 2.6, ν extends uniquely to a member of $LIM(S)$. Now $P = Aq^{-1}$ so $\nu(A) = \mu(Aq^{-1}) = \mu(P) > 0$. Therefore, $d^*(A) > 0$. □

We will need to use the following notion of size for a semigroup.

Definition 2.9. Let $(S,\cdot)$ be a semigroup. A set $A \subseteq S$ is *thick* if and only if for each $F \in \mathcal{P}_f(S)$, there is some $x \in S$ such that $Fx \subseteq A$.

The following theorem shows that thickness is characterized by having density 1 with respect to each of the three notions of density considered in this paper.

Theorem 2.10. *Let $(S,\cdot)$ be a semigroup, and let A be a subset of S.*

(1) $d_t(A) = 1$ if and only if A is thick.

(2) If S is left amenable, $d^*(A) = 1$ if and only if A is thick.

(3) If A satisfies SFC, $d(A) = 1$ if and only if A is thick.

Proof. (1) Assume that A is thick. Then, for every finite subset F of S, there exists $s \in S$ such that $Fs \subseteq A$ so that $|F \cap As^{-1}| = |F|$.

If A is not thick, there is a finite subset F of S such that, for every $s \in S$, $Fs \not\subseteq A$. So, for every $s \in F$, $|F \cap As^{-1}| \leq |F| - 1$ and $d_t(A) \leq \frac{|F|-1}{|F|} < 1$.

(2) If $d^*(A) = 1$, then by Corollary 1.18, $d_t(A) = 1$, so (1) applies.

Now assume that A is thick. The family $\mathcal{F} = \{s^{-1}A : s \in S\}$ has the finite intersection property and hence is contained in some ultrafilter $q \in \beta S$. By the choice of q, we have that $Aq^{-1} = S$. Pick $\mu \in LIM(S)$, and define

$\nu : \mathcal{P}(S) \to \mathbb{R}$ by $\nu(B) = \mu(Bq^{-1})$ for all $B \subseteq S$. By Lemma 2.6, ν extends uniquely to a member of $LIM(S)$ and $\nu(A) = \mu(S) = 1$, so $d^*(A) = 1$.

(3) This statement was proved in [12, Theorems 2.4 and 3.4]. It is also a consequence of the fact that $d(A) = d^*(A) = d_t(A)$ for every subset A of S, which we shall prove in Theorem 3.15. □

3 Density and quotients

In this section, we investigate quotients of left amenable semigroups, paying special attention to the *left cancellative quotient* produced by Lemma 3.8. And we establish that $d = d^*$ in any semigroup satisfying SFC.

Definition 3.1. Let $(S, \cdot)$ be a left amenable semigroup, and let h be a homomorphism from S onto a semigroup $(T, \cdot)$. For $\nu \in l_\infty(S)^*$, define $\nu_h \in l_\infty(T)^*$ by $\nu_h(g) = \nu(g \circ h)$ for each $g \in l_\infty(T)$.

Lemma 3.2. *Let $(S, \cdot)$ be a left amenable semigroup, let h be a homomorphism from S onto a semigroup $(T, \cdot)$, and let $\nu \in LIM(S)$. Then $\nu_h \in LIM(T)$.*

Proof. Given $g \in l_\infty(T)$, $||g \circ h||_\infty = ||g||_\infty$, so ν_h is a mean. To see that ν_h is left invariant, let $g \in l_\infty(T)$ and let $x \in T$. Pick $s \in S$ such that $h(s) = x$. Then $g\circ\lambda_x\circ h = g\circ h\circ\lambda_s$ so $\nu_h(g\circ\lambda_x) = \nu(g\circ\lambda_x\circ h) = \nu(g\circ h\circ\lambda_s) = \nu(g\circ h) = \nu_h(g)$. □

Theorem 3.3. *Let $(S, \cdot)$ be a left amenable semigroup, let h be a homomorphism from S onto a semigroup $(T, \cdot)$, and let $\mu \in LIM(T)$. There exists $\nu \in LIM(S)$ such that $\nu_h = \mu$.*

Proof. Let $E = \{f \circ h : f \in l_\infty(T)\}$. Then E is a linear subspace of $l_\infty(S)$. Define $\eta : E \to \mathbb{R}$ by $\eta(f \circ h) = \mu(f)$, noting that η is well defined. Note also that if $g \in E$ and $g(s) \geq 0$ for all $s \in S$, then $\eta(g) \geq 0$. Since $\chi_S = \chi_T \circ h$ we have that $\eta(\chi_S) = \mu(\chi_T) = 1$ so $||\eta|| = 1$.

We need to produce $\nu \in LIM(S)$ which agrees with η on E.

We claim that, for every $g \in E$ and every $s \in S$, $g \circ \lambda_s \in E$ and $\eta(g \circ \lambda_s) = \eta(g)$. To see this, let $t = h(s)$ and let $g = f \circ h$, where $f \in l_\infty(T)$. Observe that $(f \circ h \circ \lambda_s)(x) = f\big(h(sx)\big) = (f \circ \lambda_t \circ h)(x)$ for every $x \in S$. So $g \circ \lambda_s = f \circ \lambda_t \circ h$. Then $g \circ \lambda_s \in E$ and $\eta(g \circ \lambda_s) = \mu(f \circ \lambda_t) = \mu(f) = \eta(g)$.

By [6, Theorem B.14] (a version of the Hahn-Banach Theorem), there is an extension $\widetilde{\eta}$ of η to $l_\infty(S)$ with $||\widetilde{\eta}|| = 1$.

Let $X = \{\rho \in l_\infty(S)^* : ||\rho|| = 1 \text{ and } (\forall g \in E)\big(\rho(g) = \eta(g)\big)\}$. Then $\widetilde{\eta} \in X$ so $X \neq \emptyset$. We claim that if $\rho \in X$, $\rho \geq 0$ so that the members of X are all means. To see this, suppose that $\rho(g) < 0$ for some $g \geq 0$ in $l_\infty(S)$. We may suppose that $||g|| \leq 1$. Then $||\chi_S - g|| \leq 1$ and $\rho(\chi_S - g) > 1$, contradicting the assumption that $||\rho|| = 1$.

For $s \in S$ and $\rho \in X$, we define $s * \rho$ in X by $s * \rho(g) = \rho(g \circ \lambda_s)$ for each $g \in l_\infty(S)$. We want to apply Day's Fixed Point Theorem [16, Theorem 1.14]. The conclusion of that theorem is that there is some $\nu \in X$ such that for each $s \in S$, $s * \nu = \nu$. Then for each $g \in l_\infty(S)$ and each $s \in S$, $\nu(g \circ \lambda_s) = \nu(g)$, so that ν is a left invariant mean on S. Since $\nu \in X$, for each $f \in l_\infty(T)$, $\nu_h(f) = \nu(f \circ h) = \eta(f \circ h) = \mu(f)$, as required.

To apply Day's Fixed Point Theorem, we need to show

(1) for $s \in S$ and $\rho \in X$, $s * \rho \in X$;

(2) for $s, t \in S$ and $\rho \in X$, $s * (t * \rho) = (st) * \rho$;

(3) X is compact in $\times_{f \in l_\infty(S)}[-||f||_\infty, ||f||_\infty]$;

(4) X is convex; and

(5) for $\alpha \in [0,1]$, ρ_1 and ρ_2 in X, and $s \in S$, $s * (\alpha\rho_1 + (1-\alpha)\rho_2) = \alpha(s * \rho_1) + (1-\alpha)(s * \rho_2)$.

For (1), we need that $||s * \rho|| = 1$ and for each $g \in E$, $(s * \rho)(g) = \eta(g)$. Since $||s * \rho|| \leq ||\rho|| = 1$ and $s * \rho(\chi_S) = \rho(\chi_S) = 1$ we have that $||s * \rho|| = 1$. Given $g \in E$, since $g \circ \lambda_s \in E$, $(s * \rho)(g) = \rho(g \circ \lambda_s) = \eta(g \circ \lambda_s) = \eta(g)$, whereby $s * \rho \in X$.

For (2), let $g \in l_\infty(S)$. Then $s * (t * \rho)(g) = (t * \rho)(g \circ \lambda_s) = \rho(g \circ \lambda_s \circ \lambda_t) = \rho(g \circ \lambda_{st}) = (st) * \rho(g)$.

It is a routine exercise to establish (3). To verify (4), let $n \in \mathbb{N}$, let $\langle c_i \rangle_{i=1}^n$ be elements of $[0,1]$, such that $\sum_{i=1}^n c_i = 1$, and let $\langle \rho_i \rangle_{i=1}^n$ be elements of X. Given $g \in E$, $(\sum_{i=1}^n c_i\rho_i)(g) = (\sum_{i=1}^n c_i\rho_i(g)) = (\sum_{i=1}^n c_i\eta(g)) = (\sum_{i=1}^n c_i)\eta(g) = \eta(g)$. And, since the norm is additive on the set of means, $||\sum_{i=1}^n c_i\rho_i|| = \sum_{i=1}^n c_i||\rho_i|| = 1$.

Finally, the verification of (5) is a routine evaluation. □

Theorem 3.4. *Let $(S, \cdot)$ be a left amenable semigroup, let h be a homomorphism from S onto a semigroup $(T, \cdot)$, and let $B \subseteq T$. Then $d^*(B) = d^*(h^{-1}[B])$.*

Proof. Suppose first that we have some δ such that $d^*(B) > \delta > d^*(h^{-1}[B])$. By Theorem 3.3, pick $\mu \in LIM(T)$ such that $\mu(\chi_B) > \delta$ and pick $\nu \in LIM(S)$ such that $\nu_h = \mu$. Then $\delta < \mu(\chi_B) = \nu(\chi_B \circ h) = \nu(\chi_{h^{-1}[B]}) \leq d^*(h^{-1}[B])$, a contradiction.

Now suppose we have some δ such that $d^*(h^{-1}[B]) > \delta > d^*(B)$ and pick $\nu \in LIM(S)$ such that $\nu(\chi_{h^{-1}[B]}) > \delta$. Then $d^*(B) \geq \nu_h(\chi_B) > \delta$, a contradiction. □

Theorem 3.5. *Let $(S, \cdot)$ be a left amenable semigroup, let h be a homomorphism from S onto a semigroup $(T, \cdot)$, and let $A \subseteq S$. Then $d^*(h[A]) \geq d^*(A)$.*

Proof. Since $A \subseteq h^{-1}\big[h[A]\big]$, by Theorem 3.4, $d^*(A) \leq d^*\big(h^{-1}\big[h[A]\big]\big) = d^*(h[A])$. □

Denote by $\widetilde{h} : \beta S \to \beta T$ the continuous extension of h. Note that by [8, Corollary 4.22], $\widetilde{h}$ is a homomorphism of βS onto βT.

Theorem 3.6. *Let $(S, \cdot)$ be a left amenable semigroup, and let h be a homomorphism from S onto a semigroup $(T, \cdot)$. Then $\widetilde{h}[\Delta^*(S)] = \Delta^*(T)$.*

Proof. To see that $\widetilde{h}[\Delta^*(S)] \subseteq \Delta^*(T)$, let $q \in \widetilde{h}[\Delta^*(S)]$ and pick $p \in \Delta^*(S)$ such that $\widetilde{h}(p) = q$. Suppose $q \notin \Delta^*(T)$ and pick $B \in q$ such that $d^*(B) = 0$. Then $h^{-1}[B] \in p$, and by Theorem 3.4, $d^*(h^{-1}[B]) = 0$, a contradiction.

To see that $\Delta^*(T) \subseteq \widetilde{h}[\Delta^*(S)]$, let $q \in \Delta^*(T)$. Then, by Theorem 3.4, for each $B \in q$, $d^*(h^{-1}[B]) > 0$. Let $\mathcal{A} = \{h^{-1}[B] : B \in q\}$ and let $\mathcal{R} = \{A \subseteq S : d^*(A) > 0\}$. Then by [8, Theorem 3.11], there exists $p \in \beta S$ such that $\mathcal{A} \subseteq p \subseteq \mathcal{R}$. So $p \in \Delta^*(S)$ and $\widetilde{h}(p) = q$. □

It is a consequence of Lemma 1.2 that if $(S, \cdot)$ is a left amenable semigroup, then the intersection of finitely many right ideals of S is nonempty and, hence, is a right ideal.

Lemma 3.7. *Let $(S, \cdot)$ be a left amenable semigroup, let $n \in \mathbb{N}$, and let $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ be elements of S with the property that for all $i \in \{1, 2, \ldots, n\}$, there exists $x \in S$ such that $a_i x = b_i x$. There is a right ideal R of S such that $a_i u = b_i u$ for every $u \in R$ and every $i \in \{1, 2, \ldots, n\}$.*

Proof. Let $R_i = \{u \in S : a_i u = b_i u\}$. Then each R_i is a right ideal of S. Since $(S, \cdot)$ is left amenable, $R = \bigcap_{i=1}^n R_i$ is a right ideal. □

Lemma 3.8. *Let $(S, \cdot)$ be a semigroup satisfying SFC and define a relation $\sim$ on S by, for $a, b \in S$, $a \sim b$ if and only if there exists $x \in S$ such that $ax = bx$. Then $\sim$ is an equivalence relation on S and the quotient $T = S/\!\sim$ is a cancellative semigroup which satisfies SFC.*

Proof. This is [12, Lemma 3.2]. Its proof was based on the proofs of [4, Lemma 2 and Remark 3] and [14, Theorem 2.2]. □

Throughout the rest of this section we will assume that $(S, \cdot)$ is a semigroup satisfying SFC, $\sim$ is the equivalence relation of Lemma 3.8, $(T, \cdot)$ is the cancellative quotient of S, $h : S \to T$ is the projection map, and $\widetilde{h} : \beta S \to \beta T$ is the continuous extension of h.

Lemma 3.9. *Let $p \in \Delta(S)$ and let $x, y \in \beta S$. If $\widetilde{h}(x) = \widetilde{h}(y)$, then $xp = yp$.*

Proof. Assume that $\widetilde{h}(x) = \widetilde{h}(y) = q$ and suppose that $xp \neq yp$. Pick $A \in xp \setminus yp$. Since ρ_p is continuous, pick $X \in x$ and $Y \in y$ such that $\rho_p[\,\overline{X}\,] \subseteq \overline{A}$ and $\rho_p[\,\overline{Y}\,] \subseteq \overline{S \setminus A}$. Then $q \in \widetilde{h}[\,\overline{X}\,] = c\ell h[X]$ and $q \in \widetilde{h}[\,\overline{Y}\,] = c\ell h[Y]$. Since βS is extremally disconnected, $h[X] \cap h[Y] \neq \emptyset$, so pick $a \in X$ and $b \in Y$ such that $h(a) = h(b)$. By Lemma 4.4, $ap = bp$. This is a contradiction since $ap \in \overline{A}$ and $bp \in \overline{S \setminus A}$. □

Theorem 3.10. *If there is an element of $\Delta(S)$ which is right cancelable in βS, then $\widetilde{h} : \beta S \to \beta T$ is an isomorphism.*

Proof. Pick $p \in \Delta(S)$ such that p is right cancelable in βS. Since $\widetilde{h}$ is a surjective homomorphism, it sufices to show that $\widetilde{h}$ is injective. Let $x, y \in \beta S$ and assume that $\widetilde{h}(x) = \widetilde{h}(y)$. Then by Lemma 3.9, $xp = yp$ so $x = y$. □

Corollary 3.11. *If there is an element of $c\ell K(\beta S)$ which is right cancelable in βS, then $\widetilde{h} : \beta S \to \beta T$ is an isomorphism.*

Proof. By Lemma 2.3, $c\ell K(\beta S) \subseteq \Delta(S)$. □

Corollary 3.12. *If there is an element of $\Delta(S)$ which is right cancelable in βS, then S is cancellative.*

Proof. By Theorem 3.10, S and T are isomorphic, and by Lemma 3.8, T is cancellative. □

If S is cancellative and countable, $c\ell K(\beta S)$ does contain right cancelable elements ([8, Corollary 8.26]). It follows from Corollary 3.12 that, for a countable semigroup S which satisfies SFC, the existence of right cancelable elements of βS in $c\ell K(\beta S)$ is equivalent to S being cancellative.

Theorem 3.13. *The minimal left ideals of βS and βT are isomorphic.*

Proof. Let L be a minimal left ideal of βS. By [8, Exercise 1.7.3], $\widetilde{h}[L]$ is a minimal left ideal of βT. We will show that the restriction of $\widetilde{h}$ to L is an isomorphism. So let $a, b \in L$, and assume that $\widetilde{h}(a) = \widetilde{h}(b)$. Pick an idempotent p in L. By Lemma 3.9, $ap = bp$ so, since p is a right identity for L by [8, Lemma 1.30], $a = b$. □

Theorem 3.14. *Assume that T is finite. Then:*

(1) T is a finite group.

(2) T is the unique minimal left ideal of βT.

(3) The minimal left ideals of βS are isomorphic to T.

(4) βS has a unique minimal right ideal.

(5) $T = K(\beta T) = \Delta(T)$.

Proof. (1) By Lemma 3.8, T is cancellative. As a finite cancellative semigroup, T is a group.

(2) Since T has a unique idempotent, it has only one minimal left ideal. If e is the identity of T, then $T = Te$.

(3) This follows from (2) and Theorem 3.13.

(4) Let L be a minimal left ideal of βS. By (3), L is isomorphic to T, so it has a unique idempotent. If R_1 and R_2 were distinct (hence disjoint) minimal right ideals of βS, both $R_1 \cap L$ and $R_2 \cap L$, being groups, would have idempotents.

(5) By (2), Lemma 2.3, and the fact that $K(\beta T)$ is the union of the minimal left ideals of βT, $T = K(\beta T) \subseteq \Delta(T) \subseteq \beta T = T$. □

In this part of this section, we are assuming that S satisfies SFC. Since the next result is probably the most important result of the paper, we recall in its statement that SFC is the only assumption needed.

Theorem 3.15. *Let $(S,\cdot)$ be a semigroup which satisfies SFC. Then, for every subset A of S, $d(A) = d^*(A) = d_t(A)$.*

Proof. By Theorem 1.13 and Corollary 1.18 we have that $d(A) \leq d^*(A) \leq d_t(A)$ so it suffices to show that $d_t(A) \leq d(A)$. To see this we will show that if $\eta < d_t(A)$, then $d(A) \geq \eta$. So let $\eta < d_t(A)$ be given.

To see that $d(A) \geq \eta$, let $H \in \mathcal{P}_f(S)$ and $\epsilon > 0$ be given and let $\mathcal{F} = \{K \in \mathcal{P}_f(S) : (\forall s \in H)(|K \setminus sK| < \epsilon \cdot |K|)\}$. We shall show that there exists $K \in \mathcal{F}$ such that $|K \cap A| \geq \eta \cdot |K|$.

Let $g : T \to S$ be a choice function for $\sim$ and note that $h\big(g(t)\big) = t$ for every $t \in T$. Observe that g is injective and h is injective on $g[T]$. We claim that for each $s \in S$, ρ_s is injective on $g[T]$. To see this, let $a, b \in T$ and assume that $g(a)s = g(b)s$. Then $g(a) \sim g(b)$ so $a = h\big(g(a)\big) = h\big(g(b)\big) = b$.

Now $h[H] \in \mathcal{P}_f(T)$ and by Lemma 3.8, T satisfies SFC, so pick $F \in \mathcal{P}_f(T)$ such that for all $s \in H$, $|F \setminus h(s)F| < \epsilon \cdot |F|$. Now $y \in F \cap h(s)F$ if and only if $y \in F$ and $y = h(s)y'$ for some $y' \in F$. The equation $y = h(s)y'$ implies that $h\big(sg(y')\big) = h(s)h\big(g(y')\big) = h(s)y' = y = h\big(g(y)\big)$ so, by Lemma 3.7, there is a right ideal R of S such that $g(y)u = sg(y')u$ for every $u \in R$, every $y, y' \in F$, and every $s \in H$ for which $y = h(s)y'$.

For each $u \in R$, let $G_u = g[F]u$ and note that $|G_u| = |g[F]u| = |g[F]| = |F|$, because ρ_u is injective on $g[F]$ and g is injective. We claim that for each $u \in R$, $G_u \in \mathcal{F}$. To see this, let $u \in R$ and let $s \in H$. It suffices to show that $G_u \setminus sG_u \subseteq \{g(y)u : y \in F \setminus h(s)F\}$, for then $|G_u \setminus sG_u| \leq |F \setminus h(s)F| < \epsilon \cdot |F| = \epsilon \cdot |G_u|$. So let $x \in G_u \setminus sG_u$ and pick $y \in F$ such that $x = g(y)u$. If we had $y \in h(s)F$, there would be some $y' \in F$ such that $y = h(s)y'$ so $x = g(y)u = sg(y')u$ and thus $x \in sG_u$.

Choose $u \in R$. Since $d_t(A) > \eta$, we may pick $x \in S$ such that $|G_u \cap Ax^{-1}| \geq \eta \cdot |G_u|$. If $z \in G_u \cap Ax^{-1}$, then $zx \in G_u x \cap A$. So $(G_u \cap Ax^{-1})x \subseteq G_u x \cap A$. Now ρ_x is injective on G_u, because ρ_{ux} is injective on $g[F]$. So $|G_u x \cap A| \geq |G_u \cap Ax^{-1}|$ and therefore $|G_u x \cap A| \geq \eta \cdot |G_u| = \eta \cdot |G_u x|$. Since $G_u x = G_{ux} \in \mathcal{F}$, we are done. □

Corollary 3.16. *If $(S,\cdot)$ is a semigroup which satisfies SFC, then $\Delta(S)$ is a two sided ideal in βS.*

Proof. Theorems 2.8 and 3.15. □

Corollary 3.17. $\widetilde{h}[\Delta^*(S)] = \Delta(T) = \Delta^*(T)$ *and for each $B \subseteq T$, $d(B) = d^*(h^{-1}[B])$.*

Proof. By Theorem 3.15, $\Delta(S) = \Delta^*(S)$ and $\Delta(T) = \Delta^*(T)$. The conclusions are then immediate consequences of Theorems 3.4 and 3.6. □

4 When minimal left ideals are singletons

Hindman and Strauss have dealt with versions of this subject before. In [9] they had a section titled *Semigroups with isolated points in minimal left ideals*; in

[10] they had a section titled *Finitely many minimal right ideals*; and in [11] they had a section titled *Finite minimal left ideals* and showed that βS has finite minimal left ideals if and only if it has finitely many minimal right ideals.

In this paper, we arrive at this subject from a different direction and with a particular interest in density, which the aforementioned works do not address. As we noted earlier, Bergelson and Glasscock showed in [2, Theorem 3.5] that if S satisfies SFC and is either left cancellative or commutative, then for each $A \subseteq S$, $d(A) = d^*(A)$. Also, it is an old fact noted by Klawe in [14, Corollary 2.3] that if S satisfies SFC and is right cancellative, then S is also left cancellative. At that point we were actively considering the possibility of finding a counterexample to the assertion that d and d^* are always equal. So to find a semigroup for which d and d^* are not equal, we needed a semigroup which satisfies SFC and is not commutative and not left or right cancellative. The following simple observation provides such examples.

Lemma 4.1. *Let $(S, \cdot)$ be a semigroup. If S has a right zero, then S satisfies SFC.*

Proof. Let z be a right zero in S. Let $H \in \mathcal{P}_f(S)$, and let $\epsilon > 0$. Let $K = \{z\}$. Then for all $s \in H$, $sK = K$ so $|K \setminus sK| = 0 < \epsilon \cdot |K|$. □

If S is the semigroup of 2×2 matrices over the set ω of nonnegative integers, then S satisfies SFC, is not commutative, and is neither right nor left cancellative.

Lemma 4.2. *Let $(S, \cdot)$ be a semigroup which satisfies SFC, let $A \subseteq S$, and assume that for each $\nu \in LIM_0(S)$, $\nu(\chi_A) = 1$. Then $\Delta(S) \subseteq \overline{A}$.*

Proof. Suppose we have some $p \in \Delta(S) \setminus \overline{A}$. Then $S \setminus A \in p$ so $d(S \setminus A) > 0$ so by Theorem 1.13 we may pick $\nu \in LIM_0(S)$ such that $\nu(\chi_{S\setminus A}) > 0$. But since $\nu(\chi_A) = 1$, one must have $\nu(\chi_{S\setminus A}) = 0$, a contradiction. □

Lemma 4.3. *Let $(S, \cdot)$ be a semigroup which satisfies SFC, let C be a compact subset of βS, and assume that there exists $x \in S$ such that $xS \subseteq C$. Then $\Delta(S) \subseteq C$.*

Proof. Pick $x \in S$ such that $xS \subseteq C$, and let $A = xS$. By Lemma 4.2, it suffices to show that for each $\mu \in LIM_0(S)$, $\mu(\chi_A) = 1$. In fact, for each $\mu \in LIM(S)$, $\mu(\chi_A) = 1$. Since A is a right ideal of S, this follows immediately from Lemma 1.2. □

Lemma 4.4. *Let $(S, \cdot)$ be a semigroup which satisfies SFC. Let $a, b \in S$, and assume that there exists $x \in S$ such that $ax = bx$. Then for all $p \in \Delta(S)$, $ap = bp$.*

Proof. Pick $x \in S$ such that $ax = bx$. Let $C = \{p \in \beta S : ap = bp\}$. Then C is compact and $xS \subseteq C$ so by Lemma 4.3, $\Delta(S) \subseteq C$. □

The following lemma does not need the assumption that S satisfies SFC.

Lemma 4.5. *Let $(S,\cdot)$ be a semigroup. Assume that the minimal left ideals of βS are singletons and $p \in K(\beta S)$. Then for every $s \in S$, $\{t \in S : st = t\} \in p$.*

Proof. Let $s \in S$. Since $\{p\}$ is a left ideal, $sp = p$ so by [8, Theorem 3.35], $\{t \in S : st = t\} \in p$. $\square$

Observe that examples of semigroups which satisfy statement (1) of Theorem 4.6 are abundant because this class of semigroups includes all semilattices and all semigroups which have a right zero.

Theorem 4.6. *For every semigroup $(S,\cdot)$, statements (1)-(6) are equivalent. Each of statements (1)-(6) implies statements (7) and (8).*

(1) For every $a, b \in S$, there exists $x \in S$ such that $ax = bx$.

(2) For every $H \in \mathcal{P}_f(S)$, there exists $x \in S$ such that $ax = bx$ for every $a, b \in H$.

(3) The minimal left ideals of βS are singletons.

(4) For every $p \in K(\beta S)$ and every $q \in \beta S$, $qp = p$.

(5) $K(\beta S)$ is a right zero semigroup, that is, all elements of $K(\beta S)$ are right zeros of $K(\beta S)$.

(6) The semigroup S satisfies SFC and for every subset A of S, $d(A) \in \{0, 1\}$.

(7) Every member of $K(\beta S)$ is idempotent.

(8) The semigroup S satisfies SFC and $K(\beta S) = \Delta(S)$.

Proof. To show that (1) implies (2), assume that (1) holds. We shall show that (2) holds by induction on $|H|$. Assume that that $n > 2$ is an integer, and that (2) holds for every $H \in \mathcal{P}_f(S)$ for which $|H| < n$. Choose $H \in \mathcal{P}_f(S)$ with $|H| = n$, and choose any $a \in H$. There exists $x \in S$ such that $bx = cx$ for every $b, c \in H \setminus \{a\}$. Choose any $b \in H \setminus \{a\}$, and then choose $y \in S$ such that $ay = by$. There exists $z \in S$ such that $xz = yz$. So $bxz = cxz$ for every $c \in H \setminus \{a\}$, and $ayz = byz$. If $w = xz = yz$, then $cw = dw$ for every $c, d \in H$.

We now show that (2) implies (3). Assume that (2) holds. For each $H \in \mathcal{P}_f(S)$, pick $x_H \in S$ such that for all a and b in H, $ax_H = bx_H$. The relation $\subseteq$ directs $\mathcal{P}_f(S)$. Let p be a cluster point of the net $\langle x_H \rangle_{H \in \mathcal{P}_f(S)}$ in βS. Then $ap = bp$ for all a and b in S, so $|Sp| = 1$ and $\beta S p = c\ell(Sp)$ so $|\beta S p| = 1$. Thus βS has a minimal left ideal which is a singleton. Since all minimal left ideals are isomorphic, all minimal left ideals are singletons.

That (3) implies (4) and that (4) implies (5) are both trivial.

To see that (5) implies (1), pick $p \in K(\beta S)$. Let $a, b \in S$. Then ap and bp are in $K(\beta S)$ so $ap = a(pp) = (ap)p = p$ and $bp = p$, so by Lemma 4.5, $\{t \in S : at = t\} \in p$ and $\{t \in S : bt = t\} \in p$. Choose x in the intersection of these two sets so that $ax = bx$.

We have shown thus far that statements (1)-(5) are equivalent.

We now show that if (2) and (3) hold, so does (6). So assume that (2) and (3) hold. Let $H \in \mathcal{P}_f(S)$, and let $\epsilon > 0$. Pick $x \in S$ such that for all a and b in $H \cup HH$, $ax = bx$. Pick $a \in H$, and let $K = \{ax\}$. Then for each $s \in H$, $|K \setminus sK| = 0 < \epsilon \cdot |K|$. Therefore, S satisfies SFC.

Let $A \subseteq S$. If $d^*(A) = 0$, then $d(A) = 0$. If, on the other hand, $d^*(A) > 0$, we shall show that A is thick. It will then follow from [12, Theorem 3.4] that $d(A) = 1$ and that (6) holds. Assume that $d^*(A) > 0$, and choose $\lambda \in LIM(S)$ for which $\lambda(A) > 0$. Let $F \in \mathcal{P}_f(S)$ and $p \in K(\beta S)$. By Lemma 4.5, for every $s \in S$, $\{t \in S : st = t\} \in p$. So $R = \bigcap_{s \in F}\{t \in S : st = t\} \in p$. Now R is a right ideal of S and so, by Lemma 1.2, $\lambda(R) = 1$, and hence $\lambda(S \setminus R) = 0$. Since $\lambda(A) > 0$, we have $A \cap R \neq \emptyset$. Pick $t \in A \cap R$. Then $Ft = \{t\} \subseteq A$ and so A is thick.

Now we will show that (6) implies (3). A set $A \subseteq S$ such that $\overline{A} \cap K(\beta S) \neq \emptyset$ is *piecewise syndetic*. By [12, Corollary 3.5(2)], if A is piecewise syndetic, then $d(A) > 0$, so if A is a piecewise syndetic subset of S, then $d(A) = 1$.

Let L be a minimal left ideal of βS and pick an idempotent $p \in L$. Then $\beta Sp = L$. We claim that $\beta Sp = \{p\}$ for which it suffices that $Sp = \{p\}$. Suppose instead that there is some $s \in S$ such that $sp \neq p$. Choose disjoint subsets P and Q of S such that $P \in p$ and $Q \in sp$. Since $s^{-1}Q \cap P \in p$, $d(s^{-1}Q \cap P) = 1$. It follows from Theorem 1.14 that there exists $\nu \in LIM(S)$ such that $\nu(s^{-1}Q \cap P) = 1$. But this implies that $\nu(Q) = \nu(s^{-1}Q) = 1$ and $\nu(P) = 1$, which contradicts the assumption that $P \cap Q = \emptyset$. Therefore, (3) holds.

We have shown thus far that statements (1)-(6) are equivalent. Now we will show that each of (1)-(6) implies (7) and (8).

That (5) implies (7) is trivial. To see that (3) implies (8), assume that (3) holds. Since (3) implies (6) we have S satisfies SFC so by Lemma 2.3, $K(\beta S) \subseteq \Delta(S)$. To see that $\Delta(S) \subseteq K(\beta S)$, let $p \in \Delta(S)$. We shall show that for all $a \in S$, $ap = p$ so that $\beta Sp = \{p\}$ and thus $\{p\}$ is a minimal left ideal. So let $a \in S$, and pick $q \in K(\beta S)$. Then $aq \in \beta Sq = \{q\}$ so by Lemma 4.5, we may pick $s \in S$ such that $as = s$. Then $sS \subseteq \{x \in \beta S : ax = x\}$. The set $\{x \in \beta S : ax = x\}$ is compact, and so it contains $\Delta(S)$ by Lemma 4.3 so that $ap = p$, as claimed. □

We conclude this section by characterizing sets with positive density.

Theorem 4.7. *Let $(S, \cdot)$ be a semigroup for which the mimimal left ideals of βS are singletons. Define a relation $\prec$ on S by $s \prec t$ if and only if $st = t$. The relation $\preceq$ is a directed set order on S. Given $A \subseteq S$, the following statements are equivalent.*

(a) $d(A) > 0$.

(b) $d(A) = 1$.

(c) A is cofinal in S with respect to $\preceq$.

Proof. By $s \preceq t$ we mean, of course, that $s \prec t$ or $s = t$, so $\preceq$ is reflexive. To verify transitivity, assume $x \preceq y$ and $y \preceq z$. If either $x = y$ or $y = z$, then trivially $x \preceq z$. So assume that $x \prec y$ and $y \prec z$; that is $xy = y$ and $yz = z$. Then $xz = xyz = yz = z$. To complete the proof that $\preceq$ directs S, let $x, y \in S$. Pick $p \in K(\beta S)$. By Lemma 4.5, $\{t \in S : xt = t\} \cap \{t \in S : yt = t\} \in p$ so is nonempty.

It follows from Theorem 4.6 that (a) and (b) are equivalent.

To see that (c) implies (b), assume that A is cofinal in S. To see that $d(A) = 1$, let $H \in \mathcal{P}_f(S)$ and $\epsilon > 0$ be given. Using Lemma 4.5, pick $y \in S$ such that for each $s \in H$, $s \preceq y$. Pick $x \in A$ such that $y \preceq x$, and let $K = \{x\}$. Then for all $s \in H$, $K = sK$, so $|K \setminus sK| = 0 < \epsilon \cdot |K|$ and $|K \cap A| = |K|$.

To see that (a) implies (c), we assume that A is not cofinal and show that $d(A) = 0$. By Theorem 4.6, $K(\beta S) = \Delta(S)$, so it suffices to show that $\overline{A} \cap K(\beta S) = \emptyset$. Suppose instead that we have some $p \in \overline{A} \cap K(\beta S)$. Since A is not cofinal, we may pick some $s \in S$ such that there is no $t \in A$ with $s \preceq t$. But by Lemma 4.5, $\{t \in S : st = t\} \in p$. Picking $t \in A \cap \{t \in S : st = t\}$ gives a contradiction. □

Corollary 4.8. *Let $(S, \cdot)$ be a semigroup with a right zero element, let $Z = \{z \in S : z$ is a right zero in $S\}$, and let $A \subseteq S$. Then*

$$d(A) = \begin{cases} 1 & \text{if } A \cap Z \neq \emptyset \\ 0 & \text{if } A \cap Z = \emptyset \,. \end{cases}$$

Proof. Let $\preceq$ be as in Theorem 4.7. We show that A is cofinal in S if and only if $A \cap Z \neq \emptyset$. If $z \in Z \cap A$, then $\{z\}$ is cofinal in S, so A is cofinal in S. If A is cofinal in S, pick $z \in Z$ and pick $y \in A$ such that $z \preceq y$. Then either $y = z \in Z$ or $zy = y$ so that $y \in Z$. □

5 Density is determined by an arbitrary Følner net

We saw in Theorem 1.13 that if $(S, \cdot)$ satisfies SFC and $A \subseteq S$, then there is a Følner net $\langle F_\alpha \rangle_{\alpha \in D}$ in $\mathcal{P}_f(S)$ such that $d(A) = \lim_{\alpha \in D} \frac{|A \cap F_\alpha|}{|F_\alpha|}$. We will show in Theorem 5.3 that in any semigroup satisfying SFC, the density $d(A)$ – and, hence, by Theorem 3.15, the densities $d^*(A)$ and $d_t(A)$ – are determined by an arbitrary Følner net. This improves on a result of Bergelson and Glasscock [2, Corollary 3.6], who showed that if $(S, \cdot)$ is countable and right cancellative, then given any Følner sequence $\langle F_n \rangle_{n=1}^{\infty}$ in $\mathcal{P}_f(S)$ and any $A \subseteq S$, $d(A) = \lim_{n \to \infty} \sup_{m \geq n} \max_{s \in S} \frac{|A \cap F_m s|}{|F_m|}$.

Definition 5.1. Let $(S, \cdot)$ be a semigroup, $H \in \mathcal{P}_f(S)$, and $\epsilon > 0$. A set $F \in \mathcal{P}_f(S)$ is (H, ϵ)-*invariant* if for all $h \in H$, $|F \setminus hF| < \epsilon \cdot |F|$.

Lemma 5.2. *Let $(S,\cdot)$ be a semigroup, and let $A \subseteq S$. For every $\epsilon > 0$, there exists $H \in \mathcal{P}_f(S)$ such that for every $(H, \epsilon/2)$-invariant set $F \in \mathcal{P}_f(S)$,*

$$d_t(A) \le \max_{s\in S} \frac{|As^{-1} \cap F|}{|F|} < d_t(A) + \epsilon.$$

Proof. Let $\epsilon > 0$. By the definition of $d_t(A)$, there exists $H \in \mathcal{P}_f(S)$ such that

$$\max_{s\in S} \frac{|As^{-1} \cap H|}{|H|} < d_t(A) + \frac{\epsilon}{2}\,.$$

Let $F \in \mathcal{P}_f(S)$ be $(H, \epsilon/2)$-invariant, and define

$$v = \max_{s\in S} \frac{|As^{-1} \cap F|}{|F|}.$$

Let $s_0 \in S$ achieve this maximum so that $|As_0^{-1} \cap F| = v \cdot |F|$.

By the definition of $d_t(A)$, we see that $v \ge d_t(A)$. Therefore, to conclude the proof of the lemma, we need only show that $v < d_t(A) + \epsilon$.

We claim that for all $h \in H$, $|h^{-1}As_0^{-1} \cap F| > (v - \epsilon/2) \cdot |F|$. Indeed, let $h \in H$. Since

$$\big(As_0^{-1} \cap hF\big) \cup \big(F \setminus hF\big) \supseteq As_0^{-1} \cap F\,,$$

we have that $|As_0^{-1} \cap hF| \ge |As_0^{-1} \cap F| - |F \setminus hF|$. Since $\lambda_h[h^{-1}As_0^{-1} \cap F] \supseteq As_0^{-1} \cap hF$,

$$|h^{-1}As_0^{-1} \cap F| \ge |As_0^{-1} \cap hF| \ge |As_0^{-1} \cap F| - |F \setminus hF| > \left(v - \frac{\epsilon}{2}\right) \cdot |F|.$$

Now we see that

$$\begin{aligned} v - \frac{\epsilon}{2} \quad &< \frac{1}{|H|} \sum_{h\in H} \frac{|h^{-1}As_0^{-1} \cap F|}{|F|} \\ &= \frac{1}{|H|}\frac{1}{|F|} \sum_{h\in H}\sum_{f\in F} \chi_{h^{-1}As_0^{-1}}(f) \\ &= \frac{1}{|F|}\frac{1}{|H|} \sum_{f\in F}\sum_{h\in H} \chi_{A(fs_0)^{-1}}(h) \\ &= \frac{1}{|F|} \sum_{f\in F} \frac{|A(fs_0)^{-1} \cap H|}{|H|}. \end{aligned}$$

It follows that there exists $f \in F$ such that

$$v - \frac{\epsilon}{2} < \frac{|A(fs_0)^{-1} \cap H|}{|H|} \le \max_{s\in S} \frac{|As^{-1} \cap H|}{|H|} < d_t(A) + \frac{\epsilon}{2}.$$

This implies that $v < d_t(A) + \epsilon$, as was to be shown. □

Theorem 5.3. *Let $(S,\cdot)$ be a semigroup satisfying SFC, and let $\langle F_\alpha\rangle_{\alpha\in D}$ be a Følner net in $\mathcal{P}_f(S)$. For all $A\subseteq S$,*

$$d(A)=d^*(A)=d_t(A)=\lim_{\alpha\in D}\max_{s\in S}\frac{|As^{-1}\cap F_\alpha|}{|F_\alpha|}\,.$$

Proof. The first two equalities follow from Theorem 3.15. It follows from the definition of a Følner net that for any $H\in\mathcal{P}_f(S)$ and any $\epsilon>0$ there exists $\alpha\in D$ such that for all $\sigma\geq\alpha$, F_σ is (H,ϵ)-invariant, so the last equality follows from Lemma 5.2. □

We extend the result of [2, Corollary 3.6] to uncountable semigroups.

Corollary 5.4. *Let $(S,\cdot)$ be a right cancellative semigroup satisfying SFC, and let $\langle F_\alpha\rangle_{\alpha\in D}$ be a Følner net in $\mathcal{P}_f(S)$. For all $A\subseteq S$,*

$$d(A)=d^*(A)=d_t(A)=\lim_{\alpha\in D}\max_{s\in S}\frac{|A\cap F_\alpha s|}{|F_\alpha|}\,.$$

Proof. Since S is right cancellative, for any $A\subseteq S$, any $F\in\mathcal{P}_f(S)$, and any $s\in S$, $|A\cap Fs|=|As^{-1}\cap F|$. □

In the absence of right cancellation the conclusion of Corollary 5.4 can fail badly. For example, let $(S,\cdot)$ be an infinite right zero semigroup, and let $\emptyset\neq A\subseteq S$. By Theorem 4.7, $d(A)=1$. Any net in $\mathcal{P}_f(S)$ is a Følner net. If $\langle F_\alpha\rangle_{\alpha\in D}$ is a net in $\mathcal{P}_f(S)$ with each $|F_\alpha|=2$, then $\lim_{\alpha\in D}\max_{s\in S}\frac{|A\cap F_\alpha s|}{|F_\alpha|}=\frac{1}{2}$. On the other hand, $\lim_{\alpha\in D}\max_{s\in S}\frac{|A\cap F_\alpha s|}{|F_\alpha s|}=1$. So one can ask what conditions short of right cancellation guarantee that $d(A)=\lim_{\alpha\in D}\max_{s\in S}\frac{|A\cap F_\alpha s|}{|F_\alpha s|}$. The only positive answer that we have is if $d(A)=1$.

Theorem 5.5. *Let $(S,\cdot)$ be a semigroup satisfying SFC, and let $A\subseteq S$. If $d(A)=1$, then for every Følner net $\langle F_\alpha\rangle_{\alpha\in D}$ in $\mathcal{P}_f(S)$,*

$$d(A)=\lim_{\alpha\in D}\max_{s\in S}\frac{|A\cap F_\alpha s|}{|F_\alpha s|}\,.$$

Proof. Assume that $d(A)=1$, and let $\langle F_\alpha\rangle_{\alpha\in D}$ be a Følner net in $\mathcal{P}_f(S)$. Then by Theorem 2.10, A is thick. Given $\alpha\in D$, pick $s\in S$ such that $F_\alpha s\subseteq A$. Then $\frac{|A\cap F_\alpha s|}{|F_\alpha s|}=1$. □

We see now that we cannot add the case that $d(A)=0$ to the statement of Theorem 5.5. In this theorem, we deal with the semigroup $(\mathcal{P}_f(\mathbb{N}),\cup)$. In a semigroup $(S,\cdot)$, if $A\subseteq S$ and $x\in S$, we write $A\cdot x$ for $\{y\cdot x: y\in A\}$. If $\mathcal{A}\subseteq\mathcal{P}_f(\mathbb{N})$ and $M\in\mathcal{P}_f(\mathbb{N})$, then $\mathcal{A}\cup M$ already means something, so we write out what we intend, i.e. $\{A\cup M: A\in\mathcal{A}\}$. Note that because the semigroup $(\mathcal{P}_f(\mathbb{N}),\cup)$ is commutative, it satisfies SFC.

Theorem 5.6. *In the semigroup $(\mathcal{P}_f(\mathbb{N}),\cup)$, let $\mathcal{A} = \{X \in \mathcal{P}_f(\mathbb{N}) : 1 \notin X\}$. Then $d(\mathcal{A}) = 0$, but there is a Følner sequence $\langle \mathcal{F}_n \rangle_{n\in\mathbb{N}}$ in $\mathcal{P}_f\big(\mathcal{P}_f(\mathbb{N})\big)$ such that*

$$\lim_{n\in\mathbb{N}} \max_{T\in\mathcal{P}_f(\mathbb{N})} \frac{|\mathcal{A} \cap \{Z \cup T : Z \in \mathcal{F}_n\}|}{|\{Z \cup T : Z \in \mathcal{F}_n\}|} = \frac{1}{2}\,.$$

Proof. By Theorem 4.6, the minimal left ideals of $\beta\mathcal{P}_f(S)$ are singletons. Define $\prec$ on $\mathcal{P}_f(S)$ by $A \prec B$ if and only if $A \subseteq B$. Since $\mathcal{A}$ is not cofinal in $\mathcal{P}_f(S)$, by Theorem 4.7, $d(\mathcal{A}) = 0$.

For $n \in \mathbb{N}$, let $\mathcal{F}_n = \big\{\{2,3,\ldots,2n\}\big\} \cup \big\{\{1,2,\ldots,k\} : n < k \leq 2n\big\}$. Note that $|\mathcal{F}_n| = n+1$. We claim that $\langle \mathcal{F}_n \rangle_{n=1}^{\infty}$ is a Følner sequence.

Let $X \in \mathcal{P}_f(\mathbb{N})$, and let $\epsilon > 0$ be given. Pick $n \in \mathbb{N}$ such that $n > \max X$ and $\frac{1}{n+1} < \epsilon$. Let $m \geq n$. Then $\mathcal{F}_m \setminus \{X \cup Z : Z \in \mathcal{F}_m\} \subseteq \big\{\{2,3,\ldots 2m\}\big\}$ so $|\mathcal{F}_m \setminus \{X \cup Z : Z \in \mathcal{F}_m\}| \leq 1 < \epsilon \cdot (n+1) \leq \epsilon \cdot (m+1) = \epsilon \cdot |\mathcal{F}_m|$.

Now let $n \in \mathbb{N}$. We shall show that $\max\limits_{T\in\mathcal{P}_f(\mathbb{N})} \dfrac{|\mathcal{A} \cap \{Z \cup T : Z \in \mathcal{F}_n\}|}{|\{Z \cup T : Z \in \mathcal{F}_n\}|} = \dfrac{1}{2}$. So let $T \in \mathcal{P}_f(\mathbb{N})$. If $1 \in T$, then $\mathcal{A} \cap \{Z \cup T : Z \in \mathcal{F}_n\} = \emptyset$, so assume that $1 \notin T$. Then $\{2,3,\ldots,2n\} \cup T \in \mathcal{A}$ and for $n < k \leq 2n$, $\{1,2,\ldots,2n\} \cup T \notin \mathcal{A}$ so $|\mathcal{A} \cap \{Z \cup T : Z \in \mathcal{F}_n\}| = 1$. Also, $\{2,3,\ldots,2n\} \cup T \neq \{1,2,\ldots,2n\} \cup T$ so $\dfrac{|\mathcal{A} \cap \{Z \cup T : Z \in \mathcal{F}_n\}|}{|\{Z \cup T : Z \in \mathcal{F}_n\}|} \leq \dfrac{1}{2}$. And if $T = \{2,3,\ldots,2n\}$, then $\dfrac{|\mathcal{A} \cap \{Z \cup T : Z \in \mathcal{F}_n\}|}{|\{Z \cup T : Z \in \mathcal{F}_n\}|} = \dfrac{1}{2}$. □

6 Density in Product Spaces

It has been known for some time that the product of two left amenable semigroups is left amenable. This follows from a more powerful theorem, due to Maria Klawe, about the semidirect product of two left amenable semigroups [14, Proposition 3.4]. We prove this directly in Theorem 6.1, then establish in Theorem 6.2 a product property that generalizes [7, Theorem 3.4].

Theorem 6.1. *Let $(S,\cdot)$ and $(T,\cdot)$ be left amenable semigroups, and let $\mu \in LIM(S)$ and $\nu \in LIM(T)$. Then there exists $\rho \in LIM(S\times T)$ with the property that, for every $A \subseteq S$ and every $B \subseteq T$, $\rho(A \times B) = \mu(A)\nu(B)$.*

Proof. Let $f \in l_\infty(S \times T)$. For each $s \in S$, we define $f_s \in l_\infty(T)$ by $f_s(t) = f(s,t)$. Then $\langle \nu(f_s) \rangle_{s\in S} \in l_\infty(S)$. Put $\tau(f) = \mu(\langle \nu(f_s) \rangle_{s\in S})$. For $A \subseteq S$ and $B \subseteq T$, we claim that $\tau(A \times B) = \mu(A)\nu(B)$. That is, $\tau(\chi_{A\times B}) = \mu(\chi_A)\nu(\chi_B)$. To see this let $f = \chi_{A\times B}$ and let $g = \langle \nu(f_s) \rangle_{s\in S}$. For $s \in S$ and $t \in T$, $f_s(t) = \chi_B(t)$ if $s \in A$ and $f_s(t) = 0$ if $s \notin A$. So for $s \in S$, $g(s) = \nu(f_s) = \nu(\chi_B)\chi_A(s)$ so $g = \nu(\chi_B)\chi_A$. Thus $\tau(\chi_{A\times B}) = \mu(g) = \nu(\chi_B)\mu(\chi_A)$.

We claim that τ is a left invariant on $l_\infty(S\times T)$. It is clear that τ is a positive linear functional on $l_\infty(S \times T)$. Since $\tau(\chi_{S\times T}) = 1$, τ is a mean by Lemma 1.3. To see that τ is left invariant, let $f \in l_\infty(S \times T)$, and let $(a,b) \in S \times T$. In the notation from the previous paragraph, $(f \circ \lambda_{(a,b)})_s = f_{as} \circ \lambda_b$. Therefore, by the left invariance of μ and ν,

$$\begin{aligned}\tau(f \circ \lambda_{(a,b)}) &= \mu(\langle \nu((f \circ \lambda_{(a,b)})_s) \rangle_{s \in S}) \\ &= \mu(\langle \nu(f_{as} \circ \lambda_b) \rangle_{s \in S}) \\ &= \mu(\langle \nu(f_{as}) \rangle_{s \in S}) \\ &= \mu(\langle \nu(f_s) \rangle_{s \in S} \circ \lambda_a) \\ &= \mu(\langle \nu(f_s) \rangle_{s \in S}) \\ &= \tau(f),\end{aligned}$$

demonstrating the left invariance of τ. □

Theorem 6.2. *Let $(S, \cdot)$ and $(T, \cdot)$ be left amenable semigroups, let $A \subseteq S$, and let $B \subseteq T$. Then $d^*(A \times B) = d^*(A)d^*(B)$.*

Proof. It is a consequence of Theorem 6.1 that $d^*(A \times B) \geq d^*(A)d^*(B)$, so suppose that $d^*(A)d^*(B) < d^*(A \times B)$ and pick η such that $d^*(A)d^*(B) < \eta < d^*(A \times B)$. Pick $\rho \in LIM(S \times T)$ such that $\rho(A \times B) > \eta$. Note in particular that $\rho(A \times T) \geq \rho(A \times B) > 0$.

We define μ and ν mapping $\mathcal{P}(S)$ and $\mathcal{P}(T)$, respectively, to $\mathbb{R}$ by first putting $\mu(X) = \rho(X \times T)$ for every $X \subseteq S$, and $\nu(Y) = \dfrac{\rho(A \times Y)}{\rho(A \times T)}$ for every $Y \subseteq T$. These functions are finitely additive on $\mathcal{P}(S)$ and $\mathcal{P}(T)$ respectively, and $\mu(S) = \nu(T) = 1$. By Lemma 1.5, they extend uniquely to means on S and T, respectively.

We claim that these means are left invariant. To see this, observe that, for every $s \in S$, $t \in T$, $X \subseteq S$, and $Y \subseteq T$,

$$\mu(s^{-1}X) = \rho(s^{-1}X \times T) = \rho((s,t)^{-1}(X \times T)) = \rho(X \times T) = \mu(X)$$

and

$$\rho(A \cap t^{-1}Y) = \rho\big((s,t)^{-1}(A \cap t^{-1}Y)\big) = \rho\big((s,t^2)^{-1}(A \cap Y)\big) = \rho(A \cap Y),$$

whereby $\nu(t^{-1}Y) = \dfrac{\rho(A \times t^{-1}Y)}{\rho(A \times T)} = \dfrac{\rho(A \times Y)}{\rho(A \times T)} = \nu(Y)$. So, by Lemma 1.6, μ and ν are left invariant means.

Then $d^*(A)d^*(B) \geq \mu(A)\nu(B) = \rho(A \times B) > \eta$, a contradiction. □

References

[1] L. Argabright and C. Wilde, *Semigroups satisfying a strong Følner condition*, Proc. Amer. Math. Soc. **18** (1967), 587-591.

[2] V. Bergelson and D. Glasscock, *On the interplay between additive and multiplicative largeness and its combinatorial applications* J. Combin. Theory Ser. A **172** (2020), 105-203.

[3] H. Furstenberg and Y. Katznelson, *An ergodic Szemerédi theorem for IP-systems and combinatorial theory*, J. Analyse Math., **45** (1985),117-168.

[4] E. Granirer, *A theorem on amenable semigroups*, Trans. Amer. Math. Soc. **111** (1964), 367-379.

[5] J. Griesmer, *Recurrence, rigidity, and popular differences*, Ergodic Theory Dynam. Systems, **39** (2019), no. 5, 1299-1316.

[6] E. Hewitt and K. Ross, *Abstract Harmonic Analysis, I*, Springer-Verlag, Berlin, 1963.

[7] N. Hindman and D. Strauss, *Density and invariant means in left cancellative left amenable semigroips*, Topology and its Applications, **156** (2009), 2614-2628.

[8] N. Hindman and D. Strauss, *Algebra in the Stone-Čech compactification: theory and applications, second edition*, de Gruyter, Berlin, 2012.

[9] N. Hindman and D. Strauss, *Image partition regular matrices and concepts of largeness*, New York J. Math. **26** (2020), 230-260.

[10] N. Hindman and D. Strauss, *Minimal left ideals of βS with isolated points*, New York J. Math. **27** (2021), 417-436.

[11] N. Hindman and D. Strauss, *Some new results about left ideals of βS*, New York J. Math. **28** (2022), 970-992.

[12] N. Hindman and D. Strauss, *Thick sets are exactly the sets with Følner density* 1, Semigroup Forum **109** (2024), 424-432.

[13] J. Johnson and F. K. Richter, *Revisiting the nilpotent polynomial Hales-Jewett Theorem*, Adv. Math., **321** (2017), 269-286.

[14] M. Klawe, *Semidirect product of semigroups in relation to amenability, cancellation properties, and strong Følner conditions*, Pacific J. Math. **73** (1977), 91-106.

[15] I. Namioka *Følner's conditions for amenable semi-groups*, Math. Scand **15** (1964), 18-28.

[16] A. Paterson, *Amenability*, Math. Surveys Monogr. **29** American Mathematical Society, Providence, RI, 1988.