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The Research of Seven Students at Howard University

Neil Hindman

Abstract. Results from the research of my final seven Ph.D. students at Howard University are discussed.

1. Introduction

During my 37 years on the faculty of the Department of Mathematics at Howard University, I have been fortunate to be asked to serve as dissertation advisor by twenty students. From the point of view of the National Association of Mathematicians, it is probably worth noting that ten of these have been black females and eight have been black males. (I do not write "African American" because not all of them were Americans.) On the occasion of my obituary conference in 2008, when I turned 65 years old, I wrote a survey of the dissertation research of all of my students up to that time [**4**]. This paper consists of a survey of the remaining seven students. (Since I have now retired and do not currently have any students, my list of students is presumably complete.)

I will not attempt to give a summary of each of the dissertations being discussed. Rather, I will attempt to pick out results that give a flavor of the major thrust of the dissertation. This will usually mean that the strongest results of the dissertation do not even get mentioned, since these strong results tend to be quite complicated.

All of my students' dissertations have involved results in Ramsey Theory or the algebraic structure of the Stone-Čech compactification, βS, of a discrete semigroup S, or both. Most of the dissertations involving both Ramsey Theory and algebra involve applications of algebraic results to obtain results in Ramsey Theory. Two of the dissertations being discussed here have algebraic results and Ramsey Theoretic results that are essentially unrelated.

This survey is organized by subject matter. Section 2 will deal with purely combinatorial Ramsey Theory. Section 3 will involve Ramsey Theoretic results related to the algebraic structure of βS. And Section 4 will consist of purely algebraic results about S and βS. (The reason for the ordering of Sections 3 and 4 is that one of the results in Section 4 is motivated by a result in Section 3.)

2. Ramsey Theory

Ramsey Theory gets its name from the following theorem. We let $\mathbb{N}$ be the set of positive integers. Given a set X and $k \in \mathbb{N}$, $[X]^k = \{A \subseteq X : |A| = k\}$.

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Theorem 2.1 (Ramsey's Theorem). *Let X be a set, let $k,r \in \mathbb{N}$, and assume that $[X]^k = \bigcup_{i=1}^r C_i$. There exist $i \in \{1,2,\ldots,r\}$ and an infinite set $Y \subseteq X$ such that $[Y]^k \subseteq C_i$.*

Proof. [**15**, Theorem A]. □

In the alternate chromatic terminology, if $[X]^k$ is r-colored, there is an infinite set $Y \subseteq X$ such that $[Y]^k$ is *monochromatic*.

One of the major results in Ramsey Theory is the Hales-Jewett Theorem. This involves the notion of a *free semigroup*. We shall stick to an informal treatment of free semigroups. For a more formal treatment see [**6**, Definition 1.3].

Definition 2.2. Let A be a nonempty set. The *free semigroup S over the alphabet A* is the set of all finite sequences in A with the operation of concatenation. The members of S are called *words*.

For example, if $A = \{1,2,3,4\}$ and S is the free semigroup over A, then $x = 12213$ and $y = 21423$ are members of S and $xy = 1221321423$.

Definition 2.3. Let A be a nonempty set.

(a) A *variable word* over A is a word over $A \cup \{v\}$ in which v occurs, where v is a *variable* which is not a member of A.
(b) If w is a variable word over A and $a \in A$, then $w(a)$ is the result of replacing each occurrence of v in w by a.

For example, if $A = \{1,2,3,4\}$ and $w = 13v2v3$, then $w(1) = 131213$ and $w(4) = 134243$.

Theorem 2.4 (Hales-Jewett Theorem). *Let A be a finite nonempty set, let S be the free semigroup over A, let $r \in \mathbb{N}$, and let $S = \bigcup_{i=1}^r C_i$. There exist $i \in \{1,2,\ldots,r\}$ and a variable word w over A such that $\{w(a) : a \in A\} \subseteq C_i$.*

Proof. [**3**, Theorem 1]. □

The set $\{w(a) : a \in A\}$ is frequently referred to as a *combinatorial line*.

Another, by now reasonably old, result in Ramsey Theory is the Finite Products Theorem. For a set X, we let $\mathcal{P}_f(X)$ be the set of finite nonempty subsets of X. Given a semigroup $(S,\cdot)$ and a sequence $\langle x_n\rangle_{n=1}^\infty$ in S, we let $FP(\langle x_n\rangle_{n=1}^\infty) = \{\prod_{t\in F} x_t : F \in \mathcal{P}_f(\mathbb{N})\}$ where $\prod_{t\in F} x_t$ is computed in increasing order of indices. If the operation in S is denoted by $+$, we write $FS(\langle x_n\rangle_{n=1}^\infty) = \{\sum_{t\in F} x_t : F \in \mathcal{P}_f(\mathbb{N})\}$.

Theorem 2.5 (Finite Products Theorem). *Let $(S,\cdot)$ be a semigroup, let $r \in \mathbb{N}$, and let $S = \bigcup_{i=1}^r C_i$. There exist $i \in \{1,2,\ldots,r\}$ and a sequence $\langle x_n\rangle_{n=1}^\infty$ in S such that $FP(\langle x_n\rangle_{n=1}^\infty) \subseteq C_i$.*

Proof. [**6**, Corollary 5.9]. □

The last Ramsey Theoretic topic that is addressed by the students' dissertations is *image partition regularity* of matrices.

Definition 2.6. Let $u,v \in \mathbb{N}$ and let A be a $u \times v$ matrix with integer entries.

(a) The matrix A is *image partition regular over* $\mathbb{N}$ if and only if, whenever $r \in \mathbb{N}$ and $\mathbb{N} = \bigcup_{i=1}^r C_i$, there exist $i \in \{1,2,\ldots,r\}$ and $\vec{x} \in \mathbb{N}^v$ such that $A\vec{x} \in (C_i)^u$.

(b) The matrix A is *weakly image partition regular over* $\mathbb{N}$ if and only if, whenever $r \in \mathbb{N}$ and $\mathbb{N} = \bigcup_{i=1}^{r} C_i$, there exist $i \in \{1, 2, \ldots, r\}$ and $\vec{x} \in \mathbb{Z}^v$ such that $A\vec{x} \in (C_i)^u$.

Numerous characterizations of image partition regular matrices are known. See [**6**, Theorem 15.24]. To illustrate, the fact that the matrix

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \end{pmatrix}$$

is image partition regular says that whenever $\mathbb{N}$ is divided into finitely many classes, one of those classes contains a length 5 arithmetic progression. This is a special case of van der Waerden's Theorem [**17**].

Henry Jordan

Let $X = \{1, 2, 3\}^4$. We will denote the members of X without commas or parentheses. So, for example, we will write 1213 instead of $(1, 2, 1, 3)$. It was shown in [**7**] that if X is 2-colored, there is a monochromatic line. That is, there is a variable word w over $\{1, 2, 3\}$ such that $\{w(1), w(2), w(3)\}$ is monochromatic.

Definition 2.7. A set $Y \subseteq X$ is a *Hales-Jewett set* if and only whenever Y is 2-colored, there must be a monochromatic combinatorial line.

Note that X has 81 members. In his dissertation [**8**], Dr. Jordan produced by analytic methods a set with 69 members which is not a Hales-Jewett set. Specifically, if

$$Y = X \setminus \{1233, 1323, 1332, 2133, 2313, 2331, 3123, 3132, 3213, 3231, 3312, 3321\}$$

then there is a 2-coloring of Y with no monochromatic lines.

He then used computer techniques (programming in Pascal) to produce a minimal Hales-Jewett set with 44 elements. For each possible candidate, the computer searched for a 2-coloring with no monochromatic line. If none was found, then the set was a Hales-Jewett set. Since there are 2^{44} 2-colorings of a 44 element set, the program obviously can't just try every one. But, for example, if 1111 and 1112 have been assigned to the same color and one is trying to avoid a monochromatic line, then 1113 must be assigned to the other color. Using such a program repeatedly, he found that if

$$\begin{aligned} Y = \; & \{1111, 2222, 3333, 1222, 1112, 1121, 1122, 2111, 1211, 1333, 1113, 1313, 1133, \\ & 1131, 1223, 1233, 1323, 2223, 2212, 2232, 2122, 3332, 3313, 3323, 3133, 3233, \\ & 2112, 2121, 2323, 2211, 2233, 3113, 3223, 3131, 3322, 2131, 2133, 3112, 3122, \\ & 1123, 2213, 2333, 3111, 3222\} \end{aligned}$$

then (1) whenever Y is 2-colored, there must be a monochromatic combinatorial line and (2) whenever any one of the 44 elements of Y is deleted, there is a 2-coloring without a monochromatic combinatorial line.

Dev Phulara

Definition 2.8. Let $a, r \in \mathbb{N}$. Then $SP_2(a, r)$ is the first $n \in \mathbb{N}$, if such exists, such that whenever $\{a, a+1, \ldots, n\}$ is r-colored, there exist x and y with $a \leq x < y$ such that $\{x + y, xy\}$ is monochromatic. If no such n exists, then $SP_2(a, r) = \infty$.

It is an old result of Ron Graham, never published by him, that $SP_2(a,2)$ is finite for all a, and his argument allows one to compute upper bounds.

In the combinatorial portion of his dissertation [**13**], Dr. Phulara established that for all $a \in \mathbb{N}$, $SP_2(a,n) \geq a^2(a+\lfloor 2\sqrt{a}\rfloor)$ and computed some upper bounds on $SP_2(a,2)$ that were slight improvements over the ones computed using Graham's argument.

Using a sophisticated computer program written in Pascal, Dr. Phulara also computed the exact value of $SP_2(a,2)$ for every $a \in \{1,2,\ldots,105\}$. For example, $SP_2(105,2) = 1543500$. To establish this, his program needed to find a 2-coloring of $\{105,106,\ldots,$
$1543499\}$ without any x and y with $105 \leq x < y$ such that $x+y$ and xy were the same color. And it had to establish that for any 2-coloring of $\{105,106,\ldots,1543500\}$ some such x and y must exist.

One fascinating fact arose from the computation of these exact values. This was that in every computed case, $SP_2(a,2)$ is divisible by a^2. Nobody has been able to prove that $SP_2(a,2)$ is always divisible by a.

Kendra Pleasant

The notion of weakly image partition regular is weaker than the notion of image partition regular. For example, it was shown in [**5**] that the matrix

$$\begin{pmatrix} 1 & -1 \\ 3 & 2 \\ 4 & 6 \end{pmatrix}$$

is weakly partition regular but not partition regular.

In the combinatorial portion of her dissertation [**14**], Dr. Pleasant proved the following theorem. This shows that the relevant images obtained for a weakly image partition regular matrix can also be obtained for an image partition regular matrix.

THEOREM 2.9. *Let $u,v,n \in \mathbb{N}$ and let A be a $u \times v$ matrix of rank n with integer entries. There is a $u \times n$ matrix B with integer entries such that*

$$\{A\vec{k} : \vec{k} \in \mathbb{Z}^v\} \cap \mathbb{N}^u = \{B\vec{x} : \vec{x} \in \mathbb{N}^n\} \cap \mathbb{N}^u\,.$$

In particular, if A is weakly image partition regular over $\mathbb{N}$, then B is image partition regular over $\mathbb{N}$.

PROOF. [**14**, Theorem 22]. □

3. Ramsey Theory and βS

We take the Stone-Čech compactification βX of a discrete space X to be the set of ultrafilters on X, with the principal ultrafilters being identified with the points of X. Given a discrete semigroup $(S,\cdot)$, the operation extends to βS so that $(\beta S,\cdot)$ is a right topological semigroup, meaning that the function $q \mapsto q\cdot p$ from βS to itself is continuous for each $p \in \beta S$. Further, the function $q \mapsto x\cdot q$ is continuous for each $x \in S$.

Any compact Hausdorff right topological semigroup T has a smallest two sided ideal $K(T)$ and $K(T)$ has idempotents. An idempotent in $K(T)$ is said to be a *minimal* idempotent. There is an important relationship between idempotents and sets of finite products (or finite sums).

THEOREM 3.1. *Let $(S,\cdot)$ be a semigroup.*

(1) *If p is an idempotent in $(\beta S, \cdot)$ and $A \in p$, then there is a sequence $\langle x_n \rangle_{n=1}^{\infty}$ such that $FP(\langle x_n \rangle_{n=1}^{\infty}) \subseteq A$.*
(2) *If $\langle x_n \rangle_{n=1}^{\infty}$ is a sequence in S, then $\bigcap_{m=1}^{\infty} c\ell_{\beta S}\big(FP(\langle x_n \rangle_{n=m}^{\infty})\big)$ is a subsemigroup of $(\beta S, \cdot)$ and there is an idempotent in this subsemigroup. In particular, there is an idempotent $p \in \beta S$ such that $FP(\langle x_n \rangle_{n=1}^{\infty}) \in p$.*

PROOF. (1) [**6**, Theorem 5.8]. (2) [**6**, Lemma 5.11]. □

DEFINITION 3.2. Let $(S, \cdot)$ be a semigroup and let $\langle x_n \rangle_{n=1}^{\infty}$ be a sequence in S. Then $\langle y_n \rangle_{n=1}^{\infty}$ is a *product subsystem* of $\langle x_n \rangle_{n=1}^{\infty}$ if and only if there exists a sequence $\langle F_n \rangle_{n=1}^{\infty}$ in $\mathcal{P}_f(\mathbb{N})$ such that for each n, $\max F_n < \min F_{n+1}$ and $y_n = \prod_{t \in F_n} x_t$.

The analogous notion for a semigroup written additively is called a *sum subsystem*.

A very important notion, both algebraically and combinatorially, is the notion of *central* sets.

DEFINITION 3.3. Let $(S, \cdot)$ be a semigroup and let $C \subseteq S$. Then C is *central* in S if and only if there is an idempotent $p \in K(\beta S, \cdot)$ such that $C \in p$.

The original Central Sets Theorem, proved using a different but equivalent definition of central, is the following.

THEOREM 3.4 (Original Central Sets Theorem). *Let C be a central subset of $(\mathbb{N}, +)$, let $k \in \mathbb{N}$, and for each $i \in \{1, 2, \ldots, k\}$, let $\langle y_{i,n} \rangle_{n=1}^{\infty}$ be a sequence in $\mathbb{Z}$. There exists sequences $\langle a_n \rangle_{n=1}^{\infty}$ in $\mathbb{N}$ and $\langle H_n \rangle_{n=1}^{\infty}$ in $\mathcal{P}_f(\mathbb{N})$ such that*
(1) *for each $n \in \mathbb{N}$, $\max H_n < \min H_{n+1}$ and*
(2) *for each $F \in \mathcal{P}_f(\mathbb{N})$ and each $i \in \{1, 2, \ldots, k\}$, $\sum_{n \in F}(a_n + \sum_{t \in H_n} y_{i,t}) \in C$.*

PROOF. [**2**, Proposition 8.21]. □

In [**1**] the Central Sets Theorem was extended so as to handle all sequences at once in arbitrary semigroups. The extension in commutative semigroups is reasonably simple to state.

THEOREM 3.5 (Commutative Central Sets Theorem). *Let $(S, +)$ be a commutative semigroup, let $\mathbb{F}$ be the set of sequences in S, and let C be a central subset of S. There exist functions $\alpha : \mathcal{P}_f(\mathbb{F}) \to S$ and $H : \mathcal{P}_f(\mathbb{F}) \to \mathcal{P}_f(\mathbb{N})$ such that*
(1) *if $F, G \in \mathcal{P}_f(\mathbb{F})$ and $F \subsetneq G$, then $\max H(F) < \min H(G)$ and*
(2) *whenever $m \in \mathbb{N}$, $G_1, G_2, \ldots, G_m \in \mathcal{P}_f(\mathbb{F})$ such that $G_1 \subsetneq G_2 \subsetneq \ldots \subsetneq G_m$, and for each $i \in \{1, 2, \ldots, m\}$, $f_i \in G_i$, one has $\sum_{i=1}^{m} \big(\alpha(G_i) + \sum_{t \in H(G_i)} f_i(t)\big) \in C$.*

PROOF. [**1**, Theorem 2.2]. □

The Central Sets Theorem for arbitrary semigroups is more complicated because the elements a_n must be split into several parts.

DEFINITION 3.6. Let $m \in \mathbb{N}$. Then $\mathcal{I}_m = \{(H_1, H_2, \ldots, H_m) : \text{each } H_i \in \mathcal{P}_f(\mathbb{N})$ and if $i \in \{1, 2, \ldots, m-1\}$, then $\max H_i < \min H_{i+1}\}$.

THEOREM 3.7 (General Central Sets Theorem). *Let $(S, \cdot)$ be a semigroup, let $\mathbb{F}$ be the set of sequences in S, and let C be a central subset of S. There exist functions $m : \mathcal{P}_f(\mathbb{F}) \to \mathbb{N}$, $\alpha \in \times_{F \in \mathcal{P}_f(\mathbb{F})} S^{m(F)+1}$, and $H \in \times_{F \in \mathcal{P}_f(\mathbb{F})} \mathcal{I}_{m(F)}$ such that*

(1) *if* $F, G \in \mathcal{P}_f(\mathbb{F})$ *and* $F \subsetneq G$, *then* $\max H(F)\big(m(F)\big) < \min H(G)(1)$ *and*
(2) *whenever* $n \in \mathbb{N}$, $G_1, G_2, \ldots, G_n \in \mathcal{P}_f(\mathbb{F})$ *such that* $G_1 \subsetneq G_2 \subsetneq \ldots \subsetneq G_n$, *and for each* $i \in \{1, 2, \ldots, n\}$, $f_i \in G_i$, *one has* $\prod_{i=1}^{n} \big(\prod_{j=1}^{m(G_i)} \big(\alpha(G_i)(j) \cdot \prod_{t \in H(G_i)(j)} f_i(t)\big)\big)\alpha(G_i)(m(G_i)+1) \in C$.

PROOF. [**1**, Corollary 3.10]. □

Kendall Williams

The main results of Dr. Williams' dissertation [**18**] are quite complicated to state. I will present here a simple special case which should convey the flavor of some of these results.

DEFINITION 3.8. Let $k \in \mathbb{N}$, let $\langle a_i \rangle_{i=1}^{k}$ be a sequence in $\mathbb{N}$ such that, if $i < k$, $a_i \neq a_{i+1}$, and let $\langle x_n \rangle_{n=1}^{\infty}$ be a sequence in $\mathbb{N}$. Then $MT(\vec{a}, \langle x_n \rangle_{n=1}^{\infty}) = \{\sum_{i=1}^{k} a_i \sum_{t \in F_i} x_t : F_1, F_2, \ldots, F_k \in \mathcal{P}_f(\mathbb{N}) \text{ and } \max F_i < \min F_{i+1} \text{ if } i < k\}$.

The set $MT(\vec{a}, \langle x_n \rangle_{n=1}^{\infty})$ is the *Milliken-Taylor system* determined by $\vec{a}$ and $\langle x_n \rangle_{n=1}^{\infty}$. The Milliken-Taylor systems are so named because of the relationship with the Milliken-Taylor Theorem ([**10**, Theorem 2.2] and [**16**, Lemma 2.2]).

THEOREM 3.9. *Let* $k \in \mathbb{N}$, *let* $\langle a_i \rangle_{i=1}^{k}$ *be a sequence in* $\mathbb{N}$ *such that, if* $i < k$, $a_i \neq a_{i+1}$, *and let* $\langle x_n \rangle_{n=1}^{\infty}$ *be a sequence in* $\mathbb{N}$. *Let* p *be an idempotent in* $\bigcap_{m=1}^{\infty} c\ell_{\beta\mathbb{N}}\big(FS(\langle x_n \rangle_{n=m}^{\infty})\big)$, *and let* $A \in a_1 p + a_2 p + \ldots a_k p$. *There exists a sum subsystem* $\langle y_n \rangle_{n=1}^{\infty}$ *of* $\langle x_n \rangle_{n=1}^{\infty}$ *such that* $MT(\vec{a}, \langle y_n \rangle_{n=1}^{\infty}) \subseteq A$.

PROOF. [**6**, Theorem 17.31]. □

The following is a special case of one of Dr. Williams' results. If one stopped at the second term, it would be easy to prove. As is, it is not at all easy.

THEOREM 3.10. *Let* p, q *be idempotents in* $(\beta\mathbb{N}, +)$ *and* $\langle x_n \rangle_{n=1}^{\infty}$ *and* $\langle y_n \rangle_{n=1}^{\infty}$ *be sequences in* $\mathbb{N}$ *such that*

$$p \in \bigcap_{m=1}^{\infty} c\ell_{\beta\mathbb{N}}\big(FS(\langle x_n \rangle_{n=m}^{\infty})\big) \text{ and } q \in \bigcap_{m=1}^{\infty} c\ell_{\beta\mathbb{N}}\big(FS(\langle y_n \rangle_{n=m}^{\infty})\big).$$

Let $a_1, a_2, a_3 \in N$ *and* $A \in a_1 p + a_2 q + a_3 p$. *Then there exist sum subsystems* $\langle u_n \rangle_{n=1}^{\infty}$ *of* $\langle x_n \rangle_{n=1}^{\infty}$ *and* $\langle v_n \rangle_{n=1}^{\infty}$ *of* $\langle y_n \rangle_{n=1}^{\infty}$ *such that* $\{a_1 \sum_{t \in F_1} u_t + a_2 \sum_{t \in F_2} v_t + a_3 \sum_{t \in F_3} u_t : F_1, F_2, F_3 \in \mathcal{P}_f(\mathbb{N}), \max F_1 < \min F_2, \text{ and } \max F_2 < \min F_3\} \subseteq A$.

PROOF. [**18**, Theorem 3.4]. □

The difficulty of this theorem is in the choice of the subsystem $\langle u_n \rangle_{n=1}^{\infty}$ of $\langle x_n \rangle_{n=1}^{\infty}$. To get a small idea of the complexity of Dr. Williams main results, he obtains similar conclusions for expressions like $-\frac{2}{3}p_1p_3 + p_3p_2 + 3p_1p_2p_3 + p_2p_1$ where the multiplication is in $(\beta\mathbb{N}, \cdot)$ and the addition is in $(\beta\mathbb{N}, +)$.

John H. Johnson

Dr. Johnson has several results in his dissertation [**9**], but to my mind the most impressive is his immense simplification of the general Central Sets Theorem, Theorem 3.7.

DEFINITION 3.11. Let $m \in \mathbb{N}$. Then $\mathcal{J}_m = \{(t_1, t_2, \ldots, t_m) \in \mathbb{N}^m : t_1 < t_2 < \ldots < t_m\}$.

Dr. Johnson's simplification replaces the finite sets H_i by single elements.

THEOREM 3.12 (Simplified General Central Sets Theorem). *Let $(S,\cdot)$ be a semigroup, let $\mathbb{F}$ be the set of sequences in S, and let C be a central subset of S. There exist functions $m:\mathcal{P}_f(\mathbb{F})\to\mathbb{N}$, $\alpha\in \times_{F\in\mathcal{P}_f(\mathbb{F})} S^{m(F)+1}$, and $\tau\in \times_{F\in\mathcal{P}_f(\mathbb{F})}\mathcal{J}_{m(F)}$ such that*

(1) *if $F,G\in\mathcal{P}_f(\mathbb{F})$ and $F\subsetneq G$, then $\tau(F)\big(m(F)\big)<\tau(G)(1)$ and*
(2) *whenever $n\in\mathbb{N}$, $G_1,G_2,\ldots,G_n\in\mathcal{P}_f(\mathbb{F})$ such that $G_1\subsetneq G_2\subsetneq\ldots\subsetneq G_n$ and for each $i\in\{1,2,\ldots,n\}$, $f_i\in G_i$, one has $\prod_{i=1}^n\big(\prod_{j=1}^{m(G_i)}\big(\alpha(G_i)(j)\cdot f_i\big(\tau(G_i)(j)\big)\big)\big)\alpha(G_i)(m(G_i)+1)\in C$.*

If you think this is still pretty complicated, you are correct, but Theorem 3.12 is less complicated than Theorem 3.7. Notice that there is one fewer product in the computation. The product $\prod_{t\in H(G_i)(j)} f_i(t)$ is replaced by the single term $f_i\big(\tau(G_i)(j)\big)$.

Dev Phulara

In the algebraic portion of his dissertation, Dr. Phulara extended both the commutative Central Sets Theorem and the general Central Sets Theorem. Both of these results deal with arbitrary central sets, so they could be phrased as "let p be a minimal idempotent in βS and let $C\in p$." Dr. Phulara's extensions begin "let p be a minimal idempotent in βS and let $\langle C_n\rangle_{n=1}^\infty$ be a sequence of members of p." The rest of the statements remain the same except that the products which were guaranteed to be in C, are now guaranteed to be in C_k, where $k=|G_1|$.

Kendra Pleasant

In the algebraic portion of her dissertation, Dr. Pleasant extended the commutative Central Sets Theorem to *partial semigroups*. A *partial semigroup* is a pair $(S,*)$, where $*$ is a binary operation defined on some, but not necessarily all, elements of $S\times S$ with the property that for all $x,y,z\in S$, $(x*y)*z=x*(y*z)$ in the sense that if one side is defined, so is the other, and they are equal. The partial semigroup $(S,*)$ is *adequate* provided that for any $F\in\mathcal{P}_f(S)$, there is some $y\in S$ such that $x*y$ is defined for all $x\in F$. If $(S,*)$ is adequate, then $\delta S=\bigcap_{F\in\mathcal{P}_f(S)} c\ell_{\beta S}\{y\in S:(\forall x\in F)(x*y \text{ is defined})\}$ is a semigroup. As a compact Hausdorff right topological semigroup, δS has a smallest ideal so the definition of central sets as members of minimal idempotents makes sense.

Dr. Pleasant established that the commutative Central Sets Theorem remains valid in an adequate partial semigroup almost verbatim. The distinction is that instead of taking $\mathbb{F}$ as the set of all sequences in S, she defines $\mathbb{F}$ to be the set of all *adequate* sequences in S. A sequence $\langle f(n)\rangle_{n=1}^\infty$ in a partial semigroup $(S,+)$ is *adequate* if and only if (1) for each $F\in\mathcal{P}_f(\mathbb{N})$, $\sum_{t\in F} f(t)$ is defined, and (2) for each $L\in\mathcal{P}_f(S)$, there exists $m\in\mathbb{N}$ such that $x+y$ is defined for all $x\in L$ and all $y\in FS(\langle x_n\rangle_{n=m}^\infty)$. With this change in the definition of $\mathbb{F}$, the commutative Central Sets Theorem for adequate partial semigroups is verbatim the same as Theorem 3.5.

4. Algebra of βS

Two of the dissertations being surveyed deal with purely algebraic questions about S and βS, though the notion of J-sets is motivated by the Central Sets Theorem.

One of the major unsolved problems about the algebra of $(\beta\mathbb{N},+)$ is whether there is a nontrivial continuous homomorphism from $\beta\mathbb{N}$ to $\mathbb{N}^* = \beta\mathbb{N}\setminus\mathbb{N}$. (It is known that any such continuous homomorphism must have finite range.)

DEFINITION 4.1. Let $(S,\cdot)$ be a semigroup and let $\mathbb{F}$ be the set of sequences in S.

(a) Let $A\subseteq S$. Then A is a J-set in S if and only if for each $F\in\mathcal{P}_f(\mathbb{F})$, there exist $m\in\mathbb{N}$, $a\in S^{m+1}$, and $t\in\mathcal{J}_m$ such that for each $f\in F$, $a(1)\cdot f\big(t(1)\big)\cdot a(2)\cdot f\big(t(2)\big)\cdots a(m)\cdot f\big(t(m)\big)\cdot a(m+1)\in A$.

(b) $J(S)=\{p\in\beta S:(\forall A\in p)(A \text{ is a } J\text{-set})\}$.

It is known [**6**, Theorem 14.15.1] that A satisfies the conclusion of the general Central Sets Theorem (Theorem 3.12) if and only if there is an idempotent in $J(S)\cap c\ell_{\beta S}A$.

Kourtney Fulton Miller

In her dissertation [**11**], Dr. Miller dealt with the free semigroup S_ω on the distinct generators $\{a_n : n\in\mathbb{N}\}$ and for $n\in\mathbb{N}$ the free semigroup S_n on the generators $\{a_1,a_2,\ldots,a_n\}$. She also utilized the ordering of idempotents. Given a semigroup $(S,\cdot)$ and idempotents $p,q\in\beta S$, $p\leq q$ if and only if $p=p\cdot q=q\cdot p$ and $p<q$ if and only if $p\leq q$ and $p\neq q$. An idempotent p is minimal with respect to this ordering if and only if $p\in K(\beta S)$.

The following simple result allowed her to effectively address the problem of existence of continuous homomorphisms from βS_ω to $S_\omega^*=\beta S_\omega\setminus S_\omega$.

THEOREM 4.2. *There exists a sequence $\langle q_n\rangle_{n=1}^\infty$ of idempotents in S_ω^* such that for each $n\in\mathbb{N}$, $q_n\in K(\beta S_n)$ and $q_{n+1}<q_n$.*

PROOF. [**11**, Lemma 2.9]. □

Dr. Miller then showed that if T is any finite subset of $\{q_n : n\in\mathbb{N}\}$, then there is a continuous homomorphism $\varphi:\beta S_\omega\to S_\omega^*$ such that $\varphi[\beta S_\omega]=T$. Thus, unlike the situation in $\mathbb{N}$, continuous homomorphisms with finite images of any size are now known to exist. She also showed that this result does not apply to an infinite subset of $\{q_n : n\in\mathbb{N}\}$.

Dr. Miller also established that when the sequence $\langle q_n\rangle_{n=1}^\infty$ is constructed in Theorem 4.2, having chosen $\langle q_t\rangle_{t=1}^n$, there are $2^{\mathfrak{c}}$ choices for q_{n+1}, each in $K(\beta S_{n+1})$ and each less than q_n.

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We said at the start that we take βS to be a right topological with the additional property that for each $x\in S$, the function $q\mapsto x\cdot q$ is continuous. We have denoted the extended operation by the same symbol as used for the operation on S. We could alternatively have taken βS to be left topological. (In fact, that used to be my own choice.) Let us denote the left topological extension by $\odot$ to distinguish it from the right topological extension. Then one has that for each $p\in\beta S$, the function $q\mapsto p\odot q$ from βS to itself is continuous and for $x\in S$, the function $q\mapsto q\odot x$ is continuous.

If the operation on S is commutative, then $p\odot q=q\cdot p$ so $K(\beta S,\cdot)=K(\beta S,\odot)$. But if the operation is not commutative, there can be substantial differences. There are several notions of size that are relevant to the algebraic structure of βS, such as *syndetic*, *piecewise syndetic*, *thick*, and of course *central*. Each of these have both left and right versions, and in some cases it is easy to see that they are different. In my mind, the most difficult to tell apart are left and right J-sets.

Let us call the notion defined in Definition 4.1 a *right J-set*.

DEFINITION 4.3. Let $(S, \cdot)$ be a semigroup and let $\mathbb{F}$ be the set of sequences in S.

(a) Let $A \subseteq S$. Then A is a *left J-set* in S if and only if for each $F \in \mathcal{P}_f(\mathbb{F})$, there exist $m \in \mathbb{N}$, $a \in S^{m+1}$, and $t \in \mathcal{J}_m$ such that for each $f \in F$, $a(m+1) \cdot f\big(t(m)\big) \cdot a(m) \cdot f\big(t(m-1)\big) \cdot a(m-1) \cdots f\big(t(1)\big) \cdot a(1) \in A$.

(b) $(J(S), \odot) = \{p \in \beta S : (\forall A \in p)(A \text{ is a left } J\text{-set})\}$.

In her dissertation [**12**], Dr. Peters produced a subset A of the free semigroup S_ω on the distinct generators $\{a_n : n \in \mathbb{N}\}$ which is a left J-set but not a right J-set. The construction is far too complicated to be reproduced here. As a consequence, using the left-right switches of [**6**, Theorem 3.11 and Lemma 14.14.6] one also has that $J(S_\omega, \odot) \setminus J(S_\omega, \cdot) \neq \emptyset$.

5. Conclusion

In each of the seven dissertations that I have been discussing, there are a number of results that I haven't mentioned, and in some cases whole topics that I haven't hinted at. I have tried to present results that are, in my view, significant and also are reasonably easy to talk about without going into too much detail. The results of the seven dissertations extend our knowledge of Ramsey Theory and our understanding of the algebraic structure of the Stone-Čech compactification of a discrete semigroup.

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Department of Mathematics, Howard University, Washington, DC 20059
E-mail address: nhindman@aol.com