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Additive and Multiplicative Ramsey Theory in the Reals and the Rationals

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Abstract. Let a finite partition $\mathcal{F}$ of the real interval $(0,1)$ be given. We show that if every member of $\mathcal{F}$ is measurable or if every member of $\mathcal{F}$ is a Baire set, then one member of $\mathcal{F}$ must contain a sequence with all of its finite sums and products (and, in the measurable case, all of its infinite sums as well).

These results are obtained by using the algebraic structure of the Stone-Čech compactification of the real numbers with the discrete topology. And they are also obtained by elementary methods. In each case we in fact get significant strengthenings of the above stated results (with different strengthenings obtained by the algebraic and elementary methods).

Some related (although weaker) results are established for arbitrary partitions of the rationals and the dyadic rationals, and a counterexample is given to show that even weak versions of the combined additive and multiplicative results do not hold in the dyadic rationals.

1. Introduction.

The Finite Sums Theorem [8, Theorem 3.1] says that whenever the set $\mathbb{N}$ of positive integers is partitioned into finitely many classes, one of these classes must contain a sequence together with all of its finite sums taken without repetition. As an immediate corollary one obtains the corresponding statement for a sequence with all of its finite products. That is, whenever $\mathbb{N}$ is partitioned into finitely many classes, one of these classes must contain a sequence together with all of its finite products. (Simply consider the powers of 2.) For some time it was an open question as to whether or not one could always get in one cell of a partition of $\mathbb{N}$ a sequence with all of its sums *and* products. (This question was answered in the negative in [10].)

The Finite Unions Theorem [8, Corollary 3.3] is equivalent to the Finite Sums Theorem: it states that whenever the finite nonempty subsets of $\mathbb{N}$ are partitioned into

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finitely many classes, one of these classes contains all finite unions of some sequence $\langle F_n \rangle_{n=1}^{\infty}$ of pairwise disjoint sets. By way of the Finite Unions Theorem, one easily sees that similar statements hold in the real interval $(0,1)$. That is, whenever $(0,1)$ is partitioned into finitely many classes there must exist a sequence $\langle x_n \rangle_{n=1}^{\infty}$ with all of its finite sums in one class and there must exist a sequence $\langle y_n \rangle_{n=1}^{\infty}$ with all of its finite products in one class (possibly a different class). (See, for example, [11, Lemma 3.8].) The question as to whether or not the sequences $\langle x_n \rangle_{n=1}^{\infty}$ and $\langle y_n \rangle_{n=1}^{\infty}$ can be chosen to be the same remains open. In Section 5 of this paper we are able to answer the restriction of this question to the dyadic rationals in $(0,1)$ in the negative. Let us briefly remark here that, of course, any positive result about $(0,1)$ trivially implies a positive result about the entire set of non-zero reals – one works with $(0,1)$ mainly for convenience. However, the above negative result about $\mathbb{D} \cap (0,1)$ ($\mathbb{D}$ being the set of dyadic rationals) does in fact extend to a negative result about the full set of non-zero dyadics.

In Section 4 of this paper we show that given any finite partition $\mathcal{F}$ of $(0,1)$, if all of the members of $\mathcal{F}$ are measurable or if all of the members of $\mathcal{F}$ are Baire sets, then there exists one cell of the partition which contains a sequence $\langle x_n \rangle_{n=1}^{\infty}$ together with all of its finite sums and all of its finite products. This seems to be the first positive result linking addition and multiplication of the same sequence. (There was an earlier result [9] concerning special partitions of $\mathbb{N}$: if only one cell of the partition supports a sequence with finite sums, i.e., one cell is an "IP*-set", then that cell will contain a sequence $\langle x_n \rangle_{n=1}^{\infty}$ together with all of its finite sums and all of its finite products.)

In fact, the results of Section 4 are stronger than this in three directions.

Firstly, one allows the sums and products to be intermixed in a restricted fashion. (One allows expressions built up from items whose "supports" do not overlap. For example, the support of $x_1 + x_3$ is $\{1,3\}$ and $3 < 5$ so $(x_1 + x_3) \cdot x_5$ is an allowable expression as is $(x_7 + x_9) \cdot x_{11}$. Since $5 < 7$, $(x_1 + x_3) \cdot x_5 + (x_7 + x_9) \cdot x_{11}$ is an allowable expression also.)

The second strengthening is related to the (m,p,c)-systems of [6]. That is, instead of producing a sequence of numbers $\langle x_n \rangle_{n=1}^{\infty}$, one produces a sequence of finite sets $\langle G_n \rangle_{n=1}^{\infty}$ and then allows any choice of $x_n \in G_n$ in the expressions described above. The sets G_n consist of solution sets for partition regular systems of equations, either additive or multiplicative – so one could, for example, ask that each G_{2n} be a length n arithmetic progression and each G_{2n-1} be a length n geometric progression. See [7] for general background about partition regular equations.

The third strengthening concerns infinite sums. Given a sequence $\langle x_n\rangle_{n=1}^\infty$ with all of its finite sums in $(0,1)$, one has that $\Sigma_{n=1}^\infty x_n$ converges. Consequently, one may ask whether given any finite partition of $(0,1)$ there must exist one cell and a sequence $\langle x_n\rangle_{n=1}^\infty$ with all of its sums without repetition (finite *or* infinite) in that cell. It is easy to see via a standard diagonalization argument that this is false in such generality. However, Prömel and Voigt [16] showed that if one assumes that each cell of the partition is a Baire set, then one does get one cell and a sequence with all of its sums, finite or infinite, in that cell. (We remind the reader that the Baire sets are the members of the smallest σ-algebra containing the open sets and the nowhere dense sets. Thus the Baire sets are precisely those sets that can be expressed as the symmetric difference of an open set and a meager set, where a set is *meager* provided it is the countable union of nowhere dense sets.)

Later, Plewik and Voigt [15] obtained the same conclusion from the assumption that each cell of the partition is Lebesgue measurable. A simplified and unified presentation of the results in [15] and [16] is given in [4], along with several strengthenings and (counter)examples.

The third strengthening in Section 4 is to allow, as well as finite sums and products, infinite sums as well, in the measurable case. In other words, we show that given any finite partition $\mathcal{F}$ of $(0,1)$, if all of the members of $\mathcal{F}$ are measurable then there exists one cell of the partition which contains a sequence $\langle x_n\rangle_{n=1}^\infty$ together with all of its finite and infinite sums and all of its finite products. It turns out that we obtain these infinite sums with almost no extra work. Interestingly, we do not know what happens in the Baire case.

Our methods in Section 4 involve ultrafilter techniques. It is therefore natural to ask how much can be proved by "elementary" techniques (in other words, without appeal to the structure of the Stone-Čech compactifications of various spaces).

This question is addressed in Section 2. Although we are unable to recover any of the results concerning the sets G_n, we are able to prove the statements about sums and products in a Baire or measurable partition that were mentioned in the Abstract. Rather curiously, we also prove some rather strong extensions of this that we have not been able to prove by the techniques of Section 4. For example, in the Baire case, we show that given any sequence of increasing homeomorphisms $\langle \varphi_n\rangle_{n=1}^\infty$ from $(0,1)$ onto $(0,1)$ and a finite coloring of $(0,1)$ one can get a sequence $\langle y_n\rangle_{n=1}^\infty$ such that the color of sums of products of the functions applied to the terms of $\langle y_n\rangle_{n=1}^\infty$ in appropriate order depends only on the function applied to the lowest order term. (For concrete illustrations see the discussion before Theorem 2.15.) Similar results are obtained for the measurable case for a more restricted class of functions. We do not know if there are common extensions of the results of Sections 2 and 4.

In Section 5, in addition to the counterexample mentioned earlier, we establish in $\mathbb{Q} \cap (0,1)$ (and indeed in $\mathbb{D} \cap (0,1)$) separate additive and multiplicative statements involving sequences $\langle G_n \rangle_{n=1}^{\infty}$ similar to those in Section 4.

Since the methods used in Sections 4 and 5 involve topological semigroups, Section 3 consists of some topological-algebraic preliminaries. In spite of the fact that the results in Section 4 use notions based on the usual topology of the reals, we will work with the algebraic structure of the Stone-Čech compactification βX_d, where $X = (0,1)$ or $(0,1) \cap \mathbb{D}$, and the subscript indicates that one puts the discrete topology on X. We emphasise once again that the restriction to $(0,1)$ is purely for convenience.

Our notation is mostly standard. We write $\mathcal{P}_f(A)$ for $\{B : B \subseteq A,\ B \text{ is finite, and } B \neq \emptyset\}$, and we often write $c\ell$ to denote closure.

We use the notations FS, FP, and FU for "finite sums", "finite products", and "finite unions" respectively. That is given a sequence $\langle x_n \rangle_{n=1}^{\infty}$ in $\mathbb{R}$ and a sequence $\langle G_n \rangle_{n=1}^{\infty}$ in $\mathcal{P}_f(\mathbb{N})$ we write $FS(\langle x_n \rangle_{n=1}^{\infty}) = \{\Sigma_{n \in F}\, x_n : F \in \mathcal{P}_f(\mathbb{N})\}$, $FP(\langle x_n \rangle_{n=1}^{\infty}) = \{\Pi_{n \in F}\, x_n : F \in \mathcal{P}_f(\mathbb{N})\}$, and $FU(\langle G_n \rangle_{n=1}^{\infty}) = \{\bigcup_{n \in F}\, G_n : F \in \mathcal{P}_f(\mathbb{N})\}$.

2. Elementary Results.

As will be the case in Section 4, the proofs of the results for Baire sets and for measurable sets are nearly identical. We develop the corresponding notions side by side, beginning with the parallel notions of largeness that we shall need. Several of the preliminary lemmas are similar to results in [4].

For our results about measurable partitions of $(0,1)$ we use the notion of upper density near 0. We denote Lebesgue measure by μ and write $\mu^*(A)$ for the outer Lebesgue measure of the set A. In this section when we use Lebesgue measure, it will always be with measurable sets. However, in later sections we will deal with ultrafilters with the property that for every member A, its upper density $\overline{d}(A) > 0$ and we cannot assume that every member of an ultrafilter is measurable. Consequently we define $\overline{d}(A)$ in terms of the outer measure.

Let A be a subset of $\mathbb{R}$. A point x is a *density point* of A if and only if $\lim_{\epsilon \downarrow 0} \mu^*\big(A \cap (x-\epsilon, x+\epsilon)\big)/(2\epsilon) = 1$.

2.1 Definition. Let $A \subseteq (0,1)$.

(a) The *upper density near* 0 *of* A, $\overline{d}(A)$, is defined by

$$\overline{d}(A) = \limsup_{\epsilon \downarrow 0} \mu^*\big(A \cap (0,\epsilon)\big)/\epsilon \ .$$

(b) The *density near* 0 *of* A, if it exists, is $d(A) = \lim_{\epsilon \downarrow 0} \mu^*\big(A \cap (0,\epsilon)\big)/\epsilon$.

(c) $\delta(A) = \{x \in A : x \text{ is a density point of } A\}$.

Observe that if x is a density point of A, then $d(A-x) = 1$. (When we write $A - x$ in this section we mean $\{y \in (0,1) : y + x \in A\}$, that is $\{y - x : y \in A\} \cap (0,1)$.)

We now introduce a notion of largeness at 0 in terms of meager sets, that is sets that are the countable union of nowhere dense sets. (The terminology "Baire large" was also used in [4], but the notions do not coincide unless A is a Baire set.)

2.2 Definition. Let $A \subseteq (0,1)$.

(a) A is *Baire large (at* 0*)* if and only if for every $\epsilon > 0$, $A \cap (0,\epsilon)$ is not meager.

(b) A is *Baire small (at* 0*)* if and only if A is not Baire large. (Equivalently A is Baire small (at 0) if and only if there is some $\epsilon > 0$ such that $A \cap (0,\epsilon)$ is meager.)

(c) A is *Baire huge (at* 0*)* if and only if there is some $\epsilon > 0$ such that $(0,\epsilon)\backslash A$ is meager.

(d) $\delta_b(A) = \{x \in A : A - x \text{ is Baire huge}\}$.

Thus a set A is Baire huge if and only if $(0,1)\backslash A$ is Baire small.

2.3 Lemma. *Let* A *be a measurable subset of* $\mathbb{R}$. *Then* $\mu\big(A\backslash\delta(A)\big) = 0$.

Proof. This is the Lebesgue Density Theorem – see for example [14, Theorem 3.20]. $\square$

2.4 Lemma. *Let* A *be a Baire subset of* $\mathbb{R}$. *Then* $A\backslash\delta_b(A)$ *is meager.*

Proof. Pick open U and meager M such that $A = U \Delta M$. We show that $A\backslash\delta_b(A) \subseteq M$, or equivalently that $A\backslash M \subseteq \delta_b(A)$. Let $x \in A\backslash M$. Then $x \in U\backslash M$ so pick $\epsilon > 0$ such that $(x-\epsilon, x+\epsilon) \subseteq U$. Then $(0,\epsilon)\backslash(A-x) \subseteq M - x$. $\square$

Note that if A is measurable and $d(A) = 1$, then $\delta(A)$ is measurable and $d\big(\delta(A)\big) = 1$. Similarly, if A is a Baire set which is Baire huge, then $\delta_b(A)$ is a Baire set which is Baire huge. (One has deleted a meager set from a Baire set.)

We combine the Baire and measurable versions of the next two results, and omit the trivial proofs.

2.5 Lemma. *Let $A, B \subseteq (0,1)$.*

(a) If $d(A) = d(B) = 1$, then $d(A \cap B) = 1$.

(b) If A and B are Baire huge, then $A \cap B$ is Baire huge. $\square$

When we write A/x in this section we mean

$$\{y \in (0,1) : yx \in A\} = \{z/x : z \in A\} \cap (0,1)\,.$$

2.6 Lemma. *Let $A \subseteq (0,1)$ and let $x \in (0,\infty)$.*

(a) If $d(A) = 1$, then $d(A/x) = 1$. (Similarly, if $\overline{d}(A) = 1$, then $\overline{d}(A/x) = 1$.)

(b) If A is Baire huge, then A/x is Baire huge. $\square$

In one respect the results of this section are stronger for Baire partitions than for measurable partitions. (In another respect the results for measurable partitions are stronger. See the discussion before Lemma 2.11.) That is, the results in either case are stated in terms of a collection of functions from an interval $(0,\alpha)$ to $(0,\infty)$. In the case of Baire partitions, this collection is simply the increasing continuous functions which would (if extended) take 0 to 0. The collection used in the measurable case is considerably more restricted.

Our guiding principle is that we need Lemma 2.9 to hold. In the measurable case one might at first expect absolutely continuous functions to be sufficiently restrictive, but a little thought shows that they are not. (See Proposition 2.10.) We use the definition of admissible functions that we do, because it allows us to prove Lemma 2.9.

2.7 Definition. A function φ is an *admissible function* if and only if

(1) there is some $\alpha > 0$ such that $\varphi : (0,\alpha) \longrightarrow (0,\infty)$ and $\lim_{x\downarrow 0} \varphi(x) = 0$,

(2) φ is differentiable on $(0,\alpha)$ and for each $x \in (0,\alpha)$, $\varphi'(x) > 0$, and

(3) either

(a) φ' is nonincreasing on $(0,\alpha)$ and for every $\eta < 1$, $\limsup_{x\downarrow 0} \dfrac{\varphi(\eta x)}{\varphi(x)} < 1$ or

(b) φ' is nondecreasing on $(0,\alpha)$ and for every $\tau > 0$, $\liminf_{x\downarrow 0} \dfrac{\varphi(\tau x)}{\varphi(x)} > 0$.

Given any $\tau > 0$ one has that the function φ defined by $\varphi(x) = x^\tau$ is an admissible function. Other examples include the function γ defined by $\gamma(x) = e^x - 1$ and its inverse $\gamma^{-1}(x) = \log(x+1)$. On the other hand, consider the function ν defined by $\nu(x) = \frac{-1}{\log(x)}$. Then ν' is decreasing on $(0, e^{-2})$ but given any $\eta < 1$ one has $\lim_{x\downarrow 0} \dfrac{\nu(\eta x)}{\nu(x)} = 1$, so ν is not an admissible function. (And by Proposition 2.10, it could not be under any definition, given that we want Lemma 2.9 to apply to any admissible function.) Also ν^{-1} fails to be an admissible function because given any $\tau > 0$, $\lim_{x\downarrow 0} \frac{\nu^{-1}(\tau x)}{\nu^{-1}(x)} = 0$.

It is easy to check that the inverse of an admissible function is again an admissible function.

2.8 Definition. (a) $\mathcal{H} = \{\varphi :$ there is some $\alpha > 0$ such that φ is an increasing continuous map from $(0,\alpha)$ to $(0,\infty)$ and $\lim_{x\downarrow 0} \varphi(x) = 0\}$.

(b) $\mathcal{I} = \{\varphi : \varphi$ is an admissible function$\}$.

2.9 Lemma. *(a) Let $\varphi \in \mathcal{H}$ where* domain$(\varphi) = (0,\alpha)$ *and let $A \subseteq (0,\alpha)$ such that $\varphi[A]$ is Baire huge. Then A is Baire huge.*

(b) Let $\varphi \in \mathcal{I}$ where domain$(\varphi) = (0,\alpha)$ *and let $A \subseteq (0,\alpha)$ such that $d(\varphi[A]) = 1$. Then $d(A) = 1$.*

Proof. (a) Since φ is a homeomorphism (onto its image), so is φ^{-1}.

(b) Observe first that if φ' is nonincreasing, then given $a < b < \alpha - c$ one has

$$\varphi(b) - \varphi(a) = \int_a^b \varphi'(t)\ dt \geq \int_{a+c}^{b+c} \varphi'(t)\ dt = \varphi(b+c) - \varphi(a+c)\ .$$

Consequently, if $0 < \eta < 1$, $x \in (0,\alpha)$, $\langle (a_n,b_n)\rangle_{n=1}^k$ is a sequence of pairwise disjoint intervals in $(0,x)$, and $\Sigma_{n=1}^k (b_n - a_n) < \eta x$, then

(*) $\qquad \Sigma_{n=1}^k \big(\varphi(b_n) - \varphi(a_n)\big) < \varphi(\eta x)$.

(Shifting intervals to the left keeps the first sum fixed and increases the second sum. Consequently the worst possible case is when $a_1 = 0$ and for each $t \in \{1,2,\ldots,k-1\}$, $b_t = a_{t+1}$.)

Similarly, if φ' is nondecreasing, then given $c < a < b < \alpha$ one has $\varphi(b) - \varphi(a) \leq \varphi(b-c) - \varphi(a-c)$. Consequently, if $0 < \eta < 1$, $x \in (0,\alpha)$, $\langle (a_n,b_n)\rangle_{n=1}^k$ is a sequence of pairwise disjoint intervals in $(0,x)$, and $\Sigma_{n=1}^k (b_n - a_n) < \eta x$, then

(**) $\qquad \Sigma_{n=1}^k \big(\varphi(b_n) - \varphi(a_n)\big) < \varphi(x) - \varphi(x - \eta x)$.

(Shifting intervals to the right keeps the first sum fixed and increases the second sum. Consequently the worst possible case is when $b_n = x$ and for each $t \in \{1,2,\ldots,k-1\}$, $b_t = a_{t+1}$.)

To see that $\lim_{\epsilon\downarrow 0} \mu^*\big(A \cap (0,\epsilon)\big)/\epsilon = 1$, let $\eta < 1$ be given. If φ is nonincreasing, choose $\gamma < 1$ such that $\gamma > \limsup_{x\downarrow 0} \frac{\varphi(\eta x)}{\varphi(x)}$. If φ is nondecreasing, choose $\gamma < 1$ such that $1 - \gamma < \liminf_{x\downarrow 0} \frac{\varphi\big((1-\eta)x\big)}{\varphi(x)}$.

Then $\gamma < 1$ so pick $\epsilon > 0$ (with ϵ in the range of φ) such that whenever $0 < x < \epsilon$, one has that $\mu^*\big(\varphi[A] \cap (0,x)\big) > \gamma x$. We may also presume that if φ is nonincreasing, then whenever $0 < x < \epsilon$, one has $\frac{\varphi(\eta x)}{\varphi(x)} < \gamma$ and if φ is nondecreasing, then whenever $0 < x < \epsilon$, one has $\frac{\varphi\left((1-\eta)x\right)}{\varphi(x)} > 1 - \gamma$.

Now let $0 < x < \varphi^{-1}(\epsilon)$. We claim that $\mu^*\big(A \cap (0,x)\big) \geq \eta x$. So suppose instead that $\mu^*\big(A \cap (0,x)\big) < \eta x$ and pick pairwise disjoint intervals $\langle (a_n, b_n)\rangle_{n=1}^{\infty}$ such that $A \cap (0,x) \subseteq \bigcup_{n=1}^{\infty}(a_n, b_n)$ and $\Sigma_{n=1}^{\infty}(b_n - a_n) < \eta x$.

Then $\varphi[A] \cap \big(0, \varphi(x)\big) \subseteq \bigcup_{n=1}^{\infty}\big(\varphi(a_n), \varphi(b_n)\big)$ so choose $k \in \mathbb{N}$ such that $\Sigma_{n=1}^{k}\big(\varphi(b_n) - \varphi(a_n)\big) > \gamma \cdot \varphi(x)$. If φ is nonincreasing, then we have that $\gamma \cdot \varphi(x) > \varphi(\eta x)$ contradicting statement (*). So assume that φ is nondecreasing. Then we have that $\gamma \cdot \varphi(x) > \varphi(x) - \varphi\big((1-\eta)x\big)$, contradicting statement (**). $\square$

We pause now to observe that at least part of the requirement in the definition of an admissible function is necessary. Notice that we do not assume any monotonicity for φ' in the following.

2.10 Proposition. *Let $\alpha > 0$ and let φ be an increasing function from $(0,\alpha)$ to $(0,\infty)$ such that $\lim_{x\downarrow 0}\varphi(x) = 0$. If there is some $\eta < 1$ such that $\limsup_{x\downarrow 0}\dfrac{\varphi(\eta x)}{\varphi(x)} = 1$, then there is a set $A \subseteq (0,\alpha)$ such that $d(\varphi[A]) = 1$ but $d(A) \neq 1$.*

Proof. Choose a sequence $\langle b_n\rangle_{n=1}^{\infty}$ converging to 0 such that for each n, $\varphi(b_{n+1}) < \frac{\varphi(\eta b_n)}{2^n}$ and $\frac{\varphi(\eta b_n)}{\varphi(b_n)} > 1 - \frac{1}{2^n}$. Let $A = (0,\alpha)\backslash \bigcup_{n=1}^{\infty}(\eta b_n, b_n)$.

Then for each n, $\mu\big(A \cap (0, b_n)\big) < \eta b_n$ so $d(A) \neq 1$. To see that $d(\varphi[A]) = 1$, let $\epsilon > 0$ be given, pick n such that $\frac{1}{2^n} < \epsilon$, and let $x < \varphi(\eta b_n)$ be given. Pick m such that $\varphi(\eta b_m) \leq x < \varphi(\eta b_{m-1})$ and note that $m \geq n+1$.

Assume first that $x \leq \varphi(b_m)$. Then

$$(0,x)\backslash A \subseteq \big(0, \varphi(b_{m+1})\big) \cup (\varphi(\eta b_m), x)$$

so

$$\begin{array}{rcl} \mu\big((A \cap (0,x)\big) & \geq & x - \varphi(b_{m+1}) - \big(x - \varphi(\eta b_m)\big) \\ & > & \varphi(\eta b_m) \cdot (1 - \frac{1}{2^m}) \\ & > & (1 - \frac{1}{2^m})^2 \cdot \varphi(b_m) \\ & > & (1 - \frac{1}{2^n}) \cdot x \ . \end{array}$$

Next assume that $x > \varphi(b_m)$. Then

$$\begin{array}{rcl} \mu\big((A \cap (0,x)\big) & \geq & x - \varphi(b_{m+1}) - \big(\varphi(b_m) - \varphi(\eta b_m)\big) \\ & > & x - \frac{\varphi(b_m)}{2^{m-1}} \\ & > & (1 - \frac{1}{2^n}) \cdot x \ . \end{array}$$

$\square$

We do not know the precise class of functions for which Lemma 2.9 holds.

We have already seen that in one respect the results of this section for Baire partitions are stronger than for measurable partitions. In another respect, the results for the measurable case are stronger. One gets the closure of the set of all of the finite configurations contained in one set. This stronger conclusion depends on the following simple lemma.

2.11 Lemma. *Let A be a measurable subset of $(0,1)$ such that $\overline{d}(A) > 0$. Then there exists $B \subseteq A$ such that $B \cup \{0\}$ is compact and $\overline{d}(A \backslash B) = 0$.*

Proof. For each $n \in \mathbb{N}$, let $A_n = A \cap (1/2^n, 1/2^{n-1})$ and let $T = \{n \in \mathbb{N} : \mu(A_n) > 0\}$. As is well known (see [14, Definition 3.8]) given any measurable set C and any $\epsilon > 0$ there is a compact subset D of C with $\mu(D) > \mu(C) - \epsilon$. Thus for each $n \in T$, pick compact $B_n \subseteq A_n$ with $\mu(B_n) > \mu(A_n) - \frac{1}{4^{n+1}}$. Let $B = \bigcup_{n \in T} B_n$. Then $B \cup \{0\}$ is compact.

Suppose now that $\overline{d}(A \backslash B) = \alpha > 0$. Pick $m \in \mathbb{N}$ such that $\frac{1}{3 \cdot 2^m} < \alpha$. Pick $x < 1/2^m$ such that $\mu^*((A \backslash B) \cap (0,x))/x > \frac{1}{3 \cdot 2^m}$. Pick $n \in \mathbb{N}$ with $1/2^n \leq x < 1/2^{n-1}$ and note that $n > m$. Then

$$\mu((A \backslash B) \cap (0,x)) \leq \Sigma_{k=n}^{\infty} \mu(A_n \backslash B_n) < \Sigma_{k=n}^{\infty} \frac{1}{4^{k+1}} = \frac{1}{3 \cdot 4^n}$$

and $x \geq \frac{1}{2^n}$ so

$$\mu((A \backslash B) \cap (0,x)/x < \frac{1}{3 \cdot 2^n} < \frac{1}{3 \cdot 2^m},$$

a contradiction. $\square$

2.12 Lemma. *Let $r \in \mathbb{N}$ and let $(0,1) = \bigcup_{i=1}^{r} C_i$ where each C_i is measurable. Then for each $i \in \{1, 2, \ldots, r\}$ there exists $D_i \subseteq C_i$ such that $D_i \cup \{0\}$ is compact and $d(\bigcup_{i=1}^{r} D_i) = 1$.*

Proof. For each $i \in \{1, 2, \ldots, r\}$, if $\overline{d}(C_i) = 0$, let $D_i = \emptyset$ and if $\overline{d}(C_i) > 0$ pick $D_i \subseteq C_i$ as guaranteed by Lemma 2.11. Then $\overline{d}\big((0,1) \backslash \bigcup_{i=1}^{r} D_i\big) \leq \overline{d}\big(\bigcup_{i=1}^{r} (C_i \backslash D_i)\big) = 0$. $\square$

As a final preliminary, we have the following well known result. (See the discussion in [3] regarding the fact that this result is "elementary".) Given finite nonempty subsets of $\mathbb{N}$, we write $F < G$ to mean that $\max F < \min G$.

2.13 Definition. Let $\langle F_n \rangle_{n=1}^{\infty}$ and $\langle G_n \rangle_{n=1}^{\infty}$ be sequences in $\mathcal{P}_f(\mathbb{N})$. Then $\langle F_n \rangle_{n=1}^{\infty}$ is a *union subsystem* of $\langle G_n \rangle_{n=1}^{\infty}$ if and only if there is a sequence $\langle H_n \rangle_{n=1}^{\infty}$ in $\mathcal{P}_f(\mathbb{N})$ such that for each $n \in \mathbb{N}$, $H_n < H_{n+1}$ and $F_n = \bigcup_{t \in H_n} G_t$.

2.14 Lemma. *Let $\langle F_n\rangle_{n=1}^{\infty}$ be a sequence in $\mathcal{P}_f(\mathbb{N})$, let $r\in\mathbb{N}$, and let*

$$\varphi : FU(\langle F_n\rangle_{n=1}^{\infty}) \longrightarrow \{1,2,\ldots,r\}\ .$$

There exists a union subsystem $\langle G_n\rangle_{n=1}^{\infty}$ of $\langle F_n\rangle_{n=1}^{\infty}$ such that φ is constant on $FU(\langle G_n\rangle_{n=1}^{\infty})$.

Proof. This follows immediately from [3, Lemma 2.1]. $\square$

Let a partition (or coloring) of (0,1) into Baire sets, or into measurable sets, be given. Our elementary results will produce a sequence $\langle y_n\rangle_{n=1}^{\infty}$ such that all sums of products of images under nice functions (in appropriate order) have a color depending only on the function applied to the lowest order product. We allow the functions to vary within a particular product except that only one function can be applied to the lowest order product. Consider for example functions of the form $\varphi(x)=x^{\tau}$, which we have already observed are members of $\mathcal{I}$ (and of course, members of $\mathcal{H}$). We will get that the colors of ${y_7}^3\cdot{y_5}^{1/2}+y_3\cdot y_1$ and ${y_{10}}^{\sqrt{2}}\cdot y_9+y_8\cdot y_6\cdot y_2$ are the same. Also the colors of ${y_7}^3\cdot{y_5}^{1/2}+{y_3}^{100}\cdot{y_1}^{100}$, ${y_{10}}^{\sqrt{2}}\cdot y_9+{y_8}^{100}\cdot{y_6}^{100}\cdot{y_2}^{100}$ and $\log(y_{12}+1)\cdot y_8+{y_7}^{100}$ are the same. (Recall that $\log(x+1)$ defines a admissible function.)

We prove our main elementary result for Baire partitions first.

2.15 Theorem. *Let $(0,1)=\bigcup_{i=1}^{r}C_i$ where each C_i is a Baire set and let $\langle\varphi_n\rangle_{n=1}^{\infty}$ be a sequence in $\mathcal{H}$. There exist a sequence $\langle y_n\rangle_{n=1}^{\infty}$ and $\gamma:\mathbb{N}\longrightarrow\{1,2,\ldots,r\}$ such that for each $k\in\mathbb{N}$ and each $F\in\mathcal{P}_f(\mathbb{N})$ with $\min F\geq k$, $\{\Pi_{n\in F}\ \varphi_k(y_n)\}\cup\{\Sigma_{i=1}^{m}\Pi_{n\in G_i}\ \varphi_{s_n}(y_n)+\Pi_{n\in F}\ \varphi_k(y_n) : m\in\mathbb{N},\ G_1,G_2,\ldots,G_m\in\mathcal{P}_f(\mathbb{N}),\ G_1<G_2<\ldots<G_m,$ and for each $n\in\bigcup_{i=1}^{m}\ G_i,\ s_n\leq n\}\subseteq C_{\gamma(k)}$.*

Proof. We may presume the sets $C_1,C_2,\ldots,C_r$ are disjoint. For each $x\in(0,1)$, let $\psi(x)$ be the color of x (so that $x\in C_{\psi(x)}$). We inductively construct sequences $\langle B_k\rangle_{k=1}^{\infty}$, $\langle x_k\rangle_{k=1}^{\infty}$, and $\langle A_k\rangle_{k=1}^{\infty}$. Let $B_1=\bigcup_{i=1}^{r}C_i$ and let $A_1=\bigcup_{i=1}^{r}\delta_b(C_i)=\bigcup_{i=1}^{r}\delta_b(B_1\cap C_i)$. (Recall that $\delta_b(A)=\{x\in A: A-x$ is Baire huge$\}$.)

Note that by Lemma 2.4, A_1 is a Baire huge Baire set. Pick $x_1\in\varphi^{-1}[A_1]$. Then $\varphi(x_1)\in\delta_b(B_1\cap C_{\psi(\varphi(x_1))})$ so $(B_1\cap C_{\psi(\varphi(x_1))})-\varphi(x_1)$ is a Baire huge Baire set. Let $B_2=A_1\cap A_1/\varphi(x_1)\cap\big((B_1\cap C_{\psi(\varphi(x_1))})-\varphi(x_1)\big)$. Then by Lemmas 2.5 and 2.6, B_2 is a Baire huge Baire set.

Inductively, given B_k which is a Baire huge Baire set, let $A_k=\bigcup_{i=1}^{r}\delta_b(B_k\cap C_i)$. Then by Lemma 2.4, A_k is a Baire huge Baire set. By Lemma 2.9 ${\varphi_n}^{-1}[A_k]$ is Baire huge for each $n\in\{1,2,\ldots,k\}$, so by Lemma 2.5, $\bigcap_{n=1}^{k}{\varphi_n}^{-1}[A_k]$ is Baire huge, and is in particular nonempty. Pick $x_k\in\bigcap_{n=1}^{k}{\varphi_n}^{-1}[A_k]$.

For each $\ell \in \{1,2,\ldots,k\}$, let $\mathcal{H}_{\ell,k} = \{\Pi_{t\in F}\ \varphi_{s_t}(x_t) : \emptyset \neq F \subseteq \{\ell,\ell+1,\ldots,k\}$, $\min F = \ell$, $\max F = k$, and for each $t \in F$, $s_t \leq t\}$. Let

$$B_{k+1} = A_k \cap \bigcap_{\ell=1}^{k} A_k/\varphi_\ell(x_k) \cap \bigcap_{\ell=1}^{k} \bigcap_{z\in\mathcal{H}_{\ell,k}} \big((B_\ell \cap C_{\psi(z)}) - z\big)\ .$$

(Note that $\mathcal{H}_{1,1} = \{\varphi_1(x_1)\}$ so that the definition previously given of B_2 abides by this formula.) In order to show that B_{k+1} is a Baire huge Baire set, it suffices to show that for each $\ell \in \{1,2,\ldots,k\}$, $\mathcal{H}_{\ell,k} \subseteq A_\ell$. Indeed, assume we have done so. Then given $z \in \mathcal{H}_{\ell,k}$, $z \in \delta_b(B_\ell \cap C_{\psi(z)})$ so $(B_\ell \cap C_{\psi(z)}) - z$ is a Baire huge Baire set. Further, by Lemma 2.6, each $A_k/\varphi_\ell(x_k)$ is a Baire huge Baire set. Thus B_{k+1} is a finite intersection of Baire huge Baire sets, so by Lemma 2.5, B_{k+1} is a Baire huge Baire set.

So we establish by induction on $|F|$ that if $\ell \in \{1,2,\ldots,k\}$ and $x = \Pi_{t\in F}\ \varphi_{s_t}(x_t)$ where $\min F = \ell$, $\max F = k$, and each $s_t \leq t$, then $z \in A_\ell$. Assume first that $|F| = 1$ in which case $\ell = k$. Then $z = \varphi_n(x_k)$ for some $n \in \{1,2,\ldots,k\}$ so by the choice of x_k, $z \in A_k$. Now assume $|F| > 1$, let $G = F\backslash\{\ell\}$, let $v = \min G$, and let $w = \Pi_{t\in G}\ \varphi_{s_t}(x_t)$. Then $w \in \mathcal{H}_{v,k} \subseteq A_v \subseteq B_{\ell+1} \subseteq A_\ell/\varphi_{s_\ell}(x_\ell)$ so $z = \varphi_{s_\ell}(x_\ell)\cdot w \in A_\ell$ as required.

The construction of $\langle x_k\rangle_{k=1}^\infty$ being complete, we now construct the sequences $\langle y_n\rangle_{n=1}^\infty$ and $\langle\gamma(n)\rangle_{n=1}^\infty$. For each $k \in \mathbb{N}$, define $\nu_k : \mathcal{P}_f(\mathbb{N}) \longrightarrow \{1,2,\ldots,r\}$ by $\nu_k(F) = \psi\big(\Pi_{t\in F}\ \varphi_k(x_t)\big)$. By Lemma 2.14, pick a sequence $\langle F_{1,n}\rangle_{n=1}^\infty$ in $\mathcal{P}_f(\mathbb{N})$ such that for each $n \in \mathbb{N}$, $F_{1,n} < F_{1,n+1}$ and ν_1 is constant on $FU(\langle F_{1,n}\rangle_{n=1}^\infty)$. Let $\gamma(1)$ be this constant value.

Inductively, given a sequence $\langle F_{k-1,n}\rangle_{n=1}^\infty$, pick by Lemma 2.14 a subsystem $\langle F_{k,n}\rangle_{n=1}^\infty$ of $\langle F_{k-1,n}\rangle_{n=1}^\infty$ such that ν_k is constant on $FU(\langle F_{k,n}\rangle_{n=1}^\infty)$. Let $\gamma(k)$ be this constant value. For each $n \in \mathbb{N}$, let $y_n = \Pi_{t\in F_{n,n}}\ x_t$.

Then if $F \in \mathcal{P}_f(\mathbb{N})$ and $k \leq \min F$, one has that $\psi\big(\Pi_{n\in F}\ \varphi_k(y_n)\big) = \gamma(k)$, that is, $\Pi_{n\in F}\ \varphi_k(y_n) \in C_{\gamma(k)}$. To see this just observe that for each $n \in F$ there is some G_n, with $G_n < G_v$ if $n < v$, such that $F_{n,n} = \bigcup_{t\in G_n} F_{k,t}$. Thus if $H = \bigcup_{n\in F}\bigcup_{t\in G_n} F_{k,t}$, then $\nu_k(H) = \gamma(k)$ so $\psi\big(\Pi_{n\in F}\ \varphi_k(y_n)\big) = \psi\big(\Pi_{s\in H}\ \varphi_k(x_s)\big) = \gamma(k)$.

To complete the proof, let $m \in \mathbb{N}$, let $G_1 < G_2 < \ldots < G_m$ in $\mathcal{P}_f(\mathbb{N})$, and for each $n \in \bigcup_{i=1}^m G_i$, let $s_n \in \{1,2,\ldots,n\}$. Let $a = \min G_1$, $b = \max G_1$, $\ell = \min F_{a,a}$, and $k = \max F_{b,b}$. We show by induction on m that $\Sigma_{i=1}^m \Pi_{n\in G_i}\ \varphi_{s_n}(y_n) \in B_\ell$. If $m = 1$, then $\Pi_{n\in G_1}\ \varphi_{s_n}(y_n) \in \mathcal{H}_{\ell,k} \subseteq A_\ell \subseteq B_\ell$. Now assume that $m > 1$. Let $c = \min G_2$ and let $q = \min F_{c,c}$. Then $\Sigma_{i=2}^m \Pi_{n\in G_i}\ \varphi_{s_n}(y_n) \in B_q \subseteq B_{k+1} \subseteq B_\ell - \Pi_{n\in G_1}\ \varphi_{s_n}(y_n)$, so $\Sigma_{i=1}^m \Pi_{n\in G_i}\ \varphi_{s_n}(y_n) \in B_\ell$.

Now let $v = \min\bigcup_{n\in F} F_{n,n}$ and let $w = \max\bigcup_{n\in F} F_{n,n}$. Then $\Sigma_{i=1}^m \Pi_{n\in G_i}\ \varphi_{s_n}(y_n) \in B_\ell \subseteq B_{w+1} \subseteq C_{\psi\left(\Pi_{n\in F}\ \varphi_k(y_n)\right)} - \Pi_{n\in F}\ \varphi_k(y_n) = C_{\gamma(k)} - \Pi_{n\in F}\ \varphi_k(y_n)$. $\square$

2.16 Corollary. *Let* $(0,1) = \bigcup_{i=1}^{r} C_i$ *where each* C_i *is a Baire set. There exist a sequence* $\langle y_n\rangle_{n=1}^{\infty}$ *and some* $i \in \{1,2,\ldots,r\}$ *such that* $FS(\langle y_n\rangle_{n=1}^{\infty}) \cup FP(\langle y_n\rangle_{n=1}^{\infty}) \subseteq C_i$.
□

2.17 Theorem. *Let* $(0,1) = \bigcup_{i=1}^{r} C_i$ *where each* C_i *is measurable and let* $\langle \varphi_n\rangle_{n=1}^{\infty}$ *be a sequence in* $\mathcal{I}$. *There exist a sequence* $\langle y_n\rangle_{n=1}^{\infty}$ *and* $\gamma : \mathbb{N} \longrightarrow \{1,2,\ldots,r\}$ *such that for each* $k \in \mathbb{N}$ *and each* $F \in \mathcal{P}_f(\mathbb{N})$ *with* $\min F \geq k$, $c\ell\big(\{\Pi_{n\in F}\ \varphi_k(y_n)\} \cup \{\Sigma_{i=1}^{m}\Pi_{n\in G_i}\ \varphi_{s_n}(y_n) + \Pi_{n\in F}\ \varphi_k(y_n) : m \in \mathbb{N},\ G_1, G_2, \ldots, G_m \in \mathcal{P}_f(\mathbb{N}),\ G_1 < G_2 < \ldots < G_m,$ *and for each* $n \in \bigcup_{i=1}^{m}\ G_i,\ s_n \leq n\}\big) \subseteq C_{\gamma(k)} \cup \{0\}$.

Proof. For each $i \in \{1,2,\ldots,r\}$, pick $D_i \subseteq C_i$ as guaranteed by Lemma 2.12. Then proceed exactly as in the proof of Theorem 2.15 using D_i in place of C_i, with all references such as "A is a Baire huge Baire set" replaced by "A is measurable and $d(A) = 1$" and all references to δ_b replaced by δ. One then concludes that $\{\Pi_{n\in F}\ \varphi_k(y_n)\} \cup \{\Sigma_{i=1}^{m}\Pi_{n\in G_i}\ \varphi_{s_n}(y_n) + \Pi_{n\in F}\ \varphi_k(y_n) : m \in \mathbb{N},\ G_1, G_2, \ldots, G_m \in \mathcal{P}_f(\mathbb{N}),\ G_1 < G_2 < \ldots < G_m$, and for each $n \in \bigcup_{i=1}^{m}\ G_i,\ s_n \leq n\} \subseteq D_{\gamma(k)}$. The conclusion then follows since $D_{\gamma(k)} \cup \{0\}$ is compact. □

2.18 Corollary. *Let* $(0,1) = \bigcup_{i=1}^{r} C_i$ *where each* C_i *is measurable. There exist a sequence* $\langle y_n\rangle_{n=1}^{\infty}$ *and some* $i \in \{1,2,\ldots,r\}$ *such that* $c\ell\big(FS(\langle y_n\rangle_{n=1}^{\infty}) \cup FP(\langle y_n\rangle_{n=1}^{\infty})\big) \subseteq C_i \cup \{0\}$. □

3. Algebraic Preliminaries.

Recall that βX_d denotes the Stone-Čech compactification of the set X with the discrete topology. The points of βX_d are the ultrafilters on X, the principal ultrafilters being identified with the points of X. If $(X,\cdot)$ is a semigroup, then the operation $\cdot$ on X extends to βX_d so that βX_d is a right topological semigroup. That is, for each $p \in \beta X_d$, the function $\rho_p : \beta X_d \to \beta X_d$, defined by $\rho_p(q) = q \cdot p$, is continuous. Also, given any $x \in X$, the function $\lambda_x : \beta X_d \to \beta X_d$, defined by $\lambda_x(p) = x \cdot p$, is continuous. Similarly if $Y = \mathbb{R}$ or $\mathbb{Q}$ or $\mathbb{D}$, the operation $+$ extends to βY_d so that $(\beta Y_d, +)$ is a compact right topological semigroup. The operations $+$ and $\cdot$ on βY_d can be characterized as follows. Given $p, q \in \beta Y_d$ and $A \subseteq Y$ one has $A \in p \cdot q$ if and only if $\{x \in Y : x^{-1}A \in q\} \in p$, and $A \in p + q$ if and only if $\{x \in Y : -x + A \in q\} \in p$, where $x^{-1}A = \{y \in Y : xy \in A\}$ and $-x + A = \{y \in Y : x + y \in A\}$. See [12] for an introduction to $(\beta S, \cdot)$ where $(S,\cdot)$ is a discrete semigroup (with the caution that there βS is taken to be left topological rather than right topological).

The reader might wonder why we work for example with $(0,1)_d$ rather than with $(0,1)$. The reason is that it turns out that the algebraic operations on $(0,1)$ do not extend sensibly to $\beta(0,1)$.

As a compact right topological semigroup $(\beta X_d, \cdot)$ has significant known algebraic structure. In particular it has idempotents. (The fact that compact right topological semigroups have idempotents will often be used without specific mention.) Also, again as a consequence of the fact that it is a compact right topological semigroup, $(\beta X_d, \cdot)$ has a smallest two-sided ideal (that is, a two-sided ideal contained in all other two-sided ideals), which is the union of all minimal right ideals and also the union of all minimal left ideals. (Recall that in a semigroup $(S, \cdot)$ a subset A is a left (respectively right) ideal provided $SA \subseteq A$ (respectively $AS \subseteq A$).) See [5] for the basic facts about compact right topological semigroups.

Since we are working with $\beta(0,1)_d$ rather than $\beta(0,1)$, it seems that we have lost all the topology of $(0,1)$. Thus our first task is to put the topology of $(0,1)$ back in.

Let us call an ultrafilter $p \in \beta(0,1)_d$ *large at 0* if the interval $(0,\epsilon)$ belongs to p for every $0 < \epsilon < 1$. (The set $\{(0,\epsilon) : \epsilon > 0\}$ has the finite intersection property, so of course it is contained in an ultrafilter.) We shall restrict our attention to the ultrafilters that are large at 0, and in this way essentially recover the topology of $(0,1)$. Let us now introduce our main algebraic tool, namely the space O_X, showing it to be an ideal of βX_d under multiplication.

3.1 Definition. Let X be a dense subsemigroup of $\big((0,1), \cdot\big)$. Then $O_X = \{p \in \beta X_d :$ for every $\epsilon > 0$, $(0,\epsilon) \cap X \in p\}$.

Note that there are no principal ultrafilters corresponding to real numbers in O_X. It consists of "infinitesimal" ultrafilters, that is ultrafilters living in the vicinity of zero.

The set O_X is also a semigroup under addition and has interesting and intricate algebraic structure. We put off a detailed study of this structure for another day, presenting only enough here to establish our combinational results.

3.2 Lemma. *Let X be a dense subsemigroup of $\big((0,1), \cdot\big)$. Then O_X is a compact two-sided ideal of $(\beta X_d, \cdot)$. Consequently the smallest ideal of O_X is the same as the smallest ideal of βX_d. O_X is also a subsemigroup of $(\beta S_d, +)$ where S is the subsemigroup of $(\mathbb{R}, +)$ generated by X.*

Proof. First observe that $\{(0,\epsilon) \cap X : \epsilon > 0\}$ has the finite intersection property, so $O_X \neq \emptyset$. If $p \in \beta X_d \backslash O_X$, then for some $\epsilon > 0$, $(0,\epsilon) \cap X \notin p$ so $cl((\epsilon/2, 1) \cap X)$ is a neighborhood of p missing O_X and hence O_X is compact. That O_X is a two-sided ideal follows immediately from the fact that for any $\epsilon > 0$, $\big((0,1)\cap X\big)\cdot\big((0,\epsilon)\cap X\big) \subseteq (0,\epsilon)\cap X$.

Since O_X is a two-sided ideal of βX_d, it follows that the smallest ideal of βX_d is contained in O_X. Thus by [5, Corollary I.2.15], the smallest ideal of O_X is the smallest ideal of βX_d.

To see that $(O_X, +)$ is a semigroup, let $p, q \in O_X$. Let $\epsilon > 0$. Then $(0,\epsilon) \cap X \subseteq \{x \in X : -x + (0,\epsilon) \cap X \in q\}$ so $(0,\epsilon) \cap X \in p + q$. $\square$

For most of our algebraic preliminaries we will be dealing only with the multiplicative structure of O_X.

3.3 Definition. Let X be a dense subsemigroup of $\big((0,1),\cdot\big)$. Then K_X is the smallest ideal of $(O_X,\cdot)$.

As a consequence of Lemma 3.2 we obtain "for free" some important information about members of multiplicative idempotents in K_X. We first describe the "columns condition" introduced by Rado [17] in his characterization of partition regularity of homogeneous equations.

3.4 Definition. Let $u, v \in \mathbb{N}$, let C be a $u \times v$ matrix with entries from $\mathbb{R}$, and let R be a subring of $\mathbb{R}$. Then C satisfies the columns condition over R if the columns $\vec{c}_1, \vec{c}_2, \ldots, \vec{c}_v$ of C can be ordered so that there exist $m \in \mathbb{N}$ and $k_1, k_2, \ldots, k_m$ in $\mathbb{N}$ with $1 \leq k_1 < ... < k_m = v$ such that

(1) $\Sigma_{i=1}^{k_1}\vec{c}_i = \vec{0}$ and,

(2) if $m > 1$, then for every $t \in \{2, 3, \ldots, m\}$ we have $a_{1,t}$, $a_{2,t}, \ldots, a_{k+1,t}$ in R with $\Sigma_{i=k_{t-1}+1}^{k_t}\vec{c}_i = \Sigma_{i=1}^{k_{t-1}} a_{i,t}\vec{c}_i$.

3.5 Theorem. *Let X be a dense subsemigroup of $\big((0,1),\cdot\big)$ and let p be a multiplicative idempotent in K_X. Let $A \in p$ and let $D = \langle d_{ij}\rangle$ be a $u \times v$ matrix with entries from $\mathbb{Z}$.*

(a) If D satisfies the columns condition over $\mathbb{Z}$, then there exist $x_1, x_2, \ldots, x_v$ in A such that for each $i \in \{1, 2, \ldots, u\}$, $\Pi_{j=1}^{v} x_j^{d_{ij}} = 1$.

(b) If $X = (0,1)$ and D satisfies the columns condition over $\mathbb{Q}$, then there exist $x_1, x_2, \ldots, x_v$ in A such that for each $i \in \{1, 2, \ldots, u\}$, $\Pi_{j=1}^{v} x_j^{d_{ij}} = 1$.

Proof. Since p is an idempotent in K_X, which is also the smallest ideal of $(\beta X_d, \cdot)$, and $A \in p$, A is known as a "central" set in X. Then, after converting from additive to multiplicative notation, condition (a) follows immediately from [13, Theorem 2.5(a)]. Also, if $X = (0,1)$, then for any $n \in \mathbb{N}$, $(0,1) = \{x^n : x \in (0,1)\}$ so conclusion (b) follows from [13, Theorem 2.5(b)]. $\square$

As an example, note that Theorem 3.5 tells us that whenever A is a member of a multiplicative idempotent in K_X, one has that A contains arbitrarily long geometric progressions, together with their increments. For example, to see that A contains $\{r, a, ar, ar^2, ar^3\}$, consider the matrix

$$\begin{pmatrix} 1 & 1 & -1 & 0 & 0 \\ 1 & 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & 1 & -1 \end{pmatrix}.$$

We will be interested in showing that sets central in $(X, \cdot)$, i.e. members of multiplicative idempotents in K_X, also contain additive configurations (like arithmetic progressions).

3.6 Definition. Let D be a $u \times v$ matrix and let X be a dense subsemigroup of $\big((0,1), \cdot\big)$. Then $U_{X,D} = \{p \in O_X : \text{for all } A \in p \text{ there exist } x_1, x_2, \ldots, x_v \text{ in } A \text{ with } D\vec{x} = \vec{0}\}$.

3.7 Lemma. *Let D be a $u \times v$ matrix and let X be one of $\mathbb{D} \cap (0,1)$, $\mathbb{Q} \cap (0,1)$, or $(0,1)$. Then if either*

(1) the entries of D are rational and D satisfies the columns condition over $\mathbb{Q}$ or

(2) the entries of D are real, D satisfies the columns condition over $\mathbb{R}$, and $X = (0,1)$,

then $U_{X,D}$ is a two-sided ideal of $(O_X, \cdot)$ and a subsemigroup of $(O_X, +)$.

Proof. Let $V = \{p \in \beta X_d : \text{for all } A \in p, \text{ there exists } x_1, x_2, \ldots, x_v \text{ in } A \text{ such that } D\vec{x} = \vec{0}\}$ (so $U_{X,D} = V \cap O_X$). We first show that if $V \neq \emptyset$, then V is a two-sided ideal of $(\beta X_d, \cdot)$. Indeed, let $p \in V$ and let $q \in \beta X_d$. To see that $q \cdot p \in V$, let $A \in q \cdot p$ and pick $y \in X$ such that $y^{-1}A \in p$. Then pick $x_1, x_2, \ldots, x_v$ in $y^{-1}A$ such that $D\vec{x} = \vec{0}$. Then $yx_1, yx_2, \ldots, yx_v$ are in A and $Dy\vec{x} = \vec{0}$.

To see that $p \cdot q \in V$, let $A \in p \cdot q$ and pick $x_1, x_2, \ldots, x_v$ in $\{x \in X : x^{-1}A \in q\}$ with $D\vec{x} = \vec{0}$. Pick $y \in \bigcap_{i=1}^{v} x_i^{-1}A$. Then $yx_1, yx_2, \ldots, yx_v$ are in A and $Dy\vec{x} = \vec{0}$.

Thus if $V \neq \emptyset$, then $U_{X,D} = V \cap O_X$ is a two sided ideal of $(X_d, \cdot)$ and hence of $(O_X, \cdot)$, by Lemma 3.2. Thus in particular, if $V \neq \emptyset$, then $U_{X,0} \neq \emptyset$. To see that under this assumption $(U_{X,D}, +)$ is a semigroup, let $p, q \in U_{X,D}$ and let $A \in p + q$. Pick $x_1, x_2, \ldots, x_v$ in $\{x : -x + A \in q\}$ such that $D\vec{x} = \vec{0}$. Pick $y_1, y_2, \ldots, y_v \in \bigcap_{i=1}^{v}(-x_i + A)$ such that $D\vec{y} = \vec{0}$. Then $x_1 + y_1,\ x_2 + y_2, \ldots, x_v + y_v$ are in A and $D(\vec{x} + \vec{y}) = \vec{0}$.

Consequently, it suffices to show that $V \neq \emptyset$. By [7, Theorem 6.2.3] it in turn suffices to show that whenever X is partitioned into finitely many cells, ones of them contains $x_1, x_2, \ldots, x_v$ with $D\vec{x} = 0$.

If D satisfies the columns condition over $\mathbb{Q}$ then by [17, Theorem VII] D is partition regular over $\mathbb{N}$ so by compactness (see [7, Section 1.5]) given any $r \in \mathbb{N}$, there is some $n(r) \in \mathbb{N}$ so that whenever $\{1, 2, \ldots, n(r)\}$ is r-colored there is a monochrome solution to $D\vec{x} = \vec{0}$. Picking k with $2^k > n(r)$ one has $\{\frac{1}{2^k}, \frac{2}{2^k}, \ldots, \frac{n(r)}{2^k}\} \subseteq X$ and whenever $\{\frac{1}{2^k}, \frac{2}{2^k}, \ldots, \frac{n(r)}{2^k}\}$ is r-colored there must be a monochrome solution to $D\vec{x} = \vec{0}$.

The proof in case (2) is similar. Again by [17, Theorem VII] D is partition regular over $\mathbb{R}^+ = \{x \in \mathbb{R} : x > 0\}$ so given $r \in \mathbb{N}$ there is a finite subset F of $\mathbb{R}^+$ such that whenever F is r-colored there is a monochrome solution to $D\vec{x} = \vec{0}$. Pick $n > \max F$. Then whenever $\{\frac{x}{n} : x \in F\}$ is r-colored there is a monochrome solution to $D\vec{x} = \vec{0}$. □

3.8 Definition. $\mathcal{L} = \{p \in O_{(0,1)} : \text{for all } A \in p,\ \overline{d}(A) > 0\}$.

3.9 Lemma. *$\mathcal{L}$ is a left ideal of $(O_{(0,1)}, \cdot)$.*

Proof. It is an easy exercise to show that if $\overline{d}(A \cup B) > 0$ then either $\overline{d}(A) > 0$ or $\overline{d}(B) > 0$. Consequently, by [7, Theorem 6.2.3], we have $\mathcal{L} \neq \emptyset$.

Let $p \in \mathcal{L}$ and let $q \in O_{(0,1)}$. To see that $q \cdot p \in \mathcal{L}$, let $A \in q \cdot p$ and pick x such that $x^{-1}A \in p$. Another easy exercise establishes that $\overline{d}(A) = \overline{d}(x^{-1}A) > 0$. □

The next theorem, and its Baire analogue, Theorem 3.13, are the ones that allow us to obtain our combined additive and multiplicative results.

3.10 Theorem. *Let p be a multiplicative idempotent in $\mathcal{L}$ and let A be a measurable member of p. Then $\{x \in A : x^{-1}A \in p \text{ and } A - x \in p\} \in p$.*

Proof. Let $B = \{x \in A : x^{-1}A \in p\}$. Then $B \in p$ since $p = p \cdot p$. Let $C = \{y \in A : y$ is not a density point of $A\}$. By Lemma 2.3, $\mu(C) = 0$. Consequently since $p \in \mathcal{L}$, $C \notin p$ so $B \backslash C \in p$. We claim that $B \backslash C \subseteq \{x \in A : x^{-1}A \in p \text{ and } A - x \in p\}$. Indeed, given $x \notin C$ one has 0 is a density point of $A - x$ so by an easy computation, $\overline{d}\big((0,1) \backslash (A - x)\big) = 0$ so $(0,1) \backslash (A - x) \notin p$ so $A - x \in p$. □

3.11 Definition. $\mathcal{B} = \{p \in O_{(0,1)} : \text{for all A} \in p, \text{ A is Baire large}\}$.

3.12 Lemma. *$\mathcal{B}$ is a left ideal of $(O_{(0,1)}, \cdot)$.*

Proof. Since the union of finitely many meager sets is meager, one sees easily that whenever $(0, 1)$ is partitioned into finitely many sets, one of them is Baire large. Consequently, by [7, Theorem 6.2.3], it follows that $\{p \in \beta(0,1)_d : \text{for all } A \in p,\ A \text{ is Baire large}\} \neq \emptyset$. On the other hand, if $p \in \beta(0,1)_d \backslash O_{(0,1)}$ one has some $(\epsilon, 1) \in p$ and $(\epsilon, 1)$ is not Baire large. Thus $\mathcal{B} \neq \emptyset$. To see that $\mathcal{B}$ is a left ideal of $O_{(0,1)}$, let $p \in \mathcal{B}$

and $q \in O_{(0,1)}$ and let $A \in q \cdot p$. Pick $x \in (0,1)$ such that $x^{-1}A \in p$. Given $\epsilon > 0$, $x^{-1}A \cap (0,\epsilon)$ is not meager and $A \cap (0, x\epsilon) \subseteq A \cap (0,\epsilon)$. $\square$

3.13 Theorem. *Let p be an idempotent in $\mathcal{B}$ and let A be a Baire set which is a member of p. Then $\{x \in A : x^{-1}A \in p$ and $A - x \in p\} \in p$.*

Proof. Let $B = \{x \in A : x^{-1}A \in p\}$. Then since $p = p \cdot p$, $B \in p$. Also A is a Baire set so pick an open set U and a meager set M such that $A = U \Delta M$. Now $M \backslash U$ is meager so $M \backslash U \notin p$ so $U \backslash M \in p$. We claim $(U \backslash M) \cap B \subseteq \{x \in A : x^{-1}A \in p$ and $A - x \in p\}$. So let $x \in (U \backslash M) \cap B$ and pick $\epsilon > 0$ such that $(x, x+\epsilon) \subseteq U$. To see that $A - x \in p$, we observe that $(0,1) \backslash (A - x)$ is not Baire large. Indeed one has $\big((0,1)\backslash(A-x)\big) \cap (0,\epsilon) \subseteq M - x$, a meager set. (Given $y \in (0,\epsilon)$, $y + x \in U$ so, if $y + x \notin A$, then $y + x \in M$.) $\square$

We thank A. Blass for pointing out that $\mathcal{B} \cap \mathcal{L} = \emptyset$. Indeed, as is well known (see eg. [14, Theorem 1.6]), there is a set $A \subseteq (0,1)$ which is meager such that $\mu((0,1)\backslash A) = 0$. Then $\mathcal{B} \cap c\ell A = \emptyset$ and $\mathcal{L} \subseteq c\ell A$.

4. Ramsey Theory Near 0 in $(0,1)$.

We begin by defining the kinds of combined additive and multiplicative configurations that we shall produce in one cell of a measurable or Baire partition of (0,1). The notation "FSP" stands for "finite sums and products" and $\sigma(x)$ is intended to be the "support" of x. We remind the reader that $\mathcal{P}_f(\mathbb{N})$ is the set of finite nonempty subsets of $\mathbb{N}$.

4.1 Definition. Let $\langle G_n \rangle_{n=1}^{\infty}$ be a sequence of finite subsets of $(0,1)$. We define

$$FSP(\langle G_n \rangle_{n=1}^{\infty}) \text{ and } \sigma : FSP(\langle G_n \rangle_{n=1}^{\infty}) \to \mathcal{P}\big(\mathcal{P}_f(\mathbb{N})\big)$$

inductively to consist of only those objects obtainable by iteration of the following:

(1) If $m \in \mathbb{N}$ and $x \in G_m$, then $x \in FSP(\langle G_n \rangle_{n=1}^{\infty})$ and $\{m\} \in \sigma(x)$.

(2) If $x, y \in FSP(\langle G_n \rangle_{n=1}^{\infty})$, $F \in \sigma(x)$, $H \in \sigma(y)$, and $\max H < \min F$, then $\{y \cdot x, y + x\} \subseteq FSP(\langle G_n \rangle_{n=1}^{\infty})$ and $F \cup H \in \sigma(y + x)$ and $F \cup H \in \sigma(y \cdot x)$.

For example, if each $G_n = \{x_n\}$ and $z = \big((x_1 + x_3) \cdot x_5 + (x_7 + x_9) \cdot x_{11}\big) \cdot x_{12} \cdot x_{13}$ then $z \in FSP(\langle G_n \rangle_{n=1}^{\infty})$ and $\{1,3,5,7,9,11,12,13\} \in \sigma(z)$. (Of course, it is also possible that $z = x_4 + x_{12} \cdot x_{13}$, in which case also $\{4,12,13\} \in \sigma(z)$.) Note also that $(x_3 + x_5) \cdot (x_2 + x_4)$ is not, in general, a member of $FSP(\langle G_n \rangle_{n=1}^{\infty})$.

In the case that $G_n = \{x_n\}$, we write $FSP(\langle x_n \rangle_{n=1}^{\infty})$ rather than $FSP(\langle \{x_n\} \rangle_{n=1}^{\infty})$.

Given a partition of $(0,1)$ all of whose cells are Baire sets (or all of whose cells are measurable) we are after the result that we can get a sequence $\langle x_n\rangle_{n=1}^\infty$ with $FSP(\langle x_n\rangle_{n=1}^\infty)$ contained in one cell of the partition. Unfortunately, we don't quite get this result, obtaining instead arbitrarily close approximations to it. To be precise, we define below the notions $FSP_k(\langle x_n\rangle_{n=1}^\infty)$ for each $k\in\mathbb{N}\cup\{0\}$ in such a way that $FSP(\langle x_n\rangle_{n=1}^\infty)=\bigcup_{k=0}^\infty FSP_k(\langle x_n\rangle_{n=1}^\infty)$ and get one cell A of the partition so that for each $k\in\mathbb{N}$, $FSP_k(\langle x_n\rangle_{n=k}^\infty)\subseteq A$.

4.2 Definition. Let $\alpha\in\mathbb{N}\cup\{\infty\}$ and let $\langle G_n\rangle_{n=1}^\alpha$ be a sequence of finite subsets of $(0,1)$. We define

$$FSP_k(\langle G_n\rangle_{n=1}^\alpha) \text{ and } \sigma_k : FSP_k(\langle G_n\rangle_{n=1}^\alpha)\to\mathcal{P}\big((\mathcal{P}_f(\{n\in\mathbb{N}:n\le\alpha\})\big)$$

inductively to consist of only those objects obtainable by iteration of the following:

(1) $FSP_0(\langle G_n\rangle_{n=1}^\alpha)=\bigcup_{n=1}^\alpha G_n$ and if $n\in\mathbb{N}$, $n\le\alpha$, and $x\in G_n$, then $\{n\}\in\sigma_0(x)$.

(2) If $k\in\mathbb{N}\cup\{0\}$, $x\in FSP_k(\langle G_n\rangle_{n=1}^\alpha)$, and $F\in\sigma_k(x)$, then $x\in FSP_{k+1}(\langle G_n\rangle_{n=1}^\alpha)$ and $F\in\sigma_{k+1}(x)$.

(3) If $k\in\mathbb{N}\cup\{0\}$, $x\in FSP_{k+1}(\langle G_n\rangle_{n=1}^\alpha)$, $y\in FSP_k(\langle G_n\rangle_{n=1}^\alpha)$, $F\in\sigma_{k+1}(x)$, $H\in\sigma_k(y)$, and $\max H<\min F$, then $\{y\cdot x,y+x\}\subseteq FSP_{k+1}(\langle G_n\rangle_{n=1}^\alpha)$ and $F\cup H\in\sigma_{k+1}(y+x)$ and $F\cup H\in\sigma_{k+1}(y\cdot x)$.

To return to the example above, $(x_1+x_3)\cdot x_5\in FSP_2(\langle x_n\rangle_{n=1}^\infty)$ but need not be in $FSP_1(\langle x_n\rangle_{n=1}^\infty)$, while $x_1+x_3\cdot x_5\in FSP_1(\langle x_n\rangle_{n=1}^\infty)$. We leave it as an exercise to determine the first k for which $z=\big((x_1+x_3)\cdot x_5+(x_7+x_9)\cdot x_{11}\big)\cdot x_{12}\cdot x_{13}$ must be in $FSP_k(\langle x_n\rangle_{n=1}^\infty)$. At any rate, we have the following lemma whose routine proof we omit.

4.3 Lemma. *Let $\langle G_n\rangle_{n=1}^\infty$ be a sequence of subsets of $(0,1)$. Then $FSP(\langle G_n\rangle_{n=1}^\infty)=\bigcup_{k=0}^\infty FSP_k(\langle G_n\rangle_{n=1}^\infty)$.* □

4.4 Definition. $\mathcal{G}=\{\mathcal{S}:\mathcal{S}$ is a set of finite subsets of $(0,1)$ and for every $p=p\cdot p$ in $K_{(0,1)}$ and every $A\in p$ there exists $G\in\mathcal{S}$ such that $G\subseteq A.\}$

We pause to observe that $\mathcal{G}$ is a large collection.

4.5 Theorem. *Let D be a $u\times v$ matrix with real entries.*

(a) If the entries of D are integers and D satisfies the columns condition over $\mathbb{Q}$, then $\big\{\{x_1,x_2,\ldots,x_v\}$: for each $i\in\{1,2,\ldots,u\}$, $\Pi_{j=1}^v x_j^{d_{ij}}=1\big\}\in\mathcal{G}$.

(b) If D satisfies the columns condition over $\mathbb{R}$ then $\big\{\{x_1,x_2,\ldots,x_v\}$: for each $i\in\{1,2,\ldots,u\}$, $\Sigma_{j=1}^v d_{ij}\cdot x_j=0\big\}\in\mathcal{G}$.

Proof. (a) This follows immediately from Theorem 3.5(b).

(b) By Lemma 3.7 we have that $U_{(0,1),D}$ is a two sided ideal of $(O_{(0,1)}, \cdot)$ and hence $K_{(0,1)} \subseteq U_{(0,1),D}$.

Thus $p \in U_{(0,1),D}$. □

In the following theorem we choose $\mathcal{S}_n \in \mathcal{G}$. One could, for example, let for each n,

$$\mathcal{S}_{2n} = \{\{a, d, a+d, a+2d, \ldots, a+nd\} \cap (0,1) : a, d \in (0,1)\}$$

and

$$\mathcal{S}_{2n+1} = \{\{a, r, ar, ar^2, \ldots, ar^n\} : a, r \in (0,1)\}.$$

One thus obtains, in the special sets A, arithmetic and geometric progressions of every length as well as all sums and all products (and some combined sums and products) choosing at most one from each progression. (See [1], [2], and [3] for additional examples of the kinds of monochrome expressions that one can guarantee.)

4.6 Theorem. *For each $n \in \mathbb{N}$, let $\mathcal{S}_n \in \mathcal{G}$. Let $p = p \cdot p$ in $\mathcal{B} \cap K_{(0,1)}$. If $A \in p$ and A is a Baire set, then there exists a choice of $G_n \in \mathcal{S}_n$ for each n such that for each $k \in \mathbb{N}$, $FSP_k(\langle G_n \rangle_{n=k}^{\infty}) \subseteq A$.*

Proof. Let $A_{1,0} = A$ and inductively let $A_{1,t+1} = \{x \in A_{1,t} : x^{-1}A_{1,t} \in p$ and $A_{1,t} - x \in p\}$. By Theorem 3.13 $A_{1,2} \in p$ so, since $p \in K_{(0,1)}$ there is some $G_1 \in \mathcal{S}_1$ with $G_1 \subseteq A_{1,2}$. Let $A_{2,0} = A_{1,2} \cap \bigcap_{x \in G_1} \big(x^{-1}A_{1,1} \cap (A_{1,1} - x)\big)$ and note that $A_{2,0}$ is Baire and $A_{2,0} \in p$.

Inductively, given $A_{n,0}$ a Baire set such that $A_{n,0} \in p$, let for each $t \in \{0, 1, \ldots, n\}$, $A_{n,t+1} = \{x \in A_{n,t} : x^{-1}A_{n,t} \in p$ and $A_{n,t} - x \in p\}$. By Theorem 3.13 $A_{n,n+1} \in p$ so pick $G_n \in \mathcal{S}_n$ with $G_n \subseteq A_{n,n+1}$. For $r \in \{1, 2, \ldots, n\}$ and $k \in \{0, 1, \ldots, r\}$, let $\mathcal{H}_{n,k,r} = \{z \in FSP_k(\langle G_t \rangle_{t=1}^{n})$: there exists $F \in \sigma_k(z)$ such that $\max F = n$ and $\min F = r\}$. Let $A_{n+1,0} = A_{n,n+1} \cap \bigcap_{r=1}^{n} \bigcap_{k=0}^{r} \bigcap_{z \in \mathcal{H}_{n,k,r}} \big(z^{-1}A_{n,n-k} \cap (A_{n,n-k} - z)\big)$. (Observe that $FSP_1(\langle G_t \rangle_{t=1}^{1}) = FSP_2(\langle G_t \rangle_{t=1}^{1}) = G_1$ so that the definition here agrees with the definition given above for $A_{2,0}$.)

Then $A_{n+1,0}$ is a Baire set. To see that $A_{n+1,0} \in p$ it suffices to show that

(*) for each $r \in \{1, 2, \ldots, n\}$, $k \in \{0, 1, \ldots, r\}$, and $z \in \mathcal{H}_{n,k,r}$, $z \in A_{r,r-k+1}$.

We show this by induction on $|F|$ where $F \in \sigma_k(z)$, $\max F = n$, and $\min F = r$. If $|F| = 1$, then $F = \{n\}$, $r = n$ and $z \in G_n \subseteq A_{n,n+1} \subseteq A_{n,n-k+1} = A_{r,r-k+1}$. Now assume $|F| > 1$ and the claim is true for smaller values of $|F|$. We proceed by induction on k. If $k = 0$, then $F \in \sigma_0(z)$ so $F = \{n\}$, a case we have already handled. So assume $k > 0$. If $F \in \sigma_k(z)$ because of clause (2) of Definition 4.2, then $z \in \mathcal{H}_{n,k-1,r} \subseteq A_{r,r-k+2} \subseteq A_{r,r-k+1}$. So we may assume that we have some $x \in FSP_k(\langle G_t\rangle_{t=1}^n)$, $y \in FSP_{k-1}(\langle G_t\rangle_{t=1}^n)$, $L \in \sigma_k(x)$, and $H \in \sigma_{k-1}(y)$ such that $\max H < \min L$, $z \in \{y+x, y\cdot x\}$ and $F = H \cup L$. Let $\ell = \max H$ and let $v = \min L$. Then $y \in \mathcal{H}_{\ell,k-1,r}$ so $A_{\ell+1,0} \subseteq y^{-1}A_{r,r-k+1} \cap (A_{r,r-k+1} - y)$. Also, $x \in \mathcal{H}_{n,k,v}$ and since $|L| < |F|$, $\mathcal{H}_{n,k,v} \subseteq A_{v,v-k+1} \subseteq A_{\ell+1,0} \subseteq y^{-1}A_{r,r-k+1} \cap (A_{r,r-k+1} - y)$ so that $\{y+x, y\cdot x\} \subseteq A_{r,r-k+1}$ as required.

The construction of the sequence $\langle G_n\rangle_{n=1}^\infty$ is now complete. Let $k \in \mathbb{N}$ and let $z \in FSP(\langle G_n\rangle_{n=k}^\infty)$. Pick $F \in \sigma_k(z)$ and let $n = \max F$ and $r = \min F$. Then $z \in \mathcal{H}_{n,k,r} \subseteq A_{r,r-k+1} \subseteq A$. □

4.7 Corollary. *Let $r \in \mathbb{N}$ and let $(0,1) = \bigcup_{i=1}^r A_i$. If each A_i is a Baire set, then there exists $i \in \{1,2,\ldots,r\}$ such that given any choice of $\mathcal{S}_n \in \mathcal{G}_{(0,1)}$ for $n \in \mathbb{N}$ there exists a choice of $G_n \in \mathcal{S}_n$ such that for each $k \in \mathbb{N}$, $FSP_k(\langle G_n\rangle_{n=k}^\infty) \subseteq A_i$.*

Proof. By Lemma 3.12 $\mathcal{B}$ is a left ideal of $(O_{(0,1)}, \cdot)$ so (see [5, Theorem 1.3.11]) there a multiplicative idempotent $p \in \mathcal{B} \cap K_{(0,1)}$. Pick i such that $A_i \in p$ and apply Theorem 4.6. □

In particular we have the following corollary.

4.8 Corollary. *Let $r \in \mathbb{N}$ and let $(0,1) = \bigcup_{i=1}^r A_i$. If each A_i is a Baire set, then there exist $i \in \{1,2,\ldots,r\}$ and a sequence $\langle x_n\rangle_{n=1}^\infty$ such that for each $k \in \mathbb{N}$, $FSP(\langle x_n\rangle_{n=1}^\infty) \subseteq A_i$.* □

In the case of a Lebesgue measurable partition, just as with the elementary results, we will obtain the stronger conclusion that for each $k \in \mathbb{N}$, $c\ell_{\mathbb{R}} FSP_k(\langle G_n\rangle_{n=k}^\infty) \subseteq A_i \cup \{0\}$. In particular $AS(\langle G_n\rangle_{n=1}^\infty) = \{\Sigma_{n\in I} x_n : \emptyset \neq I \subseteq \mathbb{N}$ and for each $n \in I$, $x_n \in G_n\} \subseteq A_i$.

4.9 Theorem. *For each $n \in \mathbb{N}$, let $\mathcal{S}_n \in \mathcal{G}$. Let $p = p\cdot p$ in $\mathcal{L} \cap K_{(0,1)}$. If $A \in p$ and A is measurable then there exists a choice of $G_n \in \mathcal{S}_n$ for each n such that for each $k \in \mathbb{N}$ $c\ell_{\mathbb{R}} FSP_k(\langle G_n\rangle_{n=k}^\infty) \subseteq A \cup \{0\}$.*

Proof. Since $A \in p$ and $p \in \mathcal{L}$, one has $\mu(A) > 0$. Pick by Lemma 2.11 some $B \subseteq A$ such that $B \cup \{0\}$ is compact and $\overline{d}(A\backslash B) = 0$. Then $A\backslash B \notin p$ so $B \in p$.

Now proceeding identically as in the proof of Theorem 4.6, using Theorem 3.10 in place of Theorem 3.13, one obtains $\langle D_n\rangle_{n=1}^\infty$ with $FSP_k(\langle D_n\rangle_{n=k}^\infty) \subseteq B$ for each $k \in \mathbb{N}$. Since $B \cup \{0\}$ is compact, one has $c\ell_{\mathbb{R}} FSP_k(\langle D_n\rangle_{n=k}^\infty) \subseteq B \cup \{0\} \subseteq A \cup \{0\}$. □

4.10 Corollary. *Let $r \in \mathbb{N}$ and let $(0,1) = \bigcup_{i=1}^{r} A_i$. If each A_i is measurable, then there exists $i \in \{1,2,\ldots,r\}$ such that given any choice of $\mathcal{S}_n \in \mathcal{G}_{(0,1)}$ for $n \in \mathbb{N}$ there exists a choice of $G_n \in \mathcal{S}_n$ such that for each $k \in \mathbb{N}$, $c\ell_{\mathbb{R}} FSP_k(\langle G_n \rangle_{n=k}^{\infty}) \subseteq A_i \cup \{0\}$.*

Proof. By Lemma 3.9, $\mathcal{L}$ is a left ideal of $O_{(0,1)}$ so there is a multiplicative idempotent $p \in \mathcal{L} \cap K_{(0,1)}$. Pick i such that $A_i \in p$ and apply Theorem 4.9. $\square$

4.11 Corollary. *Let $r \in \mathbb{N}$ and let $(0,1) = \bigcup_{i=1}^{r} A_i$. If each A_i is measurable, then there exist $i \in \{1,2,\ldots,r\}$ and a sequence $\langle x_n \rangle_{n=1}^{\infty}$ such that $c\ell_{\mathbb{R}} FSP(\langle x_n \rangle_{n=1}^{\infty}) \subseteq A_i \cup \{0\}$.* $\square$

In contrast with the results of [4], some of the results we obtained here are weaker for Baire partitions than for measurable partitions. We do not know how much remains true for Baire partitions. For instance, we have the following question.

4.12 Question. Let $r \in \mathbb{N}$ and let $(0,1) = \bigcup_{i=1}^{r} A_i$. If each A_i is a Baire set, must there exist $i \in \{1,2,\ldots,r\}$ and a sequence $\langle x_n \rangle_{n=1}^{\infty}$ such that for each $k \in \mathbb{N}$, $c\ell_{\mathbb{R}} FSP_k(\langle x_n \rangle_{n=k}^{\infty}) \subseteq A_i \cup \{0\}$?

If A is measurable and $d(A) = 1$, then $\mathcal{L} \subseteq c\ell A$ so we get immediately from Theorem 4.9 (or, by revising the proof, from Theorem 2.17) that one can get a sequence $\langle x_n \rangle_{n=1}^{\infty}$ with $FS(\langle x_n \rangle_{n=1}^{\infty}) \cup FP(\langle x_n \rangle_{n=1}^{\infty}) \subseteq A$. On the other hand, given any $\alpha > 0$ there is a set A with $\overline{d}(A) > 1 - \alpha$ such that for any $x, y \in A$, $x \cdot y \notin A$. To see this, let $b_1 = \alpha$. Inductively, given b_n, let $a_n = b_n \cdot \alpha$ and let $b_{n+1} = {a_n}^2$. Let $A = \bigcup_{n=1}^{\infty} (a_n, b_n)$ and let $x \leq y$ be members of A. Pick n such that $a_n < x < b_n$. Then $a_n < y < b_1 = \alpha$ so $b_{n+1} = {a_n}^2 < x \cdot y < \alpha \cdot b_n = a_n$.

4.13 Question. Can one replace $c\ell_{\mathbb{R}} FSP_k(\langle G_n \rangle_{n=k}^{\infty})$ in Theorem 4.9 by $c\ell_{\mathbb{R}} FSP(\langle G_n \rangle_{n=1}^{\infty})$?

5. Ramsey Theory Near 0 in the Rationals and the Dyadic Rationals.

In this section we obtain results for $\mathbb{Q} \cap (0,1)$ (and indeed for $(\mathbb{D} \cap (0,1))$ that are much weaker than the results of Section 4 for $(0,1)$. These results yield separate sequences for sums and products. We also show that, at least in the case of $\mathbb{D} \cap (0,1)$, the stronger conclusions are not possible.

5.1 Definition. Let $\langle G_n \rangle_{n=1}^{\infty}$ be a sequence of finite subsets of $(0,1)$.

(a) $FS(\langle G_n \rangle_{n=1}^{\infty}) = \{\Sigma_{n \in F} x_n : F$ is a finite nonempty subset of $\mathbb{N}$ and for each $n \in F$, $x_n \in G_n\}$.

(b) $FP(\langle G_n\rangle_{n=1}^\infty)=\{\Pi_{n\in F}x_n : F$ is a finite nonempty subset of $\mathbb{N}$ and for each $n\in F$, $x_n\in G_n\}$.

We stated the results in Theorems 4.6 and 4.9 in terms of choices from $\mathcal{G}$ because one cannot enumerate the matrices with coefficients from $\mathbb{R}$ that satisfy the columns condition over $\mathbb{R}$. In the current context we have no such problem.

5.2 Definition. Fix an enumeration $\langle D_n\rangle_{n=1}^\infty$ of the matrices with rational coefficients that satisfy the columns condition over $\mathbb{Q}$ so that $\langle D_{2n}\rangle_{n=1}^\infty$ enumerates the matrices with integer entries that satisfy the columns condition over $\mathbb{Z}$. For each $n\in\mathbb{N}$, pick $(u(n),v(n))\in\mathbb{N}\times\mathbb{N}$ such that D_n is a $u(n)\times v(n)$ matrix. For each $n\in\mathbb{N}$, let

(a) $\mathcal{R}_n=\big\{\{x_1,x_2,\ldots,x_{v(n)}\}\subseteq\mathbb{D}:D_n\vec{x}=\vec{0}\big\}$ and

(b) $\mathcal{S}_n=\big\{\{x_1,x_2,\ldots,x_{v(2n)}\}\subseteq\mathbb{D}:$ for each $i\in\{1,2,\ldots,u(2n)\}$, $\Pi_{j=1}^{v(2n)}x_j^{d_{ij}}=1\big\}$

where $D_{2n}=\langle d_{ij}\rangle$.

In the following lemma, note that we are not yet claiming that there is an additive idempotent in $\bigcap_{n=1}^\infty U_{X,D_n}$. (Recall Definition 3.6.)

5.3 Lemma. *Let $X=\mathbb{D}\cap(0,1)$, let p be an additive idempotent in $\bigcap_{n=1}^\infty U_{X,D_n}$, and let $A\in p$. Then there is a choice of $G_n\in\mathcal{R}_n$ for each n such that $FS(\langle G_n\rangle_{n=1}^\infty)\subseteq A$.*

Proof. This is a simplified version of the proof of Theorem 4.6. Let $A_1=A$ and let $B_1=\{x\in X:-x+A_1\in p\}$. Then $B_1\in p$ and $p\in U_{X,D_n}$ so pick $G_1\in\mathcal{R}_1$ with $G_1\subseteq B_1$ and let $A_2=A_1\cap\bigcap_{x\in G_1}(-x+A_1)$. Inductively given $A_n\in p$, let $B_n=\{x\in X:-x+A_n\in p\}$ and pick $G_n\in\mathcal{R}_n$ with $G_n\subseteq B_n$ and let $A_{n+1}=A_n\cap\bigcap_{x\in G_n}(-x+A_n)$. One then shows by induction on $|F|$ that if $m=\min F$ and for each $n\in F$, $x_n\subseteq G_n$, then $\Sigma_{n\in F}x_n\in A_m$. □

5.4 Lemma. *Let $X=(0,1)\cap\mathbb{D}$, let p be a multiplicative idempotent in K_X, and let $A\in p$. Then there is a sequence $\langle H_n\rangle_{n=1}^\infty$ such that $FP(\langle H_n\rangle_{n=1}^\infty)\subseteq A$ and for each $n\in\mathbb{N}$, $H_{2n}\in\mathcal{S}_n$ and $H_{2n-1}\in\mathcal{R}_n$.*

Proof. Observe that by Theorem 3.5 one has for each $B\in p$ and each $n\in\mathbb{N}$ some $H\in\mathcal{S}_n$ with $H\subseteq B$. Further, by Lemma 3.7, one has that $\bigcap_{n=1}^\infty U_{X,D_n}$ is a two sided ideal of $(O_X,\cdot)$ so that $K_X\subseteq\bigcap_{n=1}^\infty U_{X,D_n}$.

Thus for each $B\in p$ and each $n\in\mathbb{N}$ one has some $H\in\mathcal{R}_n$ with $H\subseteq B$.

Now let $A_1=A$, let $B_1=\{x\in X:x^{-1}A_1\in p\}$, and pick $H_1\in\mathcal{R}_1$ with $H_1\subseteq B_1$. Let $A_2=A_1\cap\bigcap_{x\in H_1}(x^{-1}A_1)$. Inductively given $A_n\in p$, let $B_n=\{x\in X:x^{-1}A_1\in p\}$. If $n=2m$, pick $H_n\in\mathcal{S}_m$ with $H_n\subseteq B_n$. If $n=2m-1$, pick $H_n\in\mathcal{R}_m$ with $H_n\subseteq B_n$. Let $A_{n+1}=A_n\cap\bigcap_{x\in H_n}(x^{-1}A_n)$.

One then verifies as before that $FP(\langle H_n\rangle_{n=1}^{\infty})\subseteq A$. $\square$

Now we worry about finding additive idempotents in $\bigcap_{n=1}^{\infty}U_{X,D_n}$ that are located near K_X.

5.5 Lemma. *Let $X=(0,1)\cap\mathbb{D}$ and let $M=\{p:p+p=p$ and $p\in\bigcap_{n=1}^{\infty}U_{X,D_n}\}$. Then $c\ell M$ is a left ideal of $(O_X,\cdot)$.*

Proof. Since (by Lemma 3.7) $K_X\subseteq\bigcap_{n=1}^{\infty}U_{X,D_n}$ we have $\bigcap_{n=1}^{\infty}U_{X,D_n}\neq\emptyset$ so by Lemma 3.7 one has $\bigcap_{n=1}^{\infty}U_{X,D_n}$ is a subsemigroup of $(O_X,+)$. Since also each U_{X,D_n} is closed, as one sees easily from the form of its definition, one has $\bigcap_{n=1}^{\infty}U_{X,D_n}$ is compact and thus contains an additive idempotent. Consequently, $M\neq\emptyset$. To see that $c\ell M$ is a left ideal of $(O_X,\cdot)$ let $q\in c\ell M$ and let $r\in O_X$. Let $A\in r\cdot q$ and pick $x\in X$ such that $x^{-1}A\in q$. Since $x^{-1}A\in q$ (so $c\ell(x^{-1}A)$ is a neighborhood of q) one has some $p\in c\ell(x^{-1}A)\cap\bigcap_{n=1}^{\infty}U_{X,D_n}$ with $p+p=p$. Then $x^{-1}A\in p$ so $A\in x\cdot p=x\cdot p+x\cdot p$. Further, as was shown in the proof of Lemma 3.7, each U_{X,D_n} is a left ideal of $(\beta X_d,\cdot)$ so $x\cdot p\in\bigcap_{n=1}^{\infty}U_{X,D_n}$. Thus $c\ell A\cap M\neq\emptyset$. $\square$

The following is the main (affirmative) result of the section. Note that of course it immediately implies that corresponding statements hold for $(0,1)$ and for $(0,1)\cap\mathbb{Q}$.

5.6 Theorem. *Let $X=(0,1)\cap\mathbb{D}$, let $r\in\mathbb{N}$, and let $X=\bigcup_{i=1}^{r}A_i$. Then there exists $i\in\{1,2,\ldots,r\}$ and for each $n\in\mathbb{N}$ there exist choices of $G_n\in\mathcal{R}_n$, $H_{2n}\in\mathcal{S}_n$, and $H_{2n-1}\in\mathcal{R}_n$ with $FS(\langle G_n\rangle_{n=1}^{\infty})\cup FP(\langle H_n\rangle_{n=1}^{\infty})\subseteq A_i$.*

Proof. Let $M=\{p:p+p=p$ and $p\in\bigcap_{n=1}^{\infty}U_{X,D_n}\}$. Then by Lemma 5.5, $c\ell M$ is a left ideal of $(O_X,\cdot)$ so $c\ell M\cap K_X$ is a left ideal which thus contains a multiplicative idempotent q. Pick $i\in\{1,2,\ldots,r\}$ such that $A_i\in q$. Now $q\in c\ell M$ and $c\ell A_i$ is a neighborhood of q so pick $p=p+p\in\bigcap_{n=1}^{\infty}U_{X,D_n}$ with $A_i\in p$. Since $A_i\in p$ apply Lemma 5.3 to get the sequence $\langle G_n\rangle_{n=1}^{\infty}$. Since $A_i\in q$ apply Lemma 5.4 to get the sequence $\langle H_n\rangle_{n=1}^{\infty}$. $\square$

5.7 Corollary. *Let $X=(0,1)\cap\mathbb{D}$, let $r\in\mathbb{N}$, and let $X=\bigcup_{i=1}^{r}A_i$. Then there exists $i\in\{1,2,\ldots,r\}$ and sequences $\langle x_n\rangle_{n=1}^{\infty}$ and $\langle y_n\rangle_{n=1}^{\infty}$ in X such that $FS(\langle x_n\rangle_{n=1}^{\infty})\cup FP(\langle y_n\rangle_{n=1}^{\infty})\subseteq A_i$.* $\square$

We conclude by showing that one cannot get a combined sums and products result like those in Corollaries 4.7 and 4.10 for an arbitrary finite partition of $\mathbb{D}$.

5.8 Definition. Let $\langle x_n\rangle_{n=1}^{\infty}$ be a sequence in $\mathbb{R}$.

(a) $FS(\langle x_n\rangle_{n=1}^{\infty})=\{\Sigma_{n\in F}x_n:F$ is a finite nonempty subset of $\mathbb{N}\}$.

(b) $FP(\langle x_n\rangle_{n=1}^{\infty}) = \{\Pi_{n\in F} x_n : F$ is a finite nonempty subset of $\mathbb{N}\}$.
(c) $PP(\langle x_n\rangle_{n=1}^{\infty}) = \{x_n \cdot x_m : n, m \in \mathbb{N}$ and $n \neq m\}$.

Thus $FS(\langle x_n\rangle_{n=1}^{\infty})$ as defined in Definition 5.8 and $FS(\langle\{x_n\}\rangle_{n=1}^{\infty})$ as defined in Definition 5.1 are identical, and similarly for FP.

Our final result states that, for partitions of $\mathbb{D}\cap(0,1)$, or even of the whole of $\mathbb{D}\backslash\{0\}$, one cannot guarantee to find a sequence $\langle x_n\rangle_{n=1}^{\infty}$ with $FS(\langle x_n\rangle_{n=1}^{\infty})\cup FP(\langle x_n\rangle_{n=1}^{\infty})$ contained in one cell. In fact, one cannot even guarantee to find $FS(\langle x_n\rangle_{n=1}^{\infty})\cup PP(\langle x_n\rangle_{n=1}^{\infty})$ contained in one cell.

In the proof, when we talk of a "coloring" we mean a function to a finite set. In this case the members of the finite set will typically be k-tuples of natural numbers, for various k.

5.9 Theorem. *There exists a finite partition $\mathbb{D}\backslash\{0\} = \bigcup_{i=1}^{r} A_i$ such that there do not exist $i \in \{1, 2, \ldots, r\}$ and a sequence $\langle x_n\rangle_{n=1}^{\infty}$ with $FS(\langle x_n\rangle_{n=1}^{\infty}) \cup PP(\langle x_n\rangle_{n=1}^{\infty}) \subseteq A_i$.*

Proof. We start by giving a coloring for just $\mathbb{D}\cap(0,1)$: this will contain some of the ideas to be used in the general case. For $x \in \mathbb{D}\cap(0,1)$, write

$$x = \sum_{i\in I} 2^{-i},$$

where I is a finite subset of $\mathbb{N} = \{1, 2, 3, \ldots\}$. The *start* of x is $s = \min I$, and the *end* of x is $e = \max I$. If x is not a power of 2 (in other words, if $|I| \geq 2$) then we say that the *type* t of x is 1 if $e-1 \in I$ and 0 if $e-1 \notin I$. The *previous point* p of an x that is not a power of 2 is $\max\{1 \leq i \leq e-1 :\ i \in I\}$ if x is of type 0 and $\max\{0 \leq i \leq e-1 :\ i \notin I\}$ if x is of type 1, and the *gap length* of x is $g = e - p$. Thus we always have $g \geq 2$. Finally, if x is not a power of 2 then the *ratio* r of x is 1 if $g > s$ and 0 if $g \leq s$.

We now color $\mathbb{D}\cap(0,1)$ by giving x the color $c(x) = (t, g \bmod 2, r)$ if x is not a power of 2 and $c(x) = 0$ (say) if x is a power of 2. Thus we are coloring $\mathbb{D}\cap(0,1)$ with 9 colors.

We claim that, for the coloring c, there is no sequence $(x_n)_{n=1}^{\infty}$ with $FS((x_n)) \cup PP((x_n))$ monochromatic. Indeed, suppose to the contrary that (x_n) is such a sequence. Since all finite sums must belong to $(0,1)$, we have $x_n \to 0$ as $n \to \infty$. In particular, the x_i are not powers of 2 (since the sum of two distinct powers of 2 is not a power of 2).

Now, since $x_n \to 0$, we certainly have $s(x_n) \to \infty$ and $e(x_n) \to \infty$. We claim that, in addition, we have $g(x_n) \to \infty$. For if this is not the case then we can find infinitely many x_n with a common value of $g(x_n)$, and hence there certainly exist distinct m and n such that $g(x_m) = g(x_n)$ and either $p(x_m) - 1 \notin I(x_m)$, $p(x_n) - 1 \notin I(x_n)$ or $p(x_m) - 1 \in I(x_m)$, $p(x_n) - 1 \in I(x_n)$. However, in each case it is easy to check that we have $g(x_m x_n) = g(x_n) + 1$ (whether the type of all the x_n is 0 or 1), contradicting $c(x_m x_n) = c(x_n)$.

Because $s(x_n) \to \infty$, it follows that $r(x_1 + x_n) = 1$ for n sufficiently large. However, it is also clear that $r(x_1 x_n) = 0$ for n sufficiently large, a contradiction as required.

We now turn to the more general case of the dyadics. It is enough to give a coloring for the positive dyadics $\mathbb{D}^+$, since we may then extend to $\mathbb{D}$ by giving all negative dyadics a different color: the fact that all x_n and all $x_n x_m$ have the same color then forces all x_n to be positive. Our aim is, roughly speaking, to use new colors to force enough conditions onto a sequence (x_n) that we can somehow argue as for $\mathbb{D} \cap (0,1)$.

For a finite subset I of $\mathbb{N}$, let us put $c(I) = -1$ if I is empty. If I is not empty, put $s(I) = \min I$, and if $|I| = 1$ then put $c(I) = s \bmod 2$. If $|I| \geq 2$, we define $c(I)$ as follows. Put $e(I) = \max I$, and define $t(I)$, $p(I)$, $g(I)$ and $r(I)$ as before. Also, let the *parity* $q(I)$ of I be 1 if $1 \in I$ and 0 if $1 \notin I$. Finally, let the *zero-start* of I be $z(I) = \min\{i \in \mathbb{N} : i \notin I\}$, and let the *opposite ratio* $u(I)$ of I be 1 if $g > z$ and 0 if $g \leq z$. Define $c(I) = (s \bmod 2, e \bmod 2, t, g \bmod 2, l, q, z \bmod 2, u)$.

For $x \in \mathbb{D}^+$, write

$$x = \sum_{i \in J} 2^{i-1} + \sum_{i \in I} 2^{-i},$$

where I and J are finite subsets of $\mathbb{N}$. Color $\mathbb{D}^+$ by giving x the color $c(x) = (c(J), c(I))$. We will often write eg. $s^+(x)$ for $s(J(x))$, and similarly $s^-(x)$ for $s(I(x))$.

We claim that this is a suitable coloring of $\mathbb{D}^+$. Indeed, suppose to the contrary that there is a sequence $(x_n)_{n=1}^\infty$ in $\mathbb{D}^+$ such that the set $FS((x_n)) \cup PP((x_n))$ is monochromatic.

We cannot have $J(x_1) = \emptyset$, because then $x_n \to 0$, and so we would be done by the argument for $\mathbb{D} \cap (0,1)$. Can we have $I(x_1) = \emptyset$? If so, then we must have $s^+(x_n) \to \infty$, for otherwise we could find distinct m and n with $s^+(x_m) = s^+(x_n)$ ($= s$, say) and $s + 1 \notin J(x_m) \triangle J(x_n)$, and this implies $s(x_m + x_n) = s + 1$, a contradiction. But, given $s^+(x_n) \to \infty$, we may argue for J in a manner similar to the argument for I in the $\mathbb{D} \cap (0,1)$ case, arriving at a contradiction.

Thus we now know that $I(x_1) \neq \emptyset$ and $J(x_1) \neq \emptyset$. We must have $e^+(x_n) \to \infty$, because if $e^+(x_m) = e^+(x_n)$ then $e^+(x_m + x_n) = e^+(x_m) + 1$. We must also have $e^-(x_n) \to \infty$. Indeed, if this is not the case then we can find distinct m and n with

$e^-(x_m) = e^-(x_n)$ (=e, say) and $e-1 \notin I(x_m) \triangle I(x_n)$. But this implies $e^-(x_m+x_n) = e-1$, a contradiction.

It follows immediately that $I(x_1)$ cannot be a singleton, because $I(x_1 + x_n)$ is certainly not a singleton, for n sufficiently large. Similarly, $J(x_1)$ is not a singleton. Thus we may assume from now on that $|I(x_n)|, |J(x_n)| \geq 2$ for all n.

We now turn to the parity of $J(x_1)$. If $p^+(x_1) = 0$ then we must have $s^-(x_n) \to \infty$, for otherwise some finite sum x of the x_n would have $p^+(x) = 1$, and we must also have $s^+(x_n) \to \infty$, for otherwise we could find, as above, distinct m and n with $s^+(x_m+x_n) = s^+(x_m)+1$ (by choosing m and n with $s^+(x_m) = s^-(x_n)$ and $s^+(x_m) \notin J(x_m) \triangle J(x_n)$). Similarly, if $p^+(x_1) = 1$ then we must have $z^-(x_n) \to \infty$ and also $z^+(x_n) \to \infty$.

Now, just as for $\mathbb{D} \cap (0,1)$, we certainly have $g^-(x_n) \to \infty$. Hence in the case $p^+(x_1) = 0$ we have $r^-(x_1 + x_n) = 1$ and $r^-(x_1 x_n) = 0$ for n sufficiently large (whether the type of all the $I(x_n)$ is 0 or 1), a contradiction. And in the case $p^+(x_1) = 1$ we have $u^-(x_1 + x_n) = 1$ and $u^-(x_1 x_n) = 0$ for n sufficiently large, again a contradiction. $\square$

Unfortunately, we are not able to extend the above construction even to $\mathbb{Q}$. However we are willing to conjecture that it can be done.

5.10 Conjecture. There exists a finite partition $\mathbb{Q}\backslash\{0\} = \bigcup_{i=1}^r A_i$ such that there do not exist $i \in \{1, 2, \ldots, r\}$ and a sequence $\langle x_n \rangle_{n=1}^\infty$ with $FS(\langle x_n \rangle_{n=1}^\infty) \cup FP(\langle x_n \rangle_{n=1}^\infty) \subseteq A_i$.

At least one of the authors is less confident about the situation with respect to $\mathbb{R}$ so we conclude with the following.

5.11 Question. (a) Does there exist a finite partition $\mathbb{R}\backslash\{0\} = \bigcup_{i=1}^r A_i$ such that there do not exist $i \in \{1, 2, \ldots, r\}$ and a sequence $\langle x_n \rangle_{n=1}^\infty$ with $FSP(\langle x_n \rangle_{n=1}^\infty) \subseteq A_i$?

(b) Does there exist a finite partition $\mathbb{R}\backslash\{0\} = \bigcup_{i=1}^r A_i$ such that there do not exist $i \in \{1, 2, \ldots, r\}$ and a sequence $\langle x_n \rangle_{n=1}^\infty$ with $FS(\langle x_n \rangle_{n=1}^\infty) \cup FP(\langle x_n \rangle_{n=1}^\infty) \subseteq A_i$?

In view of Corollary 4.7, an affirmative answer to Question 5.11(a) could not be a partition into sets with the property of Baire – one would thus expect that it would involve some diagonal arguments (in other words, use of the Axiom of Choice).

We are grateful to A. Blass for making the above remark precise in a rather appealing way. There is a model M of ZF in which all sets of reals have the property of Baire. (This was constructed by Shelah [18], following related work by Solovay [19]. The essential difference in the models is in the hypotheses used to construct them–Solovay used an inaccessible cardinal while Shelah did not.) Now, of course AC fails in this model, so we cannot directly apply Corollary 4.7 in this model (since we have made heavy use of AC in our proof of Corollary 4.7). However, what Shelah actually constructed was a model of ZFC that contains the above model M as a transitive submodel with the same set of reals. It can now be checked that we may pass from M to this model, and apply Corollary 4.7 there. It follows that, in M, Question 5.11 has a negative answer. Thus any example answering Question 5.11 in the affirmative *must* involve AC.

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