This paper was published in *Topology and its Applications* **155** (2008), 433-444. To the best of my knowledge, this is the final version as it was submitted to the publisher. –NH

Almost Disjoint Large Subsets of Semigroups

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Abstract. There are several notions of largeness in a semigroup S that originated in topological dynamics. Among these are *thick*, *central*, *syndetic* and *piecewise syndetic*. Of these, central sets are especially interesting because they are partition regular and are guaranteed to contain substantial combinatorial structure. It is known that in $(\mathbb{N},+)$ any central set may be partitioned into infinitely many pairwise disjoint central sets. We extend this result to a large class of semigroups (including $(\mathbb{N},+)$) by showing that if S is a semigroup in this class which has cardinality κ then any central set can be partitioned into κ many pairwise disjoint central sets. We also show that for this same class of semigroups, if there exists a collection of μ almost disjoint subsets of any member S, then any central subset of S contains a collection of μ almost disjoint central sets. The same statement applies if "central" is replaced by "thick"; and in the case that the semigroup is left cancellative, "central" may be replaced by "piecewise syndetic". The situation with respect to syndetic sets is much more restrictive. For example there does not exist an uncountable collection of almost disjoint syndetic subsets of $\mathbb{N}$. We investigate the extent to which syndetic sets can be split into disjoint syndetic sets.

1. Introduction

Central subsets of the set $\mathbb{N}$ of positive integers were introduced by Furstenberg in [5]. They were defined in terms of notions from topological dynamics, shown to be partition regular (meaning that if a central set was divided into finitely many parts, one of these parts must be central), and shown to contain an extensive amount of combinatorial structure. For example, any central subset of $\mathbb{N}$ contains a sequence together with all of the finite sums of distinct terms and contains solutions to any partition regular system

[1]This author acknowledges support received from the National Science Foundation (USA) via grant DMS-0243586.

[2]This author thanks the University of Hull for its hospitality while this research was being conducted.

Key words: thick, central, syndetic, almost disjoint

AMS (2000) Subject Classification: Primary 54H15; Secondary 03E05; 22A15; 05D10

of homogeneous linear equations. See Chapters 14 and 15 of [7] for a detailed description of some of the structure which must exist in any central set.

The definition of "central" given by Furstenberg makes sense in an arbitrary semigroup S. In [3] and [11] that notion was shown to have a simple equivalent characterization in terms of the algebraic structure of the Stone-Čech compactification of the discrete semigroup S. (We shall present this characterization below as our definition of the notion.) Based on this characterization, it is immediate that in any semigroup, if a central set is partitioned into finitely many pieces, then one of these pieces is central. The question then arose whether an arbitrary central set could be divided into two disjoint central sets. (There is more than idle curiosity behind this question. Each of the disjoint central sets would have to contain all of the combinatorial structure guaranteed to any central set.) In the case of $(\mathbb{N}, +)$, that question was answered in the affirmative in [6, Theorem 2.12]. Of course, since a central subset of $\mathbb{N}$ can be again split into two central subsets, any central subset of $\mathbb{N}$ can be split into infinitely many pairwise disjoint central sets. That is as much as one can expect in a countable semigroup. But one can ask how many *almost disjoint* central sets a given central subset of $\mathbb{N}$ can contain.

1.1 Definition. Let X be an infinite set. A set $\mathcal{A}$ is a *set of almost disjoint subsets of* X if and only if $\mathcal{A} \subseteq \mathcal{P}(X)$, for each $A \in \mathcal{A}$, $|A| = |X|$, and for $A \neq B$ in $\mathcal{A}$, $|A \cap B| < |X|$.

We denote by ω the first infinite cardinal, and recall that $\omega = \mathbb{N} \cup \{0\}$. As is well known, there is a set $\mathcal{A}$ of $\mathfrak{c} = 2^\omega$ almost disjoint subsets of $\mathbb{N}$. Probably the simplest example of a set of $\mathfrak{c}$ almost disjoint subsets of a countably infinite set can be obtained as follows: For each $\alpha \in \mathbb{R}$, choose an increasing sequence $\langle x_{\alpha,n} \rangle_{n=0}^{\infty}$ in $\mathbb{Q}$ which converges to α. Then $\big\{\{x_{\alpha,n} : n \in \omega\} : \alpha \in \mathbb{R}\big\}$ is a set of almost disjoint subsets of $\mathbb{Q}$.

If $|S| = \kappa > \omega$, there may not exist *any* set of 2^κ almost disjoint subsets of S. (Baumgartner proved [2, Theorem 2.8] that there is always a family of κ^+ almost disjoint subsets of S, and also showed that it is consistent with ZFC that if $\kappa = \omega_1$, there is no family of 2^κ almost disjoint subsets of S.)

1.2 Definition. Let S be a semigroup. A subset A of S is a *left solution set of* S (respectively a *right solution set of* S) if and only if there exist $w, z \in S$ such that $A = \{x \in S : w = zx\}$ (respectively $A = \{x \in S : w = xz\}$).

1.3 Definition. Let S be an infinite semigroup with cardinality κ. We shall say that S is *very weakly left cancellative* if the union of fewer than κ left solution sets of S must have cardinality less than κ. We shall say that S is *very weakly right cancellative* if the union of fewer than κ right solution sets of S must have cardinality less than κ. We shall say that S is *very weakly cancellative* if it is both very weakly left and very weakly right cancellative.

We remark that if κ is regular, S is very weakly left cancellative if and only if every left solution set of S has cardinality less than κ. If κ is singular, S is very weakly left cancellative if and only if there is a cardinal less than κ which is an upper bound for the cardinalities of all left solution sets of S.

We remind the reader that S is said to be weakly left cancellative if all left solution sets of S are finite. Of course, weak left cancellativity implies very weak left cancellativity. The two notions are equivalent if $\kappa = \omega$.

The corresponding remarks are also valid for very weak right cancellativity.

Very weak left cancellativity has interesting algebraic implications. Theorem 1.7 is an example, which we shall present after introducing the necessary terminology.

We show in Section 3 that if S is an infinite semigroup which is very weakly cancellative, then whenever there exists a family of μ almost disjoint subsets of S, each central subset of S contains a family of μ almost disjoint central subsets. We also extend the theorem cited above [6, Theorem 2.12], by showing that, in an infinite very weakly cancellative semigroup with cardinality κ, every central set contains κ disjoint central sets.

There are several other notions of size in a semigroup besides "central". We shall be concerned here with four of them: *thick*, *very thick*, *syndetic*, and *piecewise syndetic*. Unlike, central sets, they each have simple elementary definitions. Given a semigroup S, a set $A \subseteq S$, and $x \in S$ we let $x^{-1}A = \{y \in S : xy \in A\}$. We also write $\mathcal{P}_f(S)$ for the set of finite nonempty subsets of S.

1.4 Definition. Let S be a semigroup and let $A \subseteq S$.

(a) A is *thick* if and only if $\big(\forall F \in \mathcal{P}_f(S)\big)(\exists x \in S)(Fx \subseteq A)$.

(b) A is *very thick* if and only if $\big(\forall F \in \mathcal{P}(S)\big)\big(|F| < |S| \Rightarrow (\exists x \in S)(Fx \subseteq A)\big)$.

(c) A is *syndetic* if and only if $\big(\exists H \in \mathcal{P}_f(S)\big)(S = \bigcup_{t\in H} t^{-1}A)$.

(d) A is *piecewise syndetic* if an only if $\big(\exists H \in \mathcal{P}_f(S)\big)\big(\forall F \in \mathcal{P}_f(S)\big)(\exists x \in S)$ $(Fx \subseteq \bigcup_{t\in H} t^{-1}A)$.

Notice that in $(\mathbb{N}, +)$ a set A is thick if and only if it contains arbitrarily long blocks; it is syndetic if and only if it has bounded gaps; and it is piecewise syndetic if and only if there is a fixed bound b and arbitrarily long blocks of $\mathbb{N}$ in which the gaps of A are bounded by b.

Also notice that a subset A of a semigroup S is syndetic in just the case that its complement is not thick.

In Section 2 we will show that the results mentioned above about central sets remain valid if "central" is replaced by "thick".

Like central sets, piecewise syndetic sets are partition regular. Also, any piecewise syndetic set has a substantial amount of combinatorial structure guaranteed to it (though significantly less than central sets). For example, any piecewise syndetic subset of $(\mathbb{N}, +)$ must contain arbitrarily long arithmetic progressions. We shall also show in Section 3 that the results mentioned above about central sets remain valid if "central" is replaced by "piecewise syndetic" and "very weakly cancellative" is replaced by "left cancellative and very weakly right cancellative".

The situation with respect to syndetic sets is quite different, and we investigate that situation in Section 4. For example, in any countable left cancellative semigroup, there does not exist an uncountable collection of almost disjoint syndetic subsets. In that section we determine several cancellation conditions that guarantee the ability to at least split a syndetic set into two syndetic sets.

We use throughout the algebraic structure of the Stone-Čech compactification of a discrete semigroup S. We present a brief overview here. Please refer to [7] for details of any unfamiliar assertions about this algebraic structure. We take the points of βS to be the ultrafilters on S, the principal ultrafilters being identified with the points of S. Given a set $A \subseteq S$, $\overline{A} = \{p \in \beta S : A \in p\}$. The set $\{\overline{A} : A \subseteq S\}$ is a basis for the open sets (as well as a basis for the closed sets) of βS.

There is a natural extension of the operation $\cdot$ of S to βS. This natural extension makes $(\beta S, \cdot)$ a compact right topological semigroup with S contained in its topological center. This says that for each $p \in \beta S$ the function $\rho_p : \beta S \to \beta S$ is continuous and for each $x \in S$, the function $\lambda_x : \beta S \to \beta S$ is continuous, where $\rho_p(q) = q \cdot p$ and $\lambda_x(q) = x \cdot q$. Given $p, q \in \beta S$ and $A \subseteq S$ one has that $A \in p \cdot q$ if and only if $\{x \in S : x^{-1}A \in q\} \in p$.

A subset U of a semigroup S is called a left ideal if it is nonempty and $S \cdot U \subseteq U$. It is called a right ideal if it is nonempty and $U \cdot S \subseteq U$. It is called a two-sided ideal, or simply an ideal, if it is both a left ideal and a right ideal. Any compact Hausdorff right topological semigroup T has a smallest two sided ideal $K(T)$ which is the union of all of the minimal left ideals of T and is also the union of all of the minimal right ideals of T. The intersection of any minimal left ideal and any minimal right ideal is a group. In particular there are idempotents in the smallest ideal. An idempotent p in T

is "minimal" if and only if $p \in K(T)$.

1.5 Definition. Let S be a semigroup and let $A \subseteq S$. Then A is *central* if and only if there is some minimal idempotent $p \in \beta S$ such that $A \in p$.

There are simple characterizations of three of the other notions of largeness in terms of the algebra of βS.

1.6 Lemma. *Let S be a semigroup and let $A \subseteq S$.*

(a) A is thick if and only if there is a left ideal of βS contained in $\overline{A}$.

(b) A is syndetic if and only if for every left ideal L of βS, $L \cap \overline{A} \neq \emptyset$.

(c) A is piecewise syndetic if and only if $\overline{A} \cap K(\beta S) \neq \emptyset$.

Proof. (a) [4, Theorem 2.9(c)].

(b) [4, Theorem 2.9(d)].

(c) [7, Theorem 4.40]. □

Notice that each of the notions of size that we are considering is one sided in its definition (if S is not commutative). This fact is obvious for "thick", "syndetic", and "piecewise syndetic". For central sets the definition can be seen to depend on the choice of continuity for βS making it a right topological rather than a left topological semigroup. Each of these notions can be prefaced by "right" (and are in [4]) and there are corresponding "left" notions.

The following theorem will not be needed in the remainder of the paper, but it provides significant information about the structure of βS when S is very weakly left cancellative. We remind the reader that any ultrafilter p on a set of cardinality κ is uniform if and only if every member of p has cardinality κ.

1.7 Theorem. *Let S be an infinite very weakly left cancellative semigroup with cardinality κ. There is a collection of 2^{2^κ} pairwise disjoint left ideals of βS. In particular, βS has 2^{2^κ} minimal idempotents.*

Proof. Enumerate the elements of S as $\langle s_\iota \rangle_{\iota<\kappa}$. Inductively construct an injective κ-sequence $\langle t_\iota \rangle_{\iota<\kappa}$ so that for all $\lambda < \mu < \kappa$, $s_\lambda t_\mu \notin \{s_\iota t_\gamma : \iota < \gamma < \mu\}$. (This is possible because S is very weakly left cancellative.)

There are 2^{2^κ} uniform ultrafilters on $T = \{t_\iota : \iota < \kappa\}$. (See [7, Theorem 3.58].) So it suffices to show that if p and q are distinct uniform ultrafilters on T, then $\beta Sp \cap \beta Sq = \emptyset$. So let p and q be distinct uniform ultrafilters on S and pick $P \in p$ and $Q \in q$. Let

$D = \{s_\iota t_\gamma : t_\gamma \in P \text{ and } \iota < \gamma\}$ and let $E = \{s_\iota t_\gamma : t_\gamma \in Q \text{ and } \iota < \gamma\}$. Then $D \cap E = \emptyset$, $\beta S p = \overline{Sp} \subseteq \overline{D}$, and $\beta S q = \overline{Sq} \subseteq \overline{E}$. $\square$

We thank the referee for providing us with several relevant references, as well as the statement and proof of Theorem 2.4.

2. Almost disjoint thick sets

In this section, we establish the existence of large almost disjoint families of thick subsets of a given thick subset for all infinite very weakly cancellative semigroups.

2.1 Lemma. *Let κ be an infinite cardinal.*

(i) If there is a family $\langle B_\iota \rangle_{\iota<\mu}$ of almost disjoint subsets of κ, then there is a family $\langle \mathcal{B}_\iota \rangle_{\iota<\mu}$ of almost disjoint subsets of $\mathcal{P}_f(\kappa)$ such that $\big(\forall F \in \mathcal{P}_f(\kappa)\big)(\forall \iota < \mu)(\exists G \in \mathcal{B}_\iota)(F \subseteq G)$.

(ii) There is a family $\langle \mathcal{C}_\iota \rangle_{\iota<\kappa}$ of pairwise disjoint subsets of $\mathcal{P}_f(\kappa)$, each with cardinality κ, such that $\big(\forall F \in \mathcal{P}_f(\kappa)\big)(\forall \iota < \kappa)(\exists G \in \mathcal{C}_\iota)(F \subseteq G)$.

Proof. (i) Enumerate $\mathcal{P}_f(\kappa)$ as $\langle F_\sigma \rangle_{\sigma<\kappa}$. For each $\iota < \mu$ inductively define an injective function $f_\iota : \mathcal{P}_f(\kappa) \to B_\iota$ so that for all $\sigma < \kappa$, $f_\iota(F_\sigma) > \max(F_\sigma)$. (Since $|\{\tau \in B_\iota : \tau > \max(F_\sigma)\}| = \kappa$, such a choice is always possible.)

For $\iota < \mu$ let $\mathcal{B}_\iota = \{F \cup \{f_\iota(F)\} : F \in \mathcal{P}_f(\kappa)\}$. Then $|\mathcal{B}_\iota| = \kappa$.

The function max takes each $\mathcal{B}_\iota$ injectively to B_ι and thus, if $\iota < \delta < \mu$, then $|\mathcal{B}_\iota \cap \mathcal{B}_\delta| \leq |B_\iota \cap B_\delta| < \kappa$.

(ii) This proof is essentially the same, using the fact that there is a family $\langle C_\iota \rangle_{\iota<\kappa}$ of pairwise disjoint subsets of κ such that $|C_\iota| = \kappa$ for every $\iota < \kappa$. $\square$

2.2 Lemma. *Let S be an infinite semigroup which is very weakly right cancellative, and let $\kappa = |S|$. If A is a thick subset of S then $|A| = \kappa$.*

Proof. Notice that for $x \in S$, $A \cap xA$ is nonempty (since there exists y such that $\{x, xx\}y \subseteq A$). We will see that this condition is enough to guarantee that A has size κ.

Argue by contradiction and assume that $|A| < \kappa$. For each $(w, z) \in A \times A$, let $T_{w,z} = \{y \in S : yw = z\}$. Since each $T_{w,z}$ is a right solution set of S,

$$|\bigcup\{T_{w,z} : (w, z) \in A \times A\}| < \kappa\,.$$

Pick $x \in S \setminus \bigcup\{T_{w,z} : (w, z) \in A \times A\}$. Then $A \cap xA = \emptyset$, a contradiction. $\square$

For groups, Theorem 2.3(ii) follows from (the left-right switch of) [8, Theorem 1] and if S is countable, the same result follows from [1, Theorem 11.5]. For countable groups Theorem 2.3(i) is in [10, p. 105].

2.3 Theorem. *Let S be an infinite semigroup which is very weakly cancellative, let $\kappa = |S|$, and let A be a thick subset of S.*

(i) If there is a family of μ almost disjoint subsets of κ, then there is a family of μ almost disjoint thick subsets of A.

(ii) There is a family of κ pairwise disjoint thick subsets of A.

Proof. (i) Enumerate $\mathcal{P}_f(S)$ as $\langle F_\sigma \rangle_{\sigma<\kappa}$ and pick by Lemma 2.1(i) a family $\langle \mathcal{B}_\iota \rangle_{\iota<\mu}$ of almost disjoint subsets of $\mathcal{P}_f(S)$ such that $\big(\forall F \in \mathcal{P}_f(S)\big)(\forall \iota < \mu)(\exists G \in \mathcal{B}_\iota)(F \subseteq G)$.

We inductively choose a κ-sequence $\langle x_\sigma \rangle_{\sigma<\kappa}$ in S such that for all $\sigma < \kappa$, $F_\sigma \cdot x_\sigma \subseteq A$, and for all $\delta < \sigma < \kappa$, $F_\sigma \cdot x_\sigma \cap F_\delta \cdot x_\delta = \emptyset$. To see that we can do this, let $\sigma < \kappa$ and assume that $\langle x_\delta \rangle_{\delta<\sigma}$ has been chosen. Let $H = \bigcup_{\delta<\sigma} F_\delta \cdot x_\delta$. Observe that $|H| \leq \max\{\omega, |\sigma|\} < \kappa$.

Given any $w \in H$ and $z \in F_\sigma$, $\{x \in S : w = zx\}$ is a left solution set of S. So since $|H \times F_\sigma| < \kappa$, we have $|\{x \in S : F_\sigma \cdot x \cap H \neq \emptyset\}| < \kappa$.

For any finite subset G of S, there is some y such that $F_\sigma G y \subseteq A$ implying that $Gy \subseteq \{x \in S : F_\sigma \cdot x \subseteq A\}$. Therefore, $\{x \in S : F_\sigma \cdot x \subseteq A\}$ is thick. By Lemma 2.2, $|\{x \in S : F_\sigma \cdot x \subseteq A\}| = \kappa$. Pick

$$x_\sigma \in \{x \in S : F_\sigma \cdot x \subseteq A\} \setminus \{x \in S : F_\sigma \cdot x \cap H \neq \emptyset\}.$$

For $\iota < \mu$, let $D_\iota = \bigcup \{F_\sigma \cdot x_\sigma : \sigma < \kappa \text{ and } F_\sigma \in \mathcal{B}_\iota\}$. Then $|D_\iota| = \kappa$.

If $\iota < \gamma < \mu$, then $D_\iota \cap D_\gamma = \bigcup\{F_\sigma \cdot x_\sigma : \sigma < \kappa \text{ and } F_\sigma \in \mathcal{B}_\iota \cap \mathcal{B}_\gamma\}$ so $|D_\iota \cap D_\gamma| < \kappa$. To see that each D_ι is thick, let $G \in \mathcal{P}_f(S)$ and pick $F_\sigma \in \mathcal{B}_\iota$ such that $G \subseteq F_\sigma$. Then $G \cdot x_\sigma \subseteq D_\iota$.

(ii) The proof is essentially the same, using Lemma 2.1(ii) instead of Lemma 2.1(i).

□

The following theorem is due to the referee. Its proof combines elements of the proofs of Lemma 2.1 and Theorem 2.3. Notice that one cannot necessarily begin by enumerating the subsets of S with cardinality less than κ as a κ-sequence. For example, if $\kappa = \omega_1$, then there are 2^ω subsets of S that are smaller than κ and one might have $\omega_1 < 2^\omega$.

2.4 Theorem. *Let S be an infinite semigroup which is very weakly cancellative, let $\kappa = |S|$, assume that κ is regular, and let A be a very thick subset of S.*

(i) If there is a family of μ almost disjoint subsets of κ, then there is a family of μ almost disjoint very thick subsets of A.

(ii) There is a family of κ pairwise disjoint very thick subsets of A.

Proof. We do both proofs at once. For (i) let $\langle B_\iota \rangle_{\iota<\mu}$ be a family of almost disjoint subsets of κ. For (ii), let $\mu = \kappa$ and let $\langle B_\iota \rangle_{\iota<\mu}$ be a family of pairwise disjoint subsets of κ with each $|B_\iota| = \kappa$.

Enumerate S as $\langle s_\sigma \rangle_{\sigma<\kappa}$. For $\sigma < \kappa$, let $I_\sigma = \{s_\delta : \delta < \sigma\}$.

We now claim that we can choose $\langle x_\sigma \rangle_{\sigma<\kappa}$ in S such that for each σ, $I_\sigma \cdot x_\sigma \subseteq A$ and for all $\delta < \sigma < \kappa$, $I_\sigma \cdot x_\sigma \cap I_\delta \cdot x_\delta = \emptyset$. So let $\sigma < \kappa$ and assume that $\langle x_\delta \rangle_{\delta<\sigma}$ has been chosen. Let $H = \bigcup_{\delta<\sigma} I_\delta \cdot x_\delta$. Since κ is regular, $|H| < \kappa$.

Given any $w \in H$ and $z \in I_\sigma$, $\{x \in S : w = zx\}$ is a left solution set of S so is smaller than κ. So since $|H \times I_\sigma| < \kappa$, we have $|\{x \in S : I_\sigma \cdot x \cap H \neq \emptyset\}| < \kappa$.

For any subset G of S with $|G| < \kappa$, there is some y such that $I_\sigma G y \subseteq A$ so that $Gy \subseteq \{x \in S : I_\sigma \cdot x \subseteq A\}$. Therefore, $\{x \in S : I_\sigma \cdot x \subseteq A\}$ is very thick (and in particular, thick). By Lemma 2.2, $|\{x \in S : I_\sigma \cdot x \subseteq A\}| = \kappa$. Pick $x_\sigma \in \{x \in S : I_\sigma \cdot x \subseteq A\} \setminus \{x \in S : I_\sigma \cdot x \cap H \neq \emptyset\}$.

For $\iota < \mu$, let $D_\iota = \bigcup_{\tau \in B_\iota} I_\tau \cdot x_\tau$. By the regularity of κ, every subset of κ of size less than κ is bounded in κ implying it is a subset of I_τ for any sufficiently large τ. This clearly implies that D_ι is very thick.

Since the sets $I_\tau \cdot x_\tau$ are pairwise disjoint, we see that $D_\iota \cap D_\gamma = \bigcup_{\tau \in B_\iota \cap B_\gamma} I_\tau \cdot x_\tau$. Therefore if $B_\iota \cap B_\gamma = \emptyset$, then $D_\iota \cap D_\gamma = \emptyset$. Moreover if $B_\iota \cap B_\gamma$ has size less than κ then so does $D_\iota \cap D_\gamma$. $\square$

3. Almost disjoint central and piecewise syndetic sets

We shall show that, if S is an infinite very weakly cancellative semigroup of cardinality κ and if κ contains μ almost disjoint sets, then every central set in S contains μ almost disjoint central subsets. The same statement holds for piecewise syndetic subsets of S if S is left cancellative and very weakly right cancellative.

3.1 Lemma. *Let S be an infinite semigroup with cardinality κ and let U denote the set of uniform ultrafilters on S. If S is very weakly left cancellative, U is a left ideal of βS. If S is very weakly cancellative, U is an ideal of βS.*

Proof. Assume first that S is very weakly left cancellative. Let $p \in U$. To show that $\beta S p \subseteq U$ it is sufficient to show that $sp \in U$ for every $s \in S$, because U is closed and $\beta Sp = cl_{\beta S}(Sp)$. Since $\{sP : P \in p\}$ is a base for sp, it is sufficient to show that $|sP| = \kappa$ if $P \in p$. Now, for every $t \in sP$, $\lambda_s^{-1}[\{t\}]$ is a left solution set of S. Since $P \subseteq \bigcup_{t \in sP} \lambda_s^{-1}[\{t\}]$, it follows that $|sP| = \kappa$.

Now suppose that S is very weakly cancellative. To show that U is a right ideal, let $p \in U$ and $q \in \beta S$. We claim that $pq \in U$. To see this we assume that, on the contrary, $pq \notin U$. Then, since U is a left ideal, $q \notin U$ and so there exists $Q \in q$ for which $|Q| < \kappa$. Since $pq \notin U$, there also exists $X \in pq$ such that $|X| < \kappa$. We may pick $P \in p$ and $Q_a \in q$ for each $a \in P$ such that $\bigcup_{a \in P} aQ_a \subseteq X$ by [7, Theorem 4.15]. Now, for each $b \in Q$ and $x \in X$, $T_{b,x} = \{s \in S : sb = x\}$ is a right solution set of S. However, $P \subseteq \bigcup_{(b,x) \in Q \times X} T_{b,x}$. (Given $a \in P$, pick $b \in Q \cap Q_a$. Then $a \in T_{b,ab}$.) This is a contradiction because $|Q \times X| < \kappa$. □

3.2 Definition. Let S be a semigroup, let p be an idempotent in βS and let $C \in p$. We put $C^\star = \{s \in C : sp \in \overline{C}\}$.

We note that $C^\star \in p$ and that, for every $s \in C^\star$, $s^{-1}C^\star \in p$ [7, Lemma 4.14].

3.3 Theorem. *Let κ be an infinite cardinal and let S be a very weakly left cancellative semigroup with cardinality κ, let p be a minimal idempotent of βS which is uniform, and let $C \in p$.*

(i) If there is a family of μ almost disjoint subsets of κ, then C contains μ almost disjoint sets each of which is a member of a uniform minimal idempotent in βS.

(ii) C contains κ disjoint sets each of which is a member of a uniform minimal idempotent in βS.

Proof. (i) For each $F \in \mathcal{P}_f(C^\star)$, let

$$S_F = \{t \in C^\star : Ft \subseteq C^\star\} = C^\star \cap \bigcap_{s \in F} s^{-1}C^\star\,.$$

We note that $S_F \in p$.

We claim that, for each $F \in \mathcal{P}_f(C^\star)$ and each $s \in S_F$, if $H = \{s\} \cup Fs$, then $sS_H \subseteq S_F$. To see this, let $t \in S_H$. Since $s \in H$, $st \in C^\star$. Also for every $r \in F$, $rs \in H$ and so $rst \in C^\star$. Thus $st \in S_H$. This shows that $sS_H \subseteq S_F$, as claimed. Let $V = \bigcap_{F \in \mathcal{P}_f(C^\star)} \overline{S}_F$. Then $p \in V$ and by [7, Theorem 4.20], V is a subsemigroup of βS.

Well order $\mathcal{P}_f(C^\star)$ as a κ-sequence and inductively choose $x_F \in S_F$ for every $F \in \mathcal{P}_f(C^\star)$ so that $Fx_F \cap Hx_H = \emptyset$ and $x_F \neq x_H$ if F and H are distinct members of $\mathcal{P}_f(C^\star)$. (Having chosen $\langle x_F \rangle_{F<H}$, one sees as in the proof of Theorem 2.3 that $|\{y \in S : Hy \cap \bigcup_{F<H} Fx_F \neq \emptyset\}| < \kappa$, while $S_H \in p$ and so $|S_H| = \kappa$.) Since $x_F \in S_F$, $Fx_F \subseteq C^\star$ implying $Fx_F \subseteq C$.

By Lemma 2.1(i), there is an almost disjoint family $\langle \mathcal{B}_\sigma \rangle_{\sigma<\mu}$ of subsets of $\mathcal{P}_f(C^\star)$ such that, for every $F \in \mathcal{P}_f(C^\star)$ and every $\sigma < \mu$, there exists $H \in \mathcal{B}_\sigma$ for which $F \subseteq H$. For each $\sigma < \mu$, put $D_\sigma = \bigcup_{F\in\mathcal{B}_\sigma} Fx_F$. Then $\langle D_\sigma \rangle_{\sigma<\mu}$ is almost disjoint and each D_σ is a subset of C. We shall show that, for each $\sigma < \mu$, D_σ is a member of a uniform minimal idempotent of βS so that the family $\langle D_\sigma \rangle_{\sigma<\mu}$ is a family with the required properties.

To this end, let $\sigma < \mu$ be given. Notice that for $F \in \mathcal{P}_f(C^\star)$, $\{H : H \in \mathcal{B}_\sigma \text{ and } F \subseteq H\}$ has cardinality κ, being a a collection of finite sets whose union is κ. Therefore, $\{x_H : H \in \mathcal{B}_\sigma \text{ and } F \subseteq H\}$ has cardinality κ. Since $\big\{\{x_H : H \in \mathcal{B}_\sigma \text{ and } F \subseteq H\} : F \in \mathcal{P}_f(C^\star)\big\}$ has the κ-uniform finite intersection property [7, Theorem 3.62], we may pick a uniform ultrafilter $q \in \beta S$ such that

$$\big\{\{x_H : H \in \mathcal{B}_\sigma \text{ and } F \subseteq H\} : F \in \mathcal{P}_f(C^\star)\big\} \subseteq q\,.$$

Given $F \in \mathcal{P}_f(C^\star)$ and $H \in \mathcal{B}_\sigma$ such that $F \subseteq H$, one has $x_H \in S_H \subseteq S_F$ so $q \in V$.

We claim now that $Vq \subseteq \overline{D_\sigma}$. We show in fact that $\overline{C^\star}q \subseteq \overline{D_\sigma}$. So let $s \in C^\star$. To see that $s^{-1}(D_\sigma) \in q$ it suffices to show that $\{x_H : s \in H \in \mathcal{B}_\sigma\} \subseteq s^{-1}D_\sigma$. So let $s \in H \in \mathcal{B}_\sigma$. Then $sx_H \in Hx_H \subseteq D_\sigma$.

We can choose a minimal idempotent r of V in the left ideal Vq of V. Since V meets $K(\beta S)$, $K(V) \subseteq K(\beta S)$ so r is also minimal in βS. Since q is uniform and since, by Lemma 3.1, the collection of uniform ultrafilters form a left ideal of βS, r is uniform.

(ii) This proof is essentially the same, using Lemma 2.1(ii), instead of Lemma 2.1(i). □

3.4 Corollary. *Let κ be an infinite cardinal and let S be a very weakly cancellative semigroup with cardinality κ. Suppose that κ contains μ almost disjoint sets. Then every central set in S contains μ almost disjoint central sets. Furthermore, every central set in S contains κ disjoint central subsets.*

Proof. Let C be a central set and pick a minimal idempotent p of βS such that $C \in p$. By Lemma 3.1 p is uniform, so Theorem 3.3 applies. □

We observe that any result about almost disjoint central subsets of an arbitrary central set in a left cancellative semigroup yields a corresponding result about piecewise syndetic sets.

3.5 Theorem. *Let S be a left cancellative semigroup, let μ be a cardinal, and assume that every central subset of S contains a family of μ almost disjoint (respectively disjoint)* *central subsets of S. Then every piecewise syndetic subset of S contains a family of μ almost disjoint (respectively disjoint) piecewise syndetic subsets of S.*

Proof. According to [7, Theorem 4.43], a subset C of S is piecewise syndetic if and only if there is some $x \in S$ such that $x^{-1}C$ is central. Let C be a piecewise syndetic subset of S. Pick some $x \in S$ such that $x^{-1}C$ is central and pick an indexed family $\langle D_\iota \rangle_{\iota<\mu}$ of almost disjoint (respectively disjoint) central subsets of $x^{-1}C$. Then for each $\iota < \mu$, $D_\iota \subseteq x^{-1}(xD_\iota)$ so xD_ι is piecewise syndetic. Also $D_\iota \subseteq x^{-1}C$ so $xD_\iota \subseteq C$. And, by left cancellativity, if $\iota < \delta < \mu$, then $|xD_\iota \cap xD_\delta| = |D_\iota \cap D_\delta|$. $\square$

4. Disjoint syndetic sets

The situation with respect to syndetic subsets of a semigroup is significantly different from that with respect to central, thick, and piecewise syndetic sets. We begin by showing that for an infinite semigroup S of cardinality κ, there can not be a family of more than κ almost disjoint syndetic subsets of S unless there is a syndetic set of size less than κ. In the latter case there are such families of size μ whenever there is an almost disjoint family of subsets of S of size μ. To see this notice that if $\mathcal{B}$ is an almost disjoint family of subsets of S and A is a subset of S of size less than κ then the collection of sets $B \cup A$ for $B \in \mathcal{B}$ form an almost disjoint family.

4.1 Theorem. *Let S be an infinite semigroup with $|S| = \kappa$. Either there is a syndetic subset of S of size less than κ or there does not exist a family of κ^+ almost disjoint syndetic subsets of S.*

Proof. Argue by contradiction.

Say that a subset X of S is small if there is some $t \in S$ such that $|tX| < \kappa$. The assumption that there is no syndetic set of size less than κ implies that S is not the union of finitely many small sets (in fact, the two conditions are equivalent).

Let $\mathcal{B}$ be a collection of κ^+ almost disjoint syndetic subsets of S. For each $B \in \mathcal{B}$, pick $F_B \in \mathcal{P}_f(S)$ such that $S = \bigcup_{t \in F_B} t^{-1}B$. Note first that we may choose $F \in \mathcal{P}_f(S)$ such that $|\{B \in \mathcal{B} : F_B = F\}| > \kappa$. (Otherwise $|\mathcal{B}| \leq \sum_{F \in \mathcal{P}_f(S)} |\{B \in \mathcal{B} : F_B = F\}| \leq \kappa \cdot \kappa = \kappa$.)

Pick $\mathcal{D} \subseteq \{B \in \mathcal{B} : F_B = F\}$ such that $|\mathcal{D}| = |F| + 1$. For $B \in \mathcal{D}$ and $s \in S$, pick $t_{s,B} \in F$ such that $(t_{s,B})s \in B$. For each $s \in S$ pick by the pigeon hole principle $B_s \neq C_s$ in $\mathcal{D}$ such that $t_{s,B_s} = t_{s,C_s}$. Pick $B \neq C$ in $\mathcal{D}$, $t \in F$, and $T \subseteq S$ such that T is not small and for all $s \in T$, $(B_s, C_s) = (B, C)$ and $t_{s,B} = t_{s,C} = t$. Let $D = tT$.

Then $D \subseteq B \cap C$ so $|D| < \kappa$. On the other hand, since T is not small, D must have size κ, a contradiction. $\square$

4.2 Corollary. *Let S be an infinite semigroup with $|S| = \kappa$. If S is very weakly left cancellative, there does not exist a family of κ^+ almost disjoint syndetic subsets of S.*

Proof. Let A be a syndetic subset of S and pick $t \in S$ such that $|t^{-1}A| = \kappa$. Then $t^{-1}A = \bigcup_{s \in A} \{y \in S : ty = s\}$ so $|A| = \kappa$. $\square$

Notice that some sort of left cancellation assumption is needed in Corollary 4.2. Indeed, in a left zero semigroup (that is a semigroup in which $ab = a$ for all a and b), every nonempty subset is syndetic. More generally, if there exist $a, b \in S$ such that $aS = \{b\}$, then any set with b as a member is syndetic.

We show now that syndetic subsets of free semigroups on infinite alphabets contain as large as possible collections of pairwise disjoint syndetic sets.

4.3 Theorem. *Let $|A| = \kappa \geq \omega$, let S be the free semigroup on A, and let B be a syndetic subset of S. There is a collection of κ pairwise disjoint syndetic subsets of B.*

Proof. For $t \in S$ let $\alpha(t)$ be the set of letters occurring in t. Pick $F \in \mathcal{P}_f(S)$ such that $S = \bigcup_{t \in F} t^{-1}B$ and let $D = \bigcup_{t \in F} \alpha(t)$. For each $a \in A \setminus D$ let $C_a = \{wav : w, v \in S$ and $\alpha(w) \subseteq D\} \cap B$. If $s \in C_a$ then a is the first letter from $A \setminus D$ occurring in s. Consequently, if a and b are distinct members of $A \setminus D$, then $C_a \cap C_b = \emptyset$.

Let $a \in A \setminus D$. We show that $S = \bigcup_{t \in Fa} t^{-1}C_a$. So let $s \in S$ and pick $t \in F$ such that $as \in t^{-1}B$. Then $tas \in C_a$ so $s \in (ta)^{-1}C_a$. $\square$

Notice that a subset of $\mathbb{N}$ with the operation $a \vee b = \max\{a, b\}$ is syndetic if and only if it is cofinite. Consequently, $(\mathbb{N}, \vee)$ is a weakly left and weakly right cancellative semigroup which does not contain disjoint syndetic subsets. We characterize now those syndetic sets which can be split into disjoint syndetic subsets.

The following lemma strengthens [7, Lemma 3.33].

4.4 Lemma. *Let A be a set and let $g : A \to A$ be a function which has no fixed points. Then A can be partitioned into three disjoint sets A_0, A_1, A_2 with the property that $g[A_0] \subseteq A_1$, $g[A_1] \subseteq A_0 \cup A_2$ and $g[A_2] \subseteq A_0$.*

Proof. Let

$$\begin{aligned}\mathcal{A} = \{f : & f \text{ is a function, range}(f) \subseteq \{0,1,2\}\,, \\ & \text{domain}(f) \subseteq A, \text{ and } \big(\forall x \in \text{domain}(f)\big) \\ & \big(g(x) \in \text{domain}(f)\,, \, (f(x) = 0 \Rightarrow f\big(g(x)\big) = 1)\,, \\ & (f(x) = 1 \Rightarrow f\big(g(x)\big) \in \{0,2\}), \text{ and } (f(x) = 2 \Rightarrow f\big(g(x)\big) = 0)\big)\}\,.\end{aligned}$$

Let g^0 be the identity function. We claim that for all $x \in A$, there is some $f \in \mathcal{A}$ such that $\text{domain}(f) = \{g^n(x) : n \in \omega\}$. If for all $k \neq n$ in ω, $g^k(x) \neq g^n(x)$, then one can define $f \in \mathcal{A}$ with $\text{domain}(f) = \{g^n(x) : n \in \omega\}$ by

$$f\big(g^n(x)\big) = \begin{cases} 0 & \text{if } n \text{ is even} \\ 1 & \text{if } n \text{ is odd.} \end{cases}$$

So assume that we have some $k < m$ such that $g^k(x) = g^m(x)$ and pick the least such m (in which case k is uniquely determined). Note that $m \geq k+2$. Note also that $\{g^n(x) : n \in \omega\} = \big\{g^n(x) : n \in \{0,1,\ldots,m-1\}\big\}$. For $i \in \{0,1,\ldots,m-2\}$,

$$\text{if } k \text{ is even let } f\big(g^i(x)\big) = \begin{cases} 0 & \text{if } i \text{ is even} \\ 1 & \text{if } i \text{ is odd} \end{cases}$$

$$\text{and if } k \text{ is odd let } f\big(g^i(x)\big) = \begin{cases} 0 & \text{if } i \text{ is odd} \\ 1 & \text{if } i \text{ is even}\,. \end{cases}$$

If $f\big(g^{m-2}(x)\big) = 0$, let $f\big(g^{m-1}(x)\big) = 1$. If $f\big(g^{m-2}(x)\big) = 1$, let $f\big(g^{m-1}(x)\big) = 2$. The claim is established.

In particular $\mathcal{A} \neq \emptyset$ and trivially the union of a chain in $\mathcal{A}$ is in $\mathcal{A}$ so pick by Zorn's Lemma a maximal member f of $\mathcal{A}$. We claim that $\text{domain}(f) = A$. Suppose instead we have some $x \in A \setminus \text{domain}(f)$. If $\{g^n(x) : n \in \omega\} \cap \text{domain}(f) = \emptyset$ pick $h \in \mathcal{A}$ such that $\text{domain}(h) = \{g^n(x) : n \in \omega\}$. Then $f \cup h \in \mathcal{A}$, a contradiction. Thus $\{g^n(x) : n \in \omega\} \cap \text{domain}(f) \neq \emptyset$ so pick the least n such that $g^n(x) \in \text{domain}(f)$. If $k, r \in \{0,1,\ldots,n\}$ and $g^k(x) = g^r(x)$, then $k = r$. (If $k < r$, then $\big\{g^m(x) : m \in \{0,1,\ldots,n\}\big\} = \big\{g^m(x) : m \in \{0,1,\ldots,r-1\}\big\}$.)

If $f\big(g^n(x)\big) \in \{0,2\}$ let for $i \in \{1,2,\ldots,n\}$

$$h(g^{n-i}(x)\big) = \begin{cases} 0 & \text{if } i \text{ is even} \\ 1 & \text{if } i \text{ is odd} \end{cases}$$

and if $f\big(g^n(x)\big) = 1$ let for $i \in \{1,2,\ldots,n\}$

$$h(g^{n-i}(x)\big) = \begin{cases} 1 & \text{if } i \text{ is even} \\ 0 & \text{if } i \text{ is odd}\,. \end{cases}$$

Then $f \cup h \in \mathcal{A}$, a contradiction.

Our claim now follows by putting $A_i = f^{-1}[\{i\}]$ for each $i \in \{0, 1, 2\}$. $\square$

According to the referee, some special cases of the following theorem with the same approach are in [1].

4.5 Theorem. *Let S be a semigroup and let $A \subseteq S$. The following statements are equivalent.*

(a) A contains disjoint syndetic subsets.

(b) $\big(\exists F \in \mathcal{P}_f(S)\big)(\forall x \in S)(\exists t \in F)(tx \in A \setminus \{x\})$.

(c) A is syndetic and $\big(\exists F \in \mathcal{P}_f(S)\big)(\forall x \in A)(\exists t \in F)(tx \in A \setminus \{x\})$.

Proof. (a) implies (b). Pick disjoint syndetic subsets B and C of A. Pick $G, H \in \mathcal{P}_f(S)$ such that $S = \bigcup_{t \in G} t^{-1}B = \bigcup_{t \in H} t^{-1}C$. Let $F = G \cup H$. Let $x \in S$. If $x \in B$ then there is $t \in H$ such that $tx \in C$ in which case $tx \in A$ and $tx \neq x$. If $x \notin B$ there is $t \in G$ such that $tx \in B$ in which case $tx \in A$ and $tx \neq x$.

(b) implies (c). This is trivial.

(c) implies (a). Pick F as guaranteed and for each $x \in A$ pick $t_x \in F$ such that $t_x x \in A \setminus \{x\}$ and let $g(x) = t_x x$. Let A_0, A_1, A_2 be the subsets of A guaranteed by Lemma 4.4.

Pick $G \in \mathcal{P}_f(S)$ such that $S = \bigcup_{t \in G} t^{-1}A$ and let $H = G \cup FG \cup FFG$. We claim that $S = \bigcup_{t \in H} t^{-1}A_0 = \bigcup_{t \in H} t^{-1}A_1$. To see this, let $x \in S$. Pick $y \in G$ such that $yx \in A$. If $yx \in A_0$, then $t_{yx}yx = g(yx) \in A_1$. If $yx \in A_1$ and $g(yx) \in A_0$, then $t_{yx}yx \in A_0$. If $yx \in A_1$ and $g(yx) \in A_2$, then $g^2(yx) \in A_0$ so $t_{g(yx)}t_{yx}yx \in A_0$. If $yx \in A_2$, then $g(yx) \in A_0$ and $g^2(yx) \in A_1$ so $t_{yx}yx \in A_0$ and $t_{g(yx)}t_{yx}yx \in A_1$. $\square$

4.6 Corollary. *Let S be a semigroup. The following statements are equivalent.*

(a) S does not contain disjoint syndetic subsets.

(b) $\big(\forall F \in \mathcal{P}_f(S)\big)(\exists x \in S)(\forall t \in F)(tx = x)$.

(c) There exists $p \in \beta S$ such that $\beta S p = \{p\}$.

(d) All minimal left ideals of βS are singletons.

Proof. (a) implies (b). Theorem 4.5.

(b) implies (c). For each $t \in S$, let $X_t = \{x \in S : tx = x\}$. Then $\{X_t : t \in S\}$ has the finite intersection property by (b) so pick $p \in \beta S$ such that $\{X_t : t \in S\} \subseteq p$. Then for each $t \in S$, λ_t is equal to the identity on a member of p so $\lambda_t(p) = p$. Therefore $Sp = \{p\}$ and thus $\beta S p = \{p\}$.

(c) implies (d). By [7, Lemma 1.62] all minimal left ideals of S are isomorphic.

(d) implies (a). Pick a minimal left ideal $L = \{p\}$ of βS. Then for any syndetic subset B of S, $p \in \overline{B}$ by Lemma 1.6. $\square$

If S is an infinite right zero semigroup no proper subset is syndetic. We see that for left cancellative semigroups, that is the only way to avoid proper syndetic subsets.

4.7 Theorem. *Let S be a left cancellative semigroup. The following statements are equivalent.*

(a) S contains no proper syndetic subsets.

(b) All elements of S are idempotents.

(c) S is a right zero semigroup.

Proof. (a) implies (b). Let $a \in S$. Then $S \setminus \{a\}$ is not syndetic so for all $F \in \mathcal{P}_f(S)$ there exists $x \in S$ such that $Fx = \{a\}$. Pick $x \in S$ such that $ax = a$ and pick $y \in S$ such that $\{a, x\}y = \{a\}$. Then $ay = ax$ so $y = x$. Therefore $ax = a$ so $axx = ax$ and thus $xx = x$. Also $xx = xy = a$ so $a = x$ and therefore $aa = a$.

(b) implies (c). Any idempotent in a left cancellative semigroup is a left identity. (If $xx = x$, then for any a, $xxa = xa$ so $xa = a$.) If all elements of S are left identities, then S is a right zero semigroup.

(c) implies (a). If $A \subseteq S$ and $a \in S \setminus A$, then $Sa \cap A = \emptyset$. $\square$

For the remainder of this section we turn our attention to finding conditions guaranteeing that any syndetic set may be split into two disjoint syndetic subsets.

4.8 Lemma. *Let S be an infinite semigroup, let A be a syndetic subset of S, and let $H \subseteq S$. If $|H| < |S|$ and S is very weakly left cancellative, then there exists $F \in \mathcal{P}_f(S \setminus H)$ such that $S \subseteq \bigcup_{t \in F} t^{-1}A$.*

Proof. Pick $F \in \mathcal{P}_f(S)$ such that $S = \bigcup_{t \in F} t^{-1}A$. Since S is very weakly left cancellative, there is some $g \in S$ such that $Fg \cap H = \emptyset$. Then $S = \bigcup_{t \in Fg} t^{-1}A$. $\square$

4.9 Theorem. *Let S be an infinite semigroup which is left cancellative and weakly right cancellative. Then any syndetic subset of S contains disjoint syndetic subsets.*

Proof. Let $T = \{t \in S : \text{for all } s \in S,\ ts \neq s\}$. and let $V = S \setminus T$. Let $E = \{t \in S : t^2 = t\}$. For $s \in S$, let $U_s = \{u \in S : us = s\}$. If $U_s \neq \emptyset$, then since S is weakly right cancellative, U_s is a finite subsemigroup of S. Since $V = \bigcup_{s \in S} U_s$ we have that every element of V has finite order.

Since S is left cancellative, any idempotent in S is a left identity for S so, since S is weakly right cancellative, E is finite.

Let A be a syndetic subset of S and pick by Lemma 4.8 some $F \in \mathcal{P}_f(S \setminus E)$ such that $S = \bigcup_{t \in F} t^{-1}A$. Suppose that A does not contain disjoint syndetic subsets and pick by Theorem 4.5 some $s \in S$ such that for all $u \in F$, $us \notin A \setminus \{s\}$. Pick $u \in F$ such that $us \in A$ so that $us = s$ and thus $u \in V \setminus E$.

Let $X = \{t \in S : ut = t\}$. We claim that $X \subseteq T$. To see this, suppose instead that we have $t \in X \cap V$. Now X is a right ideal of S, so in particular is a subsemigroup. Since $t \in V$, t has finite order so $\{t^n : n \in \mathbb{N}\}$ is a finite subsemigroup of X. Thus there is an idempotent $e \in X$. Then e is a left identity for S so $eu = u$. But $e \in X$ so $ue = e$ and thus $uu = ueu = eu = u$, a contradiction.

Since X is a right ideal of S, we have by [7, Corollary 4.18] that $\overline{X}$ is a right ideal of βS so we may pick an idempotent $q \in \overline{X} \subseteq \overline{T}$. We claim that for all $p \in \beta S$, Sqp has no points that are isolated in βSqp. Suppose instead that we have $p \in \beta S$ and $s \in S$ such that sqp is isolated in βSqp and pick $B \in sqp$ such that $\overline{B} \cap \beta Sqp = \{sqp\}$. Then $s^{-1}B \in qp = qqp$. Let $Q = \{t \in T : t^{-1}(s^{-1}B) \in qp\}$. Then $Q \in q$. Pick $t \in Q$. Then $B \in stqp$ so $stqp = sqp$ and thus by [7, Lemma 8.1] $tqp = qp$. Since $t \in T$, λ_t has no fixed points in S and so by [7, Theorem 3.34] it has no fixed points in βS, a contradiction.

Now let $p \in \beta S$ be given. Then $\overline{A} \cap \beta Sqp \neq \emptyset$ by Lemma 1.6(b) so $\overline{A} \cap Sqp \neq \emptyset$ and we may thus pick a_p and b_p in S such that a_pqp and b_pqp are distinct members of $\overline{A}$. Pick $B_p \in b_pqp \setminus a_pqp$. Then $\{t \in S : t^{-1}\big({b_p}^{-1}(A \cap B_p) \cap {a_p}^{-1}(A \setminus B_p)\big) \in p\} \in q$, so pick $t_p \in S$ such that $P_p = {t_p}^{-1}\big({b_p}^{-1}(A \cap B_p) \cap {a_p}^{-1}(A \setminus B_p)\big) \in p$.

Then $\{\overline{P_p} : p \in \beta S\}$ covers βS so pick finite $D \subseteq \beta S$ such that $\beta S = \bigcup_{p \in D} \overline{P_p}$ and in particular $S = \bigcup_{p \in D} P_p$.

Let $G = \{b_pt_p : p \in D\} \cup \{a_pt_p : p \in D\}$. By Theorem 4.5 pick $s \in S$ such that for all $x \in G$, $xs \notin A \setminus \{s\}$. Pick $p \in D$ such that $s \in P_p$. Then $b_pt_ps \in A$ so $s = b_pt_ps \in B_p$. But also $a_pt_ps \in A$ so $s = a_pt_ps \notin B_p$, a contradiction. $\square$

We see that one cannot replace "weakly right cancellative" by "very weakly right cancellative" in Theorem 4.9.

4.10 Theorem. *Let $\kappa > \omega$. There is a left cancellative semigroup S of cardinality κ which does not contain disjoint syndetic subsets and has the property that any right solution set has cardinality less than κ. In particular, if κ is regular, then S is very weakly right cancellative.*

Proof. The semigroup consisting of the ordinal κ with ordinal addition provides an example. For those unfamiliar with ordinal arithmetic, we describe an isomorphic semigroup for which the necessary properties can be verified directly.

Let S be the set of nonempty words over the alphabet κ with letters in nonincreasing order and for $x \in S$ let $\alpha(x)$ be the set of letters occurring in x. Define an operation $\cdot$ on S as follows. Let $a_1 a_2 \cdots a_n$ and $b_1 b_2 \cdots b_m$ be in S where $n, m \in \mathbb{N}$ and for each $i \in \{1, 2, \ldots, n\}$ $a_i \in \kappa$ and for each $j \in \{1, 2, \ldots, m\}$, $b_j \in \kappa$. If $a_1 < b_1$, then $(a_1 a_2 \cdots a_n) \cdot (b_1 b_2 \cdots b_m) = b_1 b_2 \cdots b_m$. If $a_1 \geq b_1$, let $t = \max\{i \in \{1, 2, \ldots, n\} : a_i \geq b_1\}$ and let $(a_1 a_2 \cdots a_n) \cdot (b_1 b_2 \cdots b_m) = a_1 a_2 \cdots a_t b_1 b_2 \cdots b_m$.

It is routine (though tedious) to verify that this operation is associative. To see that S is left cancellative, let $a_1 a_2 \cdots a_n$, $b_1 b_2 \cdots b_m$, and $c_1 c_2 \cdots c_k$ be members of S (where each a_i, b_i, and c_i is a letter) and assume that $(a_1 a_2 \cdots a_n) \cdot (b_1 b_2 \cdots b_m) = (a_1 a_2 \cdots a_n) \cdot (c_1 c_2 \cdots c_k)$. We may assume without loss of generality that $m \geq k$. If $m = k$ we have that the rightmost m letters agree and so $b_1 b_2 \cdots b_m = c_1 c_2 \cdots c_k$. So suppose $m > k$. Then the length of $(a_1 a_2 \cdots a_n) \cdot (c_1 c_2 \cdots c_k)$ is at least m so $a_1 \geq c_1$. Pick the largest t such that $a_t \geq c_1$ and let $s = t + 1 + k - m$. Then $b_1 b_2 \cdots b_m = a_s a_{s+1} \cdots a_n c_2 c_2 \cdots c_k$ so that $a_s = b_1$ and consequently the length of $(a_1 a_2 \cdots a_n) \cdot (b_1 b_2 \cdots b_m)$ is at least $s + m$ while the length of $(a_1 a_2 \cdots a_n) \cdot (c_1 c_2 \cdots c_k)$ is exactly $t + k$ so that $t + k \geq s + m = t + k + 1$, a contradiction.

To see that all right solution sets have cardinality less than κ, let $a_1 a_2 \cdots a_n$ and $b_1 b_2 \cdots b_m$ be members of S and let $T = \{x \in S : x \cdot a_1 a_2 \cdots a_n = b_1 b_2 \cdots b_m\}$. If $m < n$, then $T = \emptyset$. If $m = n$, then $T = \{x \in S : \alpha(x) \subseteq b_1\}$. (Recall that b_1 is an ordinal, so is the set of its predecessors.) If $m > n$, then $T = \{x b_1 b_2 \cdots b_{m-n} : x \in S \text{ and } \alpha(x) \subseteq b_1\}$. Consequently in any case $|T| \leq \max\{\omega, |b_1|\} < \kappa$.

Finally, suppose one has disjoint syndetic subsets B and C of S and pick finite subsets F and G of S such that $S = \bigcup_{t \in F} t^{-1}B = \bigcup_{t \in G} t^{-1}C$. Pick $a < \kappa$ such that $a > \max\{\alpha(x) : x \in F \cup G\}$. Then for any $t \in F \cup G$, $ta = a$ so $a \in B \cap C$. $\square$

As mentioned in the proof above, the example is isomorphic to the semigroup $(\kappa, +)$ (the sequence $a_1 a_2 \ldots a_n$ corresponds to $\omega^{a_1} + \omega^{a_2} \ldots + \omega^{a_n}$). The ordering of the ordinal κ corresponds to the lexicographic ordering in our example. A slight elaboration of the proof shows that a set is thick iff it is unbounded. Therefore, a set is syndetic iff it has bounded compliment. Hence, the syndetic sets form a filter.

Since every nonempty subset of a left zero semigroup is syndetic, right cancellation is not sufficient to guarantee the ability to split any syndetic subset into disjoint syndetic subsets. But it almost is.

4.11 Theorem. *Let S be a right cancellative semigroup and let A be a syndetic subset of S with at least two points. Then A contains disjoint syndetic subsets.*

Proof. Suppose the conclusion fails. Let $U = \{e \in S : e$ is a right identity for $S\}$. (Then $U = \{e \in S : ee = e\}$, but we shall not need that fact.) Let $\mathcal{F} = \{F \in \mathcal{P}_f(S) : S = \bigcup_{t\in F} t^{-1}A\}$.

We claim that for all $F \in \mathcal{F}$, $F \cap U \neq \emptyset$. So let $F \in \mathcal{F}$. By Theorem 4.5 we may pick $x \in A$ such that for all $t \in F$, $tx \notin A \setminus \{x\}$ and we may pick $t \in F$ such that $tx \in A$ so that $tx = x$. Then for all $y \in S$, $ytx = yx$ and so $yt = y$.

Since A is syndetic, pick $F \in \mathcal{F}$ and note that for all $s \in S$, $Fs \in \mathcal{F}$. Thus, for all $s \in S$ we may pick $x_s \in F$ such that $x_s s \in U$. This implies that every singleton, hence every nonempty set, is syndetic (to see this, fix $y \in S$ and let $G = yF$. For any $s \in S$, $y = yx_s s \in yFs = Gs$). Therefore, A contains disjoint syndetic sets contradicting our assumption. $\square$

4.12 Corollary. *Let S be an infinite semigroup which is right cancellative and has the property that S is not the union of any finite family of left solution sets of S. Then any syndetic subset of S contains disjoint syndetic subsets.*

Proof. By Theorem 4.11 it suffices to show that no singleton in S is syndetic. If $a \in S$ and if $\{a\}$ is syndetic, there is a finite subset F of S such that $S = \bigcup_{t\in F} t^{-1}\{a\}$, contradicting our hypothesis that S is not the union of a finite family of left solution sets. $\square$

We have already noted that our definitions of the various notions of largeness are one-sided. Of course all of the left-right switches of our results are valid. But we do not know the answers to various questions about sets which are simultaneously left-large and right-large. For example, it is shown in [1, Chapter 3] that every infinite group can be partitioned into infinitely many sets that are both left and right syndetic. We do not know whether the corresponding statement is true for any cancellative or weakly cancellative semigroup. Several results about the relations between left-large and right-large sets are in [4] and several other questions of this type are in [9].

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