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Image Partition Regularity of Affine Transformations

Neil Hindman[1]

and

Irene Moshesh

Abstract. If $u, v \in \mathbb{N}$, A is a $u \times v$ matrix with entries from $\mathbb{Q}$, and $\vec{b} \in \mathbb{Q}^u$, then $(A, \vec{b})$ determines an affine transformation from $\mathbb{Q}^v$ to $\mathbb{Q}^u$ by $\vec{x} \mapsto A\vec{x} + \vec{b}$. In 1933 and 1943 Richard Rado determined precisely when such transformations are *kernel partition regular* over $\mathbb{N}$, $\mathbb{Z}$, or $\mathbb{Q}$, meaning that whenever the nonzero elements of the relevant set are partitioned into finitely many cells, there is some element of the kernel of the transformation with all of its entries in the same cell. In 1993 the first author and Imre Leader determined when such transformations with $\vec{b} = \vec{0}$ are *image partition regular* over $\mathbb{N}$, meaning that whenever $\mathbb{N}$ is partitioned into finitely many cells, there is some element of the image of the transformation with all of its entries in the same cell. In this paper we characterize the image partition regularity of such transformations over $\mathbb{N}$, $\mathbb{Z}$, or $\mathbb{Q}$ for arbitrary $\vec{b}$.

1. Introduction

In his famous 1933 paper [8] Richard Rado studied partition regularity of systems of linear equations. That is, given a system of equations

$$\begin{array}{ccccccccc}
a_{1,1}x_1 & + & a_{1,2}x_2 & + & \ldots & + & a_{1,v}x_v & = & b_1 \\
a_{2,1}x_1 & + & a_{2,2}x_2 & + & \ldots & + & a_{2,v}x_v & = & b_2 \\
\vdots & & \vdots & & \vdots & & \vdots & & \vdots \\
a_{u,1}x_1 & + & a_{u,2}x_2 & + & \ldots & + & a_{u,v}x_v & = & b_u
\end{array}$$

and given a finite partition of the set $\mathbb{N}$ of positive integers, could one guarantee a solution set $\{x_1, x_2, \ldots, x_v\}$ contained in one cell of the partition? In alternative coloring terminology, one is asking whether, whenever $\mathbb{N}$ is finitely colored, there must be a *monochromatic* solution set.

For instance, Schur's Theorem [10], published in 1916, stated that whenever $\mathbb{N}$ is finitely colored, there must exist monochromatic x, y, and $x + y$. That is, the single equation $x + y - z = 0$ is partition regular over $\mathbb{N}$.

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In matrix notation, the question being investigated was whether, given a finite coloring of $\mathbb{N}$, one could find $\vec{x}$ with monochromatic entries such that $A\vec{x} = \vec{b}$. (We will follow the usual custom of denoting the entries of a matrix by the lower case letter corresponding to the upper case name of the matrix.) Most attention has been paid to the case where the system of equations is homogeneous, that is where $\vec{b} = \overline{0}$, and we shall address that first. In that case, the mapping $\vec{x} \mapsto A\vec{x}$ is a linear transformation.

1.1 Definition. Let $u, v \in \mathbb{N}$ and let A be a $u \times v$ matrix with entries from $\mathbb{Q}$. Let S be one of $\mathbb{N}$, $\mathbb{Z}$, or $\mathbb{Q}$. Then A is *kernel partition regular over S* (KPR/S) if and only if, whenever $S \setminus \{0\}$ is finitely colored, there must exist monochromatic $\vec{x} \in S^v$ such that $A\vec{x} = \overline{0}$.

Of course, since we are taking $\mathbb{N}$ to be the set of positive integers, coloring $\mathbb{N}$ and coloring $\mathbb{N} \setminus \{0\}$ are the same thing. Notice that the exclusion of 0 from the items being colored is necessary to avoid triviality, since otherwise any matrix would be KPR/$\mathbb{Z}$ by taking $\vec{x} = \overline{0}$.

The characterization which Rado obtained of kernel partition regularity is in terms of the following notion.

1.2 Definition. Let $u, v \in \mathbb{N}$ and let A be a $u \times v$ matrix with entries from $\mathbb{Q}$. Denote the columns of A by $\vec{c_1}, \vec{c_2}, \ldots \vec{c_v}$. Then A satisfies the *columns condition* if and only if there exist $m \in \{1, 2, \ldots, v\}$ and a partition $\{I_1, I_2, \ldots, I_m\}$ of $\{1, 2, \ldots, v\}$ such that

(a) $\sum_{i \in I_1} \vec{c_i} = \overline{0}$ and

(b) for each $t \in \{2, 3, \ldots, m\}$ (if any), $\sum_{i \in I_t} \vec{c_i}$ is a linear combination with coefficients from $\mathbb{Q}$ of $\{\vec{c_i} : i \in \bigcup_{j=1}^{t-1} I_j\}$.

1.3 Theorem (Rado). *Let $u, v \in \mathbb{N}$ and let A be a $u \times v$ matrix with entries from $\mathbb{Q}$. The following statements are equivalent.*

(a) The matrix A is KPR/$\mathbb{N}$.

(b) The matrix A is KPR/$\mathbb{Z}$.

(c) The matrix A is KPR/$\mathbb{Q}$.

(d) The matrix A satisfies the columns condition.

Proof. That (a) implies (b) and (b) implies (c) is trivial. That (c) implies (d) is [9, Theorem VI] and that (d) implies (a) is [8, Satz IV]. $\square$

Call a subset B of $\mathbb{N}$ "large" if whenever A is KPR/$\mathbb{N}$ there must exist $\vec{x}$ with entries from B such that $A\vec{x} = \overline{0}$. Rado conjectured that large sets are partition regular. That is whenever a large set is partitioned into finitely many cells, one of these must be large. Deuber [1] proved this conjecture using what he called (m,p,c)-sets.

1.4 Definition. Let $m,p,c \in \mathbb{N}$ with $p \geq c$. Then B is an (m,p,c)-*set* if and only if there exists $\vec{x} \in \mathbb{N}^m$ such that $B = \{\sum_{i=1}^{m} \lambda_i x_i$: each $\lambda_i \in \{-p,-p+1,\ldots,p-1,p\}$ and if $t = \min\{i : \lambda_i \neq 0\}$, then $\lambda_t = c\}$.

Notice that each (m,p,c)-set is the image of a *first entries matrix*.

1.5 Definition. Let $u,v \in \mathbb{N}$ and let A be a $u \times v$ matrix with entries from $\mathbb{Q}$. Then A is a *first entries matrix* if and only if

(1) no row of A is $\overline{0}$,

(2) the first nonzero entry of each row is positive, and

(3) the first nonzero entries of any two rows are equal if they occur in the same column.

Deuber's proof of Rado's conjecture involved showing that first entries matrices are *image partition regular* over $\mathbb{N}$.

1.6 Definition. Let $u,v \in \mathbb{N}$ and let A be a $u \times v$ matrix with entries from $\mathbb{Q}$.

(a) Let S be one of $\mathbb{N}$, $\mathbb{Z}$, or $\mathbb{Q}$. The matrix A is *image partition regular over* S (IPR/S) if and only if whenever $S \setminus \{0\}$ is finitely colored, there exists $\vec{x} \in S^v$ such that the entries of $A\vec{x}$ are monochromatic.

(b) The matrix A is *weakly image partition regular over* $\mathbb{N}$ (WIPR/$\mathbb{N}$) if and only if whenever $\mathbb{N}$ is finitely colored, there exists $\vec{x} \in \mathbb{Z}^v$ such that the entries of $A\vec{x}$ are monochromatic.

There are some other notions which might be considered as reasonable for image partition regularity over $\mathbb{Z}$ or $\mathbb{Q}$. See [7] for a detailed analysis of these notions. (What we are calling IPR/$\mathbb{Z}$ and IPR/$\mathbb{Q}$ were called WIPR/$\mathbb{Z}$ and WIPR/$\mathbb{Q}$ in [7], but those notions were shown to be equivalent for finite matrices, such as those we are dealing with in this paper.)

Matrices that are IPR/$\mathbb{N}$ and matrices that are WIPR/$\mathbb{N}$ were characterized in [3], and several additional characterizations for matrices that are IPR/$\mathbb{N}$ were found in [5]. Some of the known characterizations of WIPR/$\mathbb{N}$ will be given in Theorem 2.4 along with some new ones.

Some of the characterizations of the following theorem refer to the notion of a *central* subset of a semigroup. We shall define this notion later in this introduction. For now it suffices to note that if a semigroup is partitioned into finitely many classes, at least one of these classes must be central.

1.7 Theorem. *Let $u, v \in \mathbb{N}$ and let A be a $u \times v$ matrix with entries from $\mathbb{Q}$. The following statements are equivalent.*

(a) The matrix A is IPR/$\mathbb{N}$.

(b) Given any central subset C of $\mathbb{N}$, $\{\vec{x} \in \mathbb{N}^v : A\vec{x} \in C^u\}$ is central in $\mathbb{N}^v$.

(c) There exist $m \in \{1, 2, \ldots, v\}$ and a $u \times m$ first entries matrix B with the property that for each $\vec{y} \in \mathbb{N}^m$ there exists $\vec{x}$ in $\mathbb{N}^v$ such that $A\vec{x} = B\vec{y}$.

(d) For any row $\vec{r} \in \mathbb{Q}^v \setminus \{\overline{0}\}$ there exists $b \in \mathbb{Q}$ such that $b > 0$ and $\begin{pmatrix} A \\ b\vec{r} \end{pmatrix}$ is IPR/$\mathbb{N}$.

(e) Given any central subset C of $\mathbb{N}$, there exists $\vec{x} \in \mathbb{N}^v$ such that $A\vec{x} \in C^u$, all entries of $\vec{x}$ are distinct, and entries of $A\vec{x}$ corresponding to distinct rows of A are distinct.

Proof. [3, Theorem 3.1] and [5, Theorem 2.10]. □

We now turn our attention to the case that $\vec{b} \neq \overline{0}$, in which case the mapping $\vec{x} \mapsto A\vec{x} + \vec{b}$ is an affine transformation.

1.8 Definition. Let $u, v \in \mathbb{N}$, let A be a $u \times v$ matrix with entries from $\mathbb{Q}$, and let $\vec{b} \in \mathbb{Q}^u$. Let S be one of $\mathbb{N}$, $\mathbb{Z}$, or $\mathbb{Q}$. Then $(A, \vec{b})$ is *kernel partition regular over S* (KPR/S) if and only if, whenever $S \setminus \{0\}$ is finitely colored, there must exist monochromatic $\vec{x} \in S^v$ such that $A\vec{x} + \vec{b} = \overline{0}$.

Notice that $(A, \overline{0})$ is KPR/S if and only if A is KPR/S. If $\vec{b} \neq \overline{0}$, then the assumption that $S \setminus \{0\}$ is finitely colored can be replaced by the assumption that S is finitely colored. (To see this, assign 0 to its own color. If $\vec{x}$ is monochromatic in this color, that is if $\vec{x} = \overline{0}$, then $A\vec{x} + \vec{b} = \vec{b} \neq \overline{0}$.)

As we remarked earlier, Rado also characterized completely those pairs for which $(A, \vec{b})$ is kernel partition regular over $\mathbb{N}$, $\mathbb{Z}$, or $\mathbb{Q}$.

1.9 Theorem (Rado). *Let $u, v \in \mathbb{N}$, let A be a $u \times v$ matrix with entries from $\mathbb{Q}$, and let $\vec{b} \in \mathbb{Q}^u \setminus \{\overline{0}\}$.*

(a) The pair $(A, \vec{b})$ is KPR/$\mathbb{Z}$ if and only if there exists $k \in \mathbb{Z}$ such that $A\overline{k} + \vec{b} = \overline{0}$.

(b) The pair $(A, \vec{b})$ is KPR/$\mathbb{Q}$ if and only if there exists $k \in \mathbb{Q}$ such that $A\overline{k} + \vec{b} = \overline{0}$.

(c) The pair $(A, \vec{b})$ is KPR/$\mathbb{N}$ if and only if either

(i) there exists $k \in \mathbb{N}$ such that $A\overline{k} + \vec{b} = \overline{0}$ or

(ii) there exists $k \in \mathbb{Z}$ such that $A\overline{k} + \vec{b} = \overline{0}$ and A satisfies the columns condition.

Proof. (a) [8, Satz VIII].

(b) The ideas needed for the proof are in [9]. See [4, Theorem 2.5] for the details.

(c) [8, Satz V]. □

At least in the cases of $\mathbb{Z}$ and $\mathbb{Q}$, one sees why the case $\vec{b} \neq \overline{0}$ has received less attention; the pair $(A, \vec{b}\,)$ is monochromatic if and only if it has a trivial solution. Notice also that the equivalence between KPR/$\mathbb{N}$ and KPR/$\mathbb{Z}$ is lost.

In [4], Imre Leader and the first author of this paper addressed *nonconstant* kernel partition regularity of $(A, \vec{b}\,)$.

1.10 Definition. Let $u, v \in \mathbb{N}$, let A be a $u \times v$ matrix with entries from $\mathbb{Q}$, and let $\vec{b} \in \mathbb{Q}^u$. Let S be one of $\mathbb{N}$, $\mathbb{Z}$, or $\mathbb{Q}$. Then $(A, \vec{b}\,)$ is *nonconstantly kernel partition regular over* S (NCKPR/S) if and only if, whenever S is finitely colored, there must exist monochromatic nonconstant $\vec{x} \in S^v$ such that $A\vec{x} + \vec{b} = \overline{0}$.

Notice that, regardless of whether $\vec{b} = \overline{0}$, this definition is equivalent to one that only requires that $S \setminus \{0\}$ be colored. Indeed, given a finite coloring of $S \setminus \{0\}$, extend it to S by giving 0 its own color. Any nonconstant vector cannot be contained in $\{0\}$.

One motivation for considering nonconstant kernel partition regularity was provided by van der Waerden's Theorem [11] which says that whenever $\mathbb{N}$ is finitely colored, there exist arbitrarily long monochromatic arithmetic progressions. The length five case of this theorem is precisely the assertion that $(A, \overline{0})$ is nonconstantly kernel partition regular, where

$$A = \begin{pmatrix} 1 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 1 & -2 & 1 \end{pmatrix}.$$

Another motivation was the possibility of eliminating the trivialities from Theorem 1.9.

1.11 Theorem. *Let $u, v \in \mathbb{N}$ and let A be a $u \times v$ matrix with entries from $\mathbb{Q}$. The following statements are equivalent.*

(a) The pair $(A, \overline{0})$ is NCKPR/$\mathbb{N}$.

(b) The pair $(A, \overline{0})$ is NCKPR/$\mathbb{Z}$.

(c) The pair $(A, \overline{0})$ is NCKPR/$\mathbb{Q}$.

(d) The matrix A satisfies the columns condition and there exists nonconstant $\vec{x} \in \mathbb{Q}^v$ such that $A\vec{x} = \overline{0}$.

(e) *The matrix A satisfies the columns condition and if the sum of the columns of A is $\overline{0}$, then there exists nonempty $B \subsetneq \{1, 2, \ldots, v\}$ and for each $j \in B$ there exists $\alpha_j \in \mathbb{Q} \setminus \{0\}$ such that $\sum_{j\in B} \alpha_j \vec{c_j} = \overline{0}$, where $\vec{c_j}$ is column j of A.*

Proof. [4, Theorem 3.2]. $\square$

1.12 Theorem. *Let $u, v \in \mathbb{N}$, let A be a $u \times v$ matrix with entries from $\mathbb{Q}$, and let $\vec{b} \in \mathbb{Q}^u \setminus \{\overline{0}\}$. The following statements are equivalent.*

(a) *The pair $(A, \vec{b}\,)$ is nonconstantly kernel partition regular over $\mathbb{Q}$.*

(b) *There exists $k \in \mathbb{Q}$ such that $A\overline{k} + \vec{b} = \overline{0}$, A satisfies the columns condition, and there exists nonconstant $\vec{x} \in \mathbb{Q}^v$ such that $A\vec{x} + \vec{b} = \overline{0}$.*

Proof. [4, Theorem 3.3]. $\square$

1.13 Theorem. *Let $u, v \in \mathbb{N}$, let A be a $u \times v$ matrix with entries from $\mathbb{Q}$, and let $\vec{b} \in \mathbb{Q}^u \setminus \{\overline{0}\}$. The following statements are equivalent.*

(a) *The pair $(A, \vec{b}\,)$ is NCKPR/$\mathbb{N}$.*

(b) *The pair $(A, \vec{b}\,)$ is NCKPR/$\mathbb{Z}$.*

(c) *There exists $k \in \mathbb{Z}$ such that $A\overline{k} + \vec{b} = \overline{0}$, A satisfies the columns condition, and there exists nonconstant $\vec{x} \in \mathbb{Q}^v$ such that $A\vec{x} + \vec{b} = \overline{0}$.*

Proof. [4, Theorem 3.4]. $\square$

In this paper we address image partition regularity and nonconstant image partition regularity of the affine transformation $\vec{x} \mapsto A\vec{x} + \vec{b}$ when $\vec{b} \neq \overline{0}$.

1.14 Definition. Let $u, v \in \mathbb{N}$, let A be a $u \times v$ matrix with entries from $\mathbb{Q}$, and let $\vec{b} \in \mathbb{Q}^u \setminus \{\overline{0}\}$.

(a) The pair $(A, \vec{b}\,)$ is *weakly image partition regular over* $\mathbb{N}$ (WIPR/$\mathbb{N}$) if and only if whenever $\mathbb{N}$ is finitely colored, there exists $\vec{x} \in \mathbb{Z}^v$ such that the entries of $A\vec{x} + \vec{b}$ are monochromatic.

(b) Let S be any of $\mathbb{N}$, $\mathbb{Z}$, or $\mathbb{Q}$. The pair $(A, \vec{b}\,)$ is *image partition regular over S* (IPR/S) if and only if whenever S is finitely colored, there exists $\vec{x} \in S^v$ such that the entries of $A\vec{x} + \vec{b}$ are monochromatic.

When $\vec{b} \neq \overline{0}$ there seems to be no good reason to forbid coloring 0, so in Definition 1.14(b) we allow all of S to be colored. Notice, however, that if one applied Definition 1.14(b) to the pair $(A, \overline{0}\,)$ for $S = \mathbb{Z}$ or $S = \mathbb{Q}$, one would obtain a statement which is not equivalent to the assertion that the matrix A is IPR/S. This difficulty disappears when one is dealing with nonconstant image partition regularity so we allow $\vec{b} = \overline{0}$ in the following definition.

1.15 Definition. Let $u, v \in \mathbb{N}$, let A be a $u \times v$ matrix with entries from $\mathbb{Q}$, and let $\vec{b} \in \mathbb{Q}^u$.

(a) The pair $(A, \vec{b}\,)$ is *nonconstantly weakly image partition regular over* $\mathbb{N}$ (NCWIPR/$\mathbb{N}$) if and only if whenever $\mathbb{N}$ is finitely colored, there exists $\vec{x} \in \mathbb{Z}^v$ such that the entries of $A\vec{x} + \vec{b}$ are monochromatic and nonconstant.

(b) Let S be any of $\mathbb{N}$, $\mathbb{Z}$, or $\mathbb{Q}$. The pair $(A, \vec{b}\,)$ is *nonconstantly image partition regular over* S (NCWIPR/S) if and only if whenever S is finitely colored, there exists $\vec{x} \in S^v$ such that the entries of $A\vec{x} + \vec{b}$ are monochromatic and nonconstant.

Section 2 of this paper consists of preliminary results.

In Section 3 we present characterizations of WIPR/$\mathbb{N}$, IPR/$\mathbb{Z}$, and IPR/$\mathbb{Q}$ for pairs $(A, \vec{b}\,)$ with $\vec{b} \neq \overline{0}$ as well as the nonconstant versions of each of these notions. In Section 4 we characterize IPR/$\mathbb{N}$ and nonconstantly IPR/$\mathbb{N}$ for such pairs. The material in Sections 2 and 3 is taken from the second author's doctoral dissertation.

We conclude this introduction with a brief description of *central* sets. Central sets were introduced by Furstenberg [2] and defined in terms of notions of topological dynamics. These sets enjoy very strong combinatorial properties. (See [2, Proposition 8.21] or [6, Chapter 14].) They have a nice characterization in terms of the algebraic structure of $\beta\mathbb{N}$, the Stone-Čech compactification of $\mathbb{N}$. We shall present this characterization below, after introducing the necessary background information.

Let $(S, +)$ be an infinite discrete semigroup. We take the points of βS to be the ultrafilters on S, the principal ultrafilters being identified with the points of S. Given a set $A \subseteq S$, $\overline{A} = \{p \in \beta S : A \in p\}$. The set $\{\overline{A} : A \subseteq S\}$ is a basis for the open sets (as well as a basis for the closed sets) of βS.

There is a natural extension of the operation $+$ of S to βS, making βS a compact right topological semigroup with S contained in its topological center. This says that for each $p \in \beta S$ the function $\rho_p : \beta S \to \beta S$ is continuous and for each $x \in S$, the function $\lambda_x : \beta S \to \beta S$ is continuous, where $\rho_p(q) = q + p$ and $\lambda_x(q) = x + q$. See [6] for an elementary introduction to the semigroup βS. The reader should be cautioned that even if the semigroup $(S, +)$ is commutative (which we are not assuming), the semigroup $(\beta S, +)$ seldom is. In particular, the center of $(\beta\mathbb{N}, +)$ is $\mathbb{N}$.

Any compact Hausdorff right topological semigroup $(T, +)$ has a smallest two sided ideal $K(T)$ which is the union of all of the minimal left ideals of T, each of which is closed

[6, Theorem 2.8] and any compact right topological semigroup contains idempotents. Since the minimal left ideals are themselves compact right topological semigroups, this says in particular that there are idempotents in the smallest ideal. There is a partial ordering of the idempotents of T determined by $p \leq q$ if and only if $p = p+q = q+p$. An idempotent p is minimal with respect to this order if and only if $p \in K(T)$ [6, Theorem 1.59]. Such an idempotent is called simply "minimal"

1.16 Definition. Let $(S, +)$ be an infinite discrete semigroup. A set $A \subseteq S$ is *central* if and only if there is some minimal idempotent p in βS such that $A \in p$.

Notice that whenever S is divided into finitely many classes, some one of these classes must be central.

Notice also that if S is a cancellative semigroup, then by [6, Theorem 4.36] $\beta S \setminus S$ is an ideal of βS and consequently $K(\beta S) \subseteq \beta S \setminus S$. In particular, no singleton subset of S can be central.

2. Preliminary Results

In this section we present some technical results which will be needed later.

2.1 Definition. Let $u, v \in \mathbb{N}$ and let A be a $u \times v$ matrix with entries from $\mathbb{Q}$. Let $\operatorname{rank}(A) = l < u$. Assume that the first l rows of A are linearly independent and denote the rows of A by $\vec{r_1}, \vec{r_2}, \ldots, \vec{r_u}$. For $i \in \{1, 2, \ldots, u-l\}$ and $j \in \{1, 2, \ldots, l\}$, let $\gamma_{l+i,j} \in \mathbb{Q}$ be determined by, $\overrightarrow{r_{l+i}} = \sum_{j=1}^{l} \gamma_{l+i,j} \cdot \vec{r_j}$. Then $D(A)$ is the $(u-l) \times u$ matrix such that, for $i \in \{1, 2, \ldots, u-l\}$ and $j \in \{1, 2, \ldots, u\}$,

$$d_{i,j} = \begin{cases} \gamma_{l+i,j} & \text{if } j \leq l \\ -1 & \text{if } j = l+i \\ 0 & \text{otherwise.} \end{cases}$$

2.2 Lemma. *Let $u, v \in \mathbb{N}$ and let A be a $u \times v$ matrix with entries from $\mathbb{Q}$ and assume that $l = \operatorname{rank}(A) < u$. Assume also that the first l rows of A are linearly independent and let $D = D(A)$. Then $DA = \mathbf{O}$.*

Proof. For $i \in \{1, 2, \ldots, u-l\}$ and $j \in \{1, 2, \ldots, v\}$, let $\alpha_{i,j}$ be the entry in row i and column j of the matrix DA. Then,

$$\begin{aligned} \alpha_{i,j} &= \textstyle\sum_{r=1}^{u} d_{i,r} \cdot a_{r,j} \\ &= \textstyle\sum_{r=1}^{l} \gamma_{l+i,r} \cdot a_{r,j} - a_{l+i,j} \\ &= a_{l+i,j} - a_{l+i,j} \\ &= 0\,. \end{aligned}$$

□

Lemma 2.3. *Let $u, v \in \mathbb{N}$ and let A be a $u \times v$ matrix with entries from $\mathbb{Q}$. Let $\vec{b} \in \mathbb{Q}^u \setminus \{\overline{0}\}$. If all the rows of A are identical, then $(A, \vec{b}\,)$ is not NCIPR/$\mathbb{Q}$.*

Proof. Assume that the rows of A are identical. If $\vec{b} \in \mathbb{Q}^u \setminus \{\overline{0}\}$ is a constant column vector, then for any $\vec{x} \in \mathbb{Q}^v$, $A\vec{x}+\vec{b} = \overline{k}$ for some $k \in \mathbb{Q}$. Hence $(A, \vec{b}\,)$ is not (NCIPR/$\mathbb{Q}$). Thus we may assume that $\vec{b} \in \mathbb{Q}^u \setminus \{\overline{0}\}$ is a nonconstant column vector. So, pick $i, j \in \{1, 2, \ldots, u\}$ such that $b_j < b_i$. Let φ be a finite coloring of $\mathbb{Q}$ defined by, for $x \in \mathbb{Q}$,

$$\varphi(x) = \begin{cases} 0 & \text{if } \left\lfloor \frac{x}{b_i - b_j} \right\rfloor \text{ is even} \\ 1 & \text{if } \left\lfloor \frac{x}{b_i - b_j} \right\rfloor \text{ is odd.} \end{cases}$$

Suppose $(A, \vec{b}\,)$ is NCIPR/$\mathbb{Q}$. Pick $\vec{x} \in \mathbb{Q}^v$ such that the entries of $A\vec{x} + \vec{b}$ are ψ-monochromatic and nonconstant. Since the rows of A are equal, $A\vec{x} = \overline{k}$ for some $k \in \mathbb{Q}$. Therefore, $\psi(k + b_i) = \psi(k + b_j)$. Therefore, there exist $l, m \in \mathbb{Z}$ such that

$$l \leq \frac{k + b_i}{b_i - b_j} < l + 1 \text{ and,}$$
$$m \leq \frac{k + b_j}{b_i - b_j} < m + 1$$

where l and m are either both even or both odd. Then

$$m(b_i - b_j) \leq k + b_j < (m+1)(b_i - b_j) \text{ and}$$
$$(m+1)(b_i - b_j) \leq k + b_i < m + 2(b_i - b_j), \text{ so}$$
$$m + 1 \leq \frac{k + b_i}{b_i - b_j} < m + 2.$$

Therefore, $l = m + 1$, which is a contradiction. □

Notice that if A has only one row, then trivially A is IPR/$\mathbb{N}$.

Theorem 2.4. *Let $u, v \in \mathbb{N}$ and let A be a $u \times v$ matrix with entries from $\mathbb{Q}$ and assume that A has at least two distinct rows. If $l = \operatorname{rank}(A) < u$, assume that the first l rows of A are linearly independent. The following statements are equivalent.*

(a) The matrix A is WIPR/$\mathbb{N}$.

(b) There exists $m \in \mathbb{N}$ and a $u \times m$ first entries matrix B such that for each $\vec{y} \in \mathbb{Z}^m$ there exists $\vec{x} \in \mathbb{Z}^v$ such that $A\vec{x} = B\vec{y}$.

(c) Either $\operatorname{rank}(A) = u$ or $D(A)$ satisfies the columns condition.

(d) For each $\vec{r} \in \mathbb{Q}^v \setminus \{\overline{0}\}$, there exists $b \in \mathbb{Q} \setminus \{0\}$ such that $\begin{pmatrix} A \\ b\vec{r} \end{pmatrix}$ is WIPR/$\mathbb{N}$.

(e) For every central set C in $\mathbb{N}$, there exists $\vec{x} \in \mathbb{Z}^v$ such that $A\vec{x} \in C^u$.

(f) *Whenever $m \in \mathbb{N}$ and $\phi_1, \phi_2, \ldots, \phi_m$ are nonzero linear mappings from $\mathbb{Q}^v$ to $\mathbb{Q}$, there exists $\vec{b} \in (\mathbb{Q} \setminus \{0\})^m$ such that whenever C is central in $\mathbb{N}$, there exists $\vec{x} \in \mathbb{Z}^v$ for which $A\vec{x} \in C^u$ and for each $i \in \{1, 2, \ldots, m\}$, $b_i\phi_i(\vec{x}) \in C$ and in particular $\phi_i(\vec{x}) \neq 0$.*

(g) *For every central set C in $\mathbb{N}$, there exists $\vec{x} \in \mathbb{Z}^v$ such that $\vec{y} = A\vec{x} \in C^u$, all entries of $\vec{x}$ are distinct and for all $i, j \in \{1, 2, \ldots, u\}$ if rows i and j of A are not equal then $y_i \neq y_j$.*

(h) *The pair $(A, \overline{0}\,)$ is NCWIPR/$\mathbb{N}$.*

(i) *The pair $(A, \overline{0}\,)$ is NCIPR/$\mathbb{Z}$.*

(j) *The pair $(A, \overline{0}\,)$ is NCIPR/$\mathbb{Q}$.*

(k) *The matrix A is IPR/$\mathbb{Q}$.*

(l) *The matrix A is IPR/$\mathbb{Z}$.*

Proof. That (a), (b), (c), and (d) are equivalent is part of [3, Theorem 2.2].

(b) $\Rightarrow$ *(e)*. Let B be a $u \times m$ first entries matrix as guaranteed by (b). Let C be a central set in $\mathbb{N}$. By [5, Lemma 2.8], pick $\vec{y} \in \mathbb{N}^v$ such that $B\vec{y} \in C^u$. By assumption, pick $\vec{x} \in \mathbb{Z}^v$ such that $A\vec{x} = B\vec{y}$.

Since in any finite partition of $\mathbb{N}$ one cell is central, it is trivial that (e) implies (a).

(d) $\Rightarrow$ *(f)*. Let $m \in \mathbb{N}$ and let $\phi_1, \phi_2, \ldots, \phi_m$ be nonzero linear mappings from $\mathbb{Q}^v$ to $\mathbb{Q}$. For each $i \in \{1, 2, \ldots, m\}$, there exists $\vec{r}_i \neq \overline{0}$ such that $\phi_i(\vec{x}) = \vec{r}_i \cdot \vec{x}$ for all $\vec{x} \in \mathbb{Q}^v$. By assumption, pick $b_1 \in \mathbb{Q} \setminus \{0\}$ such that $\begin{pmatrix} A \\ b_1\vec{r}_1 \end{pmatrix}$ is WIPR/$\mathbb{N}$. Repeating this process $m - 1$ times, using at each stage the fact that (a) implies (d), we obtain $b_2, b_3, \ldots, b_m \in \mathbb{Q} \setminus \{0\}$ such that

$$\begin{pmatrix} A \\ b_1\vec{r}_1 \\ \vdots \\ b_m\vec{r}_m \end{pmatrix}$$

is WIPR/$\mathbb{N}$. Let C be a central set in $\mathbb{N}$. Now, using the fact that (a) implies (e), pick $\vec{x} \in \mathbb{Q}^v$ such that

$$\begin{pmatrix} A \\ b_1\vec{r}_1 \\ \vdots \\ b_m\vec{r}_m \end{pmatrix} \vec{x} \in C^{u+m}\,.$$

(f) $\Rightarrow$ *(g)*. For $i \neq j$ in $\{1, 2, \ldots, v\}$, let $\phi_{i,j}$ be the linear mapping from $\mathbb{Q}^v$ to $\mathbb{Q}$ taking $\vec{x}$ to $x_i - x_j$. For $i \neq j$ in $\{1, 2, \ldots, u\}$, if rows i and j of A are unequal, let $\psi_{i,j}$ be the linear mapping from $\mathbb{Q}^v$ to $\mathbb{Q}$ taking $\vec{x}$ to $\sum_{t=1}^{v}(a_{i,t} - a_{j,t}) \cdot x_t$. Applying

statement (f) to the set $\big\{\phi_{i,j} : i \neq j$ in $\{1,2,\ldots,v\}\big\} \cup \big\{\psi_{i,j} : i \neq j$ in $\{1,2,\ldots,u\}$ and rows i and j of A are unequal$\big\}$, we reach the desired conclusion.

$(g) \Rightarrow (h)$. Given a finite coloring of $\mathbb{N}$, one of the color classes is central.

It is trivial that each of (h), (i), (j), and (k) implies the next.

$(k) \Rightarrow (l)$. [7, Theorem 2.4].

$(l) \Rightarrow (a)$. Let $\psi : \mathbb{N} \to \{1,2,\ldots,r\}$. Define $\varphi : \mathbb{Z} \setminus \{0\} \to \{1,2,\ldots,2r\}$ by, for $x \in \mathbb{N}$, $\varphi(x) = \psi(x)$ and $\varphi(-x) = \psi(x) + r$. Since A is IPR/$\mathbb{Z}$, pick $\vec{x} \in \mathbb{Z}^v$ such that the entries of $A\vec{x}$ are φ-monochromatic. If $A\vec{x} \in (\varphi^{-1}[\{i\}])^u$ for $i \in \{r+1, r+2\ldots, 2r\}$, then $A(\overrightarrow{-x}) = -A\vec{x} \in (\psi^{-1}[\{i-r\}])^u$. $\square$

2.5 Lemma. *Let $a, v \in \mathbb{N}$. For every central subset E of $\mathbb{N}^v$,*

$$E \cap \{\vec{x} \in \mathbb{N}^v : \text{ for all } i \in \{1,2,\ldots,v\}\,,\, x_i \geq a\} \neq \emptyset\,.$$

Proof. Let E be central in $\mathbb{N}^v$ and let $W = \{\vec{x} \in \mathbb{N}^v : \text{for all } i \in \{1,2,\ldots,v\},\ x_i \geq a\}$. Then W is an ideal of $\mathbb{N}^v$ so by [6, Corollary 4.18] $\overline{W}$ is an ideal of $\beta(\mathbb{N}^v)$ and thus $K\big(\beta(\mathbb{N}^v)\big) \subseteq \overline{W}$. Pick a minimal idempotent p of $\beta\mathbb{N}^v$ such that $E \in p$. Then $p \in K\big(\beta(\mathbb{N}^v)\big)$ so $W \in p$ so $E \cap W \neq \emptyset$. $\square$

One can in fact show that the set W in the proof of the above lemma is in fact a member of *every* idempotent in $\beta(\mathbb{N}^v)$, not just every minimal idempotent.

3. Weak Image Partition Regularity over $\mathbb{N}$ and Image Partition Regularity over $\mathbb{Z}$ and $\mathbb{Q}$

In this section we obtain characterizations of WIPR/$\mathbb{N}$, NCWIPR/$\mathbb{N}$, IPR/$\mathbb{Z}$, NCIPR/$\mathbb{Z}$, IPR/$\mathbb{Q}$, and NCIPR/$\mathbb{Q}$ for affine transformations with $\vec{b} \neq \overline{0}$.

Notice that both of the equivalent characterizations given by the following theorem are computable.

3.1 Theorem. *Let $u, v \in \mathbb{N}$ and let A be a $u \times v$ matrix with entries from $\mathbb{Q}$. Let $\vec{b} \in \mathbb{Q}^u \setminus \{\overline{0}\}$. Let $l = \mathrm{rank}(A)$. If $l < u$ assume that the first l rows of A are linearly independent and let $D = D(A)$. The following statements are equivalent.*

(a) The pair $(A, \vec{b}\,)$ is IPR/$\mathbb{Q}$.

(b) Either

(i) $l = u$, or

(ii) $l < u$ and there exists $k \in \mathbb{Q}$ such that $D\overline{k} = D\vec{b}$.

(c) There exists $k \in \mathbb{Q}$ and $\vec{x} \in \mathbb{Q}^v$ such that $A\vec{x} + \vec{b} = \overline{k}$.

Proof. $(a) \Rightarrow (b)$. Assume that $l < u$. If $D\vec{b} = \overline{0}$, let $k = 0$. So assume that $D\vec{b} \neq \overline{0}$ We claim that $(D, -D\vec{b})$ is KPR/$\mathbb{Q}$ so that by Theorem 1.9(b), there exists $k \in \mathbb{Q}$ such that $D\overline{k} - D\vec{b} = 0$. To this end, let ψ be a finite coloring of $\mathbb{Q}$ and pick $\vec{x} \in \mathbb{Q}^v$ such that $\vec{z} = A\vec{x} + \vec{b}$ is monochromatic with respect to ψ. Then $DA\vec{x} + D\vec{b} = D\vec{z}$ and by Lemma 2.2 $DA = \mathbf{O}$ so $D\vec{z} - D\vec{b} = \overline{0}$ as required.

$(b) \Rightarrow (c)$. If $u = l$, then the dimension of the column space of A is u so for any $k \in \mathbb{Q}$, $\overline{k} - \vec{b}$ is in the column space of A. So assume that $l < u$ and pick $k \in \mathbb{Q}$ such that $D\overline{k} = D\vec{b}$. We are already assuming that the first l rows are linearly independent. Assume now also that the first l columns of A are linearly independent. Let C be the upper left $l \times l$ corner of A. Then C is invertible. Recall that $\gamma_{l+i,j} \in \mathbb{Q}$ is determined by, $\overrightarrow{r_{l+i}} = \sum_{j=1}^{l} \gamma_{l+i,j} \cdot \vec{r_j}$ for $i \in \{1, 2, \ldots, u-l\}$ and $j \in \{1, 2, \ldots, l\}$.

Let $\vec{b'}$ consist of the first l entries of $\vec{b}$. Let $\vec{y} = C^{-1}\overline{k} - C^{-1}\vec{b'}$. Then $\vec{y} \in \mathbb{Q}^l$ and $C\vec{y} + \vec{b'} = \overline{k}$. Let $\vec{x} \in \mathbb{Q}^v$ such that

$$x_i = \begin{cases} y_i & \text{for } i \leq l \\ 0 & \text{otherwise.} \end{cases}$$

We now show that $A\vec{x} + \vec{b} = \overline{k}$. For $i \in \{1, 2, \ldots, l\}$,

$$\begin{aligned} \sum_{j=1}^{v} a_{i,j} \cdot x_j + b_i &= \sum_{j=1}^{l} a_{i,j} \cdot x_j + b_i \\ &= \sum_{j=1}^{l} c_{i,j} \cdot y_j + b'_i \\ &= k\,. \end{aligned}$$

For $i \in \{l+1, l+2, \ldots, u\}$,

$$\begin{aligned} \sum_{j=1}^{v} a_{i,j} \cdot x_j + b_i &= \sum_{j=1}^{l} a_{i,j} \cdot x_j + b_i \\ &= \sum_{j=1}^{l} x_j \cdot \sum_{r=1}^{l} \gamma_{i,r} \cdot a_{r,j} + b_i \\ &= \sum_{r=1}^{l} \gamma_{i,r} \cdot \sum_{j=1}^{l} a_{r,j} \cdot x_j + b_i \\ &= \sum_{r=1}^{l} \gamma_{i,r}(k - b_r) + b_i \\ &= \sum_{r=1}^{l} \gamma_{i,r} \cdot k - \sum_{r=1}^{l} \gamma_{i,r} \cdot b_r + b_i \\ &= \sum_{r=1}^{l} d_{i-l,r} \cdot k - \sum_{r=1}^{l} d_{i-l,r} \cdot b_r + b_i\,. \end{aligned}$$

Since one has

$$\sum_{r=1}^{u} d_{i-l,r} \cdot k = \sum_{r=1}^{l} d_{i-l,r} \cdot k - k \text{ and } \sum_{r=1}^{u} d_{i-l,r} \cdot b_r = \sum_{r=1}^{l} d_{i-l,r} \cdot b_r - b_i\,,$$

$$\begin{aligned} \sum_{j=1}^{v} a_{i,j} \cdot x_j + b_i &= \sum_{r=1}^{u} d_{i-l,r} \cdot k + k - (\sum_{r=1}^{u} d_{i-l,r} \cdot b_r + b_i) + b_i \\ &= \sum_{r=1}^{u} d_{i-l,r} \cdot k - \sum_{r=1}^{u} d_{i-l,r} \cdot b_r + k \\ &= k \end{aligned}$$

since $D \cdot \overline{k} = D \cdot \vec{b}$.

It is trivial that (c) implies (a). $\square$

For IPR/$\mathbb{Z}$ we obtain a characterization nearly identical to that given by Theorem 3.1(c) for IPR/$\mathbb{Q}$. We thank Dona Strauss for providing the proof of the necessity in the following theorem. This proof significantly shortens our original proof.

3.2 Theorem. *Let $u, v \in \mathbb{N}$ and let A be a $u \times v$ matrix with entries from $\mathbb{Q}$. Let $\vec{b} \in \mathbb{Q}^u \setminus \{\vec{0}\}$. Then the pair $(A, \vec{b}\,)$ is IPR/$\mathbb{Z}$ if and only if there exists $k \in \mathbb{Z}$ and $\vec{x} \in \mathbb{Z}^v$ such that $A\vec{x} + \vec{b} = \overline{k}$.*

Proof. The sufficiency is trivial. For the necessity let $l = \text{rank}(A)$. We proceed by induction on $u - l$. Assume first that $u - l = 0$. Without loss of generality, assume that the first u columns of A are linearly independent and let C consist of the first u columns of A. Then C is invertible. Let d be an element of $\mathbb{N}$ such that the entries of dC^{-1} are integers. Let ψ be a d-coloring of $\mathbb{Z}$ such that for $x \in \mathbb{Z}$, $\psi(x) \equiv x \,(\text{mod}\, d)$. By assumption, pick $\vec{y} \in \mathbb{Z}^v$, $k \in \{1, 2, \ldots, d\}$ and $\vec{w} \in \mathbb{Z}^u$ such that $A\vec{y} + \vec{b} = \overline{k} + d\vec{w}$. Let $\vec{z'} = dC^{-1}\vec{w}$. Then $\vec{z'} \in \mathbb{Z}^u$ and $C\vec{z'} = d\vec{w}$. Let $\vec{z} \in \mathbb{Z}^v$ such that,

$$z_i = \begin{cases} {z'}_i & \text{for } i \leq u \\ 0 & \text{otherwise.} \end{cases}$$

Let $\vec{x} = \vec{y} - \vec{z}$. Then $\vec{x} \in \mathbb{Z}^v$. We now show that $A\vec{x} + \vec{b} = \overline{k}$. For $i \in \{1, 2, \ldots, u\}$,

$$\begin{aligned} \textstyle\sum_{j=1}^{v} a_{i,j} x_j + b_i &= \textstyle\sum_{j=1}^{v} a_{i,j} y_j + b_i - \sum_{j=1}^{u} c_{i,j} z'_j \\ &= k + dw_i - dw_i \\ &= k \end{aligned}$$

Now assume that $u - l > 0$ and the conclusion holds for smaller values of $u - l$. Assume without loss of generality that the first l rows of A are linearly independent. Let $A' = (\, A \quad \overline{1}\,)$, the $u \times (v+1)$ matrix obtained by adding a column of 1's to A. For $i \in \{1, 2, \ldots, u\}$ denote the i^{th} row of A by $\vec{r_i}$ and the i^{th} row of A' by $\vec{r_i}'$. If $\vec{r_u}'$ is linearly independent of $\vec{r_1}', \vec{r_2}', \ldots, \vec{r_l}'$, then since $u - (l+1) < u - l$ we may pick by the induction hypothesis some $k \in \mathbb{Z}$ and some $\vec{x} \in \mathbb{Z}^{v+1}$ such that $A'\vec{x} + \vec{b} = \overline{k}$. If we let $\vec{y}$ consist of the first v entries of $\vec{x}$, we then have that $A\vec{y} + \vec{b} = \overline{k - x_{v+1}}$ as required. So we assume that $\vec{r_u}'$ is a linearly combination of $\vec{r_1}', \vec{r_2}', \ldots, \vec{r_l}'$. Since we already know that $\vec{r_u} = \sum_{j=1}^{l} \gamma_{u,j} \cdot \vec{r_j}$, where $\gamma_{u,j}$ is as given by Definition 2.1 in the definition of $D = D(A)$, we have that $\vec{r_u}' = \sum_{j=1}^{l} \gamma_{u,j} \cdot \vec{r_j}'$. By considering the last entry of each of these vectors, we see that $\sum_{j=1}^{l} \gamma_{u,j} = 1$.

Since the pair $(A,\vec{b}\,)$ is IPR/$\mathbb{Q}$, pick by Theorem 3.1 some $s\in\mathbb{Q}$ such that $D\overline{s}=D\vec{b}$. The last entry of $D\overline{s}$ is $\sum_{j=1}^{l}\gamma_{u,j}\cdot s-s=0$ and the last entry of $D\vec{b}$ is $\sum_{j=1}^{l}\gamma_{u,j}\cdot b_j-b_u$ and thus $\sum_{j=1}^{l}\gamma_{u,j}\cdot b_j=b_u$.

Let B consist of the first $u-1$ rows of A. Since $(u-1)-l<u-l$ we may pick by the induction hypothesis some $k\in\mathbb{Z}$ and some $\vec{x}\in\mathbb{Z}^{v+1}$ such that $B\vec{x}+\vec{b}=\overline{k}$. We claim that $A\vec{x}+\vec{b}=\overline{k}$. We have directly that for $i\in\{1,2,\ldots,u-1\}$, $\sum_{j=1}^{v}a_{i,j}\cdot x_j+b_i=k$. Finally

$$\begin{aligned}
\textstyle\sum_{j=1}^{v}a_{u,j}\cdot x_j+b_u &= \textstyle\sum_{j=1}^{v}x_j\cdot\sum_{i=1}^{l}\gamma_{u,i}\cdot a_{i,j}+b_u\\
&= \textstyle\sum_{i=1}^{l}\gamma_{u,i}\cdot\sum_{j=1}^{v}a_{i,j}\cdot x_j+b_u\\
&= \textstyle\sum_{i=1}^{l}\gamma_{u,i}\cdot(k-b_i)+b_u\\
&= \textstyle\sum_{i=1}^{l}\gamma_{u,i}\cdot k-\sum_{i=1}^{l}\gamma_{u,i}\cdot b_i+b_u\\
&= k-b_u+b_u\,.
\end{aligned}$$

□

The following theorem, which is analogous to Theorem 1.9(c), establishes that again things get more interesting when one is talking about partition regularity over $\mathbb{N}$.

3.3 Theorem. *Let $u,v\in\mathbb{N}$ and let A be a $u\times v$ matrix with entries from $\mathbb{Q}$. The pair $(A,\vec{b}\,)$ is WIPR/$\mathbb{N}$ if and only if either*

(i) there exists $k\in\mathbb{N}$ and $\vec{x}\in\mathbb{Z}^v$ such that $A\vec{x}+\vec{b}=\overline{k}$ or,

(ii) there exists $k\in\mathbb{Z}$ and $\vec{x}\in\mathbb{Z}^v$ such that $A\vec{x}+\vec{b}=\overline{k}$ and the matrix A is WIPR/$\mathbb{N}$.

Proof. Necessity. By Theorem 3.2, pick $k\in\mathbb{Z}$ and $\vec{x}\in\mathbb{Z}^v$ such that $A\vec{x}+\vec{b}=\overline{k}$. If $k\in\mathbb{N}$, we are done. Assume $k\in\mathbb{Z}\setminus\mathbb{N}$. To see that A is WIPR/$\mathbb{N}$ let $r\in\mathbb{N}$ and let ψ be an r-coloring of $\mathbb{N}$. Let φ be an r-coloring of $\mathbb{N}$ such that for $x\in\mathbb{N}$, $\varphi(x)=\psi(x-k)$. Pick $\vec{y}\in\mathbb{Z}^v$ such that $A\vec{y}+\vec{b}=\vec{d}$, where the entries of $\vec{d}$ are monochromatic with respect to φ. Let $\vec{z}=\vec{y}-\vec{x}$. Then $\vec{z}\in\mathbb{Z}^v$. Therefore, $A\vec{z}=\vec{d}-\vec{b}-(\overline{k}-\vec{b}\,)=\vec{d}-\overline{k}$. And $\psi(d_i-k)=\varphi(d_i)$, for $i\in\{1,2,\ldots,u\}$.

Sufficiency. If (i) holds, we are done, so assume that (i) does not hold. Pick $k\in\mathbb{Z}\setminus\mathbb{N}$ and $\vec{x}\in\mathbb{Z}^v$ such that $A\vec{x}+\vec{b}=\overline{k}$. Let $r\in\mathbb{N}$ and let $\psi:\mathbb{N}\to\{1,2,\ldots,r\}$. Let φ be an $(r-k)$-coloring of $\mathbb{N}$ such that for $x\in\mathbb{N}$,

$$\varphi(x)=\begin{cases}\psi(x+k) & \text{if } x>-k\\ r+x & \text{if } x\le -k.\end{cases}$$

Since no singleton is central in $\mathbb{N}$, pick $t\in\{1,2,\ldots,r\}$ such that $\varphi^{-1}[\{t\}]$ is central in $\mathbb{N}$. By Theorem 2.4(e), pick $\vec{y}\in\mathbb{Z}^v$ such that $A\vec{y}=\vec{d}\in(\varphi^{-1}[\{t\}])^u$. So for $i\in\{1,2,\ldots,u\}$, $d_i>-k$ and $\varphi(d_i)=t$. Let $\vec{z}=\vec{x}+\vec{y}$. Then $\vec{z}\in\mathbb{Z}^v$ and $A\vec{z}+\vec{b}=\vec{d}+\overline{k}$. And

$\psi(d_i + k) = \varphi(d_i) = t$, for $i \in \{1, 2, \ldots, u\}$. $\square$

We see now that, as in the case of kernel partition regularity, when we demand nonconstancy the triviality of solutions disappears.

3.4 Theorem. *Let $u, v \in \mathbb{N}$, let A be a $u \times v$ matrix with entries from $\mathbb{Q}$, let $\vec{b} \in \mathbb{Q}^u \setminus \{\overline{0}\}$, and let $l = \mathrm{rank}(A)$. If $l < u$ assume that the first l rows of A are linearly independent and let $D = D(A)$. The following statements are equivalent.*

(a) The pair $(A, \vec{b})$ is NCIPR/$\mathbb{Q}$.

(b) Either

(i) $l = u \geq 2$ or

(ii) $l < u$ and $(D, -D\vec{b})$ is NCKPR/$\mathbb{Q}$.

(c) either

(i) $l = u \geq 2$ or

(ii) $l < u$, D satisfies the columns condition, there exists nonconstant $\vec{z} \in \mathbb{Q}^u$ such that $D\vec{z} = D\vec{b}$, and there exists $k \in \mathbb{Q}$ such that $D\overline{k} = D\vec{b}$.

(d) There exist $k \in \mathbb{Q}$ and $\vec{y} \in \mathbb{Q}^v$ such that $A\vec{y} + \vec{b} = \overline{k}$, A is IPR/$\mathbb{Q}$, and A has at least two distinct rows.

Proof. $(a) \Rightarrow (b)$. By Lemma 2.3, A has at least two distinct rows. Consequently $u \geq 2$. If $l = u$ we are done. Assume that $l < u$. To see that $(D, -D\vec{b})$ is NCKPR/$\mathbb{Q}$ let ψ be a finite coloring of $\mathbb{Q}$. By assumption, pick $\vec{x} \in \mathbb{Q}^v$ such that $A\vec{x} + \vec{b} = \vec{z}$, where the entries of $\vec{z}$ are monochromatic with respect to ψ and nonconstant. Since by Lemma 2.2, $DA = \mathbf{O}$, $D\vec{z} - D\vec{b} = \overline{0}$ as required.

$(b) \Rightarrow (c)$. Assume that (b)(ii) holds. If $D\vec{b} = \overline{0}$ we have by Theorem 1.11 that D satisfies the columns condition and there is a nonconstant $\vec{z} \in \mathbb{Q}^u$ such that $D\vec{z} = \overline{0}$. In this case $D\overline{0} = D\vec{b}$. Assume then that $D\vec{b} \neq \overline{0}$. Then by Theorem 1.12 condition (c)(ii) holds.

$(c) \Rightarrow (d)$. Assume first that $l = u \geq 2$. Then the dimension of the column space of A is u so for any $k \in \mathbb{Q}$ there is some $\vec{y} \in \mathbb{Q}^v$ such that $A\vec{y} + \vec{b} = \overline{k}$ and by Theorem 2.4, A is IPR/$\mathbb{Q}$. Since $l \geq 2$, A has at least two distinct rows.

Now assume that condition (c)(ii) holds. To see that A has at least two distinct rows suppose instead that all rows of A are identical. Then for any $\vec{z} \in \mathbb{Q}^u$, $D\vec{z} = \begin{pmatrix} z_1 - z_2 \\ z_1 - z_3 \\ \vdots \\ z_1 - z_u \end{pmatrix}$.

In particular, for any $k \in \mathbb{Q}$, $D\overline{k} = \overline{0}$ and so $D\vec{b} = \overline{0}$. But given any nonconstant $\vec{z} \in \mathbb{Q}^u$, $D\vec{z} \neq \overline{0}$ and so $D\vec{b} \neq D\vec{z}$, a contradiction.

By Theorem 2.4 A is IPR/$\mathbb{Q}$, and by Theorem 3.1 there exist $k \in \mathbb{Q}$ and $\vec{y} \in \mathbb{Q}^v$ such that $A\vec{y} + \vec{b} = \overline{k}$.

$(d) \Rightarrow (a)$. Since A is IPR/$\mathbb{Q}$ and A has at least two distinct rows, we have by Theorem 2.4 that $(A, \overline{0}\,)$ is NCIPR/$\mathbb{Q}$. Pick $k \in \mathbb{Q}$ and $\vec{y} \in \mathbb{Q}^v$ such that $A\vec{y} + \vec{b} = \overline{k}$. To see that $(A, \vec{b}\,)$ is NCIPR/$\mathbb{Q}$, let ψ be an r-coloring of $\mathbb{Q}$ and define an r-coloring φ of $\mathbb{Q}$ by $\varphi(x) = \psi(x + k)$. Pick $\vec{z} \in \mathbb{Q}^v$ such that the entries of $\vec{w} = A\vec{z}$ are nonconstant and monochromatic with respect to φ. Let $\vec{x} = \vec{z} + \vec{y}$. Then $A\vec{x} + \vec{b} = A\vec{z} + A\vec{y} + \vec{b} = \vec{w} + \overline{k}$. Since the entries of $\vec{w}$ are nonconstant, so are the entries of $\vec{w} + \overline{k}$. And the entries of $\vec{w} + \overline{k}$ are monochromatic with respect to ψ. $\square$

The situation with respect to NCIPR/$\mathbb{Z}$ is very similar to the description of NCKPR/$\mathbb{Z}$ provided by Theorem 1.13.

Theorem 3.5. *Let $u, v \in \mathbb{N}$, let A be a $u \times v$ matrix with entries from $\mathbb{Q}$, and let $\vec{b} \in \mathbb{Q}^u \setminus \{\overline{0}\}$. The following statements are equivalent.*

(a) The pair $(A, \vec{b}\,)$ is NCWIPR/$\mathbb{N}$.

(b) The pair $(A, \vec{b}\,)$ is NCIPR/$\mathbb{Z}$.

(c) There exist $k \in \mathbb{Z}$ and $\vec{y} \in \mathbb{Z}^v$ such that $A\vec{y} + \vec{b} = \overline{k}$, A is IPR/$\mathbb{Z}$, and A has at least two distinct rows.

(d) There exist $k \in \mathbb{Z}$ and $\vec{y} \in \mathbb{Z}^v$ such that $A\vec{y} + \vec{b} = \overline{k}$, A is WIPR/$\mathbb{N}$, and A has at least two distinct rows.

Proof. It is trivial that (a) implies (b).

$(b) \Rightarrow (c)$. By Lemma 2.3 A has at least two distinct rows. Since $(A, \vec{b}\,)$ is NCIPR/$\mathbb{Z}$, $(A, \vec{b}\,)$ is IPR/$\mathbb{Z}$ so by Theorem 3.2 pick $k \in \mathbb{Z}$ and $\vec{y} \in \mathbb{Z}^v$ such that $A\vec{y} + \vec{b} = \overline{k}$. To see that A is IPR/$\mathbb{Z}$, let $r \in \mathbb{N}$ and let ψ be an r-coloring of $\mathbb{Z} \setminus \{0\}$. Let φ be an $(r+1)$-coloring of $\mathbb{Z}$ such that, for $x \in \mathbb{Z}$,

$$\varphi(x) = \begin{cases} \psi(x - k) & \text{if } x \neq k \\ r + 1 & \text{if } x = k\,. \end{cases}$$

Pick $\vec{x} \in \mathbb{Q}^v$ such that $A\vec{x} + \vec{b} = \vec{d}$, where the entries of $\vec{d}$ are φ-monochromatic and nonconstant. Since the entries are nonconstant one has $d_i \neq k$ for $i \in \{1, 2, \ldots, u\}$. Let $\vec{z} = \vec{x} - \vec{y}$. Then $A\vec{z} = \vec{d} - \vec{b} - (\overline{k} - \vec{b}\,) = \vec{d} - \overline{k}$ so the entries of $A\vec{z}$ are ψ-monochromatic.

$(c) \Rightarrow (d)$. By Theorem 2.4 A is WIPR/$\mathbb{N}$.

$(d) \Rightarrow (a)$. Let $r \in \mathbb{N}$ and let ψ be an r-coloring of $\mathbb{N}$. Pick $k \in \mathbb{Z}$ and $\vec{y} \in \mathbb{Z}^v$ such that $A\vec{y} + \vec{b} = \overline{k}$. Define a coloring φ of $\mathbb{N}$ by

$$\varphi(x) = \begin{cases} \psi(x+k) & \text{if } x > -k \\ r + x & \text{if } x \leq -k\,. \end{cases}$$

(If $k \geq 0$, then φ uses r colors. If $k < 0$, then φ uses $r - k$ colors.) Since A is WIPR/$\mathbb{N}$ we have by Theorem 2.4 that $(A, \overline{0}\,)$ is NCWIPR/$\mathbb{N}$. Pick $\vec{x} \in \mathbb{Z}^v$ such that the entries of $\vec{d} = A\vec{x}$ are φ-monochromatic and nonconstant. Then, $d_i > -k$ for each $i \in \{1, 2, \ldots, u\}$. Let $\vec{z} = \vec{x} + \vec{y}$. Then $A\vec{z} + \vec{b} = \vec{d} + \overline{k}$ and the entries of $\vec{d} + \overline{k}$ are ψ-monochromatic and nonconstant. □

4. Image Partition Regularity over $\mathbb{N}$

In this section we characterize those pairs $(A, \vec{b}\,)$ with $\vec{b} \neq \overline{0}$ which are image partition regular over $\mathbb{N}$ and those which are nonconstantly image partition regular over $\mathbb{N}$.

We need some new characterizations of image partition regularity of A. The ideas needed for the proof are contained in the proof of [5, Theorem 2.10].

4.1 Theorem. *Let $u, v \in \mathbb{N}$ and let A be a $u \times v$ matrix with entries from $\mathbb{Q}$. The following statements are equivalent.*

(a) The matrix A is IPR/$\mathbb{N}$.

(b) Given any column $\vec{c} \in \mathbb{Q}^u$, the matrix $(\,A \quad \vec{c}\,)$ is image partition regular.

(c) Whenever $m \in \mathbb{N}$, $\phi_1, \phi_2, \ldots, \phi_m$ are nonzero linear mappings from $\mathbb{Q}^v$ to $\mathbb{Q}$, and C is a central subset of $\mathbb{N}$, there exist positive $b_1, b_2, \ldots, b_m$ in $\mathbb{Q}$ such that $\{\vec{x} \in \mathbb{N}^v : A\vec{x} \in C^u \text{ and for each } i \in \{1, 2, \ldots, m\}\,, b_i\phi_i(\vec{x}) \in C\}$ is central in $\mathbb{N}^v$.

(d) Whenever $m \in \mathbb{N}$ and C is a central subset of $\mathbb{N}$, $\{\vec{x} \in \mathbb{N}^v : A\vec{x} \in C^u$, all entries of $\vec{x}$ are distinct, and entries of $A\vec{x}$ corresponding to distinct rows of A are distinct$\}$ is central in $\mathbb{N}^v$.

Proof. That (b) implies (a) is trivial. To see that (a) implies (b), let $\vec{c} \in \mathbb{Q}^u$ be given and let $A' = (\,A \quad \vec{c}\,)$. Pick by Theorem 1.7(c) some $m \in \{1, 2, \ldots, v\}$ and a $u \times m$ first entries matrix B with the property that for each $\vec{y} \in \mathbb{N}^m$ there exists $\vec{x}$ in $\mathbb{N}^v$ such that $A\vec{x} = B\vec{y}$. Let $B' = (\,B \quad \vec{c}\,)$. Then B' is a first entries matrix. We claim that for each $\vec{z} \in \mathbb{N}^{m+1}$ there exists $\vec{w}$ in $\mathbb{N}^{v+1}$ such that $A'\vec{w} = B'\vec{z}$. So let $\vec{z} \in \mathbb{N}^{m+1}$ be given and let $\vec{y}$ consist of the first m entries of $\vec{z}$. Pick $\vec{x}$ in $\mathbb{N}^v$ such that $A\vec{x} = B\vec{y}$ and let the first v entries of $\vec{w}$ consist of the entries of $\vec{x}$ and let $w_{v+1} = z_{m+1}$.

To see that (a) implies (c), let $m \in \mathbb{N}$ be given and let $\phi_1, \phi_2, \ldots, \phi_m$ be nonzero linear mappings from $\mathbb{Q}^v$ to $\mathbb{Q}$, For each $i \in \{1, 2, \ldots, m\}$, pick $\vec{r_i} \in \mathbb{Q}^v \setminus \{\overline{0}\}$ such that for each $\vec{x} \in \mathbb{Q}^v$, $\phi_i(\vec{x}) = \vec{r_i} \cdot \vec{x}$. Applying Therorem 1.7(d) m times in succession, pick positive $b_1, b_2, \ldots, b_m$ in $\mathbb{Q}$ such that

$$B = \begin{pmatrix} A \\ b_1\vec{r_1} \\ \vdots \\ b_m\vec{r_m} \end{pmatrix}$$

is IPR/$\mathbb{N}$. Now let C be a central subset of $\mathbb{N}$. By Theorem 1.7(b), $\{\vec{x} \in \mathbb{N}^v : B\vec{x} \in C^u\}$ is central in $\mathbb{N}^v$.

Trivially (d) implies (a) so to complete the proof we show that (c) implies (d). For $i \neq j$ in $\{1, 2, \ldots, v\}$, let $\overrightarrow{\phi_{i,j}}$ be the linear mapping from $\mathbb{Q}^v$ to $\mathbb{Q}$ taking $\vec{x}$ to $x_i - x_j$. For $i \neq j$ in $\{1, 2, \ldots, u\}$, if row i of A is not equal to row j of A, let $\overrightarrow{\psi_{i,j}}$ be the linear mapping from $\mathbb{Q}^v$ to $\mathbb{Q}$ taking $\vec{x}$ to $\sum_{t=1}^{v}(a_{i,t} - a_{j,t}) \cdot x_t$. Applying statement (d) to the set $\big\{\phi_{i,j} : i \neq j \text{ in } \{1, 2, \ldots, v\}\big\} \cup \big\{\psi_{i,j} : i \neq j \text{ in } \{1, 2, \ldots, u\}$ and row i of A is not equal to row j of $A\big\}$, we reach the desired conclusion. $\square$

On our way to characterizing pairs $(A, \vec{b})$ which are IPR/$\mathbb{N}$, we need to characterize a stronger condition. Recall that, given any finite partition of $\mathbb{N}$, some cell is central. Also, no singleton in $\mathbb{N}$ is central. (This is true for any cancellative semigroup, but in the case of $\mathbb{N}$ it is true for the trivial reason that $\mathbb{N}$ has no idempotents.)

4.2 Lemma. *Let $u, v \in \mathbb{N}$, let A be a $u \times v$ matrix with entries from $\mathbb{Q}$, and let $\vec{b} \in \mathbb{Q}^u \setminus \{\overline{0}\}$. The following statements are equivalent.*

(a) For all $a \in \mathbb{N}$ and every finite coloring φ of $\mathbb{N}$ there exists $\vec{x} \in \mathbb{N}^v$ such that

- *(i) for each $i \in \{1, 2, \ldots, v\}$, $x_i \geq a$,*
- *(ii) the entries of $A\vec{x} + \vec{b}$ are nonconstant and monochromatic with respect to φ, and*
- *(iii) entries of $A\vec{x} + \vec{b}$ corresponding to distinct rows of A are distinct.*

(b) For all $a \in \mathbb{N}$ and every finite coloring φ of $\mathbb{N}$ there exists $\vec{x} \in \mathbb{N}^v$ such that

- *(i) for each $i \in \{1, 2, \ldots, v\}$, $x_i \geq a$ and*
- *(ii) the entries of $A\vec{x} + \vec{b}$ are nonconstant and monochromatic with respect to φ.*

(c) There exist $k \in \mathbb{Z}$ and $\vec{y} \in \mathbb{Z}^v$ such that $A\vec{y} + \vec{b} = \overline{k}$, the matrix A is IPR/$\mathbb{N}$, and A has at least two distinct rows.

Proof. That (a) implies (b) is trivial. To see that (b) implies (c), note that $(A, \vec{b})$ is IPR/$\mathbb{Z}$. Pick by Theorem 3.2 some $k \in \mathbb{Z}$ and $\vec{y} \in \mathbb{Z}^v$ such that $A\vec{y} + \vec{b} = \overline{k}$. By Lemma 2.3 A has at least two distinct rows.

To see that A is IPR/$\mathbb{N}$, let $r \in \mathbb{N}$ and let $\varphi : \mathbb{N} \to \{1, 2, \ldots, r\}$. Let $s = \max\{r, r + k\}$ and define $\psi : \mathbb{N} \to \{1, 2, \ldots, s\}$ by, for $x \in \mathbb{N}$,

$$\psi(x) = \begin{cases} \varphi(x - k) & \text{if } x > k \\ r + x & \text{if } x \leq k\,. \end{cases}$$

Let $a = \max\big(\{1\} \cup \{y_i : i \in \{1, 2, \ldots, v\}\}\big)$ and pick $\vec{x} \in \mathbb{N}^v$ such that for each $i \in \{1, 2, \ldots, v\}$, $x_i \geq a$ and the entries of $A\vec{x} + \vec{b}$ are nonconstant and monochromatic with respect to ψ. Pick $t \in \{1, 2, \ldots, s\}$ such that for each entry z of $A\vec{x} + \vec{b}$, $\psi(z) = t$. Since the entries of $A\vec{x} + \vec{b}$ are nonconstant, $t \leq r$. Then the entries of $A\vec{x} + \vec{b} - \overline{k}$ are nonconstant and monochromatic with respect to φ and $A\vec{x} + \vec{b} - \overline{k} = A(\vec{x} - \vec{y})$. Since each entry of $\vec{x}$ is greater than a, we have that $(\vec{x} - \vec{y}) \in \mathbb{N}^v$.

To see that (c) implies (a), pick $k \in \mathbb{Z}$ and $\vec{y} \in \mathbb{Z}^v$ such that $A\vec{y} + \vec{b} = \overline{k}$. Let $a, r \in \mathbb{N}$ and let $\varphi : \mathbb{N} \to \{1, 2, \ldots, r\}$. Let $s = \max\{r, r - k\}$ and define $\psi : \mathbb{N} \to \{1, 2, \ldots, s\}$ by, for $x \in \mathbb{N}$,

$$\psi(x) = \begin{cases} \varphi(x + k) & \text{if } x > -k \\ r + x & \text{if } x \leq -k\,. \end{cases}$$

Pick $t \in \{1, 2, \ldots, s\}$ such that $\psi^{-1}[\{t\}]$ is central in $\mathbb{N}$ and note that, since no singleton is central in $\mathbb{N}$, $t \leq r$. Let

$$B = \{\vec{x} \in \mathbb{N}^v : A\vec{x} \in (\psi^{-1}[\{t\}])^u \text{ and entries of } A\vec{x} \text{ corresponding}$$
$$\text{to distinct rows of } A \text{ are distinct}\}\,.$$

By Theorem 4.1 B is central in $\mathbb{N}^v$. Let $b = \max\big(\{1\} \cup \{-y_i : i \in \{1, 2, \ldots, v\}\}\big)$. Pick by Lemma 2.5 some $\vec{x} \in B$ such that for all $i \in \{1, 2, \ldots, v\}$, $x_i > a + b$. Since the entries of $A\vec{x}$ are monochromatic with respect to ψ, the entries of $A\vec{x} + \overline{k}$ are monochromatic with respect to φ. And since A has at least two distinct rows, the entries of $A\vec{x}$, and consequently of $A\vec{x} + \overline{k}$, are nonconstant. Now $A\vec{x} + \overline{k} = A(\vec{x} + \vec{y}) + \vec{b}$. Since each entry of $\vec{x}$ is bigger than $a + b$, each entry of $\vec{x} + \vec{y}$ is bigger than a. □

4.3 Lemma. *Let $u, v \in \mathbb{N}$, let A be a $u \times v$ matrix with entries from $\mathbb{Q}$, and let $\vec{b} \in \mathbb{Q}^u \setminus \{\vec{0}\}$. If $(A, \vec{b})$ is NCIPR/$\mathbb{N}$, then for all $a \in \mathbb{N}$ and every finite coloring φ of $\mathbb{N}$ there exists $\vec{x} \in \mathbb{N}^v$ such that for each $i \in \{1, 2, \ldots, v\}$, $x_i \geq a$ and the entries of $A\vec{x} + \vec{b}$ are nonconstant and monochromatic with respect to φ.*

Proof. We proceed by induction on v, so first assume that $v = 1$. Let $a, r \in \mathbb{N}$ and let $\varphi : \mathbb{N} \to \{1, 2, \ldots, r\}$. Let $d = \max\big(\{1\} \cup \{a_{1,1}x + b_1 : x \in \{1, 2, \ldots, a\}\}\big)$ and define a finite coloring ψ of $\mathbb{N}$ by, for $x \in \mathbb{N}$,

$$\psi(x) = \begin{cases} \varphi(x) & \text{if } x > d \\ r + x & \text{if } x \leq d\,. \end{cases}$$

Pick $x \in \mathbb{N}$ such that the entries of $Ax + \vec{b}$ are nonconstant and monochromatic with respect to ψ. Pick $t \in \{1, 2, \ldots, r+d\}$ such that $Ax + \vec{b} \in (\psi^{-1}[\{t\}])^u$. Since the entries of $Ax + \vec{b}$ are nonconstant, $t \leq r$. Now $\psi(a_{1,1}x + b_1) = t \leq r$ so $a_{1,1}x + b_1 > d$ so $x > a$.

Now let $v \in \mathbb{N}$ and assume that the lemma is true for every $u \times v$ matrix and every $\vec{b} \in \mathbb{Q}^u$. Let A be a $u \times (v+1)$ matrix with entries in $\mathbb{Q}$ and let $\vec{b} \in \mathbb{Q}^u \setminus \{\overline{0}\}$. Assume that $(A, \vec{b})$ is NCIPR/$\mathbb{N}$. Suppose the conclusion fails and pick $a \in \mathbb{N}$ and a finite coloring φ of $\mathbb{N}$ such that whenever $\vec{x} \in \mathbb{N}^{v+1}$ and the entries of $A\vec{x} + \vec{b}$ are nonconstant and monochromatic with respect to φ, there is some $t \in \{1, 2, \ldots, v+1\}$ such that $x_t < a$. We claim that

(∗) there exist $t \in \{1, 2, \ldots, v+1\}$ and $d \in \{1, 2, \ldots, a-1\}$ such that for every finite coloring φ of $\mathbb{N}$ there exists $\vec{x} \in \mathbb{N}^{v+1}$ such that $A\vec{x} + \vec{b}$ has nonconstant entries that are monochromatic with respect to φ and $x_t = d$.

Suppose instead that (∗) fails and for each $t \in \{1, 2, \ldots, v+1\}$ and $d \in \{1, 2, \ldots, a-1\}$ pick a finite coloring $\psi_{t,d}$ of $\mathbb{N}$ such that whenever $\vec{x} \in \mathbb{N}^{v+1}$ and the entries of $A\vec{x} + \vec{b}$ are nonconstant and monochromatic with respect to $\psi_{t,d}$, one has that $x_t \neq d$.

Let μ be a finite coloring of $\mathbb{N}$ with the property that whenever $x, y \in \mathbb{N}$ and $\mu(x) = \mu(y)$, one has $\varphi(x) = \varphi(y)$ and for each $t \in \{1, 2, \ldots, v+1\}$ and $d \in \{1, 2, \ldots, a-1\}$, $\psi_{t,d}(x) = \psi_{t,d}(y)$. Pick $\vec{x} \in \mathbb{N}^{v+1}$ such that the entries of $A\vec{x} + \vec{b}$ are nonconstant and monochromatic with respect to μ. Then the entries of $A\vec{x} + \vec{b}$ are nonconstant and monochromatic with respect to φ so pick $t \in \{1, 2, \ldots, v+1\}$ such that $x_t < a$ and let $d = x_t$. We then get a contradiction because the entries of $A\vec{x} + \vec{b}$ are nonconstant and monochromatic with respect to $\psi_{t,d}$.

Pick t and d as guaranteed by (∗) and let $\vec{c_t}$ be column t of A. Let A' be the $u \times v$ matrix obtained by deleting column t from A and let $\vec{b'} = \vec{b} + d\vec{c_t}$. We claim that $(A', \vec{b'})$ is nonconstantly image partition regular over $\mathbb{N}$. To this end let δ be a finite coloring of $\mathbb{N}$ and pick $\vec{x} \in \mathbb{N}^{v+1}$ such that the entries of $A\vec{x} + \vec{b}$ are nonconstant and monochromatic with respect to δ. Define $\vec{z} \in \mathbb{N}^v$ by, for $i \in \{1, 2, \ldots, v\}$,

$$z_i = \begin{cases} x_i & \text{if } i < t \\ x_{i+1} & \text{if } i \geq t\,. \end{cases}$$

Then $A'\vec{z} + b' = A'\vec{z} + d\vec{c_t} + \vec{b} = A\vec{x} + \vec{b}$ so the entries of $A'\vec{z} + b'$ are nonconstant and monochromatic with respect to δ.

We claim now that A' is IPR/$\mathbb{N}$. If $\vec{b'} = \overline{0}$, we have this directly so assume that $\vec{b'} \neq \overline{0}$. Then by the induction hypothesis we have that for all $a \in \mathbb{N}$ and every finite coloring δ of $\mathbb{N}$ there exists $\vec{x} \in \mathbb{N}^v$ such that for each $i \in \{1, 2, \ldots, v\}$, $x_i \geq a$ and the entries of $A'\vec{x} + \vec{b'}$ are nonconstant and monochromatic with respect to δ. Thus by

Lemma 4.2 the matrix A' is IPR/$\mathbb{N}$. Consequently by Theorem 4.1(b), A is IPR/$\mathbb{N}$.

By Lemma 2.3 A has at least two distinct rows and by Theorem 3.2 there exist $k \in \mathbb{Z}$ and $\vec{y} \in \mathbb{Z}^v$ such that $A\vec{y}+\vec{b} = \overline{k}$. Applying Lemma 4.2, we have that for all $a \in \mathbb{N}$ and every finite coloring δ of $\mathbb{N}$ there exists $\vec{x} \in \mathbb{N}^v$ such that for each $i \in \{1, 2, \ldots, v\}$, $x_i \geq a$ and the entries of $A\vec{x} + \vec{b}$ are nonconstant and monochromatic with respect to δ as required. $\square$

4.4 Theorem. *Let $u, v \in \mathbb{N}$, let A be a $u \times v$ matrix with entries from $\mathbb{Q}$, and let $\vec{b} \in \mathbb{Q}^u \setminus \{\overline{0}\}$. The following statements are equivalent.*

(a) The pair $(A, \vec{b}\,)$ is NCIPR/$\mathbb{N}$.

(b) There exist $k \in \mathbb{Z}$ and $\vec{y} \in \mathbb{Z}^v$ such that $A\vec{y} + \vec{b} = \overline{k}$, the matrix A is IPR/$\mathbb{N}$, and A has at least two distinct rows.

Proof. Lemmas 4.2 and 4.3. $\square$

We also obtain immediately a characterization of IPR/$\mathbb{N}$.

4.5 Theorem. *Let $u, v \in \mathbb{N}$, let A be a $u \times v$ matrix with entries from $\mathbb{Q}$, and let $\vec{b} \in \mathbb{Q}^u \setminus \{\overline{0}\}$. The following statements are equivalent.*

(a) The pair $(A, \vec{b}\,)$ is IPR/$\mathbb{N}$.

(b) Either

(i) there exist $k \in \mathbb{N}$ and $\vec{y} \in \mathbb{N}^v$ such that $A\vec{y} + \vec{b} = \overline{k}$ or

(ii) there exist $k \in \mathbb{Z}$ and $\vec{y} \in \mathbb{Z}^v$ such that $A\vec{y} + \vec{b} = \overline{k}$, the matrix A is IPR/$\mathbb{N}$, and A has at least two distinct rows.

Proof. Trivially (b)(i) implies (a) and by Theorem 4.4, (b)(ii) implies (a).

To see that (a) implies (b) assume that $(A, \vec{b}\,)$ is IPR/$\mathbb{N}$ and suppose that (b)(ii) fails. Then by Theorem 4.4, $(A, \vec{b}\,)$ is not nonconstantly image partition regular over $\mathbb{N}$ so pick a finite coloring φ of $\mathbb{N}$ such that there is no $\vec{x} \in \mathbb{N}^v$ with the entries of $A\vec{x} + \vec{b}$ nonconstant and monochromatic with respect to φ. Since $(A, \vec{b}\,)$ is IPR/$\mathbb{N}$, there is some $\vec{y} \in \mathbb{N}^v$ with the entries of $A\vec{y} + \vec{b}$ monochromatic with respect to φ, so these entries must be constant. $\square$

We note that we have established the exact pattern of implications that hold among the various notions of image partition regularity of nonlinear affine transformations over $\mathbb{N}$, $\mathbb{Z}$, or $\mathbb{Q}$.

Theorem 4.6. *Let $u, v \in \mathbb{N}$, let A be a $u \times v$ matrix with entries from $\mathbb{Q}$, and let $\vec{b} \in \mathbb{Q}^u \setminus \{\vec{0}\}$. All of the implications in the following diagram hold and the only implications that hold among these notions are those shown or ones that follow by transitivity.*

$$\begin{array}{ccccccc}
\mathrm{NCIPR}/\mathbb{N} & \Rightarrow & \mathrm{NCWIPR}/\mathbb{N} & \Leftrightarrow & \mathrm{NCIPR}/\mathbb{Z} & \Rightarrow & \mathrm{NCIPR}/\mathbb{Q} \\
\Downarrow & & \Downarrow & & \Downarrow & & \Downarrow \\
\mathrm{IPR}/\mathbb{N} & \Rightarrow & \mathrm{WIPR}/\mathbb{N} & \Rightarrow & \mathrm{IPR}/\mathbb{Z} & \Rightarrow & \mathrm{IPR}/\mathbb{Q}
\end{array}$$

Proof. All of the diagramed implications are trivial except for the fact that NCIPR/$\mathbb{Z}$ $\Rightarrow$ NCWIPR/$\mathbb{N}$ which is part of Theorem 3.5. To establish that none of the other implications are valid, it suffices to show that

(a) IPR/$\mathbb{N}$ $\not\Rightarrow$ NCIPR/$\mathbb{Q}$,
(b) WIPR/$\mathbb{N}$ $\not\Rightarrow$ IPR/$\mathbb{N}$,
(c) IPR/$\mathbb{Z}$ $\not\Rightarrow$ WIPR/$\mathbb{N}$,
(d) IPR/$\mathbb{Q}$ $\not\Rightarrow$ IPR/$\mathbb{Z}$,
(e) NCIPR/$\mathbb{Z}$ $\not\Rightarrow$ IPR/$\mathbb{N}$, and
(f) NCIPR/$\mathbb{Q}$ $\not\Rightarrow$ IPR/$\mathbb{Z}$.

(a) Let $A = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$ and $\vec{b} = \begin{pmatrix} 4 \\ 6 \end{pmatrix}$. If $\vec{x} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, then $A\vec{x} + \vec{b} = \begin{pmatrix} 5 \\ 5 \end{pmatrix}$. Therefore, the pair $(A, \vec{b}\,)$ is IPR/$\mathbb{N}$. Since $D(A) = (\,-1 \quad -1\,)$, which does not satisfy the columns condition, we have that A is not IPR/$\mathbb{Q}$ so by Theorem 3.4, $(A, \vec{b}\,)$ is not NCIPR/$\mathbb{Q}$.

(b) The pair $\big((-2), (1)\big)$ is WIPR/$\mathbb{N}$ but not IPR/$\mathbb{N}$.

(c) Let $A = \begin{pmatrix} 2 & -1 \\ -2 & 1 \\ -1 & \frac{1}{2} \end{pmatrix}$ and $\vec{b} = \begin{pmatrix} -5 \\ 3 \\ 1 \end{pmatrix}$. Then $(A, \vec{b}\,)$ is IPR/$\mathbb{Z}$ but not WIPR/$\mathbb{N}$. To see that it is IPR/$\mathbb{Z}$ note that if $\vec{x} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$, then $A\vec{x} + \vec{b} == \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix}$. This is the only constant image and $D(A) = (\,-\frac{1}{2} \quad 0 \quad -1\,)$, which does not satisfy the columns condition so by Theorem 3.3 $(A, \vec{b}\,)$ is not WIPR/$\mathbb{N}$.

(d) The pair $\big((1), (\frac{1}{2})\big)$ is IPR/$\mathbb{Q}$ but not IPR/$\mathbb{Z}$.

(e) Let $A = \begin{pmatrix} -2 & -1 \\ -1 & 0 \\ -4 & -3 \end{pmatrix}$ and $\vec{b} = \begin{pmatrix} -6 \\ -9 \\ 0 \end{pmatrix}$. If $\vec{x} \in \mathbb{N}^2$, then $-x_1 - 9 \notin \mathbb{N}$. Therefore, $(A, \vec{b}\,)$ is not IPR/$\mathbb{N}$. If $\vec{x} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, then $A\vec{x} + \vec{b} = \begin{pmatrix} -10 \\ -10 \\ -10 \end{pmatrix}$. Also, $D(A) = (\,3 \quad -2 \quad -1\,)$.

Since $D(A)$ satisfies the columns conditions A is IPR/$\mathbb{Z}$. Since A has 3 distinct rows, by Theorem 3.5, $(A, \vec{b}\,)$ is NCIPR/$\mathbb{Z}$.

(f) Let $A = \begin{pmatrix} -2 & -1 \\ -1 & 0 \\ -4 & -3 \end{pmatrix}$ and $\vec{b} = \begin{pmatrix} \frac{13}{3} \\ \frac{4}{3} \\ \frac{31}{3} \end{pmatrix}$. If $\vec{x} \in \mathbb{Z}^2$, then $-x_1 + \frac{4}{3} \notin \mathbb{Z}$. Therefore, $(A, \vec{b}\,)$ is not IPR/$\mathbb{Z}$. If $\vec{x} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, then $A\vec{x} + \vec{b} = \begin{pmatrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{pmatrix}$. Also, $D(A) = \begin{pmatrix} 3 & -2 & -1 \end{pmatrix}$. Since $D(A)$ satisfies the columns conditions A is IPR/$\mathbb{Q}$. Since A has 3 distinct rows, by Theorem 3.4 $(A, \vec{b}\,)$ is NCIPR/$\mathbb{Q}$. $\square$

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* (Currently available at `http://members.aol.com/nhindman/`.)

Neil Hindman
Department of Mathematics
Howard University
Washington, DC 20059
`nhindman@howard.edu`

Irene Moshesh
Department of Mathematics
Howard University
Washington, DC 20059
`diatomic@mindspring.com`