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All-Sums Sets in (0,1] – Category and Measure

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Abstract. We provide a unified and simplified proof that for any partition of $(0,1]$ into sets that are measurable or have the property of Baire, one cell will contain an infinite sequence together with all of its sums (finite or infinite) without repetition. In fact any set which is *large around 0* in the sense of measure or category will contain such a sequence. We show that sets with 0 as a density point have very rich structure. Call a sequence $\langle t_n\rangle_{n=1}^{\infty}$ and its resulting all-sums set *structured* provided for each n, $t_n \geq \sum_{k=n+1}^{\infty} t_k$. We show further that structured all-sums sets with positive measure are not partition regular even if one allows shifted all-sums sets. That is, we produce a two cell measureable partition of $(0,1]$ such that neither set contains a translate of any structured all-sums set with positive measure.

1. Introduction.

In [1] it was shown that whenever the set $\mathbb{N}$ of positive integers is partitioned into finitely many classes, some one of these contains an infinite sequence together with all of its finite sums (without repetition). That is, if one defines as usual the set of *finite sums* of a sequence by $FS(\langle t_n\rangle_{n=1}^{\infty}) = \{\sum_{n\in F}\ t_n : \emptyset \neq F \subseteq \mathbb{N}$ and F is finite$\}$, then whenever $r \in \mathbb{N}$ and $\mathbb{N} = \bigcup_{i=1}^{r} A_i$, there exist $i \in \{1,2,\ldots,r\}$ and a sequence $\langle t_n\rangle_{n=1}^{\infty}$ in $\mathbb{N}$ with $FS(\langle t_n\rangle_{n=1}^{\infty}) \subseteq A_i$.

Given a sequence $\langle t_n\rangle_{n=1}^{\infty}$ in $(0,1]$, such that $\sum_{n=1}^{\infty} t_n$ converges, define the set of *all sums* of the sequence by $AS(\langle t_n\rangle_{n=1}^{\infty}) = \{\sum_{n\in F}\ t_n : \emptyset \neq F \subseteq \mathbb{N}\}$. In [6], Prömel and Voigt considered the question: If $(0,1] = \bigcup_{i=1}^{r} A_i$, must there exist $i \in \{1,2,\ldots,r\}$ and a sequence $\langle t_n\rangle_{n=1}^{\infty}$ in $(0,1]$ with $AS(\langle t_n\rangle_{n=1}^{\infty}) \subseteq A_i$?

As they pointed out, one easily sees (using the Axiom of Choice) that the answer is "no" by a standard diagonalization argument. (There are continuum many sequences in $(0,1]$ with convergent sums. Well order them. At stage $\sigma < \mathfrak{c} = 2^{\aleph_0}$ of the construction

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chose previously unassigned elements of the all-sums set of the $\sigma^{\rm th}$ sequence – which has $\mathfrak{c}$ members – putting one in each of the two cells being constructed.) At the end of this introduction we will however observe that one can always get i and $\langle t_n\rangle_{n=1}^\infty$ with $FS(\langle t_n\rangle_{n=1}^\infty)\subseteq A_i$ and $\sum_{n=1}^\infty t_n\in A_i$.

Prömel and Voigt showed that if one adds the requirement that each A_i has the property of Baire, then the answer becomes "yes". (The sets with the property of Baire are the members of the smallest σ-algebra containing the open sets and the nowhere dense sets.) In [5] Plewik and Voigt reached the same conclusion in the event that each A_i is assumed to be Lebesgue measurable. It is well known (assuming choice) that there are Baire sets which are not measurable and measurable sets which are not Baire. (See [4].) Consequently neither of these results is stronger than the other.

In Section 2 we present a unified and simplified proof of both of these results. In addition to using a common method of argument our approach has the advantage that the terms of our sequence are chosen and fixed inductively, so that at stage n of the induction one has at hand the first n terms of the sequence. (In [6] and [5] one chooses at stage n a new n-term sequence and concludes at the end that the desired infinite sequence must exist.)

In Sections 3, 4, and 5 we restrict our attention to sets with positive measure. In Section 3 we show that if a set has 0 as an upper density point, it contains a very rich additive structure. (The point 0 is a *density point* of $A\subseteq(0,1]$ if and only if

$$\lim_{\epsilon\downarrow 0}\frac{\mu\big(A\cap(0,\epsilon)\big)}{\epsilon}=1\ .$$

It is an *upper density point* if and only if

$$\limsup_{\epsilon\downarrow 0}\frac{\mu\big(A\cap(0,\epsilon)\big)}{\epsilon}=1\ .$$

A point $x\in(0,1)$ is a density point of A if and only if

$$\lim_{\epsilon\downarrow 0}\frac{\mu\big(A\cap(x-\epsilon,x+\epsilon)\big)}{2\epsilon}=1\ .)$$

As a consequence any set of positive measure contains many translates of sets with this very rich additive structure since almost all points are points of density.

The results of Sections 2 and 3 raise the question of the possible partition regularity of all-sums sets with positive measure. That is, one wonders if whenever an all-sums set with positive measure is partitioned into finitely many measurable sets, must one of these contain an all-sums set with positive measure, or at least a translate of such a set? We show in Section 4 that, at least for sufficiently well behaved sequences, the ones we call *structured*, this is not the case. (See Question 4.13.)

It is an important property of the kind of sequences considered in Section 4 that if the measure of the all-sums set is positive, then it has 0 as a density point (Theorem 5.2). We show in Section 5 that there is a sequence (obviously not of the kind considered in Section 4) whose all-sums set has measure as close to 1 as we please (in fact contains an interval of length close to 1) but has 0 as a density point of its complement.

As promised earlier, we show now that any finite partition of $(0,1]$ must contain a sequence with all of its finite sums as well as the sum of all of its terms in one cell.

1.1 Lemma. *Let $\langle t_n\rangle_{n=1}^\infty$ be a sequence in $(0,1]$ such that $\sum_{n=1}^\infty t_n$ converges, let $r\in\mathbb{N}$, and let $\varphi : AS(\langle t_n\rangle_{n=1}^\infty) \longrightarrow \{1,2,\ldots,r\}$. Then one (or both) of the following statements holds.*

(1) There exists a sequence $\langle s_n\rangle_{n=1}^\infty$ such that φ is (defined and) constant on $FS(\langle s_n\rangle_{n=1}^\infty)\cup\{\sum_{n=1}^\infty s_n\}$.

(2) There exist $\langle z_n\rangle_{n=1}^\infty$ and $j\in\{1,2,\ldots,r\}$ such that $AS(\langle z_n\rangle_{n=1}^\infty)\subseteq AS(\langle t_n\rangle_{n=1}^\infty)$ and for all $x\in AS(\langle z_n\rangle_{n=1}^\infty)$, $\varphi(x)\neq j$.

Proof. Let $\mathcal{F}=\{F\subseteq\mathbb{N} : F\neq\emptyset$ and F is finite$\}$. Define $\tau:\mathcal{F}\longrightarrow\{1,2,\ldots,r\}$ by $\tau(F)=\varphi(\sum_{n\in F} t_n)$. By the Finite Union Theorem [1, Corollary 3.3] pick a pairwise disjoint sequence $\langle F_n\rangle_{n=1}^\infty$ in $\mathcal{F}$ and $j\in\{1,2,\ldots,r\}$ such that $\varphi(\bigcup_{n\in G} F_n)=j$ for every $G\in\mathcal{F}$. For each $n\in\mathbb{N}$, let $w_n=\sum_{i\in F_n} t_i$.

Case 1. For some infinite $H\subseteq\mathbb{N}$, $\varphi(\sum_{n\in H} w_n)=j$. Then let $\langle s_n\rangle_{n=1}^\infty$ enumerate $\langle w_n\rangle_{n\in H}$ and one sees that conclusion (1) holds.

Case 2. For each infinite $H\subseteq\mathbb{N}$, $\varphi(\sum_{n\in H} w_n)\neq j$. Let $\langle H_n\rangle_{n=1}^\infty$ be a pairwise disjoint sequence of infinite subsets of $\mathbb{N}$ and for each n let $z_n=\sum_{i\in H_n} w_i$. Then conclusion (2) holds. □

1.2 Theorem. *Let $r\in\mathbb{N}$, let $\langle s_n\rangle_{n=1}^\infty$ be a sequence in $(0,1]$ such that $\sum_{n=1}^\infty s_n\leq 1$, and $AS(\langle s_n\rangle_{n=1}^\infty)=\bigcup_{i=1}^r A_i$, then there exist some $\langle t_n\rangle_{n=1}^\infty$ and some $i\in\{1,2,\ldots,r\}$ with $FS(\langle t_n\rangle_{n=1}^\infty)\cup\{\sum_{n=1}^\infty t_n\}\subseteq A_i$. In particular, if $(0,1]=\bigcup_{i=1}^r A_i$, then there exist $i\in\{1,2,\ldots,r\}$ and some sequence $\langle t_n\rangle_{n=1}^\infty$ such that $FS(\langle t_n\rangle_{n=1}^\infty)\cup\{\sum_{n=1}^\infty t_n\}\subseteq A_i$.*

Proof. We proceed by induction on r. The case $r=1$ is trivial. Let $r\in\mathbb{N}$ and assume the result for r. Let $AS(\langle s_n\rangle_{n=1}^\infty)=\bigcup_{i=1}^{r+1} A_i$ and define $f:AS(\langle s_n\rangle_{n=1}^\infty)\longrightarrow\{1,2,\ldots,r+1\}$ by

$$f(\textstyle\sum_{n\in F} s_n)=\min\{i\in\{1,2,\ldots,r+1\}:\sum_{n\in F} s_n\in A_i\}\ .$$

Now apply Lemma 1.1. If conclusion (1) holds, we are done, while if conclusion (2) holds, the induction hypothesis applies.

The "in particular" conslusion follows by taking $s_n = \frac{1}{2^n}$. $\square$

2. All-sums sets contained in large Baire sets or measurable sets.

We present here a unified proof that "large" Baire sets or (Lebesgue) measurable sets contain all-sums sets. We write $\mu(A)$ for the Lebesgue measure of A. Both notions of "large" involve being large close to 0.

2.1 Definition. Let $A \subseteq (0,1]$.

(a) The set A is *measurably large* (at 0) if and only if A is measurable and for every $\epsilon > 0$, $\mu\big(A \cap (0,\epsilon)\big) > 0$.

(b) The set A is *Baire large* (at 0) if and only if there exist an open set U and a meager set M such that $U \backslash M \subseteq A$ and for every $\epsilon > 0$, $U \cap (0,\epsilon) \neq \emptyset$. ("Meager" = "First category" = "Countable union of nowhere dense sets".)

Recall that 0 is a *density point* of $A \subseteq (0,1]$ if and only if $\lim_{\epsilon \downarrow 0} \dfrac{\mu\big(A \cap (0,\epsilon)\big)}{\epsilon} = 1$. One can similarly formulate the notions of 0 being a point of positive upper or lower density of A. Observe that the statement that A is measurably large is weaker than the statement that 0 is a point of positive upper density of A. In fact, if $A = \bigcup_{n=1}^{\infty}(1/2^{2^n}, 2/2^{2^n})$, then A is measurably large but 0 is a density point of $(0,1] \backslash A$.

We thank the referees and the editor for suggesting a simplification of the proof of the following lemma.

2.2 Lemma. *Let $A \subseteq (0,1]$ be measurably large. There exist (many) $t \in A$ such that $A \cap (A - t)$ is measurably large.*

Proof. We first establish two simple observations.

(1) If E is a measurable subset of $\mathbb{R}$ and $\mu(E) > 0$ then there is some $\epsilon > 0$ such that for all $x \in (0,\epsilon)$, $\mu\big(E \cap (E - x)\big) > 0$.

(2) If E is a measurable subset of $\mathbb{R}$, $\mu(E) > 0$, and 0 is an accumulation point of $T \subseteq (0,\infty)$, then there is some $t \in E$ such that 0 is an accumulation point of $T \cap (E - t)$.

To verify (1) pick any density point t of E with $t > 0$. (By the Lebesgue Density Theorem [4, Theorem 3.20] almost every point of E is a density point of E.) Pick $\epsilon > 0$

such that $\mu\big(E\cap(t-\epsilon,t+\epsilon)\big)>\frac{3}{2}\epsilon$. Then given any $x\in(0,\epsilon)$,

$$\mu\big((E-x)\cap(t-x-\epsilon,t-x+\epsilon)\big)>\frac{3}{2}\epsilon\ ,$$

so $\mu\big(E\cap(E-x)\cap(t-\epsilon,t-x+\epsilon)\big)>\frac{3}{2}\epsilon+\frac{3}{2}\epsilon-(2\epsilon+x)>0$.

Now we verify (2). Notice that we may pick a compact subset D of E such that $\mu(D)>0$ so we may presume that E is compact. Let $E_1=E$ and by observation (1) pick $x_1\in T$ such that $\mu\big(E_1\cap(E_1-x_1)\big)>0$, let $E_2=E_1\cap(E_1-x_1)$ and note that E_2 is compact. Inductively, given E_n compact with positive measure, pick $x_n\in T$ with $x_n<\frac{x_{n-1}}{2}$ such that $\mu\big(E_n\cap(E_n-x_n)\big)>0$ and let $E_{n+1}=E_n\cap(E_n-x_n)$.

Clearly $\lim\limits_{n\to\infty}x_n=0$. And $\{E_n:n\in\mathbb{N}\}$ is a nested collection of compact subsets of $\mathbb{R}$ so pick $t\in\bigcap_{n=1}^{\infty}E_n$. Then $t\in E$ and for each $n\in\mathbb{N}$, $x_n\in T\cap(E-t)$.

Now we apply observation (2) with $E=T=\{x\in A: x\text{ is a density point of }A\}$. By the Lebesgue Density Theorem $\mu(E)=\mu(A)>0$. Choose $t\in E$ such that 0 is an accumulation point of $E\cap(E-t)$. To see that $A\cap(A-t)$ is measurably large, let $\epsilon>0$ be given. Pick $x\in E\cap(E-t)$ such that $x<\epsilon/2$. Now x and $x+t$ are density points of A so pick $\delta<\epsilon/2$ such that $\mu\big(A\cap(x-\delta,x+\delta)\big)>\frac{3}{2}\delta$ and $\mu\big(A\cap(x+t-\delta,x+t+\delta)\big)>\frac{3}{2}\delta$. Then also $\mu\big((A-t)\cap(x-\delta,x+\delta)\big)>\frac{3}{2}\delta$, so $\mu\big(A\cap(A-t)\cap(x-\delta,x+\delta)\big)>\frac{3}{2}\delta+\frac{3}{2}\delta-2\delta>0$. Since $x+\delta<\epsilon$, we have $\mu\big(A\cap(A-t)\cap(0,\epsilon)\big)>0$ as required. □

We remark that if we had used the stronger notion of measurably large which requires that 0 be a point of positive upper density of A, the result of Lemma 2.2 would be trivial. Simply take any density point of A and observe that 0 is a point of positive upper density of $A\cap(A-t)$.

Now we have the Baire version of Lemma 2.2.

2.3 Lemma. *Let $A\subseteq(0,1]$ be Baire large. There exist (many) $t\in A$ such that $A\cap(A-t)$ is Baire large.*

Proof. Pick an open set U and a meager set M such that $U\backslash M\subseteq A$ and $U\cap(0,\epsilon)\neq\emptyset$ for each $\epsilon>0$. Choose sequences $\langle a_n\rangle_{n=1}^{\infty}$ and $\langle b_n\rangle_{n=1}^{\infty}$ such that for each n, $b_{n+1}<a_n<b_n<\frac{1}{n}$ and $\bigcup_{n=1}^{\infty}(a_n,b_n)\subseteq U$. We show that for each $t\in\bigcup_{n=1}^{\infty}(a_n,b_n)\backslash M$, $A\cap(A-t)$ is Baire large. So let $n\in\mathbb{N}$ and let $t\in(a_n,b_n)\backslash M$. Let $\delta=b_n-t$ and pick m such that $b_m<\delta$. Let $V=\bigcup_{k=m}^{\infty}(a_k,b_k)$ and let $\tilde{M}=M\cup(M-t)$. Then $\tilde{M}$ is meager and for every $\epsilon>0$, $V\cap(0,\epsilon)\neq\emptyset$. We claim that $V\backslash\tilde{M}\subseteq A\cap(A-t)$. Let $x\in V\backslash\tilde{M}$ and pick $k\geq m$ such that $x\in(a_k,b_k)$. Then $x\in U\backslash M$ so $x\in A$. Also $x<b_k\leq b_m<\delta$ so $a_n<t<x+t<\delta+t=b_n$ so $x+t\in U\backslash M$. □

2.4 Lemma. *Let $A \subseteq (0,1]$ and let $\epsilon > 0$.*

(a) If A is measurably large, then there exists $t \in A \cap (0,\epsilon)$ such that $A \cap (A-t)$ is measurably large.

(b) If A is Baire large, then there exists $t \in A \cap (0,\epsilon)$ such that $A \cap (A-t)$ is Baire large.

Proof. In either case $A \cap (0,\epsilon)$ is large so apply Lemma 2.2 or Lemma 2.3 to $A \cap (0,\epsilon)$. □

We need one more preliminary result, namely part (b) of the following lemma. Part (a) will be used in the next section.

2.5 Lemma. *Let $A \subseteq (0,1]$ be measurable. There is a subset B of A such that $B \cup \{0\}$ is compact and*

(a) if 0 is an upper density point of A, then 0 is an upper density point of B;

(b) if A is measurably large, then B is measurably large.

Proof. Recall that given any measurable set C and any $\epsilon > 0$ there is a compact subset D of C with $\mu(D) > \mu(C) - \epsilon$ [4, Definition 3.8]. Now for each $n \in \mathbb{N}$, let $A_n = A \cap (1/2^n, 1/2^{n-1})$ and let $T = \{n \in \mathbb{N} : \mu(A_n) > 0\}$. For each $n \in T$, pick compact $B_n \subseteq A_n$ with $\mu(B_n) > \mu(A_n) - \min\{\frac{\mu(A_n)}{2}, \frac{1}{4^{n+1}}\}$. Let $B = \bigcup_{n \in T} B_n$. Then immediately $B \cup \{0\}$ is compact and conclusion (b) holds.

We observe now that given any $\alpha \in (0,1)$, if $n \in \mathbb{N}$ and $1/2^n \leq \alpha < 1/2^{n-1}$, then $\mu\big(B \cap (0,\alpha)\big) > \mu\big(A \cap (0,\alpha)\big) - \frac{1}{3 \cdot 4^n}$ from which fact conclusion (a) follows immediately. To see this, we first note that

$$\begin{aligned} \mu\big(B_n \cap (1/2^n, \alpha)\big) &= \mu(B_n) - \mu\big(B_n \cap (\alpha, 1/2^{n-1})\big) \\ &\geq \mu(B_n) - \mu\big(A_n \cap (\alpha, 1/2^{n-1})\big) \\ &> \mu(A_n) - 1/4^{n+1} - \mu\big(A_n \cap (\alpha, 1/2^{n-1})\big) \\ &= \mu\big(A_n \cap (1/2^n, \alpha)\big) - 1/4^{n+1}\ . \end{aligned}$$

Consequently

$$\begin{aligned} \mu\big(B \cap (0,\alpha)\big) &= \textstyle\sum_{k=n+1}^{\infty} \mu(B_k) + \mu\big(B_n \cap (1/2^n, \alpha)\big) \\ &> \textstyle\sum_{k=n+1}^{\infty} (\mu(A_k) - 1/4^{k+1}) + \mu\big(A_n \cap (1/2^n, \alpha)\big) - 1/4^{n+1} \\ &= \mu\big(A \cap (0,\alpha)\big) - \frac{1}{3 \cdot 4^n}\ . \end{aligned}$$

□

2.6 Theorem. *Let $A \subseteq (0,1]$ be measurably large. There is a sequence $\langle t_n \rangle_{n=1}^{\infty}$ in $(0,1]$ such that $AS(\langle t_n \rangle_{n=1}^{\infty}) \subseteq A$.*

Proof. Pick by Lemma 2.5(b) some $B \subseteq A$ such that $B \cup \{0\}$ is compact and B is measurably large. Let $B_1 = B$ and pick by Lemma 2.4(a) some $t_1 \in B_1 \cap (0, 1/2)$ such that $B_1 \cap (B_1 - t_1)$ is measurably large. Let $B_2 = B_1 \cap (B_1 - t_1)$. Inductively, given $B_{n+1} = B_n \cap (B_n - t_n)$, choose by Lemma 2.4(a) some $t_{n+1} \in B_{n+1} \cap (0, t_n/2)$ such that $B_{n+1} \cap (B_{n+1} - t_{n+1})$ is measurably large.

Now we show by induction on $|F|$ that whenever F is a finite nonempty subset of $\mathbb{N}$ and $r = \min F$, one has $\sum_{n\in F} t_n \in B_r$. If $|F| = 1$, this is immediate, so assume $|F| > 1$, let $G = F\backslash\{r\}$, and let $m = \min G$. Then $\sum_{n\in G} t_n \in B_m \subseteq B_{r+1} \subseteq (B_r - t_r)$ so $\sum_{n\in F} t_n \in B_r$ as required.

Since every finite sum from $\langle t_n\rangle_{n=1}^{\infty}$ is in B and $B \cup \{0\}$ is compact we have that all infinite sums from $\langle t_n\rangle_{n=1}^{\infty}$ are also in $B \cup \{0\}$. Since none of these sums is 0, we are done. $\square$

2.7 Theorem. *Let $A \subseteq (0, 1]$ be Baire large. There is a sequence $\langle t_n\rangle_{n=1}^{\infty}$ in $(0, 1]$ such that $AS(\langle t_n\rangle_{n=1}^{\infty}) \subseteq A$.*

Proof. Pick an open set U and a meager set M such that $U\backslash M \subseteq A$ and for every $\epsilon > 0$, $U \cap (0, \epsilon) \neq \emptyset$. Since there is no harm in replacing M by a larger meager set we may presume $M = (0, 1]\backslash \bigcap_{n=1}^{\infty} D_n$ where each D_n is open and dense in $(0, 1)$ and each $D_{n+1} \subseteq D_n$. Then $U \cap \bigcap_{n=1}^{\infty} D_n \subseteq A$. Pick sequences $\langle a_n\rangle_{n=1}^{\infty}$, $\langle b_n\rangle_{n=1}^{\infty}$, $\langle c_n\rangle_{n=1}^{\infty}$, and $\langle d_n\rangle_{n=1}^{\infty}$ such that for each n, $b_{n+1} < a_n < c_n < d_n < b_n < 1/n$ and $\bigcup_{n=1}^{\infty}(a_n, b_n) \subseteq U$. Let $U_1 = \bigcup_{n=1}^{\infty}(c_n, d_n)$ and let $A_1 = U_1\backslash M$. Then A_1 is Baire large so by Lemma 2.4(b), pick some $t_1 \in A_1 \cap (0, 1/2)$ such that $A_1 \cap (A_1 - t_1)$ is Baire large. Let $A_2 = A_1 \cap (A_1 - t_1)$.

Inductively let $n > 1$ be given and assume $A_n = A_{n-1} \cap (A_{n-1} - t_{n-1})$ is Baire large. Now as in the proof of Theorem 2.6 we see that for each nonempty $F \subseteq \{1, 2, \ldots, n-1\}$, $\sum_{k\in F} t_k \in A_1$, and hence $\sum_{k\in F} t_k \in D_n$. For each such F pick $\epsilon_{n,F} > 0$ such that $(\sum_{k\in F} t_k - \epsilon_{n,F}, \sum_{k\in F} t_k + \epsilon_{n,F}) \subseteq D_n$. Let

$$\epsilon_n = \min(\{t_{n-1}\} \cup \{\epsilon_{n,F} : \emptyset \neq F \subseteq \{1, 2, \ldots, n-1\}\}) \ .$$

Pick $t_n \in A_n \cap (0, \epsilon_n/2)$ such that $A_n \cap (A_n - t_n)$ is Baire large.

Now we have (as in the proof of Theorem 2.6) that

$$FS(\langle t_n\rangle_{n=1}^{\infty}) \subseteq A_1 \subseteq \bigcup_{n=1}^{\infty}[c_n, d_n]$$

so $AS(\langle t_n\rangle_{n=1}^{\infty}) \subseteq \bigcup_{n=1}^{\infty}[c_n, d_n] \subseteq U$ (since $\{0\}\cup\bigcup_{n=1}^{\infty}[c_n, d_n]$ is compact). Thus we only need show that $AS(\langle t_n\rangle_{n=1}^{\infty}) \subseteq \bigcap_{n=1}^{\infty} D_n$. To this end, let I be a nonempty subset of $\mathbb{N}$

and let $n \in \mathbb{N}$ be given. Since for each k, $D_{k+1} \subseteq D_k$ we may presume that $n > \min I$. Let $F = I \cap \{1, 2, \ldots, n-1\}$ and let $H = I \backslash F$. Now

$$(\sum_{k\in F}\ t_k - \epsilon_n, \sum_{k\in F}\ t_k + \epsilon_n) \subseteq (\sum_{k\in F}\ t_k - \epsilon_{n,F}, \sum_{k\in F}\ t_k + \epsilon_{n,F}) \subseteq D_n\ .$$

If $H = \emptyset$ we have $\sum_{k\in I}\ t_k \in D_n$ as required. Otherwise, we have

$$\sum_{k\in H}\ t_k \leq \sum_{k=n}^{\infty} t_k < t_n \text{ (since } t_k < \epsilon_k/2 \leq T_{k-1}/2 < \epsilon_{k-1}/4 \ldots\ ),$$

so $\sum_{k\in I}\ t_k = \sum_{k\in F}\ t_k + \sum_{k\in H}\ t_k \in (\sum_{k\in F}\ t_k, \sum_{k\in F}\ t_k + \epsilon_n) \subseteq D_n$. □

2.8 Theorem. *Let $r \in \mathbb{N}$ and let $(0,1] = \bigcup_{i=1}^{r} A_i$. If each A_i is Baire or each A_i is measurable, then there exist i and a sequence $\langle t_n\rangle_{n=1}^{\infty}$ such that $AS(\langle t_n\rangle_{n=1}^{\infty}) \subseteq A_i$.*

Proof. If each A_i is measurable, one of them must be measurably large. If each A_i is Baire, one of them must be Baire large. □

3. Sums with repetition.

As was observed in [5], it is easy to prevent repetition of sums in Theorem 2.8. For example, let $A_1 = \bigcup_{n=1}^{\infty}(1/2^{2n-1}, 1/2^{2n-2}]$ and let $A_2 = \bigcup_{n=1}^{\infty}(1/2^{2n}, 1/2^{2n-1}]$. Then given any $t < 1/2$ and $i \in 1, 2$, if $t \in A_i$, then $2t \notin A_i$ and A_1 and A_2 are both measurably large and Baire large. We shall see in this section that if 0 is an upper density point of a measurable set A, one can obtain substantial repetitions. (Notice that having 0 as a point of positive upper density is not good enough, as is established by the sets A_1 and A_2 above, each of which has upper density $\frac{2}{3}$ at 0.)

3.1 Lemma. *Let A be a measurable subset of $(0,1]$ and let $\alpha > 0$.*

(a) If 0 is an upper density point of A, then 0 is an upper density point of A/α.

(b) If $t \in (0,1)$ is a density point of A/α, then $t\alpha$ is a density point of A.

(c) If F is a finite subset of $(0,\infty)$ and 0 is an upper density point of A, then 0 is an upper density point of $\bigcap_{\alpha\in F}\ A/\alpha$.

Proof. Assertions (a) and (b) are established by routine computations based on the fact that for $0 \leq x < y \leq 1$, $\mu\big(A/\alpha \cap (x,y)\big) = (1/\alpha)\cdot\mu\big(A \cap (x\alpha, y\alpha)\big)$.

To establish (c), we proceed by induction on $|F|$, the case $|F| = 1$ being conclusion (a). Assume $|F| > 1$ and let $\epsilon, \nu > 0$. We show that there exists positive $\delta < \nu$ such that

$$\frac{\mu\big((0,\delta) \cap \bigcap_{\alpha\in F}\ A/\alpha\big)}{\delta} \geq 1 - \epsilon\ .$$

Let $x = \max F$ and let $y = \max(F\backslash\{x\})$. Pick positive $\gamma < 1$ such that $\gamma - (1-\gamma)\frac{x}{y} \geq 1 - \epsilon$. Let $C = \bigcap_{\alpha\in F\backslash\{x\}} A/\alpha$. By the induction hypothesis, 0 is an upper density point of C so pick $\delta < \nu$ such that $\mu\big((0,\delta)\cap C\big) \geq \gamma\cdot\delta$.

Now $C \subseteq A/y$ so $\frac{y}{x}C \subseteq A/x$ so $\mu\big((0,\delta\frac{y}{x})\cap A/x\big) \geq \mu\big((0,\delta\frac{y}{x})\cap \frac{y}{x}C\big) \geq \gamma\cdot\delta\cdot\frac{y}{x}$. Also, $(0,\delta\frac{y}{x})\cap C = \big((0,\delta)\cap C\big)\backslash[\delta\frac{y}{x},\delta)$ so $\mu\big((0,\delta\frac{y}{x})\cap C\big) \geq \gamma\cdot\delta - (\delta - \delta\frac{y}{x})$. Thus

$$\begin{array}{rcl} \mu\big((0,\delta\frac{y}{x})\cap C\cap A/x\big) & \geq & \gamma\cdot\delta - (\delta-\delta\frac{y}{x}) + \gamma\cdot\delta\cdot\frac{y}{x} - \delta\frac{y}{x} \\ & = & \big(\gamma - (1-\gamma)\frac{x}{y}\big)\cdot\delta\cdot\frac{y}{x} \\ & \geq & (1-\epsilon)\cdot\delta\cdot\frac{y}{x} \end{array}$$

and $\delta\cdot\frac{y}{x} < \delta < \nu$. □

The statement of the following theorem is somewhat complicated. As a simple example, let $f(\langle t_i\rangle_{i=1}^{n-1}) = \{1,2,\ldots,n\}$. Then given any measurable set A with 0 as a density point one can get a sequence $\langle t_n\rangle_{n=1}^{\infty}$ with

$$\big\{\textstyle\sum_{n\in G} kt_n : \emptyset \neq G \subseteq \mathbb{N} \text{ and for each } n\in G, k\in\{1,2,\ldots,n\}\big\} \subseteq A\,.$$

We show in the following result that we can get all sums of linear combinations of a sequence in $(0,1]$ where the coefficients of the linear combination can vary over any prespecified finite set (depending only on the earlier terms).

3.2 Theorem. *Let f be a function taking the set of finite sequences in $(0,1)$ to the set of finite subsets of $(0,\infty)$ and let A be a measurable subset of $(0,1]$ such that 0 is an upper density point of A. There is a sequence $\langle t_n\rangle_{n=1}^{\infty}$ such that, given any $\varphi \in \times_{n=1}^{\infty} f(\langle t_i\rangle_{i=1}^{n-1})$, $\{\sum_{n\in G}\varphi_n\cdot t_n : \emptyset\neq G\subseteq\mathbb{N}\}\subseteq A$.*

Proof. Pick by Lemma 2.5(a) a subset B of A such that $B\cup\{0\}$ is compact and 0 is an upper density point of B. As in the proof of Theorem 2.6 it suffices to produce $\langle t_n\rangle_{n=1}^{\infty}$ such that, given any $\varphi \in \times_{n=1}^{\infty} f(\langle t_i\rangle_{i=1}^{n-1})$ and any finite nonempty $G\subseteq\mathbb{N}$, $\sum_{n\in G}\varphi_n\cdot t_n \in B$.

Let $B_1 = B$ and let $F_1 = f(\emptyset) = f(\langle t_i\rangle_{i=1}^{0})$. By Lemma 3.1(c), 0 is an upper density point of $\bigcap_{\alpha\in F_1}(B_1/\alpha)$. In particular, for each $\epsilon > 0$, $\mu\big((0,\epsilon)\cap\bigcap_{\alpha\in F_1}(B_1/\alpha)\big) > 0$. Let $\delta_1 = \max F_1$ and pick a density point t_1 of $\bigcap_{\alpha\in F_1}(B_1/\alpha)$ with $t_1\in\bigcap_{\alpha\in F_1}(B_1/\alpha)$ such that $0 < \delta_1 t_1 < 1/2$ (using Lebesgue's Density Theorem [4, Theorem 3.20]). By Lemma 3.1(b), for each $\alpha\in F_1$, αt_1 is a density point of B_1 so 0 is an upper density point of $B_2 = B_1\cap\bigcap_{\alpha\in F_1}(B_1 - \alpha t_1)$.

Inductively, let B_n be given with 0 as an upper density point of B_n. Then as above, let $F_n = f(\langle t_i\rangle_{i=1}^{n-1})$. By Lemma 3.1(c), 0 is an upper density point of $\bigcap_{\alpha\in F_n}(B_n/\alpha)$.

and hence for each $\epsilon > 0$, $\mu\big((0,\epsilon) \cap \bigcap_{\alpha\in F_n} (B_n/\alpha)\big) > 0$. Let $\delta_n = \max F_n$ and pick a density point t_n of $\bigcap_{\alpha\in F_n} (B_n/\alpha)$ with $t_n \in \bigcap_{\alpha\in F_n} (B_n/\alpha)$ and $0 < \delta_n t_n < 1/2^n$. By Lemma 3.1(b) for each $\alpha \in F_n$, αt_n is a density point of B_n so 0 is an upper density point of $B_{n+1} = B_n \cap \bigcap_{\alpha\in F_n} (B_n - \alpha t_n)$.

Now let $\varphi \in \times_{n=1}^{\infty} F_n$ be given. As in the proof of Theorem 2.6, we show by induction on $|G|$ that if $r = \min G$, then $\sum_{n\in G} \varphi_n \cdot t_n \in B_r$. Again, if $|G| = 1$ the conclusion is immediate, so assume $|G| > 1$, let $H = G\backslash\{r\}$ and let $m = \min H$. Then $\sum_{n\in H} \varphi_n \cdot t_n \in B_m \subseteq B_{r+1} \subseteq B_r - \varphi_r \cdot t_r$ so $\sum_{n\in G} \varphi_n \cdot t_n \in B_r$ as required. □

3.3 Corollary. *Let $\langle F_n\rangle_{n=1}^{\infty}$ be a sequence of finite subsets of $(0,\infty)$ and let A be a measurable subset of $(0,1]$ with $\mu(A) > 0$. Then given any finite set H of density points of A there is a sequence $\langle t_n\rangle_{n=1}^{\infty}$ such that given any $\varphi \in \times_{n=1}^{\infty} F_n$ and any $a \in H$, $a + \{\sum_{n\in G} \varphi_n \cdot t_n : \emptyset \neq G \subseteq \mathbb{N}\} \subseteq A$.*

Proof. Given any sequence $\langle t_n\rangle_{n=1}^{\infty}$ and any $n \in \mathbb{N}$ define $f(\langle t_i\rangle_{i=1}^{n-1}) = F_n$. We have that 0 is a density point of $\bigcap_{a\in H} (A - a)$ so Theorem 3.2 applies. □

4. Counterexamples to partition regularity of fat sets of sums and their translates.

In the case of finite sums of integers it is well known that the sets of finite sums are themselves partition regular. That is, whenever $r \in \mathbb{N}$, $\langle t_n\rangle_{n=1}^{\infty}$ is a sequence in $\mathbb{N}$, and $FS(\langle t_n\rangle_{n=1}^{\infty}) = \bigcup_{i=1}^{r} A_i$, there exist i and $\langle s_n\rangle_{n=1}^{\infty}$ with $FS(\langle s_n\rangle_{n=1}^{\infty}) \subseteq A_i$. (As in the proof of Lemma 1.1, if $FS(\langle t_n\rangle_{n=1}^{\infty}) = \bigcup_{i=1}^{r} A_i$, let for each $i \in \{1,2,\ldots,r\}$, $B_i = \{F : \emptyset \neq F \subseteq \mathbb{N}$ and F is finite and $\sum_{n\in F} t_n \in A_i\}$ and apply [1, Corollary 3.3].) Likewise, both of our largeness notions in Section 2 are partition regular.

Similarly, as was pointed out in [6], all-sums sets are partition regular with respect to Borel partitions. Thus in view of Theorem 2.8, it is natural to ask whether it is true that whenever $\langle t_n\rangle_{n=1}^{\infty}$ is a sequence in $(0,1]$ with $\sum_{n=1}^{\infty} t_n \leq 1$, $r \in \mathbb{N}$, and $AS(\langle t_n\rangle_{n=1}^{\infty}) = \bigcup_{i=1}^{r} A_i$ with each A_i (Lebesgue) measurable, then for some i and some $\langle s_n\rangle_{n=1}^{\infty}$, one has $AS(\langle s_n\rangle_{n=1}^{\infty}) \subseteq A_i$. This is however easily seen to be false. Simply take any sequence $\langle t_n\rangle_{n=1}^{\infty}$ such that $\mu\big(AS(\langle t_n\rangle_{n=1}^{\infty})\big) = 0$, (for example $t_n = \frac{1}{3^n}$). Then let B_1 and B_2 be (nonmeasurable) sets obtained by a diagonal argument (see the introduction) such that $(0,1] = B_1 \cup B_2$ and for no sequence $\langle s_n\rangle_{n=1}^{\infty}$ and no $i \in \{1,2\}$ is $AS(\langle s_n\rangle_{n=1}^{\infty}) \subseteq B_i$. Finally let $A_i = B_i \cap AS(\langle t_n\rangle_{n=1}^{\infty})$ for $i \in \{1,2\}$. Then $\mu(A_1) = \mu(A_2) = 0$ so both are measurable.

But that is certainly a cheap way out. If $\mu\big(AS(\langle t_n\rangle_{n=1}^{\infty})\big) > 0$, then this counterexample is blocked. In fact, we have the following easy result.

4.1 Theorem. *Let $\langle t_n\rangle_{n=1}^\infty$ be a sequence with $\mu\big(AS(\langle t_n\rangle_{n=1}^\infty)\big)>0$. Then $AS(\langle t_n\rangle_{n=1}^\infty)$ is measurably large. Consequently, if $r\in\mathbb{N}$ and $AS(\langle t_n\rangle_{n=1}^\infty)=\bigcup_{i=1}^r A_i$ with each A_i measurable, then there exist $i\in\{1,2,\ldots,r\}$ and a sequence $\langle s_n\rangle_{n=1}^\infty$ such that $AS(\langle s_n\rangle_{n=1}^\infty)\subseteq A_i$.*

Proof. Given any $k\in\mathbb{N}$ we have that

$$AS(\langle t_n\rangle_{n=1}^\infty)=\bigcup_{F\subseteq\{1,2,\ldots,k\}}\big(\textstyle\sum_{n\in F}\ t_n+AS(\langle t_n\rangle_{n=k+1}^\infty)\big)$$

so that $0<\mu\big(AS(\langle t_n\rangle_{n=1}^\infty)\big)\leq 2^k\mu\big(AS(\langle t_n\rangle_{n=k+1}^\infty)\big)$ and hence $\mu\big(AS(\langle t_n\rangle_{n=k+1}^\infty)\big)>0$. Let $\epsilon>0$ be given and pick $k\in\mathbb{N}$ such that $\sum_{n=k+1}^\infty t_n<\epsilon$. Then

$$\mu\big(AS(\langle t_n\rangle_{n=1}^\infty)\cap(0,\epsilon)\big)\geq\mu\big(AS(\langle t_n\rangle_{n=k+1}^\infty)\big)>0\ .$$

Thus $AS(\langle t_n\rangle_{n=1}^\infty)$ is measurably large. The remaining conclusion of the theorem now follows from Theorem 2.6. $\square$

Notice that Theorem 4.1 does not establish the partition regularity of anything. That is we assume that the all sums set of $\langle t_n\rangle_{n=1}^\infty$ is fat, and we do not guarantee that the all sums set of $\langle s_n\rangle_{n=1}^\infty$ is fat. One is thus naturally led to ask whether fat all sums sets are partition regular. That is, is it true that whenever $\langle t_n\rangle_{n=1}^\infty$ is a sequence in $(0,1]$ with $\sum_{n=1}^\infty t_n\leq 1$ and $\mu\big(AS(\langle t_n\rangle_{n=1}^\infty)\big)>0$ and $AS(\langle t_n\rangle_{n=1}^\infty)=\bigcup_{i=1}^r A_i$ with each A_i measurable, there must exist i and $\langle s_n\rangle_{n=1}^\infty$ with $AS(\langle s_n\rangle_{n=1}^\infty)\subseteq A_i$ and $\mu\big(AS(\langle s_n\rangle_{n=1}^\infty)\big)>0$? We see now that the answer to this question is "no" as well.

4.2 Theorem. *Let $A_1=\bigcup_{n=1}^\infty(\frac{1}{2^{2n-1}},\frac{1}{2^{2n-2}}]$, let $A_2=\bigcup_{n=1}^\infty(\frac{1}{2^{2n}},\frac{1}{2^{2n-1}}]$, and for each n let $t_n=\frac{1}{2^n}$. Then $AS(\langle t_n\rangle_{n=1}^\infty)=(0,1]=A_1\cup A_2$ but there do not exist $i\in\{1,2\}$ and $\langle s_n\rangle_{n=1}^\infty$ with $AS(\langle s_n\rangle_{n=1}^\infty)\subseteq A_i$ and $\mu\big(AS(\langle s_n\rangle_{n=1}^\infty)\big)>0$.*

Proof. Assume essentially without loss of generality that we have a sequence $\langle s_n\rangle_{n=1}^\infty$ with $AS(\langle s_n\rangle_{n=1}^\infty)\subseteq A_2$. We show that $\mu\big(AS(\langle s_n\rangle_{n=1}^\infty)\big)=0$. Each interval $(\frac{1}{2^{2n}},\frac{1}{2^{2n-1}}]$ contains at most one term from the sequence so (assuming $\langle s_n\rangle_{n=1}^\infty$ is decreasing) we have each $s_n\leq\frac{1}{2^{2n-1}}$. Then by [3, Theorem 1] $\mu\big(AS(\langle s_n\rangle_{n=1}^\infty)\big)=0$. (Alternatively let for each k, $a_k=\frac{1}{3\cdot 2^{2k-1}}$. Then for each k,

$$AS(\langle s_n\rangle_{n=1}^\infty)\subseteq\bigcup\big\{[\textstyle\sum_{n\in F}\ s_n,(\sum_{n\in F}\ s_n)+a_k]:\emptyset\neq F\subseteq\{1,2,\ldots,k\}\big\}$$

so $\mu\big(AS(\langle s_n\rangle_{n=1}^\infty)\big)\leq(2^k-1)\cdot a_k<\frac{1}{3\cdot 2^{k-1}}$.) $\square$

Each cell of the partition of Theorem 4.2 contains intervals, hence will contain a translate of a "fat" all-sums set (i.e. one with positive measure). Corollary 3.3 then suggests that one might be able to get translates of such fat sets of sums in one cell of such a partition, or even in any set of positive measure. If one could establish the latter assertion one would have that the set of translates of fat sets of sums is partition regular just as the ordinary sets of finite sums are.

We present in this section counterexamples to both of these assertions which are, unfortunately, valid only for a restricted class of sequences, ones we call "structured" sequences. (These are the sequences considered by Menon in [3].) See Question 4.13. In the following definition we follow Menon by suppressing the dependence of R_n on the sequence $\langle t_n\rangle_{n=1}^\infty$.

4.3 Definition. Let $\langle t_n\rangle_{n=1}^\infty$ be a sequence in $(0,1]$ such that $\sum_{n=1}^\infty t_n$ converges. For each $n\in\mathbb{N}$ let $R_n=\sum_{k=n+1}^\infty t_k$. The sequence $\langle t_n\rangle_{n=1}^\infty$ is *structured* if and only if $\sum_{n=1}^\infty t_n\leq 1$ and for each n, $t_n\geq R_n$.

If $\langle t_n\rangle_{n=1}^\infty$ is a structured sequence, $\mu\big(AS(\langle t_n\rangle_{n=1}^\infty)\big)>0$, and $a\in(0,1]$, then $a+AS(\langle t_n\rangle_{n=1}^\infty)$ is also known as a "symmetric Cantor set of positive measure". We thank the editor, David Preiss, and one of the referees for suggesting the proof that we present (in Theorem 4.12) that there is a partition of $(0,1]$ into two Borel sets, neither of which contains a symmetric Cantor set of positive measure. This proof is based on an example of Talagrand's which was presented in [2]. This proof in fact provides a stronger result than our original proof of Theorem 4.12 in that one of the sets has measure 1. Our original proof did not depend on [2], but it was longer and (even) more cumbersome.

Given a decreasing sequence $\langle t_n\rangle_{n=1}^\infty$ in $(0,1]$ such that $\sum_{n=1}^\infty t_n$ converges, it is an easy exercise to show that there is some $\epsilon>0$ with $(0,\epsilon)\subseteq AS(\langle t_n\rangle_{n=1}^\infty)$ if and only if eventually $t_n\leq R_n$. Consequently, badly nonstructured sequences (i.e. those for which eventually $t_n\leq R_n$ and frequently $t_n<R_n$) cause us no problem; any measurable set which contains no interval could not contain $a+AS(\langle t_n\rangle_{n=1}^\infty)$ for any such sequence. Our problem arises with sequences for which infinitely often $t_n>R_n$ and infinitetly often $t_n<R_n$. We will return to this issue in Section 5.

4.4 Lemma. *Let $\langle t_n\rangle_{n=1}^\infty$ be a structured sequence. Then*

$$\mu\big(AS(\langle t_n\rangle_{n=1}^\infty)\big)=\lim_{n\to\infty}2^n\cdot R_n\ .$$

Proof. This is precisely [3, Theorem 1] except that it is assumed there that $\sum_{n=1}^\infty t_n=1$. However, if $\sum_{n=1}^\infty t_n<1$, we may replace t_1 by $1-\sum_{n=2}^\infty t_n$ and apply [3, Theorem 1]. □

The following lemma provides a convenient characterization of structured sequences with fat all-sums sets.

4.5 Lemma. *Let $\langle t_n\rangle_{n=1}^{\infty}$ be a structured sequence with $\mu\big(AS(\langle t_n\rangle_{n=1}^{\infty})\big)=\gamma>0$. Then there is a sequence $\langle \delta_n\rangle_{n=1}^{\infty}$ such that*

(1) for each n, $\delta_n\geq\delta_{n+1}\geq 0$,

(2) $\lim\limits_{n\to\infty}\delta_n=0$,

(3) $\gamma+\delta_1\leq 1$,

(4) for each n, $t_{n+1}=(\gamma-\delta_{n+1})\cdot\frac{1}{2^{n+1}}+\delta_n\cdot\frac{1}{2^n}$,

(5) for each n, $R_n=(\gamma+\delta_n)\cdot\frac{1}{2^n}$, and

(6) for each n, $t_{n+1}-R_{n+1}=(\delta_n-\delta_{n+1})\cdot\frac{1}{2^n}$.

Conversely, if $\gamma>0$ and $\langle\delta_n\rangle_{n=1}^{\infty}$ is any sequence satisfying statements (1), (2), and (3), $t_1=1-(\gamma+\delta_1)/2$, and for each n, $t_{n+1}=(\gamma-\delta_{n+1})\cdot\frac{1}{2^{n+1}}+\delta_n\cdot\frac{1}{2^n}$, then $\langle t_n\rangle_{n=1}^{\infty}$ is a structured sequence and $\mu\big(AS(\langle t_n\rangle_{n=1}^{\infty})\big)=\gamma$.

Proof. For each n let $\delta_n=2^n\cdot R_n-\gamma$. Then (5) holds directly and (2) holds by Lemma 4.4. Also, given n, $t_{n+1}=R_n-R_{n+1}=(\gamma-\delta_{n+1})\cdot\frac{1}{2^{n+1}}+\delta_n\cdot\frac{1}{2^n}$ as required by (4). Statement (6) follows from statements (4) and (5), while statement (1) follows from statements (6) and (2) and the fact that $t_{n+1}\geq R_{n+1}$.

To verify the converse, let $\gamma>0$, let $\langle\delta_n\rangle_{n=1}^{\infty}$ be any sequence satisfying statements (1), (2), and (3), let $t_1=1-(\gamma+\delta_1)/2$, and for each n, let $t_{n+1}=(\gamma-\delta_{n+1})\cdot\frac{1}{2^{n+1}}+\delta_n\cdot\frac{1}{2^n}$. Then for each n,

$$\begin{aligned} R_n &= \textstyle\sum_{k=n+1}^{\infty} t_k \\ &= \textstyle\sum_{k=n+1}^{\infty}(\gamma-\delta_k)\cdot\frac{1}{2^k}+\delta_{k-1}\cdot\frac{1}{2^{k-1}} \\ &= (\gamma\cdot\textstyle\sum_{k=n+1}^{\infty}\frac{1}{2^k})+\delta_n\cdot\frac{1}{2^n} \\ &= (\gamma+\delta_n)\cdot\frac{1}{2^n} \end{aligned}$$

so (5) holds.

Then immediately statement (6) holds, so the sequence $\langle t_n\rangle_{n=1}^{\infty}$ is structured and hence by Lemma 4.4, $\mu\big(AS(\langle t_n\rangle_{n=1}^{\infty})\big)=\gamma$. □

4.6 Lemma. *Let $\langle t_n\rangle_{n=1}^{\infty}$ be a structured sequence, let $a\in[0,1]$, and let $b,c\in\mathbb{R}$ with $b<c$. If $(b,c)\cap\big(a+AS(\langle t_n\rangle_{n=1}^{\infty})\big)=\emptyset$, then one of the following holds.*

(a) $c\leq a$.

(b) $b\geq a+\sum_{n=1}^{\infty}t_n$.

(c) There exist $k\in\mathbb{N}$ and $F\subseteq\{1,2,\ldots,k-1\}$ such that

(i) $a+\sum_{\ell\in F}\,t_\ell+\sum_{\ell=k+1}^{\infty}t_\ell\leq b$ and

(ii) $a+\sum_{\ell\in F}\,t_\ell+t_k\geq c$.

Proof. Assume that neither conclusion (a) nor conclusion (b) holds. Notice that $a < b$. (For if $a \geq b$ then for some n, $a + t_n \in (b, c)$.) Let $n(1) = \min\{k \in \mathbb{N} : a + t_k \leq b\}$. Inductively, having chosen $n(1), n(2), \ldots, n(\ell)$, if $b = a + \sum_{k=1}^{\ell} t_{n(k)}$, then stop and let $H = \{n(1), n(2), \ldots, n(\ell)\}$. (We will see that in fact this case cannot happen.) Otherwise, let $n(\ell+1) = \min\{k \in \mathbb{N} : k > n(\ell) \text{ and } a + \sum_{i=1}^{\ell} t_{n(i)} + t_k \leq b\}$.

When the induction is complete, let $H = \{n(i) : i \in \mathbb{N}\}$. Then $a + \sum_{n\in H} t_n \leq b$. Pick m such that $t_m < c - b$. Then in fact $\{m, m+1, m+2, \ldots\} \subseteq H$. To see this suppose instead that for some $\ell \geq m$ one has $\ell \notin H$. Then $a + \sum_{n\in H} t_n + t_\ell \in (b, c) \cap AS(\langle t_n \rangle_{n=1}^{\infty})$, a contradiction. Pick the first $k \in \mathbb{N} \cup \{0\}$ such that $\{k+1, k+2, k+3, \ldots\} \subseteq H$ and notice that $k > 0$ since we have excluded the possibility that $b \geq a + \sum_{n=1}^{\infty} t_n$. Let $F = H \cap \{1, 2, \ldots, k-1\}$. Since $k \notin H$ we have $a + \sum_{n\in F} t_n + t_k > b$ so $a + \sum_{n\in F} t_n + t_k \geq c$. Also $a + \sum_{\ell\in F} t_\ell + \sum_{\ell=k+1}^{\infty} t_\ell = a + \sum_{\ell\in H} t_\ell \leq b$. □

We now introduce some notation from [2, pages 259, 266, and 268]. The only change that we make from their definition is that we require the set C to be bounded. We do this because we will be working with bounded sets and it keeps us from worrying about infinite values for $\alpha_n(x|C)$, which is introduced in the following definition.

4.7 Definition. Let C be a bounded subset of $\mathbb{R}$. For $x \in \mathbb{R}$ and $n \in \mathbb{N}$,

$$\alpha_n(x|C) = \inf\{|x - \frac{b+c}{2}| : b, c \in \mathbb{R},\ [b,c] \cap C = \emptyset, \text{ and } \frac{1}{2^{n+1}} < \frac{c-b}{2} \leq \frac{1}{2^n}\}\ .$$

4.8 Lemma. *Let $\langle t_n \rangle_{n=1}^{\infty}$ be a structured sequence such that $\mu\big(AS(\langle t_n \rangle_{n=1}^{\infty})\big) = \gamma > 0$, let $a \in [0,1]$, and let $C = a + AS(\langle t_n \rangle_{n=1}^{\infty})$. There is a dense subset D of C such that, for every $x \in D$,*

$$\sum_{n=1}^{\infty} \frac{1}{2^n \cdot \alpha_n(x|C)} < \infty\ .$$

Proof. Let $D = \big\{a + \sum_{\ell\in F} t_\ell + \sum_{\ell=k}^{\infty} t_{2\ell} : k \in \mathbb{N} \text{ and } F \subseteq \{1, 2, \ldots, 2k-1\}\big\}$. Then trivially D is dense in C. Fix $x \in D$ and pick $k \in \mathbb{N}$ and $F \subseteq \{1, 2, \ldots, 2k-1\}$ such that $x = a + \sum_{\ell\in F} t_\ell + \sum_{\ell=k}^{\infty} t_{2\ell}$.

Pick a sequence $\langle \delta_n \rangle_{n=1}^{\infty}$ as guaranteed by Lemma 4.5. If eventually $t_n = R_n$, then by Lemma 4.5(6), the sequence $\langle \delta_n \rangle_{n=1}^{\infty}$ is eventually constant, hence by Lemma 4.5(2) is eventually 0 and thus (by Lemma 4.5(4)), eventually $t_n = \frac{\gamma}{2^n}$. Then x is interior to an interval which is contained in C so the conclusion is trivial. Thus we assume that infinitely often $t_n > R_n$.

Fix $q \in \mathbb{N}$ such that there is some $m > 2k$ such that $t_m - R_m > \frac{1}{2^q}$. For each $n \geq q$ in $\mathbb{N}$, let $m(n) = \max\{m \in \mathbb{N} : t_m - R_m > \frac{1}{2^n}\}$. We claim that it suffices to show:

(1) $$\sum_{n=q}^{\infty} \frac{2^{m(n)}}{2^n} \text{ converges}$$

and

(2) $$\text{For each } n \geq q,\ \alpha_n(x|C) \geq \sum_{\ell=1}^{\infty} t_{m(n)+2\ell} + \frac{1}{2^{n+1}}$$

(In fact, equality holds in (2), but we won't need that.)

So suppose that we have established (1) and (2). Given $n \geq q$, we have

$$\begin{array}{rcl} 2\sum_{\ell=1}^{\infty} t_{m(n)+2\ell} & > & \sum_{\ell=1}^{\infty} t_{m(n)+2\ell} + \sum_{\ell=1}^{\infty} t_{m(n)+2\ell+1} \\ & = & R_{m(n)+1} \end{array}$$

and, since $m(n)+1 > m(n)$, $t_{m(n)+1} - R_{m(n)+1} \leq \frac{1}{2^n}$. Thus for each $n \geq q$ one has

$$\begin{array}{rcl} \alpha_n(x|C) & \geq & \sum_{\ell=1}^{\infty} t_{m(n)+2\ell} + \frac{1}{2^{n+1}} \\ & > & \frac{1}{2} R_{m(n)+1} + \frac{1}{2^{n+1}} \\ & \geq & \frac{1}{2} t_{m(n)+1} \ . \end{array}$$

Thus it suffices to show that $\sum_{n=q}^{\infty} \frac{1}{2^n \cdot t_{m(n)+1}}$ converges.

By Lemma 4.5(4), we have for each $n \geq q$ that

$$\frac{1}{2^n \cdot t_{m(n)+1}} = \frac{2^{m(n)}}{2^n(\frac{1}{2}\gamma - \frac{1}{2}\delta_{m(n)+1} + \delta_{m(n)})} \ .$$

Since eventually $\frac{1}{2}\gamma - \frac{1}{2}\delta_{m(n)+1} + \delta_{m(n)} \geq \frac{1}{4}\gamma$ one has that eventually $\frac{1}{2^n \cdot t_{m(n)+1}} \leq \frac{2^{m(n)+2}}{2^n \gamma}$ so (1) applies.

To establish (1), let $M = \{m(n) : n \in \{q, q+1, q+2, \ldots\}\}$ and for each $m \in M$, let $a(m) = \min\{n : m = m(n)\}$ and let $b(m) = \min\{n : m = m(n)\}$. Then

$$\begin{array}{rcl} \sum_{n=q}^{\infty} \frac{2^{m(n)}}{2^n} & = & \sum_{m \in M} \ 2^m \sum_{\ell=a(m)}^{b(m)} \frac{1}{2^\ell} \\ & < & \sum_{m \in M} \ 2^m \sum_{\ell=a(m)}^{\infty} \frac{1}{2^\ell} \\ & = & \sum_{m \in M} \frac{2^m}{2^{a(m)-1}} \ . \end{array}$$

For each $m \in M$, $t_m - R_m > \frac{1}{2^{a(m)}}$ so by Lemma 4.5(6), $\frac{2^m}{2^{a(m)-1}} < 4(\delta_{m-1} - \delta_m)$ and hence, if $\ell = \min M$, then

$$\begin{array}{rcl} \sum_{m \in M} \frac{2^m}{2^{a(m)-1}} & < & \sum_{m \in M} \ 4(\delta_{m-1} - \delta_m) \\ & \leq & 4\delta_{\ell-1} \ . \end{array}$$

To complete the proof, we establish (2). So let $n \geq q$ and suppose instead that $\alpha_n(x|C) < \sum_{\ell=1}^{\infty} t_{m(n)+2\ell} + \frac{1}{2^{n+1}}$. Pick by the definition of $\alpha_n(x|C)$ some $b, c \in \mathbb{R}$ with $b < c$ such that $[b,c] \cap C = \emptyset$, $\frac{1}{2^{n+1}} < \frac{c-b}{2} \leq \frac{1}{2^n}$, and $|x - \frac{b+c}{2}| < \sum_{\ell=1}^{\infty} t_{m(n)+2\ell} + \frac{1}{2^{n+1}}$.

Since $x \in C$, we have that $b > x$ or $c < x$. So assume first that $b > x$. Then

$$\sum_{\ell=1}^{\infty} t_{m(n)+2\ell} + \frac{1}{2^{n+1}} > |x - \tfrac{b+c}{2}| = (b - x) + \tfrac{c-b}{2} > b - x + \tfrac{1}{2^{n+1}}$$

so $b < x + \sum_{\ell=1}^{\infty} t_{m(n)+2\ell} = a + \sum_{\ell\in F}\ t_\ell + \sum_{\ell=k}^{\infty} t_{2\ell} + \sum_{\ell=1}^{\infty} t_{m(n)+2\ell}$.

Now if $m(n)$ is odd, this tells us directly that $b < a + \sum_{\ell\in F}\ t_\ell + \sum_{\ell=2k}^{\infty} t_\ell$ while if $m(n)$ is even, one uses the fact that $m(n) > 2k$ and $t_{m(n)+2\ell} < t_{m(n)+2\ell-1}$ to again conclude that $b < a + \sum_{\ell\in F}\ t_\ell + \sum_{\ell=2k}^{\infty} t_\ell$.

In particular, $b < a + \sum_{\ell=1}^{\infty} t_\ell$. Thus, by Lemma 4.6, pick some $p \in \mathbb{N}$ and some $H \subseteq \{1, 2, \ldots, p-1\}$ such that $a + \sum_{\ell\in H}\ t_\ell + \sum_{\ell=p+1}^{\infty} \le b$ and $a + \sum_{\ell\in H}\ t_\ell + t_p \ge c$. Since neither b nor c is in C, both inequalities are strict. Then $c - b < t_p - \sum_{\ell=p+1}^{\infty} t_\ell = t_p - R_p$ and $c - b > \frac{1}{2^n}$ so $p \le m(n)$.

Letting $v = \lfloor \frac{m(n)}{2} \rfloor$, one thus obtains (again using the fact that $t_{m(n)+2\ell} < t_{m(n)+2\ell-1}$ in the case that $m(n)$ is even)

$$\begin{array}{rcl} a + \sum_{\ell\in H}\ t_\ell + \sum_{\ell=p+1}^{\infty} t_\ell & < & b \\ & < & x + \sum_{\ell=1}^{\infty} t_{m(n)+2\ell} \\ & \le & a + \sum_{\ell\in F}\ t_\ell + \sum_{\ell=k}^{v} t_{2\ell} + \sum_{\ell=m(n)+1}^{\infty} t_\ell \end{array}$$

and hence $\sum_{\ell\in H}\ t_\ell + \sum_{\ell=p+1}^{\infty} t_\ell < \sum_{\ell\in F}\ t_\ell + \sum_{\ell=k}^{v} t_{2\ell} + \sum_{\ell=m(n)+1}^{\infty} t_\ell$.

Since $\langle t_n\rangle_{n=1}^{\infty}$ is structured one has whenever $\sum_{\ell\in G}\ t_\ell < \sum_{\ell\in L}\ t_\ell$ that $\min(G\Delta L) \in L$. Thus, letting

$$r = \min\big((F \cup \{2k, 2k+2, \ldots, 2v\} \cup \{m(n)+1, m(n)+2, \ldots\})\Delta(H \cup \{p+1, p+2, \ldots\})\big)\ ,$$

one has that $r \in F \cup \{2k, 2k+2, \ldots, 2v\} \cup \{m(n)+1, m(n)+2, \ldots\}$ and in fact, since $p \le m(n)$, $r \in F \cup \{2k, 2k+2, \ldots, 2v\}$.

Also, $a + \sum_{\ell\in F}\ t_\ell + \sum_{\ell=k}^{\infty} t_{2\ell} = x < b < c < a + \sum_{\ell\in H}\ t_\ell + t_p$ so letting

$$s = \min\big((F \cup \{2k, 2k+2, \ldots\})\Delta(H \cup \{p\})\big)$$

one has that $s \in H \cup \{p\}$ (and in particular $s \le p$).

Now $r \in F \cup \{2k, 2k+2, \ldots, 2v\}$ while $s \notin F \cup \{2k, 2k+2, \ldots\}$ so $r \ne s$. Suppose that $r < s$. Then $r \in (F \cup \{2k, 2k+2, \ldots, 2v\})\backslash(H \cup \{p+1, p+2, \ldots\})$ and since $r < s$, $r \notin (F \cup \{2k, 2k+2, \ldots\})\backslash(H \cup \{p\}$, so $r = p$. But then $p \ge s > r = p$, a contradiction.

Thus we must have that $s < r$. Then $s \in (H \cup \{p\})\backslash(F \cup \{2k, 2k+2, \ldots\})$ and $s \notin (H \cup \{p+1, p+2, \ldots\})\backslash(F \cup \{2k, 2k+2, \ldots, 2v\} \cup \{m(n)+1, m(n)+2, \ldots\})$ and $s < r \le 2v$ so $s = p$. Also $r \notin \{p+1, p+2, \ldots\}$ so $p \ge r > s = p$, a contradiction.

This completes the case that $b > x$. So now assume that $c < x$. Then $b < c < x < a + \sum_{\ell=1}^{\infty} t_\ell$. Also,

$$\textstyle\sum_{\ell=1}^{\infty} t_{m(n)+2\ell} + \frac{1}{2^{n+1}} > |x - \frac{b+c}{2}| = x - c + \frac{c-b}{2} > x - c + \frac{1}{2^{n+1}}$$

so

$$c > x - \sum_{\ell=1}^{\infty} t_{m(n)+2\ell} = a + \sum_{\ell\in F}\ t_\ell + \sum_{\ell=k}^{\infty} t_{2\ell} - \sum_{\ell=1}^{\infty} t_{m(n)+2\ell} > a\ .$$

Thus, by Lemma 4.6, pick some $p \in \mathbb{N}$ and some $H \subseteq \{1, 2, \ldots, p-1\}$ such that $a + \sum_{\ell\in H}\ t_\ell + \sum_{\ell=p+1}^{\infty} \leq b$ and $a + \sum_{\ell\in H}\ t_\ell + t_p \geq c$. Since neither b nor c is in C, both inequalities are strict. Then $c - b < t_p - \sum_{\ell=p+1}^{\infty} t_\ell = t_p - R_p$ and $c - b > \frac{1}{2^n}$ so $p \leq m(n)$.

Now $a + \sum_{\ell\in H}\ t_\ell + t_p > c > x - \sum_{\ell=1}^{\infty} t_{m(n)+2\ell}$ so $a + \sum_{\ell\in H}\ t_\ell + t_p + \sum_{\ell=1}^{\infty} t_{m(n)+2\ell} > x$. Let $v = \lfloor \frac{m(n)}{2} \rfloor$. Then

$$x < a + \sum_{\ell\in H}\ t_\ell + t_p + \sum_{\ell=1}^{\infty} t_{2v+2\ell}$$

where, if $m(n)$ is odd, we use the fact that $t_{m(n)+2\ell} < t_{m(n)+2\ell-1}$.

Thus

$$\sum_{\ell\in F}\ t_\ell + \sum_{\ell=k}^{\infty} t_{2\ell} < \sum_{\ell\in H}\ t_\ell + t_p + \sum_{\ell=1}^{\infty} t_{2v+2\ell}\ .$$

Also, $x > c > b > a + \sum_{\ell\in H}\ t_\ell + \sum_{\ell=p+1}^{\infty} t_\ell$ so

$$\sum_{\ell\in F}\ t_\ell + \sum_{\ell=k}^{\infty} t_{2\ell} > \sum_{\ell\in H}\ t_\ell + \sum_{\ell=p+1}^{\infty} t_\ell\ .$$

Let

$$r = \min\big((F \cup \{2k, 2k+2, \ldots\})\Delta(H \cup \{p+1, p+2, \ldots\})\big)$$

and

$$s = \min\big((H \cup \{p\} \cup \{2v+2, 2v+4, \ldots\})\Delta(F \cup \{2k, 2k+2, \ldots\})\big)\ .$$

Then

$$r \in (F \cup \{2k, 2k+2, \ldots\})\backslash(H \cup \{p+1, p+2, \ldots\})$$

and

$$s \in (H \cup \{p\} \cup \{2v+2, 2v+4, \ldots\})\backslash(F \cup \{2k, 2k+2, \ldots\})\ .$$

One sees immediately that $s \neq r$. Since $r \notin \{p+1, p+2, \ldots\}$, $r \leq p$. Suppose that $s < r$. Then $s < r \leq p \leq m(n) \leq 2v+1$, so $s \in H\backslash(F \cup \{2k, 2k+2, \ldots\})$, contradicting the minimality of r.

Consequently $r < s$. Now $r \in (F \cup \{2k, 2k+2, \ldots\})\backslash H$ and $r \leq p < 2v+2$ so if $r \neq p$, then $r \in (F \cup \{2k, 2k+2, \ldots\})\backslash(H \cup \{p\} \cup \{2v+2, 2v+4, \ldots\})$ contradicting the minimality of s. Thus $p = r < s$. Since $s \in (H \cup \{p\} \cup \{2v+2, 2v+4, \ldots\})$ one has $s \in \{2v+2, 2v+4, \ldots\}$. Since $2v+2 > m(n) > 2k$ we have that $s \in \{2k, 2k+2, \ldots\}$, a contradiction. □

Now we are ready for our final preliminary result.

4.9 Lemma. *Let $\epsilon > 0$. There is a closed nowhere dense set $E \subseteq [0,1]$ such that $\mu(E) > 1-\epsilon$ and for every $x \in E$,*

$$\textstyle\sum_{n=1}^{\infty} \frac{1}{2^n \cdot \alpha_n(x|E)} = \infty$$

.

Proof. On pages 269 through 271 of [2] it is proved that:
For any $\delta > 0$ and any $K > 0$ there is a set $Z \subseteq [0,1]$ such that $\mu(Z) > 1-\delta$ and for almost all x in Z, $\sum_{n=1}^{\infty} \frac{1}{2^n \cdot \alpha_n(x|Z)} \geq K$.

Notice that whenever $B \subseteq C$, $x \in \mathbb{R}$, and $n \in \mathbb{N}$, one has $\alpha_n(x|B) \leq \alpha_n(x|C)$. By discarding a set of measure 0 one may presume that $\sum_{n=1}^{\infty} \frac{1}{2^n \cdot \alpha_n(x|Z)} \geq K$ for all $x \in Z$. Also, since as we have already noted, given any measurable set C and any $\gamma > 0$ there is a compact subset D of C with $\mu(D) > \mu(C) - \gamma$ [4, Definition 3.8], one may also presume that Z is compact.

Now for each $j \in \mathbb{N}$ choose compact $Z_j \subseteq [0,1]$ such that

$$\mu(Z_j) > 1 - \frac{\epsilon}{2^j} \text{ and for each } x \in Z_j,\ \textstyle\sum_{n=1}^{\infty} \frac{1}{2^n \cdot \alpha_n(x|Z_j)} \geq j\ .$$

Let $E = \bigcap_{j=1}^{\infty} Z_j$. Then E is clearly as required, except possibly for the assertion that E is nowhere dense. So suppose E has nonempty interior. Given x in the interior of E one has some $\nu > 0$ such that $\alpha_n(x|E) \geq \nu$ for every n and hence $\sum_{n=1}^{\infty} \frac{1}{2^n \cdot \alpha_n(x|E)} < \infty$, a contradiction. □

We show now that one cannot be guaranteed a translate of some fat set of sums in a set of positive measure, even quite large measure.

4.10 Theorem. *Let $\epsilon > 0$. There is a compact set $E \subseteq [0,1]$ with $\mu(E) > 1-\epsilon$ such that there do not exist any structured sequence $\langle t_n \rangle_{n=1}^{\infty}$ and any $a \in [0,1)$ with $\mu\big(AS(\langle t_n \rangle_{n=1}^{\infty})\big) > 0$ and $a + AS(\langle t_n \rangle_{n=1}^{\infty}) \subseteq E$.*

Proof. Let E be as guaranteed by Lemma 4.9. Then Lemma 4.8 together with the observation that whenever $B \subseteq C$, $x \in \mathbb{R}$, and $n \in \mathbb{N}$, one has $\alpha_n(x|B) \leq \alpha_n(x|C)$, establish the conclusion. (For this, one only needs that the set D of Lemma 4.8 is nonempty.) □

Notice that since the set E of Theorem 4.10 is compact, one has in fact that there do not exist any structured sequence $\langle t_n \rangle_{n=1}^{\infty}$ and any $a \in [0,1)$ with $\mu\big(AS(\langle t_n \rangle_{n=1}^{\infty})\big) > 0$ and $a + FS(\langle t_n \rangle_{n=1}^{\infty}) \subseteq E$. In this respect we can contrast the situation in which $\mu(E) = 1$.

4.11 Theorem. *Let $A \subseteq (0,1]$ with $\mu(A) = 1$. Then for each $\epsilon > 0$ there is a structured sequence $\langle t_n \rangle_{n=1}^{\infty}$ with $\mu\big(AS(\langle t_n \rangle_{n=1}^{\infty})\big) > 1 - \epsilon$ and $FS(\langle t_n \rangle_{n=1}^{\infty}) \subseteq A$.*

Proof. Pick $\gamma \in \mathbb{N}$ with $\frac{1}{2^{\gamma-2}} < \epsilon$. Let $A_1 = A$. Now $A_1 \cap (\frac{1}{2} - \frac{1}{2^\gamma}, \frac{1}{2}) \neq \emptyset$ so pick $t_1 \in A_1 \cap (\frac{1}{2} - \frac{1}{2^\gamma}, \frac{1}{2})$ and let $A_2 = A_1 \cap (A_1 - t_1)$ and observe that $\mu\big(A_2 \cap (0, \frac{1}{2})\big) = \frac{1}{2}$. Inductively, given A_k with $\mu\big(A_k \cap (0, \frac{1}{2^{k-1}})\big) = \frac{1}{2^{k-1}}$ and given

$$t_{k-1} \in (\frac{1}{2^{k-1}} - \frac{1}{2^{\gamma+k-3}} + \frac{1}{2^{\gamma+2k-4}}, \frac{1}{2^{k-1}})$$

observe that $A_k \cap (\frac{t_{k-1}}{2} - \frac{1}{2^{\gamma+2k-2}}, \frac{t_{k-1}}{2}) \neq \emptyset$ and pick $t_k \in A_k \cap (\frac{t_{k-1}}{2} - \frac{1}{2^{\gamma+2k-2}}, \frac{t_{k-1}}{2})$. Let $A_{k+1} = A_k \cap (A_k - t_k)$ and note that, since $t_{k-1} < \frac{1}{2^{k-1}}$ we have $\mu\big(A_{k+1} \cap (0, \frac{1}{2^k})\big) = \frac{1}{2^k}$. Observe also that $t_k < \frac{t_{k-1}}{2} < \frac{1}{2^k}$ and

$$\begin{aligned} t_k &> \tfrac{t_{k-1}}{2} - \tfrac{1}{2^{\gamma+2k-2}} \\ &> \tfrac{1}{2^k} - \tfrac{1}{2^{\gamma+k-2}} + \tfrac{1}{2^{\gamma+2k-3}} - \tfrac{1}{2^{\gamma+2k-2}} \\ &= \tfrac{1}{2^k} - \tfrac{1}{2^{\gamma+k-2}} + \tfrac{1}{2^{\gamma+2k-2}} \ . \end{aligned}$$

As in the proof of Theorem 2.6 one sees that $FS(\langle t_n \rangle_{n=1}^{\infty}) \subseteq A$. Since for each $k > 1$, $t_k < \frac{t_{k-1}}{2}$, the sequence $\langle t_n \rangle_{n=1}^{\infty}$ is structured. Also

$$\begin{aligned} R_n &= \textstyle\sum_{k=n+1}^{\infty} t_k > \sum_{k=n+1}^{\infty} (\frac{1}{2^k} - \frac{1}{2^{\gamma+k-2}} + \frac{1}{2^{\gamma+2k-2}}) \\ &> \textstyle\sum_{k=n+1}^{\infty} (\frac{1}{2^k} - \frac{1}{2^{\gamma+k-2}}) \\ &= \tfrac{1}{2^n} - \tfrac{1}{2^{\gamma+n-2}} \end{aligned}$$

so $2^n R_n > 1 - \frac{1}{2^{\gamma-2}} > 1 - \epsilon$. Consequently, by Lemma 4.4, $\mu\big(AS(\langle t_n \rangle_{n=1}^{\infty})\big) > 1 - \epsilon$. $\square$

The following result shows, however, that one can get a set of measure 1 which does not contain a translate of a fat all-sums set. Theorem 4.10 leaves open the possibility that translates of fat sets of sums of structured sequences could be partition regular. That possibility is also eliminated by the next result.

4.12 Theorem. *There exist disjoint measurable (in fact Borel) sets A and B such that $\mu(A) = 1$, $A \cup B = [0,1]$, and for no structured sequence $\langle t_n \rangle_{n=1}^{\infty}$ with $\mu\big(AS(\langle t_n \rangle_{n=1}^{\infty})\big) > 0$ and for no $a \in [0,1)$ is $a + AS(\langle t_n \rangle_{n=1}^{\infty}) \subseteq A$ (and of course, since $\mu(B) = 0$, for no structured sequence $\langle t_n \rangle_{n=1}^{\infty}$ with $\mu\big(AS(\langle t_n \rangle_{n=1}^{\infty})\big) > 0$ and for no $a \in [0,1)$ is $a + AS(\langle t_n \rangle_{n=1}^{\infty}) \subseteq B$).*

Proof. For each $n \in \mathbb{N}$ pick a set E_n as guaranteed by Lemma 4.9 for $\epsilon = \frac{1}{n}$. Let $A = \bigcup_{n=1}^{\infty} E_n$ and let $B = [0,1] \backslash A$.

Now suppose one has a sequence $\langle t_n \rangle_{n=1}^{\infty}$ with $\mu\big(AS(\langle t_n \rangle_{n=1}^{\infty})\big) > 0$ and $a \in [0,1)$ with $a + AS(\langle t_n \rangle_{n=1}^{\infty}) \subseteq A$. Let $C = a + AS(\langle t_n \rangle_{n=1}^{\infty})$ and pick by Lemma 4.8 a dense subset D of C such that, for every $x \in D$, $\sum_{n=1}^{\infty} \frac{1}{2^n \cdot \alpha_n(x|C)} < \infty$. We claim that for each $m \in \mathbb{N}$, $C \backslash E_m$ is dense in C. Indeed, suppose instead that there is some open set $U \subseteq \mathbb{R}$ such that $\emptyset \neq U \cap C \subseteq E_m$. Pick $x \in D \cap U$. Since U is open, one either has some $\gamma > 0$ such that for all $n \in \mathbb{N}$, $\alpha_n(x|U \cap C) \geq \gamma$, or for sufficiently large n, $\alpha_n(x|U \cap C) = \alpha_n(x|C)$. In either case we have $\sum_{n=1}^{\infty} \frac{1}{2^n \cdot \alpha_n(x|U \cap C)} < \infty$. On the other hand, for each n, $\alpha_n(x|U \cap C) \leq \alpha_n(x|E_m)$ so $\sum_{n=1}^{\infty} \frac{1}{2^n \cdot \alpha_n(x|U \cap C)} = \infty$. This contradiction establishes that $C \backslash E_m$ is dense as claimed.

Consequently, by the Baire Category Theorem, one has $C \backslash A \neq \emptyset$, a contradiction.

□

4.13 Question. *Does Theorem* 4.12 *remain valid with the word "structured" removed?*

5. A fat all-sums set with 0 as a density point of its complement.

We saw in Theorem 4.1 that whenever $\mu\big(AS(\langle t_n \rangle_{n=1}^{\infty})\big) > 0$, one has that $AS(\langle t_n \rangle_{n=1}^{\infty})$ is measurably large (at 0). We will show in Theorem 5.2 that if the sequence $\langle t_n \rangle_{n=1}^{\infty}$ is structured, much more is true. That is, 0 is a density point of $AS(\langle t_n \rangle_{n=1}^{\infty})$. By way of contrast, we show in Corollary 5.7 that it is possible to have a (necessarily nonstructured) sequence $\langle t_n \rangle_{n=1}^{\infty}$ such that $\mu\big(AS(\langle t_n \rangle_{n=1}^{\infty})\big) > 0$ and 0 is a density point of the complement of $AS(\langle t_n \rangle_{n=1}^{\infty})$.

We start with a preliminary lemma.

5.1 Lemma. *Let $\langle t_n \rangle_{n=1}^{\infty}$ be a structured sequence such that $\mu\big(AS(\langle t_n \rangle_{n=1}^{\infty})\big) > 0$. Then for any $F \in \mathcal{P}_f(\mathbb{N})$,*

$$\mu\big(AS(\langle t_n \rangle_{n=1}^{\infty}) \cap (0, \textstyle\sum_{n \in F} t_n)\big) = \textstyle\sum_{m \in F} \frac{\mu\big(AS(\langle t_n \rangle_{n=1}^{\infty})\big)}{2^m} .$$

Proof. We proceed by induction on $|F|$. If $F = \{m\}$, we have $AS(\langle t_n \rangle_{n=1}^{\infty}) \cap (0, t_m) = AS(\langle t_n \rangle_{n=m+1}^{\infty})$ so the result is simply Lemma 4.4 applied to the sequence $\langle s_n \rangle_{n=1}^{\infty}$, where $s_n = t_{m+n}$. So assume that $|F| > 1$, let $m = \min F$ and let $G = F \backslash \{m\}$. Then we notice that

$$AS(\langle t_n \rangle_{n=1}^{\infty}) \cap (0, \textstyle\sum_{n \in F} t_n) =$$
$$\big(AS(\langle t_n \rangle_{n=1}^{\infty}) \cap (0, t_m]\big) \cup \Big(t_m + \big(AS(\langle t_n \rangle_{n=1}^{\infty}) \cap (0, \textstyle\sum_{n \in G} t_n)\big)\Big) .$$

This suffices, since then we have

$$\mu\big(AS(\langle t_n \rangle_{n=1}^{\infty}) \cap (0, \textstyle\sum_{n \in F} t_n)\big) = \frac{\mu\big(AS(\langle t_n \rangle_{n=1}^{\infty})\big)}{2^m} + \textstyle\sum_{k \in G} \frac{\mu\big(AS(\langle t_n \rangle_{n=1}^{\infty})\big)}{2^k} .$$

$\square$

5.2 Theorem. *Let $\langle t_n\rangle_{n=1}^\infty$ be a structured sequence such that $\mu\big(AS(\langle t_n\rangle_{n=1}^\infty)\big) > 0$. Then 0 is a density point of $AS(\langle t_n\rangle_{n=1}^\infty)$.*

Proof. Let $\epsilon > 0$ be given and pick $\ell \in \mathbb{N}$ such that $\frac{2^\ell - 1}{2^\ell} > 1 - \epsilon$. Let $\gamma = \mu\big(AS(\langle t_n\rangle_{n=1}^\infty)\big)$. We claim that it suffices to establish

(*) Whenever $F \subseteq \{0,1,\ldots,\ell\}$ with $0 \in F$, $k = \max\{-1,0,1,\ldots,\ell\}\backslash F$, and $G = (F \cap \{0,1,\ldots,k-1\}) \cup \{k\}$, one has

$$\lim_{m\to\infty} \frac{\sum_{n\in F}\ (\gamma/2^{m+n})}{\sum_{n\in G}\ t_{m+n}} > \frac{2^\ell - 1}{2^\ell}\ .$$

Indeed, assume that we have established (*). Since there are only finitely many subsets of $\{0,1,\ldots,\ell\}$, pick $p \in \mathbb{N}$ such that whenever $m > p$ and F and G are as in (*), one has $\frac{\sum_{n\in F}\ (\gamma/2^{m+n})}{\sum_{n\in G}\ t_{m+n}} > \frac{2^\ell-1}{2^\ell}$. We show that whenever $0 < \alpha < t_p$ we have $\mu\big(AS(\langle t_n\rangle_{n=1}^\infty) \cap (0,\alpha)\big) > (1-\epsilon)\cdot\alpha$. So let α with $0 < \alpha < t_p$ be given, pick m such that $t_m \leq \alpha < t_{m-1}$, and note that $m > p$. Pick $F \subseteq \{0,1,\ldots,\ell\}$ with $0 \in F$ and $\sum_{n\in F}\ 2^{\ell-n}$ a maximum among all

$$\{\textstyle\sum_{n\in H}\ 2^{\ell-n} : 0 \in H \subseteq \{0,1,\ldots,\ell\} \text{ and } \sum_{n\in H} t_{m+n} \leq \alpha\}\ .$$

(Note that $H = \{0\}$ satisfies the requirements so the listed set is nonempty.) Now let G be as in (*), so that $\sum_{n\in G} 2^{\ell-n} = \sum_{n\in F} 2^{\ell-n} + 1$. Now if $F = \{0,1,\ldots,\ell\}$, then $G = \{-1\}$ so $\sum_{n\in G}\ t_{m+n} = t_{m-1} > \alpha$; if $F \neq \{0,1,\ldots,\ell\}$, then $G \subseteq \{0,1,\ldots,\ell\}$ and $\sum_{n\in G} 2^{\ell-n} > \sum_{n\in F} 2^{\ell-n}$. That is, we have $\sum_{n\in F}\ t_{m+n} \leq \alpha < \sum_{n\in G}\ t_{m+n}$.

Now we have

$$\mu\big(AS(\langle t_n\rangle_{n=1}^\infty) \cap (0,\alpha)\big) \geq \mu\big(AS(\langle t_n\rangle_{n=1}^\infty) \cap (0, \textstyle\sum_{n\in F}\ t_{m+n})\big) = \sum_{n\in F}\ \frac{\gamma}{2^{m+n}}$$

by Lemma 5.1. Thus by (*) we have

$$\mu\big(AS(\langle t_n\rangle_{n=1}^\infty) \cap (0,\alpha)\big) > \frac{2^\ell - 1}{2^\ell} \cdot \textstyle\sum_{n\in G}\ t_{m+n} > (1-\epsilon)\cdot\alpha$$

as required.

Now to establish (*), first observe that

$$\frac{\sum_{n\in F}\ 2^{\ell-n}}{\sum_{n\in G}\ 2^{\ell-n}} = \frac{\sum_{n\in F}\ 2^{\ell-n}}{\sum_{n\in F}\ 2^{\ell-n} + 1} > \frac{2^\ell - 1}{2^\ell}$$

so it suffices to show that

$$\lim_{m\to\infty} \frac{\sum_{n\in F}\ (\gamma/2^{m+n})}{\sum_{n\in G}\ t_{m+n}} = \frac{\sum_{n\in F}\ 2^{\ell-n}}{\sum_{n\in G}\ 2^{\ell-n}}\ .$$

Pick a sequence $\langle \delta_n\rangle_{n=1}^{\infty}$ as guaranteed by Lemma 4.5. Then we have that

$$\lim_{m\to\infty} \frac{\gamma}{2^{m+\ell}\cdot R_{m+\ell}} = \lim_{m\to\infty} \frac{\gamma}{2^{m+\ell}\cdot(\gamma+\delta_{m+\ell})\cdot(1/2^{m+\ell})} = 1\ .$$

Further, for any $i\in\mathbb{N}$, we have

$$\lim_{m\to\infty} \frac{t_m}{R_{m+i}} = \lim_{m\to\infty} \frac{(\gamma-\delta_m)/2^m+\delta_{m-1}/2^{m-1}}{(\gamma+\delta_{m+i})/2^{m+i}} = 2^i\ .$$

Thus we have

$$\lim_{m\to\infty} \frac{\sum_{n\in G}\ t_{m+n}}{R_{m+\ell}} = \sum_{n\in G} \lim_{m\to\infty} \frac{t_{m+n}}{R_{m+\ell}} = \sum_{n\in G}\ 2^{\ell-n}$$

and hence

$$\lim_{m\to\infty} \frac{\sum_{n\in G}\ 2^{\ell-n}\cdot R_{m+\ell}}{\sum_{n\in G}\ t_{m+n}} = 1\ .$$

Putting these limits together, we have

$$\lim_{m\to\infty} \frac{\sum_{n\in F}\ (\gamma/2^{m+n})}{\sum_{n\in G}\ t_{m+n}} =$$

$$\frac{\sum_{n\in F}\ 2^{\ell-n}}{\sum_{n\in G}\ 2^{\ell-n}}\cdot \lim_{m\to\infty} \frac{\sum_{n\in G}\ 2^{\ell-n}\cdot R_{m+\ell}}{\sum_{n\in G}\ t_{m+n}}\cdot \lim_{m\to\infty} \frac{\gamma}{2^{m+\ell}\cdot R_{m+\ell}} =$$

$$\frac{\sum_{n\in F}\ 2^{\ell-n}}{\sum_{n\in G}\ 2^{\ell-n}}$$

as required. □

What happens to the first few terms of a sequence is clearly irrelevant, so if eventually $t_n \geq R_n$ (so that, in a manner of speaking, $\langle t_n\rangle_{n=1}^{\infty}$ is eventually a structured sequence) and $\mu\big(AS(\langle t_n\rangle_{n=1}^{\infty})\big) > 0$, then 0 is a density point of $AS(\langle t_n\rangle_{n=1}^{\infty})$. Also, as we have previously remarked, (before Lemma 4.4) there is some $\epsilon > 0$ with $(0,\epsilon) \subseteq AS(\langle t_n\rangle_{n=1}^{\infty})$ if and only if eventually $t_n \leq R_n$.

We show in this section that one can get sequences $\langle t_n\rangle_{n=1}^{\infty}$ with $\mu\big(AS(\langle t_n\rangle_{n=1}^{\infty})\big)$ as close to 1 as we desire, in fact with $AS(\langle t_n\rangle_{n=1}^{\infty})$ containing an interval of length as close to 1 as we desire, such that the density of $AS(\langle t_n\rangle_{n=1}^{\infty})$ at 0 is 0, i.e. 0 is a density point of $(0,1]\backslash AS(\langle t_n\rangle_{n=1}^{\infty})$. (Necessarily of course, often $t_n > R_n$ and often $t_n < R_n$.)

Given an infinite set B we write $AS(B) = \{\sum F : \emptyset \neq F \subseteq B\}$ so that if $\langle t_n\rangle_{n=1}^{\infty}$ enumerates B then $AS(B) = AS(\langle t_n\rangle_{n=1}^{\infty})$.

5.3 Definition. Let $r \in \mathbb{N}$. Define sequences $\langle T_n \rangle_{n=0}^{\infty}$, $\langle L_n \rangle_{n=1}^{\infty}$, and $\langle N_n \rangle_{n=1}^{\infty}$ by: $T_0 = 20$, $L_1 = r$, and for each $n \in \mathbb{N}$, $N_n = 2^{T_{n-1}} - 2$, $T_n = 2^{T_{n-1}/4} - 4$, and $L_{n+1} = L_n + N_n$. Let

$$B_r = \{2^{-1}, 2^{-2}, \ldots, 2^{-r}\} \cup \bigcup_{n=1}^{\infty} \{2^{-L_n} + 2^{-L_n - t} : t \in \{1, 2, \ldots, N_n\}\} .$$

5.4 Theorem. *Let $r \in \mathbb{N}$ and let $A = AS(B_r)$. The density of A at 0 is 0. That is* $\lim_{\delta \downarrow 0} \frac{\mu\big(A \cap (0,\delta)\big)}{\delta} = 0$.

Proof. We first observe that for any $n \in \mathbb{N}$,

(*) $$\sum_{\ell=n+1}^{\infty} \sum_{t=1}^{N_\ell} (2^{-L_\ell} + 2^{-L_\ell - t}) < 2^{-L_{n+1} + T_n}.$$

Indeed

$$\begin{aligned} \sum_{\ell=n+1}^{\infty} \sum_{t=1}^{N_\ell} (2^{-L_\ell} + 2^{-L_\ell - t}) &= \sum_{\ell=n+1}^{\infty} (N_\ell \cdot 2^{-L_\ell} + 2^{-L_\ell} - 2^{-L_{\ell+1}}) \\ &= \sum_{\ell=n+1}^{\infty} \big((2^{T_{\ell-1}} - 2) \cdot 2^{-L_\ell} + 2^{-L_\ell} - 2^{-L_{\ell+1}}\big) \\ &= \sum_{\ell=n+1}^{\infty} (2^{-L_\ell + T_{\ell-1}} - 2^{-L_\ell} - 2^{-L_{\ell+1}}) \\ &< 2^{-L_{n+1} + T_n} . \end{aligned}$$

Now let $\epsilon > 0$ be given and pick $p \in \mathbb{N}$ such that $p > 1$ and $1/2^{T_{p-1}/6} < \epsilon/4$. We will show that for any $\delta < 1/2^{L_{p-1}}$, $\frac{\mu\big(A \cap (0,\delta)\big)}{\delta} < \epsilon$. So let $\delta < 1/2^{L_{p-1}}$ be given. Pick $n \in \mathbb{N}$ with $2^{-L_{n-1}} \geq \delta > 2^{-L_n}$ (so $n \geq p$) and pick the largest $k \in \{0, 1, \ldots, T_{n-1} - 1\}$ with $2^{-L_n + k} \leq \delta$.

Now observe that if $x \in (0,\delta) \cap A$, then $x < 2^{-L_n + k + 1}$. (If $k \leq T_{n-1} - 2$ this is because $\delta < 2^{-L_n + k + 1}$, so assume that $k = T_{n-1} - 1$. Then $x < \delta \leq 2^{-L_{n-1}}$ so $x = \sum C$ for some

$$C \subseteq \bigcup_{\ell=n}^{\infty} \{2^{-L_\ell} + 2^{-L_\ell - t} : t \in \{1, 2, \ldots, N_\ell\}\}$$

so $x \leq \sum_{\ell=n}^{\infty} \sum_{t=1}^{N_\ell} (2^{-L_\ell} + 2^{-L_\ell - t}) < 2^{-L_n + T_{n-1}} = 2^{-L_n + k + 1}$ by (*).)

Now let

$$\begin{aligned} H = \{m \cdot 2^{L_n} + \textstyle\sum_{t \in F} 2^{-t} : \quad & m \in \{0, 1, \ldots, 2^{k+1} - 1\}, \\ & F \subseteq \{L_n + 1, L_n + 2, \ldots, L_{n+1} - T_n\}, \text{ and} \\ & m - T_n \leq |F| \leq m|\} . \end{aligned}$$

We claim that it suffices to establish the following two statements:

(**) $$|H| \leq 2^{N_n - T_n + k + 1 - T_{n-1}/6} \text{ and}$$

(***) $$(0,\delta) \cap A \subseteq \bigcup_{z \in H} [z, z + 2^{-L_{n+1} + T_n + 1}].$$

Indeed, assume for the moment that we have established (**) and (***). Then

$$\begin{aligned}\frac{\mu\big((0,\delta)\cap A\big)}{\delta} &\le \frac{|H|\cdot 2^{-L_{n+1}+T_n+1}}{2^{-L_n+k}}\\ &= \frac{|H|}{2^{N_n-T_n+k-1}} \le 2^{2-T_{n-1}/6}\\ &\le 2^{2-T_{p-1}/6} < \epsilon\end{aligned}$$

as required.

To establish (**) observe that by Stirling's formula we have for any even $M > 0$ and any $t \in \{0,1,\ldots,M\}$, $\begin{pmatrix} M \\ t \end{pmatrix} < \frac{2^M}{\sqrt{M}}$. Observe also that

$$\begin{aligned}\frac{(T_n+1)}{\sqrt{N_n-T_n}} &= \frac{2^{T_{n-1}/4}-3}{\sqrt{2^{T_{n-1}}-2-2^{T_{n-1}/4}+4}}\\ &< \frac{2^{T_{n-1}/4}}{\sqrt{2^{T_{n-1}}-2^{T_{n-1}/4}}}\\ &< \frac{1}{2^{T_{n-1}/6}}\ .\end{aligned}$$

Now for each $m \in \{0,1,\ldots,2^{k+1}-1\}$, let

$$\begin{aligned}H_m = \{m\cdot 2^{-L_n} + \textstyle\sum_{t\in F}\ 2^{-t}:\quad & F \subseteq \{L_n+1, L_n+2,\ldots,L_{n+1}-T_n\} \text{ and}\\ & m-T_n \le |F| \le m\}\ .\end{aligned}$$

Then $|H| \le \sum_{m=0}^{2^{k+1}-1} |H_m|$. Let $M = N_n - T_n = |\{L_n+1, L_n+2,\ldots,L_{n+1}-T_n\}|$. Now, if $m \le T_n$, then

$$H_m = \textstyle\sum_{i=0}^{m} \begin{pmatrix} M \\ i \end{pmatrix} < \sum_{i=0}^{m} \frac{2^M}{\sqrt{M}} = \frac{(m+1)\cdot 2^M}{\sqrt{M}} \le \frac{(T_n+1)\cdot 2^M}{\sqrt{M}}\ .$$

If $m > T_n$, then

$$H_m = \textstyle\sum_{i=m-T_n}^{m} \begin{pmatrix} M \\ i \end{pmatrix} < \sum_{i=m-T_n}^{m} \frac{2^M}{\sqrt{M}} = \frac{(T_n+1)\cdot 2^M}{\sqrt{M}}\ .$$

Thus

$$|H| < \frac{2^{k+1}\cdot(T_n+1)\cdot 2^M}{\sqrt{M}} = \frac{2^{N_n-T_n+k+1}\cdot(T_n+1)}{\sqrt{N_n-T_n}} < 2^{N_n-T_n+k+1-T_{n-1}/6}\ .$$

Thus (**) holds.

Finally, we verify (***). Let $x \in (0,\delta)\cap A$ be given. For each $\ell \in \mathbb{N}$ with $\ell \ge n$, pick $G(\ell) \subseteq \{1,2,\ldots,N_\ell\}$ such that $x = \sum_{\ell=n}^{\infty}\sum_{t\in G(\ell)}(2^{-L_\ell} + 2^{-L_\ell - t})$. Let

$y = \sum_{t\in G(n)}(2^{-L_n} + 2^{-L_n-t})$ (so that $y = 0$ if $G(n) = \emptyset$). Let $m = |G(n)|$. Then $y = m \cdot 2^{-L_n} + \sum_{t\in G(n)} 2^{-L_n-t}$. Since $y \le x < 2^{-L_n+k+1}$ we have $m \le 2^{k+1} - 1$. Let

$$F = \{L_n + t : t \in G(n) \text{ and } t \le N_n - T_n\}$$

and let $z = m \cdot 2^{-L_n} + \sum_{t\in F} 2^{-t}$. Now $m \in \{0, 1, \ldots, 2^{k+1} - 1\}$, $F \subseteq \{L_n + 1, L_n + 2, \ldots, L_n + N_n - T_n\}$, $|F| \le |G(n)| = m$, and $|F| \ge |G(n)| - T_n$ so $z \in H$.

Certainly $z \le x$. Let $K = \{L_n + t : t \in G(n) \text{ and } t > N_n - T_n\}$. Then $y = z + \sum_{t\in K} 2^{-t}$ so

$$\begin{array}{rcll} x & = & z + \sum_{t\in K} 2^{-t} + \sum_{\ell=n+1}^{\infty} \sum_{t\in G(\ell)}(2^{-L_\ell} + 2^{-L_\ell-t}) & \\ & \le & z + \sum_{t=N_n-T_n+1}^{N_n} 2^{-L_n-t} + \sum_{\ell=n+1}^{\infty} \sum_{t=1}^{N_\ell}(2^{-L_\ell} + 2^{-L_\ell-t}) & \\ & < & z + 2^{-L_n-N_n+T_n} + 2^{-L_{n+1}+T_n} & \text{(by (*))} \\ & = & z + 2^{-L_{n+1}+T_n+1} & \end{array}$$

so (***) holds. □

We now set out to show that $AS(B_r)$ contains intervals. The following lemma shows that we can write certain terminating binary expressions as members of $AS(B_r)$.

5.5 Lemma. *Let $r, n \in \mathbb{N}$ and let $\sum_{\ell=1}^{n} \sum_{t=1}^{N_\ell}(2^{-L_\ell} + 2^{-L_\ell-t}) \le x < 1$ such that $2^{L_{n+1}} \cdot x \in \mathbb{N}$. Then there exist $G_0 \subseteq \{1, 2, \ldots, r\}$ and for each $\ell \in \{1, 2, \ldots, n\}$ some $G_\ell \subseteq \{1, 2, \ldots, N_\ell\}$ such that*

$$x = \sum_{t\in G_0} 2^{-t} + \sum_{\ell=1}^{n} \sum_{t\in G_\ell}(2^{-L_\ell} + 2^{-L_\ell-t}) \ .$$

Proof. We proceed by induction on n. First let $n = 1$. We have that

$$2^{L_2} \cdot x \in \{1, 2, \ldots, 2^{L_2} - 1\}$$

so pick $F \subseteq \{0, 1, \ldots, L_2 - 1\}$ such that $2^{L_2} \cdot x = \sum_{t\in F} 2^t$. Then $x = \sum_{t\in F} 2^{t-L_2}$. Let $G_1 = \big\{N_1 - t : t \in F \cap \{0, 1, \ldots, N_1 - 1\}\big\}$ and let $H = F \cap \{N_1, N_1 + 1, \ldots, L_2 - 1\}$. Let $y = x - \sum_{t\in G_1}(2^{-L_1} + 2^{-L_1-t})$. Then $y = \sum_{t\in H} 2^{t-L_2} - |G_1| \cdot 2^{-L_1}$ so $2^{L_1} \cdot y$ is an integer. Also $y \ge x - \sum_{t=1}^{N_1}(2^{-L_1} + 2^{-L_1-t}) \ge 0$. Finally $y \le x < 1$ so $2^{L_1} \cdot y \in \{0, 1, \ldots, 2^{L_1} - 1\}$. Choose $K \subseteq \{0, 1, \ldots, L_1 - 1\}$ such that $2^{L_1} \cdot y = \sum_{t\in K} 2^t$ and let $G_0 = \{L_1 - t : t \in K\}$. Then $x = \sum_{t\in G_0} 2^{-t} + \sum_{t\in G_1}(2^{-L_1} + 2^{-L_1-t})$ as required.

Now let $n > 1$ be given and let $\sum_{\ell=1}^{n} \sum_{t=1}^{N_\ell}(2^{-L_\ell} + 2^{-L_\ell-t}) \le x < 1$ with $2^{L_{n+1}} \cdot x \in \mathbb{N}$. Pick $F \subseteq \{0, 1, \ldots, L_{n+1} - 1\}$ such that $2^{L_{n+1}} \cdot x = \sum_{t\in F} 2^t$. Let

$$G_n = \big\{N_n - t : t \in F \cap \{0, 1, \ldots, N_n - 1\}\big\}$$

and let $H = F \cap \{N_n, N_n+1, \ldots, L_{n+1}-1\}$. Let $y = x - \sum_{t\in G_n}(2^{-L_n} + 2^{-L_n-t})$. Then $y = \sum_{t\in H}\ 2^{t-L_{n+1}} - |G_n| \cdot 2^{-L_n}$ so $2^{L_n} \cdot y$ is an integer. Also

$$\begin{array}{rcl} \sum_{\ell=1}^{n-1}\sum_{t=1}^{N_\ell}(2^{-L_\ell} + 2^{-L_\ell-t}) & \leq & x - \sum_{t=1}^{N_n}(2^{-L_n} + 2^{-L_n-t}) \\ & \leq & x - \sum_{t\in G_n}(2^{-L_n} + 2^{-L_n-t}) \\ & = & y \leq x < 1\ . \end{array}$$

Thus y satisfies the induction hypothesis so pick $G_0 \subseteq \{1, 2, \ldots, r\}$ and for each $\ell \in \{1, 2, \ldots, n\}$ some $G_\ell \subseteq \{1, 2, \ldots, N_\ell\}$ such that

$$y = \sum_{t\in G_0}\ 2^{-t} + \sum_{\ell=1}^{n-1}\sum_{t\in G_\ell}(2^{-L_\ell} + 2^{-L_\ell-t})\ .$$

□

5.6 Theorem. *Let $r \in \mathbb{N}$ and let $\gamma = \sum_{\ell=1}^{\infty}\sum_{t=1}^{N_\ell}(2^{-L_\ell} + 2^{-L_\ell-t})$. Then $(\gamma, 1) \subseteq AS(B_r)$.*

Proof. Let $x \in (\gamma, 1)$ and pick $H \subseteq \mathbb{N}$ with $x = \sum_{t\in H}\ 2^{-t}$. For each n let $F_n = H \cap \{1, 2, \ldots, L_{n-1}\}$ and let $x_n = \sum_{t\in F_n}\ 2^{-t}$. Then $x_n \leq x < 1$. We claim that also $x_n \geq \sum_{\ell=1}^{n}\sum_{t=1}^{N_\ell}(2^{-L_\ell} + 2^{-L_\ell-t})$. Indeed, $x > \sum_{\ell=1}^{\infty}\sum_{t=1}^{N_\ell}(2^{-L_\ell} + 2^{-L_\ell-t})$ and $x - x_n \leq \sum_{t=L_{n+1}+1}^{\infty} 2^{-t} = 2^{-L_{n+1}}$. Thus

$$\begin{array}{rcl} x_n & > & \sum_{\ell=1}^{n}\sum_{t=1}^{N_\ell}(2^{-L_\ell} + 2^{-L_\ell-t}) + \sum_{\ell=n+1}^{\infty}\sum_{t=1}^{N_\ell}(2^{-L_\ell} + 2^{-L_\ell-t}) - 2^{-L_{n+1}} \\ & > & \sum_{\ell=1}^{n}\sum_{t=1}^{N_\ell}(2^{-L_\ell} + 2^{-L_\ell-t})\ . \end{array}$$

Since also $2^{L_{n+1}} \cdot x_n \in \mathbb{N}$ we have x_n satisfies the hypotheses of Lemma 5.5 so pick $G_0(n) \subseteq \{1, 2, \ldots, r\}$ and for each $\ell \in \{1, 2, \ldots, n\}$ pick $G_\ell(n) \subseteq \{1, 2, \ldots, N_\ell\}$ such that $x_n = \sum_{t\in G_0(n)}\ 2^{-t} + \sum_{\ell=1}^{n}\sum_{t\in G_\ell(n)}(2^{-L_\ell} + 2^{-L_\ell-t})$. Then $x_n \in AS(B_r)$. It suffices to show that $\{0\} \cup AS(B_r)$ is compact. For then, since $\lim_{n\to\infty} x_n = x$, we have $x \in AS(B_r)$.

To verify that $\{0\} \cup AS(B_r)$ is compact, let $b \in c\ell AS(B_r)$ and note that $b < \infty$. Enumerate B_r as $\langle t_n\rangle_{n=1}^{\infty}$. For each $m \in \mathbb{N}$ choose a sequence $\langle \alpha(m,n)\rangle_{n=1}^{\infty}$ in $\{0, 1\}$ such that $|\sum_{n=1}^{\infty}\alpha(m,n)\cdot t_n - b| < \frac{1}{m}$. Choose infinite $A_1 \subseteq \mathbb{N}$ and $\sigma(1) \in \{0, 1\}$ such that for all $m \in A_1$, $\alpha(m, 1) = \sigma(1)$. Inductively, given A_k, choose infinite $A_{k+1} \subseteq A_k$ and $\sigma(k+1) \in \{0, 1\}$ such that for all $m \in A_{k+1}$, $\alpha(m, k+1) = \sigma(k+1)$. We claim that $\sum_{n=1}^{\infty}\sigma(n)\cdot t_n = b$, so suppose instead that $\epsilon = |\sum_{n=1}^{\infty}\sigma(n)\cdot t_n - b| > 0$. Pick k such that $\sum_{n=k+1}^{\infty} t_n < \frac{\epsilon}{3}$ and pick $m \in A_k$ such that $\frac{1}{m} < \frac{\epsilon}{3}$. Then

$$\begin{array}{rl} & |\sum_{n=1}^{\infty}\sigma(n)\cdot t_n - b| \\ = & |\sum_{n=1}^{k}\sigma(n)\cdot t_n + \sum_{n=k+1}^{\infty}\sigma(n)\cdot t_n - b| \\ = & |\sum_{n=1}^{\infty}\alpha(m,n)\cdot t_n - \sum_{n=k+1}^{\infty}\alpha(m,n)\cdot t_n + \sum_{n=k+1}^{\infty}\sigma(n)\cdot t_n - b| \\ \leq & |\sum_{n=1}^{\infty}\alpha(m,n)\cdot t_n - b| + |\sum_{n=k+1}^{\infty}\alpha(m,n)\cdot t_n| + |\sum_{n=k+1}^{\infty}\sigma(n)\cdot t_n| \\ < & \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3}\ . \end{array}$$

□

5.7 Corollary. *Let $\epsilon > 0$ be given. There is a decreasing sequence $\langle t_n\rangle_{n=1}^{\infty}$ such that $\sum_{n=1}^{\infty} t_n \leq 1$ and there exist $a < b < 1$ such that $b - a > 1 - \epsilon$, $(a,b) \subseteq AS(\langle t_n\rangle_{n=1}^{\infty})$, and 0 is a density point of $(0,1]\backslash AS(\langle t_n\rangle_{n=1}^{\infty})$.*

Proof. Given $r \in \mathbb{N}$, define sequences $\langle T_n\rangle_{n=0}^{\infty}$, $\langle L_n\rangle_{n=1}^{\infty}$, and $\langle N_n\rangle_{n=1}^{\infty}$ as in Definition 5.3 and let $\gamma_r = \sum_{\ell=1}^{\infty}\sum_{t=1}^{N_\ell}(2^{-L_\ell} + 2^{-L_\ell - t})$. Choose r such that $\dfrac{1-\gamma_r}{1+\gamma_r} > 1-\epsilon$. Then $AS(B_r) \subseteq (0, 1+\gamma_r)$, and by Theorem 5.6, $(\gamma_r, 1) \subseteq AS(B_r)$. By Theorem 5.4, 0 is a density point of $(0,1+\gamma_r)\backslash AS(B_r)$. Let $C = \dfrac{1}{1+\gamma_r}\cdot B_r$ and let $\langle t_n\rangle_{n=1}^{\infty}$ enumerate C in decreasing order. Then $AS(\langle t_n\rangle_{n=1}^{\infty}) \subseteq (0,1)$ and 0 is a density point of $(0,1)\backslash AS(\langle t_n\rangle_{n=1}^{\infty})$. Let $a = \dfrac{\gamma_r}{1+\gamma_r}$ and let $b = \dfrac{1}{1+\gamma_r}$. Then $(a,b) \subseteq AS(\langle t_n\rangle_{n=1}^{\infty})$ and $b - a = \dfrac{1-\gamma_r}{1+\gamma_r} > 1 - \epsilon$. □

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