Almost large subsets of a semigroup

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Abstract

We investigate a notion of largeness introduced by Bergelson and Robertson. Given a notion R of largeness in a semigroup, a set is an *almost* R set if it differs from an R set by a set with Banach density zero. We investigate almost large sets for several notions of largeness, establishing the exact relationships among many of these sets for subsets of the set $\mathbb{N}$ of positive integers.

1 Introduction

The notion of a subset of a semigroup which *almost* has a property R was introduced by Bergelson and Robertson for R as IP* in [3] and for R as IP_r^* in [4]. (See Sections 2 and 3 for the definitions of IP* and IP_r^*, as well as $IP_{<\omega}^*$ which is mentioned later in this paragraph.) In [3, Theorem 1.6], Bergelson and Robertson showed that a specified subset of an algebraic number field is a translate of a set which was almost an IP* set; in [4, Theorem 1.2], they showed that for any countable field F and any $n \in \mathbb{N}$, a specified subset of F^n has the property that there is some $r \in \mathbb{N}$ for which the set is almost an IP_r^* set. In [2, Corollary 1.7] Bergelson and Leibman showed that certain subsets of $\mathbb{Z}$ are almost $IP_{<\omega}^*$. In each case almost having the specified property was sufficient to obtain combinatorial consequences.

The notions of "almost large" are based on the Banach density of a subset of a left amenable semigroup, which we introduce now. Let $(S,\cdot)$ be a semigroup. Let $l_\infty(S)$ be the set of bounded real valued functions on S with the supremum norm, denoted by $\|\ \|_\infty$. Let $l_\infty(S)^*$ be the set of continuous real valued linear functionals on $l_\infty(S)$ with the dual norm $||\mu|| = \sup\{\mu(f) : f \in l_\infty(S)$ and $||f||_\infty \leq 1\}$. A *mean* on S is an element of $l_\infty(S)^*$ such that $||\mu|| = 1$ and $\mu \geq 0$, that is, whenever $g \in l_\infty(S)$ and for all $s \in S$, $g(s) \geq 0$, one has that $\mu(g) \geq 0$. A *left invariant mean* on S is a mean μ such that for all $s \in S$ and all $g \in l_\infty(S)$, $\mu(g \circ \lambda_s) = \mu(g)$ where for $s,t \in S$, $\lambda_s(t) = s \cdot t$. The semigroup S is defined to be *left amenable* if and only if there exists a left invariant mean on S.

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Definition 1.1. Let $(S, \cdot)$ be a left amenable semigroup, and let $A \subseteq S$. The *Banach density of* A is defined by $d(A) = \sup\{\lambda(\chi_A) : \lambda$ is a left invariant mean on $S\}$.

The only properties of Banach density that we will use in this paper are: (1) if $d(A) = d(B) = 0$, then $d(A \cup B) = 0$ and (2) if $A \subseteq B$ and $d(B) = 0$, then $d(A) = 0$. We note that if the semigroup S satisfies the *Strong Følner Condition* (SFC), then S is left amenable. See [8, Section 3] for a detailed introduction to SFC and historical references, including for the important fact that all commutative semigroups satisfy SFC.

By [6, Theorem 2.15], if S satisfies SFC, and $A \subseteq S$, then

$$d(A) = \sup\{\alpha \in [0,1] : \big(\forall H \in \mathcal{P}_f(S)\big)(\forall \epsilon > 0)\big(\exists K \in \mathcal{P}_f(S)\big) \\ \big((\forall s \in H)(|K \setminus sK| < \epsilon \cdot |K|) \text{ and } |A \cap K| \geq \alpha \cdot |K|\big)\},$$

where $\mathcal{P}_f(S)$ is the set of finite nonempty subsets of S.

Definition 1.2. Let $(S, \cdot)$ be a semigroup. We say that R is a *notion of largeness for* S provided that R is a property which may be possessed by subsets of S, $\emptyset$ is not an R set, S is an R set, and if A is an R set and $A \subseteq B \subseteq S$, then B is an R set.

Definition 1.3. Let $(S, \cdot)$ be a left amenable semigroup, let R be a notion of largeness for S, and let $A \subseteq S$. The set A is an αR set if and only if there exists an R set B such that $d(A \triangle B) = 0$.

The notation is intended to indicate that if A is an αR set, then A is "almost" an R set.

Since we are concerned in this paper with the almost large sets, all hypothesized semigroups will be assumed to be left amenable.

Theorem 1.4. *Let S be a semigroup, let R be a notion of largeness, and let $A \subseteq S$. The following statements are equivalent.*

1. *A is an αR set.*
2. *There is an R set B such that $d(B \setminus A) = 0$.*
3. *There exist an R set D and a set $C \subseteq S$ such that $d(C) = 0$ and $A = D \setminus C$.*
4. *There exists a set $E \subseteq S$ such that $d(E) = 0$ and $A \cup E$ is an R set.*

Proof. Trivially (1) implies (2). To see that (2) implies (3), pick an R set B such that $d(B \setminus A) = 0$. Let $D = A \cup B$ and let $C = B \setminus A$.

To see that (3) implies (4) pick an R set D and a set $C \subseteq S$ such that $d(C) = 0$ and $A = D \setminus C$ and let $E = C \cap D$. Then $D = A \cup E$.

To see that (4) implies (1), pick a set $E \subseteq S$ such that $d(E) = 0$ and $A \cup E$ is an R set. Let $B = A \cup E$. Then $A \triangle B = (A \setminus B) \cup (B \setminus A) = (B \setminus A) \subseteq E$. □

Using Theorem 1.4(2) one easily sees that if R is a notion of largeness and the empty set is not an αR set, then αR is a notion of largeness.

We note that if R is any notion of largeness, then $\alpha(\alpha R) = \alpha R$. That is, if A is an $\alpha(\alpha R)$, then A is an αR set. To see this, let A be an $\alpha(\alpha R)$ set and pick an αR set B and a zero density set C such that $A = B \setminus C$. Pick an R set D and a zero density set E such that $B = D \setminus E$. Then $C \cup E$ is a zero density set and $A = D \setminus (C \cup E)$.

If R is a notion of largeness, there is the corresponding notion of R* defined by the fact that A is an R* set if and only if for every R set B, $A \cap B \neq \emptyset$. Since notions of largeness are closed under passage to supersets, one has that A is an R* set if and only if $S \setminus A$ is not an R set. Also, if R and T are notions of largeness, then $(R \Rightarrow T)$ if and only if $(T^* \Rightarrow R^*)$. When we write αR^* we mean $\alpha(R^*)$, not $(\alpha R)^*$.

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2 Notions of largeness

We will utilize the algebraic structure of the Stone-Čech compactification of a discrete semigroup $(S, \cdot)$. We give a very brief introduction to this structure now. For a detailed introduction see [11, Part I].

We let $\beta S = \{p : p \text{ is an ultrafilter on } S\}$, identifying the principal ultrafilters on S with the points of S so that we may assume that $S \subseteq \beta S$. Given $A \subseteq S$, $\overline{A} = \{p \in \beta S : A \in p\}$. We choose $\{\overline{A} : A \subseteq S\}$ as a basis for the topology of βS. Then $\overline{A}$ is the closure of A in βS.

The operation $\cdot$ on S extends to an operation, also denoted $\cdot$, on βS so that $(\beta S, \cdot)$ is a right topological semigroup with S contained in the topological center of βS. That is, for each $p \in \beta S$, the function $\rho_p : \beta S \to \beta S$ defined by $\rho_p(q) = q \cdot p$ is continuous and for each $x \in S$, the function $\lambda_x : \beta S \to \beta S$ defined by $\lambda_x(q) = x \cdot q$ is continuous. Given $p, q \in \beta S$ and $A \subseteq S$, $A \in p \cdot q$ if and only if $\{x \in S : x^{-1}A \in q\} \in p$, where $x^{-1}A = \{y \in S : x \cdot y \in A\}$. (There is no suggestion that x has an inverse. However it is true that if S has an identity and x^{-1} is a two sided inverse of x, then $x^{-1}A = \{x^{-1} \cdot a : a \in A\}$.)

As does any compact Hausdorff right topological semigroup, βS has idempotents and a smallest two sided ideal, denoted $K(\beta S)$, which is the union of all of the minimal left ideals of βS and also the union of all of the minimal right ideals of βS. An idempotent in βS is an element of $K(\beta S)$ if and only if it is mimimal with respect to the ordering of idempotents wherein $p \leq q$ if and only if $p \cdot q = q \cdot p = p$. Such idempotents are simply said to be minimal. Minimal left ideals of βS are closed. The intersection of any minimal left ideal with any minimal right ideal is a group, and any two such groups are isomorphic.

In [8] we considered 52 notions of largeness. These began with 15 basic definitions, which we shall present next. Thirteen of these had distinct versions resulting from a left-right switch. (The notions P and WP are two sided notions.) And for any one of them, say R, there is the notion R^*. (If the curious reader is counting she may note that comes to 56 notions, not 52. The reason is that we counted both thick and syndetic, while syndetic is thick*.) We will differ in one respect from the full listing of (right) notions considered in [8]; we will only use one version of progressions, and call it P while it was called WP in [8].

As we define the notions, we will occasionally give equivalent characterizations. For the proofs of the equivalences (or references to the proofs) see [8].

Definition 2.1. Let $(S,\cdot)$ be a semigroup and let $A\subseteq S$.

(1) A is a *Q set* if and only if there exists a sequence $\langle x_n\rangle_{n=1}^\infty$ in S such that whenever $m<n$, $x_n\in x_m\cdot A$.

(2) A is an *IP set* if and only if there exists a sequence $\langle x_n\rangle_{n=1}^\infty$ in S such that $FP(\langle x_n\rangle_{n=1}^\infty)\subseteq A$, where $FP(\langle x_n\rangle_{n=1}^\infty)=\{\prod_{n\in F}x_n : F\in\mathcal{P}_f(\mathbb{N})\}$ and for $F\in\mathcal{P}_f(\mathbb{N})$, $\prod_{n\in F}x_n$ is the product in increasing order of indices. Equivalently, A is an IP set if and only if there is an idempotent $p\in\beta S$ such that $A\in p$.

(3) A is a P set if and only if for each $k\in\mathbb{N}$, there exist $m\in\mathbb{N}$, $a\in S^{m+1}$, and $d\in S$ such that $\big\{a(1)d^ta(2)d^t\cdots a(m)d^ta(m+1) : t\in\{1,2,\ldots,k\}\big\}\subseteq A$.

(4) A is a J set if and only if for each $F\in\mathcal{P}_f({}^{\mathbb{N}}S)$, there exist $m\in\mathbb{N}$, $a\in S^{m+1}$, and $t(1)<t(2)<\ldots<t(m)$ in $\mathbb{N}$ such that for each $f\in F$, $a(1)f\big(t(1)\big)a(2)f\big(t(2)\big)\cdots a(m)f\big(t(m)\big)a(m+1)\in A$. (Here ${}^{\mathbb{N}}S$ is the set of sequences in S.)

(5) A is a C set if and only if there is an idempotent in $\overline{A}\cap J(S)$, where $J(S)=\{p\in\beta S : (\forall B\in p)(B \text{ is a J set})\}$.

(6) A is a B set if and only if $d(A)>0$.

(7) A is a D set if and only if there is an idempotent in $\overline{A}\cap\Delta^*(S)$, where $\Delta^*(S)=\{p\in\beta S : (\forall B\in p)(d(B)>0)\}$.

(8) A is *piecewise syndetic*, that is a PS set, if and only if $\overline{A}\cap K(\beta S)\neq\emptyset$.

(9) A is *quasi central*, that is a QC set, if an only if there is an idempotent in $\overline{A}\cap c\ell K(\beta S)$.

(10) A is *central* if an only if there is an idempotent in $\overline{A}\cap K(\beta S)$.

(11) A is *syndetic* if and only if for every left ideal L of βS, $\overline{A}\cap L\neq\emptyset$.

(12) A is *strongly central*, that is an SC set, if and only if for every left ideal L of βS, there is an idempotent in $\overline{A}\cap L$.

(13) A is *thick* if and only if for each $F\in\mathcal{P}_f(S)$ there exists $x\in S$ such that $Fx\subseteq A$. Equivalently, A is *thick* if and only if there exists a left ideal L of βS such that $L\subseteq\overline{A}$.

(14) A is *strongly piecewise syndetic*, that is a SPS set, if and only if there exists $H \in \mathcal{P}_f(S)$ such that $\bigcup_{t \in H} At^{-1}$ is thick.

The names Q, P, and IP come from "quotient", "progression", and "infinite dimensional parallelepiped" respectively. The names C, J, B, and D have no particular significance.

We show now in Theorems 2.2, 2.3, and 2.5 that for all but five of the notions R that we have defined, either $\emptyset$ is an αR set, so that αR is not a notion of largeness, or every αR set is an R set, so that αR is not of separate interest.

We remind the reader that we are assuming that all hypothesized semigroups are left amenable.

Theorem 2.2. *Let $(S, \cdot)$ be a semigroup.*

(a) If R is a notion of largeness for S, then $\emptyset$ is an αR set if and only if there is an R set A such that $d(A) = 0$.

(b) If $A \subseteq S$ and $d(A) = 0$, then for any notion of largeness R for S, $S \setminus A$ is an αR^* set.

(c) If R is a notion of largeness for S, A is an R set in S, and $d(A) = 0$, then $S \setminus A$ is an αR^* set which is not an R^* set.

Proof. (a) For any $A \subseteq S$, $d(\emptyset \triangle A) = d(A)$.

(b) S is an R^* set and $(S \setminus A) \triangle S = A$ so $S \setminus A$ is an αR^* set.

(c) Since A is an R set, $S \setminus A$ is not an R^* set. □

In [7, Theorem 2.1] it was shown that there is a subset A of $\mathbb{N}$ with $d(A) = 0$ which is a C set. Consequently, since C implies each of J, IP, P and Q, if R is any of C, J, IP, P, or Q, then $\emptyset$ is an αR set and there is an αR^* set which is not an R^* set. It is a consequence of the next two theorems that C^*, J^*, IP^*, P^*, and Q^* are the only of the notions that we have defined whose almost versions are distinct from them, at least in $(\mathbb{N}, +)$.

Recall that a notion of largenss R is partition regular provided that if the union of two sets is an R set, then one of them is an R set.

Theorem 2.3. *Let $(S, \cdot)$ be a semigroup and let R be a partition regular notion of largeness for S such that every R set has positive density. Then every αR set is an R set and every αR^* set is an R^* set.*

Proof. Suppose A is an αR set which is not an R set. Pick an R set C such that $d(A \triangle C) = 0$. Then $C \subseteq A \cup (C \setminus A)$. Since R is partition regular, $(C \setminus A)$ is an R set and so $d(C \setminus A) > 0$, a contradiction.

Suppose A is an αR^* set which is not an R^* set. Pick an R^* set C such that $d(A \triangle C) = 0$. Since A is not an R^* set, $S \setminus A$ is an R set and $(S \setminus A) = (C \setminus A) \cup \big(S \setminus (A \cup C)\big)$. Since $\big(S \setminus (A \cup C)\big) \cap C = \emptyset$, $S \setminus (A \cup C)$ is not an R set. Since R is partition regular, $C \setminus A$ is an R set so that $d(C \setminus A) > 0$, a contradiction. □

The partition regular notions considered in [8] with the property that all sets satisfying those notions have positive density include central, QC, PS, D, and B so each of central, QC, PS, D, B, central*, QC*, PS*, D*, and B* are identical with their almost versions.

Lemma 2.4. *Let $(S,\cdot)$ be a semigroup and let A be a piecewise syndetic subset of S. Then $d(A) > 0$.*

Proof. By [6, Theorem 2.8], $\Delta^*(S)$ is a two sided ideal of βS so $K(\beta S) \subseteq \Delta^*(S)$. Since A is piecewise syndetic, $\overline{A} \cap K(\beta S) \neq \emptyset$ so $\overline{A} \cap \Delta^*(S) \neq \emptyset$. □

Theorem 2.5. *Let $(S,\cdot)$ be a semigroup and let R be a notion of largeness for S. Assume that for any R set C and any subset A of S which is not an R set, $A \triangle C$ is piecewise syndetic. Then every αR set is an R set and every αR^* set is an R^* set.*

Proof. Suppose that A is an αR set which is not an R set. Pick an R set C such that $d(A \triangle C) = 0$. Then $A \triangle C$ is piecewise syndetic so by Lemma 2.4, $d(A \triangle C) > 0$, a contradiction.

Now suppose that A is an αR^* set which is not an R^* set. Then $S \setminus A$ is an R set. Pick an R^* set C such that $d(A \triangle C) = 0$. Then $S \setminus C$ is not an R set so $(S \setminus A) \triangle (S \setminus C)$ is piecewise syndetic. That is, $A \triangle C$ is piecewise syndetic so $d(A \triangle C) > 0$, a contradiction. □

The remaining properties considered in [8] that we have not yet determined whether they and their almost versions agree are SPS, SPS^*, thick, syndetic, SC, and SC^*. Since syndetic is thick*, it suffices now to verify that SPS, thick, and SC satisfy the hypotheses of Theorem 2.5. For the verification for each of these properties we will assume that $(S,\cdot)$ is a semigroup.

Let C be an SPS set and let A be a subset of S which is not an SPS set. Since C is an SPS set, pick $H \in \mathcal{P}_f(S)$ such that $\bigcup_{t\in H} Ct^{-1}$ is thick and pick a minimal left ideal L of βS such that $L \subseteq \overline{\bigcup_{t\in H} Ct^{-1}}$. Since A is not an SPS set, $\bigcup_{t\in H} At^{-1}$ is not thick so $L \setminus \overline{\bigcup_{t\in H} At^{-1}} \neq \emptyset$. Pick $p \in L \setminus \overline{\bigcup_{t\in H} At^{-1}}$. Since $p \in L$, pick $t \in H$ such that $Ct^{-1} \in p$. Then $At^{-1} \notin p$ so $A \notin pt$ while $C \in pt$ and thus $C \setminus A \in pt$. Since $pt \in K(\beta S)$, $C \setminus A$ is piecewise syndetic so $A \triangle C$ is piecewise syndetic.

Let C be a thick set and let A be a subset of S which is not thick. Pick a minimal left ideal L of βS such that $L \subseteq \overline{C}$. Since A is not thick, $L \setminus \overline{A} \neq \emptyset$ so pick $p \in L \setminus \overline{A}$. Then $C \setminus A \in p$ so $C \setminus A$ is piecewise syndetic and thus $A \triangle C$ is piecewise syndetic.

Let C be an SC set and let A be a subset of S which is not an SC set. Since A is not an SC set, pick a minimal left ideal L of βS such that there is no idempotent in $L \cap \overline{A}$. Since C is an SC set, pick an idempotent $p \in L \cap \overline{C}$. Then $C \setminus A \in p$ so $C \setminus A$ is piecewise syndetic and thus $A \triangle C$ is piecewise syndetic.

We have established that if R is any of C^*, J^*, IP^*, P^*, or Q^*, then in $(\mathbb{N}, +)$ there is an αR set which is not an R set. If R is any other of the notions we have defined, then in any (left amenable) semigroup, either $\emptyset$ is an αR set or any αR set is an R set. In other words, in this study of almost large subsets of a semigroup, the notions C^*, J^*, IP^*, P^*, and Q^* are of most interest.

Lemma 2.6. *Let $(S, \cdot)$ be a semigroup, let R be a notion of largeness for S, and let A be a B^* set in S. Then A is an αR^* set.*

Proof. Since A is a B^* set, $S \setminus A$ is not a B set so $d(S \setminus A) = 0$. Then S is an R* set and $d(A \triangle S) = 0$ so A is an αR^* set. □

We note now that if S is commutative, then the notions αP^* and αJ^* are each equivalent to B^*.

Theorem 2.7. *Let $(S, \cdot)$ be a commutative semigroup and let $A \subseteq S$. The following statements are equivalent.*

(1) A is an αP^* set.

(2) A is an αJ^* set.

(3) A is an αB^* set.

(4) A is a B^* set.

Proof. It was established in [8] that in commutative semigroups $(B \Rightarrow J)$ and $(J \Rightarrow P)$ so (1) implies (2), and (2) implies (3). By Theorem 2.3, (3) and (4) are equivalent. By Lemma 2.6, (4) implies (1). □

3 IP_r sets and SIP_r sets

In [4], following [5], the authors define an IP_r set as a set which contains $FS(\langle x_t \rangle_{t=1}^r) = \left\{ \sum_{t \in F} x_t : \emptyset \neq F \subseteq \{1, 2, \ldots, r\} \right\}$ for some $\langle x_t \rangle_{t=1}^r$. (If the operation is denoted by $\cdot$, then $FP(\langle x_t \rangle_{t=1}^r)$ is defied analogously.) In [1], an IP_r set is defined as one which, whenever it is finitely colored, there is monochromatic $FS(\langle x_t \rangle_{t=1}^r)$ for some $\langle x_t \rangle_{t=1}^r$. These are different notions so we introduce separate terminology.

Definition 3.1. Let $(S, \cdot)$ be a semigroup, and let $A \subseteq S$.

(1) For $r \in \mathbb{N}$, A is an IP_r set if and only if there exist $x_1, x_2, \ldots, x_r$ in S such that $FP(\langle x_t \rangle_{t=1}^r) \subseteq A$.

(2) For $r \in \mathbb{N}$, A is an SIP_r set if and only if whenever A is finitely colored, there exist $x_1, x_2, \ldots, x_r$ in S such that $FP(\langle x_t \rangle_{t=1}^r)$ is monochromatic.

(3) A is an $IP_{<\omega}$ set if and only if A is an IP_r set for every $r \in \mathbb{N}$.

(4) For $r \in \mathbb{N}$, $S_r(S) = \{p \in \beta S : (\forall A \in p)(A \text{ is an } IP_r \text{ set in } S)\}$.

Note that if A is an $IP_{<\omega}$ set, then it is also true that for each $n \in \mathbb{N}$, A is an SIP_n set; given r and n in $\mathbb{N}$ a standard compactness argument establishes that there is a sufficiently large k so that whenever an IP_k set is r-colored, there is a monochromatic IP_n set. (See [11, Section 5.5] for an introduction to compactness arguments.)

The notation S_r is from [10], where it was noted that for each $r \in \mathbb{N}$, $S_r(\mathbb{N},+)$ is a compact subsemigroup of $(\beta\mathbb{N},+)$ containing the idempotents.

By Theorem 2.2, if R is any of the notions in Definition 3.1, then $\emptyset$ is an αR set and there is an αR^* set which is not an R^* set.

We establish now some algebraic facts about $S_r(S)$.

Theorem 3.2. *Let $(S,\cdot)$ be a semigroup and let $r \in \mathbb{N} \setminus \{1\}$.*

(a) $S_r(S)$ is a compact subset of $(\beta S,\cdot)$ containing the idempotents, $S_r(S) = \{p \in \beta S : (\forall A \in p)(A \text{ is an } SIP_r \text{ set in } S)\}$, and for every $A \subseteq S$, A is an SIP_r set in S if and only if $\overline{A} \cap S_r(S) \neq \emptyset$.

(b) If S is commutative, then $S_r(S)$ is a subsemigroup of βS.

Proof. (a) Trivially $S_r(S)$ is compact. If p is an idempotent in S, then every member of p is an IP set, hence an IP_r set. Let $p \in S_r(S)$ and let $A \in p$. To see that A is an SIP_r set, let A be finitely colored. Then one color class is a member of p, hence an IP_r set. The final conclusion is an immediate consequence of [11, Theorem 3.11] and the fact that SIP_r is a partition regular property.

(b) Assume that S is commutative, let p and q be members of $S_r(S)$, and let $A \in p \cdot q$. Then $\{x \in S : x^{-1}A \in q\} \in p$ so pick $\langle y_t \rangle_{t=1}^r$ in S such that $FP(\langle y_t \rangle_{t=1}^r) \subseteq \{x \in S : x^{-1}A \in q\}$. Let $B = \bigcap\{x^{-1}A : x \in FP(\langle y_t \rangle_{t=1}^r)\}$. Then $B \in q$ so pick $\langle z_t \rangle_{t=1}^r$ in S such that $FP(\langle z_t \rangle_{t=1}^r) \subseteq B$. Then $FP(\langle y_t \cdot z_t \rangle_{t=1}^r) \subseteq A$. □

Theorem 3.3. *(a) For every $n \in \mathbb{N}\setminus\{1\}$, $n\mathbb{N}$ is an SIP_2^* set in $(\mathbb{N},+)$, hence an SIP_m^* set for every $m \geq 2$ in $\mathbb{N}$.*

(b) For every $r \in \mathbb{N} \setminus \{1\}$, $S_r(\mathbb{N},+)$ is an ideal of $(\beta\mathbb{N},\cdot)$. In particular, every piecewise syndetic subset of $(\mathbb{N},\cdot)$ is an $IP_{<\omega}$ set in $(\mathbb{N},+)$.

Proof. (a) Let $n \in \mathbb{N}$. Then $\mathbb{N} \setminus n\mathbb{N} = \bigcup_{i=1}^{n-1}(n\mathbb{N} - i)$ and for each $i \in \{1,2,\ldots,n-1\}$, $n\mathbb{N} - i$ does not contain any $\{x, y, x+y\}$.

(b) This is [10, Theorem 4.3]. For the "in particular" conclusion, let A be a piecewise syndetic subset of $(\mathbb{N},\cdot)$. Then $\overline{A} \cap K(\beta\mathbb{N},\cdot) \neq \emptyset$. For each $r \in \mathbb{N}\setminus\{1\}$, $K(\beta\mathbb{N},\cdot) \subseteq S_r(\mathbb{N},+)$ so $\overline{A} \cap S_r(\mathbb{N},+) \neq \emptyset$ and so Theorem 3.2(a) applies. □

Theorem 3.4. *If $(S,\cdot)$ is a left cancellative semigroup, and A is a Q set in S, then A is an SIP_2 set.*

Proof. Assume that A is a Q set and choose a sequence $\langle s_n \rangle_{n=1}^\infty$ in S with $s_m \in s_n \cdot A$ whenever $n < m$. For each such $n < m$ let $t_{n,m}$ be the unique member of A such that $s_m = s_n \cdot t_{n,m}$. Given $F \subseteq A$, let $B(F) = \big\{\{n,m\} : n < m \text{ and } t_{n,m} \in F\big\}$. Given a finite partition $\mathcal{F}$ of A, one has that $\{B(F) : F \in \mathcal{F}\}$

is a finite partition of the set of two element subsets of $\mathbb{N}$, so pick by Ramsey's Theorem $k < n < m$ and $F \in \mathcal{F}$ with $\{k,n\}, \{k,m\}, \{n,m\} \in B(F)$. Then $s_m = s_n \cdot t_{n,m} = s_k \cdot t_{k,n} \cdot t_{n,m}$ and $s_m = s_k \cdot t_{k,m}$ and so $t_{k,m} = t_{k,n} \cdot t_{n,m}$. □

We see now that one cannot weaken the assumption of left cancellation in Theorem 3.4 to weakly left cancellative, even if one adds the assumption of commutativity. (Recall that S is weakly left cancellative if and only if for all $u, v \in S$ $\{x \in S : ux = v\}$ is finite and S is weakly right cancellative if and only if for all $u, v \in S$ $\{x \in S : xu = v\}$ is finite.)

Theorem 3.5. *There exist a countable, commutative, and weakly cancellative semigroup $(S, *)$ and a Q set $A \subseteq S$ such that A is not an SIP_2 set. In fact, there do not exist X and Y in S such that $X * Y \in A$.*

Proof. Let $S = \mathcal{P}_f(\mathbb{N})$ and for $X, Y \in S$, let $X * Y = \{\max(X \cup Y)\}$. Given $X, Y, Z \in S$, $(X \cup Y) \cup Z = X \cup (Y \cup Z)$, so $*$ is associative. It is easy to verify that for $U, V \in S$, $\{X \in S : U * X = V\}$ is finite, so S is weakly cancellative.

Let $A = \{X \in S : |X| = 2\}$. Then for any $X, Y \in S$, $X * Y \notin A$. To see that A is a Q set, let for each $n \in \mathbb{N}$, $X_n = \{n\}$. If $m < n$ in $\mathbb{N}$, then $X_n = X_m * \{1, n\}$ and $\{1, n\} \in A$. □

Note that the proof of Theorem 3.5 works equally well if $S = \{X \in \mathcal{P}_f(\mathbb{N}) : |X| \leq 2\}$, in which case the sizes of some of the solution sets are reduced.

We turn our attention now to characterizing αR^* sets for partition regular notions of largeness.

Definition 3.6. Let S be a semigroup and let R be a notion of largeness.

(a) $\mathcal{B}_R = \{B \subseteq S : B \text{ is an } R^* \text{ set in } S\}$.

(b) $M_R = \bigcap\{\overline{E} : E \in \mathcal{B}_R\}$.

Lemma 3.7. *Let R be a partition regular notion of largeness in a semigroup S.*

(a) $\mathcal{B}_R$ is closed under finite intersections.

(b) For all $B \subseteq S$, B is an R set if and only if $\overline{B} \cap M_R \neq \emptyset$.

(c) For all $B \subseteq S$, B is an R^* set if and only if $M_R \subseteq \overline{B}$.

Proof. (a) Let B and C be R^* sets. If $B \cap C \notin \mathcal{B}_R$, then $S \setminus (B \cap C) = (S \setminus B) \cup (S \setminus C)$ is an R set so either $(S \setminus B)$ or $S \setminus C$ is an R set.

(b) Necessity. Assume B is an R set. If $C \in \mathcal{B}_R$, then $C \cap B \neq \emptyset$ so it follows from (a) that $\mathcal{B}_R \cup \{B\}$ has the finite intersection property so pick $p \in \beta S$ such that $\mathcal{B}_R \cup \{B\} \subseteq p$. Then $p \in \overline{B} \cap M_R$.

Sufficiency. Assume that $\overline{B} \cap M_R \neq \emptyset$ and pick $p \in \overline{B} \cap M_R$. Then $S \setminus B \notin p$ so $S \setminus B \notin \mathcal{B}_R$ so B is an R set.

(c) Since $M_R \subseteq \overline{B}$ if and only if $\overline{S \setminus B} \cap M_R = \emptyset$, this follows from (b). □

Theorem 3.8. *Let S be a semigroup, let R be a partition regular notion of largeness, and let $A \subseteq S$. Then A is an αR^* set if and only if $\Delta^*(S) \cap M_R \subseteq \overline{A}$.*

Proof. Necessity. Assume that A is an αR^* set and let $q \in \Delta^*(S) \cap M_R$. Pick by Theorem 1.4(4), $C \subseteq S$ such that $d(C) = 0$ and $A \cup C$ is an R^* set. Since $q \in M_R$, $A \cup C \in q$. Since $q \in \Delta^*(S)$, $C \notin q$. So $A \in q$.

Sufficiency. Assume that $\Delta^*(S) \cap M_R \subseteq \overline{A}$. We claim that there is some $E \in \mathcal{B}_R$ such that $\Delta^*(S) \cap \overline{E} \subseteq \overline{A}$. Suppose not, and for $E \in \mathcal{B}_R$, let $C_E = \Delta^*(S) \cap \overline{E} \cap \overline{S \setminus A}$. Then $\{C_E : E \in \mathcal{B}_R\}$ is a collection of nonempty compact subsets of βS which is closed under finite intersections by Lemma 3.7(a). So $\emptyset \neq \bigcap_{E \in \mathcal{B}_R} C_E = \Delta^*(S) \cap M_R \cap \overline{S \setminus A}$, contradicting the fact that $\Delta^*(S) \cap M \subseteq \overline{A}$.

So pick $E \in \mathcal{B}_R$ such that $\Delta^*(S) \cap \overline{E} \subseteq \overline{A}$ and thus $\overline{E \setminus A} \cap \Delta^*(S) = \emptyset$. Thus $d(E \setminus A) = 0$ so by Theorem 1.4(2), A is an αR^* set. □

We will be concerned in the next section with determining which notions of largeness imply which other notions.

Theorem 3.9. *Let R and T be partition regular notions of largeness in a semigroup S. The following statements are equivalent.*

(1) Every T^* set in S is an αR^* set.

(2) $\Delta^*(S) \cap M_R \subseteq M_T$.

Proof. (1) implies (2). Assume that every T^* set in S is an αR^* set, let $p \in \Delta^*(S) \cap M_R$ and suppose that $p \notin M_T$. Since M_T is compact, pick $B \in p$ such that $\overline{B} \cap M_T = \emptyset$. Then $M_T \subseteq \overline{S \setminus B}$ so by Lemma 3.7(c), $S \setminus B$ is a T^* set so by assumption $S \setminus B$ is an αR^* set. Then by Theorem 3.8, $\Delta^*(S) \cap M_R \subseteq \overline{S \setminus B}$. But then $S \setminus B \in p$, a contradiction.

(2) implies (1). Assume that $\Delta^*(S) \cap M_R \subseteq M_T$ and let A be a T^* set. It suffices by Theorem 3.8 to show that $\Delta^*(S) \cap M_R \subseteq \overline{A}$ so let $p \in \Delta^*(S) \cap M_R$ and suppose that $p \notin \overline{A}$. Then $S \setminus A \in p$ and since $\Delta^*(S) \cap M_R \subseteq M_T$, $p \in M_T$ so $\overline{S \setminus A} \cap M_T \neq \emptyset$. Consequently by Lemma 3.7(b), $S \setminus A$ is a T set so A is not a T^* set, a contradiction. □

Given $r \in \beta\mathbb{N}$, $-r \in \beta\mathbb{Z}$ is defined to be the ultrafilter on $\mathbb{Z}$ generated by $\{-A : A \in r\}$. So if $r, p \in \beta\mathbb{N}$ and $A \subseteq \mathbb{Z}$, then $A \in -r + p$ if and only if $\{x \in \mathbb{Z} : -x + A \in p\} \in -r$ where $-x + A = \{y \in \mathbb{Z} : x + y \in A\}$.

Lemma 3.10. *Let $p \in K(\beta\mathbb{N})$ and let $r \in \beta\mathbb{N}$. Then $-r+p \in K(\beta\mathbb{N}) \subseteq \Delta^*(\mathbb{N})$.*

Proof. By [11, Exercise 4.3.5] $-r + p \in \mathbb{N}^*$. By [11, Exercise 4.3.8], $K(\beta\mathbb{N},+) \cup -K(\beta\mathbb{N},+) = K(\beta\mathbb{Z},+)$ so $-r + p \in K(\beta\mathbb{Z},+) \cap \mathbb{N}^* = K(\beta\mathbb{N},+)$. By [6, Theorem 2.8] $\Delta^*(\mathbb{N},+)$ is an ideal of $(\beta\mathbb{N},+)$ so $K(\beta\mathbb{N},+) \subseteq \Delta^*(\mathbb{N},+)$. □

In the proof of the following theorem we use the algebraic structure of $(\beta\mathbb{N},+)$ and of $(\beta\mathbb{N},\cdot)$.

Theorem 3.11. *Let $A \subseteq \mathbb{N}$ and assume that for every minimal idempotent $p \in (\beta\mathbb{N}, +)$, $A \in -p + p$. Then A is $IP_{<\omega}$ in $(\mathbb{N}, +)$ and for any $C \subseteq \mathbb{N}$ such that $d(C) = 0$, $(\mathbb{N} \setminus A) \cup C$ is not $IP^*_{<\omega}$ in $(\mathbb{N}, +)$.*

Proof. Pick a minimal idempotent $p \in (\beta\mathbb{N}, +)$ and let $D = \{-sp + sp : s \in \mathbb{N}\}$. By [11, Lemma 5.19.2] for any $s \in \mathbb{N}$, sp is a minimal idempotent in $(\beta\mathbb{N}, +)$ so $c\ell D \subseteq \overline{A}$. We claim that $c\ell D$ is a left ideal of $(\beta\mathbb{N}, \cdot)$. To see this, let $q \in c\ell D$, let $r \in \beta\mathbb{N}$, and let $B \in r \cdot q$. Pick $x \in \mathbb{N}$ such that $x^{-1}B \in q$. Pick $s \in \mathbb{N}$ such that $-sp + sp \in \overline{x^{-1}B}$. Then $x^{-1}B \in -sp + sp$ so $B \in x(-sp + sp)$ and by [11, Lemma 13.1] $x(-sp + sp) = x(-s)p + xsp$. It is an easy exercise to show that $x(-s)p = -(xs)p$ so $-xsp + xsp \in D \cap \overline{B}$.

Since $c\ell D$ is a left ideal of $(\beta\mathbb{N}, \cdot)$, pick $q \in c\ell D \cap K(\beta\mathbb{N}, \cdot)$. Then $q \in c\ell D \subseteq \overline{A}$. Given $s \in \mathbb{N}$, $sp \in K(\beta\mathbb{N})$ so by Lemma 3.10, $-sp + sp \in \Delta^*(\mathbb{N})$ so $D \subseteq \Delta^*(\mathbb{N})$.

For each $m \in \mathbb{N} \setminus \{1\}$, $S_m(\mathbb{N}) = \{q \in \beta\mathbb{N} : (\forall E \in q)(\exists \langle x_t \rangle_{t=1}^m)(FS(\langle x_t \rangle_{t=1}^m) \subseteq E)\}$. By Theorem 3.3(b) each $S_m(\mathbb{N})$ is an ideal of $(\beta\mathbb{N}, \cdot)$ so $q \in K(\beta\mathbb{N}, \cdot) \subseteq \bigcap_{m=2}^\infty S_m(\mathbb{N})$. Since $q \in \bigcap_{m=2}^\infty S_m(\mathbb{N})$, every member of q is an $IP_{<\omega}$ set in $(\mathbb{N}, +)$ and in particular A is an $IP_{<\omega}$ set.

For the second conclusion of the theorem, let $C \subseteq \mathbb{N}$ such that $d(C) = 0$, and suppose that $(\mathbb{N} \setminus A) \cup C$ is $IP^*_{<\omega}$ in $(\mathbb{N}, +)$. Then $(\mathbb{N} \setminus A) \cup C$ meets every member of q so $(\mathbb{N} \setminus A) \cup C \in q$. But $\mathbb{N} \setminus A \notin q$ and since $q \in \Delta^*(\mathbb{N}, +)$, $C \notin q$. □

4 Implications among notions of largeness

Given notions of largeness R and T for subsets of a semigroup S, we will abbreviate the statement "if A is a subset of S and A is an R set in S, then A is a T set in S" by writing "R implies T".

Figure 1 shows implications involving the almost versions of all of the notions of largeness R that we have been considering for which $\emptyset \notin \alpha R$ and $\alpha R \neq R$, as well as B^* and D^*. The only known implications that are missing from the diagram are the facts that R^* implies αR^* for R as IP_n, SIP_n, and $IP_{<\omega}$.

All of the implications in Figure 1 follow from implications that were established in [8] or in results presented earlier in this paper. For the implications involving D^*, we have that IP^* implies D^* so αIP^* implies αD^* which is equivalent to D^*. Similarly, if S is commutative, then C^* implies D^* so αC^* implies D^*.

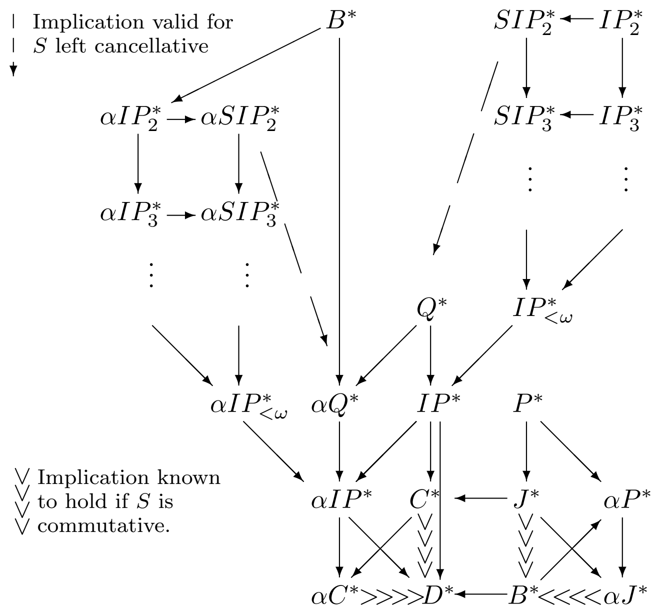

Figure 1

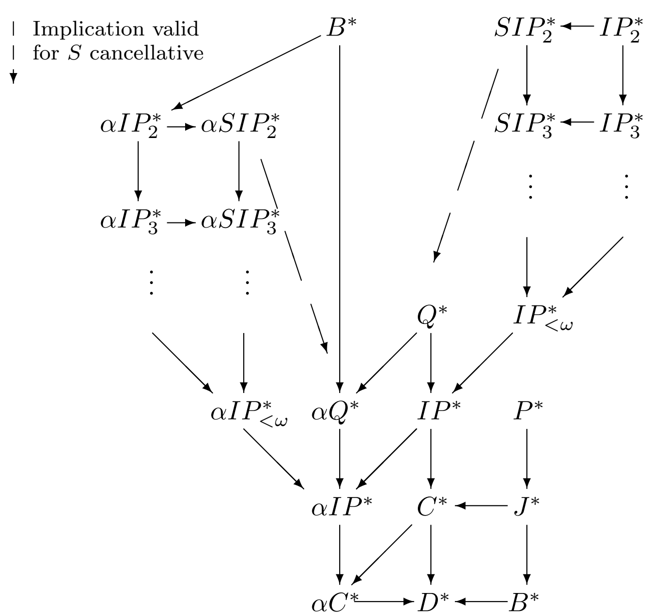

Figure 2
(S commutative)

Figure 2 is Figure 1 under the assumption that S is commutative (in which case αP^* and αJ^* both disappear because by Theorem 2.7 they are the same as B^* and the fact that αIP^* implies D^* is omitted since it follows from the facts that αIP^* implies αC^* and αC^* implies D^*).

Note that B^* appears twice in both Figure 1 and Figure 2. So, for example, the fact that P^* implies αQ^* follows from the implications shown in Figure 2.

In Figure 3 we display the implications that are known to hold among SIP_m^*, SIP_n^*, IP_m^*, IP_n^* for αSIP_m^*, αSIP_n^*, αIP_m^*, and αIP_n^* for $1 < m < n$ in $\mathbb{N}$.

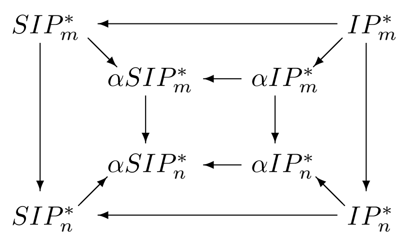

Figure 3

All of the implications listed in Figure 3 hold trivially.

Theorem 4.1. *Let R and T be notions of largeness in a semigroup S. The following statements are equivalent.*

(1) Every R^* subset of S is an αT^* set.

(2) Every αR^* subset of S is an αT^* set.

Proof. That (2) implies (1) is trivial. To see that (1) implies (2), assume that (1) holds and let A be an αR^* set in S. By Theorem 1.4(3), pick an R set B and a subset C of S such that $d(C) = 0$ and $B = A \cup C$. Since B is an R^* set, B is an αT^* set so pick a T^* set D and a subset E of S such that $d(E) = 0$ and $D = B \cup E$. Then $B = A \cup (C \cup E)$ so A is an αT^* set. □

Theorem 4.2. *Let R be a notion of largeness for a semigroup S. If there is an R subset A of S such that $d(A) = 0$, then for any notion T of largeness in S, $S \setminus A$ is an αT^* which is not an R^* set. In particular, if R is any of C, J, IP, P, Q, $IP_{<\omega}$, or IP_n or SIP_n for some $n \in \mathbb{N} \setminus \{1\}$, then for any notion of largeness T, αT^* does not imply R^* in $(\mathbb{N}, +)$.*

Proof. Pick such A. By Theorem 2.2(b), $S \setminus A$ is an αT^* set. For the "in particular" conclusions, all follow from the fact shown in [7, Theorem 2.1] that there is a C set in $\mathbb{N}$ which has density 0. □

In the remainder of this paper we set out to determine, as far as possible, whether any of the missing implications in Figures 2 or 3 are valid in $(\mathbb{N}, +)$.

Theorem 4.3. *Let $r \in \mathbb{N} \setminus \{1,2\}$. Then $r\mathbb{N}$ is an IP_r^* set but not an αIP_{r-1}^* set.*

Proof. To see that $r\mathbb{N}$ is an IP_r^* set, let $\langle x_t \rangle_{t=1}^r$ be a sequence in $\mathbb{N}$. For each $t \in \{1,2,\ldots,r\}$ pick $s_t \in \{0,1,\ldots,r-1\}$ such that $\sum_{i=1}^t x_i \equiv s_t \pmod r$. If any $s_t = 0$, we are done so assume that each $s_t \in \{1,2,\ldots,r-1\}$. Pick $j < t$ in $\{1,2,\ldots,r\}$ such that $s_j = s_t$. Then $\sum_{i=j+1}^t x_i \in r\mathbb{N}$.

Suppose $r\mathbb{N}$ is αIP_{r-1}^* and pick subsets B and C of $\mathbb{N}$ such that B is IP_{r-1}^*, $d(C) = 0$, and $r\mathbb{N} = B \setminus C$. For $n \in \omega$ and $t \in \{0,1,\ldots,r-2\}$ let $x_{n,t} = r(r-1)n + tr + 1$ and let $K_n = FS(\langle x_{n,t} \rangle_{t=0}^{r-2})$. Since B is IP_{r-1}^*, for each $n \in \omega$, $K_n \cap B \neq \emptyset$. Note that if $\emptyset \neq F \subseteq \{0,1,\ldots,r-2\}$, then $\sum_{t\in F} x_{n,t} \equiv |F| \pmod r$. In particular, for each $n \in \omega$, $K_n \cap r\mathbb{N} = \emptyset$. Further, if $x \in B \cap K_n$, then $x \in C$, since otherwise, $x \in B \setminus C = r\mathbb{N}$.

Next we note that if $m < n$ in ω, then $K_m \cap K_n = \emptyset$. To see this let $F, G \in \mathcal{P}_f(\{0,1,\ldots,r-2\})$ and suppose that $\sum_{t\in F} x_{m,t} = \sum_{t\in G} x_{n,t}$. Then $|F| \equiv \sum_{t\in F} x_{m,t} = \sum_{t\in G} x_{n,t} \equiv |G| \pmod r$ so $|F| = |G|$.

Let $l = |F|$. Then $\sum_{t\in F} x_{m,t} \leq l x_{m,r-2} = lr(r-1)m + l(r-2)r + l$ and $\sum_{t\in G} x_{n,t} \geq \sum_{t\in G} x_{m+1,t} \geq l x_{m+1,0} = l(r(r-1)(m+1)+1) = lr(r-1)m + l(r-1)r + l$ so $\sum_{t\in F} x_{m,t} < \sum_{t\in G} x_{n,t}$, a contradiction.

Let $m \in \mathbb{N}$. Note that $\max K_m = (r-1)^2 rm + \dfrac{r^3 - 3r^2 + 4r - 2}{2}$.

Now $m + 1 \leq |C \cap \bigcup_{n=0}^m K_n| \leq |C \cap \{1,2,\ldots,\max K_m\}|$ so

$$|C \cap \{1,2,\ldots,\max K_m\}| \geq \frac{1}{(r-1)^2 r} \cdot \max K_m \,.$$

Thus the upper asymptotic density of C is at least $\frac{1}{(r-1)^2 r}$ so $d(C) \geq \frac{1}{(r-1)^2 r}$, a contradiction. □

Lemma 4.4. *Let*

$$A = \mathbb{N} \setminus \left(\{2^{2n} + m2^n + 1 : m,n \in \mathbb{N} \text{ and } m < n\} \cup \{\textstyle\sum_{n\in F} 2^{2n} : F \in \mathcal{P}_f(\mathbb{N})\}\right).$$

Then A is a J^* set in $\mathbb{N}$ and is neither a P^* set nor an IP^* set.

Proof. Let $B = \{2^{2n} + m2^n + 1 : m,n \in \mathbb{N}$ and $m < n\}$ and let $C = \{\sum_{n\in F} 2^{2n} : F \in \mathcal{P}_f(\mathbb{N})\}$. Since B is a P set and C is an IP set, A is neither a P^* set nor an IP^* set. By [9, Lemma 4.3], B is not a J set. Since C does not contain any three term arithmetic progressions, C is not a P set so not a J set. Since J is a partition regular notion by [11, Lemma 14.14.6], $B \cup C$ is not a J set so A is a J^* set. □

Lemma 4.5. *There is a subset of $\mathbb{N}$ which is a Q^* set and is not an $IP_{<\omega}^*$ set.*

Proof. Choose a sequence $\langle B_n \rangle_{n=1}^\infty$ in $\mathcal{P}_f(\mathbb{N})$ such that for each $n \in \mathbb{N}$, $|B_n| = n$ and $\max FS(\langle 2^t \rangle_{t\in B_n}) < \min FS(\langle 2^t \rangle_{t\in B_{n+1}})$. Let $A = \bigcup_{n=1}^\infty FS(\langle 2^t \rangle_{t\in B_n})$. Then A is an $IP_{<\omega}$ set. We claim that A is not a Q set, so that $\mathbb{N} \setminus A$ is as required by the lemma.

Suppose instead we have a sequence $\langle x_n\rangle_{n=1}^{\infty}$ in $\mathbb{N}$ with $\{x_n - x_m : n,m \in \mathbb{N}$ and $m < n\} \subseteq A$. Notice that $\langle x_n\rangle_{n=1}^{\infty}$ is increasing. Pick $n \in \mathbb{N}$ and nonempty $F \subseteq B_n$ such that $x_2 - x_1 = \sum_{t\in F} 2^t$. Pick $m \in \mathbb{N}$ such that $x_m > x_2 + \sum_{t\in B_n} 2^t$. Pick $k \in \mathbb{N}$ and nonempty $G \subseteq B_k$ such that $x_m - x_2 = \sum_{t\in G} 2^t$. Then $\sum_{t\in G} 2^t > x_m > \sum_{t\in B_n} 2^t$ so $k > n$. Therefore $F \cap G = \emptyset$ so $x_m - x_1 = \sum_{t\in F\cup G} 2^t \notin A$, a contradiction. □

We now set out to show in Theorem 4.15 that IP* does not imply αQ^* in $(\mathbb{N}, +)$.

Definition 4.6. Given $x \in \mathbb{N}$, $m(x)$ is the number of blocks of 1's in the binary expansion of x. We let $\langle \alpha_i(x)\rangle_{i=1}^{m(x)}$ and $\langle \delta_i(x)\rangle_{i=1}^{m(x)}$ be the increasing sequences in ω defined by $x = \sum_{i=1}^{m(x)} \sum_{t=\alpha_i(x)}^{\delta_i(x)} 2^t$ where for $i > 1$ (if any) $\alpha_i(x) > \delta_{i-1}(x)+1$.

Thus $\alpha_i(x)$ and $\delta_i(x)$ are respectively the start and end positions of the ith block of 1's in the expansion of x.

Definition 4.7. Define $f : \mathbb{N} \to \{0,1\}$ by $f(x) \equiv m(x) \pmod 2$ and let $\widetilde{f} : \beta\mathbb{N} \to \{0,1\}$ be the continuous extension of f.

Lemma 4.8. *Let A be a subset of $\mathbb{N}$. A is a Q set if and only if $A \in -p+p$ for some $p \in \mathbb{N}^*$.*

Proof. First assume that $p \in \mathbb{N}^*$ and that $A \in -p+p$. So, if $C = \{x \in \mathbb{N} : x + A \in p\} \in p$, then $C \in p$. Choose $x_1 \in C$. Given $n \in \mathbb{N}$, having chosen $\langle x_t\rangle_{t=1}^{n}$ in C, pick $x_{n+1} \in C \cap \bigcap_{t=1}^{n}(x_t + A)$. Then $\langle x_n\rangle_{n=1}^{\infty}$ is as required for A to be a Q set.

Now assume that A is a Q set. Choose $\langle x_n\rangle_{n=1}^{\infty}$ such that $x_n \in x_m + A$ whenever $m < n$. Let p be any member of $\mathbb{N}^*$ such that, for every $m \in \mathbb{N}$, $\{-x_m + x_n : n \in \mathbb{N}$ and $n > m\} \in p$. Then $-x_m + p \in \overline{A}$ for every $m \in \mathbb{N}$, and so $-p+p \in \overline{A}$. □

The following corollary is immediate.

Corollary 4.9. *The property of being a Q subset of $\mathbb{N}$ is partition regular.*

Corollary 4.10. *A subset A of $\mathbb{N}$ is a Q^* set if and only if $A \in -p+p$ for every $p \in \mathbb{N}^*$.*

Proof. Suppose that A is a Q^* set. If $p \in \mathbb{N}^*$, every member of $-p+p$ is a Q set, by Lemma 4.8. So A meets every member of $-p+p$ and hence $A \in -p+p$.

Now suppose that A is a member of $-p+p$ for every $p \in \mathbb{N}^*$. If B is a Q subset of $\mathbb{N}$, B is a member $-p+p$ for some $p \in \mathbb{N}^*$, by Lemma 4.8. So $A \cap B \neq \emptyset$. □

Corollary 4.11. *In the case in which $S = \mathbb{N}$, $M_Q = c\ell_{\beta\mathbb{N}}(\{-p+p : p \in \mathbb{N}^*\})$.*

Proof. By Corollary 4.10, $c\ell_{\beta\mathbb{N}}(\{-p+p : p \in \mathbb{N}^*\}) \subseteq M_Q$. For the reverse inclusion, assume that $q \in M_q$, suppose that $q \notin c\ell_{\beta\mathbb{N}}(\{-p+p : p \in \mathbb{N}^*\})$, and pick $E \in q$ such that $\overline{E} \cap \{-p+p : p \in \mathbb{N}^*\} = \emptyset$. By Corollary 4.10, $\mathbb{N} \setminus E$ is a Q^* set so $q \in \overline{\mathbb{N} \setminus E}$, a contradiction. □

Lemma 4.12. *Let p be an idempotent in $\beta\mathbb{N}$. Then $\widetilde{f}(p) = 0$ and for each $r \in \mathbb{N}$, $\{x \in \mathbb{N} : m(x) > r\} \in p$.*

Proof. Let $A = \{x \in \mathbb{N} : f(x) = 1\}$ and suppose that $\widetilde{f}(p) = 1$. Then $A \in p$. By [11, Lemma 4.14] $A^\star = \{x \in A : -x + A \in p\} \in p$ so pick $x \in A^\star$. Let $k = \delta_{m(x)}(x)$. Then $2^{k+2}\mathbb{N} \in p$ so pick $y \in 2^{k+2}\mathbb{N} \cap A \cap (-x + A)$. Then $f(x+y) = f(x) = f(y) = 1$ while $m(x+y) = m(x)+m(y)$, which is impossible.

For the second assertion suppose we have some $r \in \mathbb{N}$ such that $B = \{x \in \mathbb{N} : m(x) = r\} \in p$. Pick $x \in B$ such that $-x + B \in p$ and pick $y \in (-x + B) \cap 2^{m(x)+2}\mathbb{N}$. Then $m(x + y) = m(y) + r$. □

Lemma 4.13. *Let $J_0 = \{x \in \mathbb{N} : \delta_1(x) = \alpha_1(x)\}$ and let $J_1 = \{x \in \mathbb{N} : \delta_1(x) > \alpha_1(x)\}$. Let $x, y \in \mathbb{N}$ and assume that $\alpha_1(y) \geq \delta_{m(x)}(x) + 2$, $m(x) \geq 4$, and $m(y) \geq 4$. If $x, y \in J_0$, then $m(y - x) = m(x) + m(y) - 1$. If $x, y \in J_1$, then $m(y - x) = m(x) + m(y) + 1$.*

Proof. For $i \in \{1, 2, \ldots, m(x)\}$ let $a_i = \alpha_i(x)$ and let $b_i = \delta_i(x)$. For $i \in \{1, 2, \ldots, m(y)\}$ let $c_i = \alpha_i(y)$ and let $d_i = \delta_i(y)$.

Assume first that $x, y \in J_0$. For $i \in \{1, 2, \ldots, m(x) + m(y) - 1\}$ define e_i and f_i as follows:

$e_1 = a_1$ and $f_1 = a_2 - 1$.

For $i \in \{2, 3, \ldots, m(x) - 1\}$, $e_i = b_i + 1$ and $f_i = a_{i+1} - 1$.

$e_{m(x)} = b_{m(x)} + 1$ and $f_{m(x)} = c_1 - 1$.

For $i \in \{2, 3, \ldots, m(y)\}$, $e_{m(x)+i-1} = c_i$ and $f_{m(x)+i-1} = d_i$.

Then $\sum_{i=1}^{m(x)+m(y)-1} \sum_{t=e_i}^{f_i} 2^t =$
$\sum_{t=a_1}^{a_2-1} 2^t + \sum_{i=2}^{m(x)-1} \sum_{t=b_i+1}^{a_{i+1}-1} 2^t + \sum_{t=b_{m(x)}+1}^{c_1-1} 2^t + \sum_{i=2}^{m(y)} \sum_{t=c_i}^{d_i} 2^t = y - x.$

Since for each $i \in \{1, 2, \ldots, m(x) + m(y) - 2\}$, $f_i + 1 < e_{i+1}$, we have that $m(y - x) = m(x) + m(y) - 1$ as required.

Now assume that $x, y \in J_1$. For $i \in \{1, 2, \ldots, m(x) + m(y) + 1\}$ define e_i and f_i as follows:

$e_1 = a_1$ and $f_1 = a_1$.

For $i \in \{2, 3, \ldots, m(x)\}$, $e_i = b_{i-1} + 1$ and $f_i = a_i - 1$.

$e_{m(x)+1} = b_{m(x)} + 1$ and $f_{m(x)+1} = c_1 - 1$.

$e_{m(x)+2} = c_1 + 1$ and $f_{m(x)+2} = d_1$.

For $i \in \{2, 3, \ldots, m(y)\}$, $e_{m(x)+i+1} = c_i$ and $f_{m(x)+i+1} = d_i$.

Then $\sum_{i=1}^{m(x)+m(y)+1} \sum_{t=e_i}^{f_i} 2^t =$
$2^{a_1} + \sum_{i=2}^{m(x)} \sum_{t=b_{i-1}+1}^{a_i-1} 2^t + \sum_{t=b_{m(x)}+1}^{c_1-1} 2^t + \sum_{t=c_1+1}^{d_1} 2^t + \sum_{i=2}^{m(y)} \sum_{t=c_i}^{d_i} 2^t = y - x.$

Since for each $i \in \{1, 2, \ldots, m(x) + m(y)\}$, $f_i + 1 < e_{i+1}$, we have that $m(y - x) = m(x) + m(y) + 1$ as required. □

Lemma 4.14. *Let $A = \{x \in \mathbb{N} : f(x) = 1\}$ and let p be an idempotent in $\beta\mathbb{N}$. Then $A \in -p + p$.*

Proof. Let J_0 and J_1 be as in Lemma 4.13 and pick $i \in \{0, 1\}$ such that $J_i \in p$. By Lemma 4.12 $\{x \in J_i : m(x)$ is even and $m(x) \geq 4\} \in p$. We claim that $\{x \in J_i : m(x)$ is even and $m(x) \geq 4\} \subseteq \{x \in \mathbb{N} : x + A \in p\}$ which will suffice. So let $x \in J_i$ with $m(x)$ even and $m(x) \geq 4$. Recalling that $2^k\mathbb{N} \in p$ for each $k \in \mathbb{N}$, we have that $C = \{y \in J_i : m(y)$ is even, $m(y) \geq 4$, and $\alpha_1(y) \geq \delta_{m(x)} + 2\} \in p$. To see that $C \subseteq x + A$, let $y \in C$. By Lemma 4.13, $m(y-x)$ is odd so $y \in x + A$. $\square$

Theorem 4.15. *Let $B = \{x \in \mathbb{N} : f(x) = 0\}$. Then B is IP* and is not αQ^*.*

Proof. By Lemma 4.12, B is IP*. Pick an idempotent $p \in K(\beta\mathbb{N})$ and let $q = -p + p$. By Lemma 4.14, $B \notin q$. By Lemma 3.10 $q \in \Delta^*(\mathbb{N})$. By Lemma 4.8 if E is a Q^* set, then $E \in q$. By Theorem 3.8, B is not an αQ^* set. $\square$

Theorem 4.16. *Let $B = \{x \in \mathbb{N} : f(x) = 0\}$. Then B is IP* and is not $\alpha IP^*_{<\omega}$.*

Proof. By Lemma 4.12, B is IP*. By Lemma 4.14, for every idempotent $p \in \beta\mathbb{N}$, $\mathbb{N} \setminus B \in -p + p$. Suppose that B is $\alpha IP^*_{<\omega}$ and pick by Theorem 1.4(4) $C \subseteq \mathbb{N}$ such that $d(C) = 0$ and $B \cup C$ is $IP_{<\omega}$. This contradicts Theorem 3.11. $\square$

The equivalences in the following questions are consequences of Theorems 4.1 and 3.9.

Question 4.17. *(1) Does Q^* imply $\alpha IP^*_{<\omega}$ in $\mathbb{N}$? Equivalently does αQ^* imply $\alpha IP^*_{<\omega}$ in $\mathbb{N}$? Equivalently is $\Delta^*(\mathbb{N}) \cap M_{IP_{<\omega}} \subseteq M_Q$?*

(2) Does C^* imply αIP^* in $\mathbb{N}$? Equivalently does αC^* imply αIP^* in $\mathbb{N}$? Equivalently is $\Delta^*(\mathbb{N}) \cap M_{IP} \subseteq M_C$?

(3) Does $IP^*_{<\omega}$ imply αQ^* in $\mathbb{N}$? Equivalently does $\alpha IP^*_{<\omega}$ imply αQ^* in $\mathbb{N}$? Equivalently is $\Delta^*(\mathbb{N}) \cap M_Q \subseteq M_{IP_{<\omega}}$?

(4) Let $m \in \mathbb{N} \setminus \{1,2\}$. Does IP^*_m imply αQ^* in $\mathbb{N}$? Equivalently does αIP^*_m imply αQ^* in $\mathbb{N}$?

(5) Let $m \in \mathbb{N} \setminus \{1,2\}$. Does SIP^*_m imply αQ^* in $\mathbb{N}$? Equivalently does αSIP^*_m imply αQ^* in $\mathbb{N}$? Equivalently is $\Delta^*(\mathbb{N}) \cap M_Q \subseteq M_{SIP_m}$?

(6) Let $m \in \mathbb{N} \setminus \{1\}$. Does Q^* imply αSIP^*_m in $\mathbb{N}$? Equivalently does αQ^* imply αSIP^*_m in $\mathbb{N}$? Equivalently is $\Delta^*(\mathbb{N}) \cap M_{SIP_m} \subseteq M_Q$?

(7) Let $m \in \mathbb{N} \setminus \{1\}$. Does $IP^*_{<\omega}$ imply αSIP^*_m in $\mathbb{N}$? Equivalently does $\alpha IP^*_{<\omega}$ imply αSIP^*_m in $\mathbb{N}$? Equivalently is $\Delta^*(\mathbb{N}) \cap M_{SIP_m} \subseteq M_{IP_{<\omega}}$?

(8) Let $m, n \in \mathbb{N}$ with $1 < m < n$. Does IP^*_n imply αSIP^*_m in $\mathbb{N}$? Equivalently does αIP^*_n imply αSIP^*_m in $\mathbb{N}$?

(9) Let $m, n \in \mathbb{N}$ with $1 < m < n$. Does SIP^*_n imply αSIP^*_m in $\mathbb{N}$? Equivalently does αSIP^*_n imply αSIP^*_m in $\mathbb{N}$? Equivalently is $\Delta^*(\mathbb{N}) \cap M_{SIP_m} \subseteq M_{SIP_n}$?

(10) Does D^* imply αC^* in $\mathbb{N}$? Equivalently is $\Delta^*(\mathbb{N}) \cap M_C \subseteq M_D$?

Note that the parts of Question 4.17 are not independent. For example if Question (7) has a positive answer then so do Questions (8) and (9).

Theorem 4.18. *Question* 4.17 *contains all of the things that are not known about implications among the notions listed in Figures* 2 *and* 3 *for subsets of* $\mathbb{N}$.

Proof. We divide the notions from Figures 2 and 3 into two sets. Let $\Gamma = \{Q^*, \alpha Q^*, IP^*, \alpha IP^*, C^*, \alpha C^*, IP^*_{<\omega}, \alpha IP^*_{<\omega}, P^*, J^*, B^*, D^*\}$, and let $\Theta = \bigcup_{m=2}^{\infty}\{SIP^*_m, \alpha SIP^*_m, IP^*_m, \alpha IP^*_m\}$.

We present four tables, namely $\Gamma \times \Gamma$, $\Theta \times \Gamma$, $\Gamma \times \Theta$, and $\Theta \times \Theta$.

If R and T are notions of largeness, then the entry in row R^* and column T^* of one of these tables is

(i) + if the fact that R^* implies T^* in $\mathbb{N}$ follows from the fact that $IP^*_{<\omega}$ implies $\alpha IP^*_{<\omega}$ and the implications shown in Figures 2 and 3;

(ii) X, where X is a capital letter referring to an example showing that R^* does not imply T^* in $\mathbb{N}$; or

(iii) Qn, where the question whether R^* implies T^* is Question 4.17(n).

We begin now the listing of the examples.

(A) By Lemma 4.5 there is a subset A of $\mathbb{N}$ which is Q^* and not $IP^*_{<\omega}$. Then A is also αQ^*, IP^*, αIP^*, C^*, αC^*, and D^* and A is neither IP^*_m or SIP^*_m for any $m \in \mathbb{N} \setminus \{1\}$.

(B) Given x_1 x_2 in $\mathbb{N}$, $\{x_1, x_2, x_1 + x_2\} \cap 2\mathbb{N} \neq \emptyset$. So the set $2\mathbb{N}$ satisfies all of the listed notions except P^*, J^*, and B^*.

$\Gamma \times \Gamma$

	Q^*	αQ^*	IP^*	αIP^*	C^*	αC^*	$IP^*_{<\omega}$	$\alpha IP^*_{<\omega}$	P^*	J^*	B^*	D^*
Q^*	+	+	+	+	+	+	A	$Q1$	B	B	B	+
αQ^*	C	+	C	+	C	+	A	$Q1$	B	B	B	+
IP^*	D	D	+	+	+	+	A	E	B	B	B	+
αIP^*	C	D	C	+	C	+	A	E	B	B	B	+
C^*	D	D	F	$Q2$	+	+	A	E	B	B	B	+
αC^*	C	D	C	$Q2$	C	+	A	E	B	B	B	+
$IP^*_{<\omega}$	G	$Q3$	+	+	+	+	+	+	B	B	B	+
$\alpha IP^*_{<\omega}$	C	$Q3$	C	+	C	+	C	+	B	B	B	+
P^*	H	+	H	+	+	+	H	+	+	+	+	+
J^*	H	+	F	+	+	+	H	+	F	+	+	+
B^*	H	+	H	+	I	+	H	+	F	I	+	+
D^*	D	D	F	J	I	$Q10$	I	J	B	B	B	+

(C) By Theorem 4.2, for any notion of largeness T, αT^* does not imply any of Q^*, IP^*, C^*, $IP^*_{<\omega}$, or IP^*_m or SIP^*_m for any $m \in \mathbb{N} \setminus \{1\}$.

(D) By Theorem 4.15 there is a subset of $\mathbb{N}$ which is IP^* and not αQ^*. This set is also αIP^*, C^*, αC^*, and D^* and is also not Q^*.

(E) By Theorem 4.16 there is a subset of $\mathbb{N}$ which is IP^* and therefore αIP^*, C^*, and αC^* and is not $\alpha IP^*_{<\omega}$ and therefore not αSIP^*_m for any $m \in \mathbb{N} \setminus \{1\}$.

(F) By Lemma 4.4 there is a subset of $\mathbb{N}$ which is J^* and neither P^* nor IP^*. This set is also C^*, B^*, and D^* and is also neither of IP^*_m or SIP^*_m for any $m \in \mathbb{N} \setminus \{1\}$.

(G) The set $\{2^{2n} - 2^{2m} : m < n \text{ in } \mathbb{N}\}$ is a Q set and it is easy to see that it is not an IP_3 set. So $\mathbb{N} \setminus \{2^{2n} - 2^{2m} : m < n \text{ in } \mathbb{N}\}$ is IP^*_3, hence also $IP^*_{<\omega}$ and IP^*_m and SIP^*_m for $m \geq 3$, and is not Q^*.

$\Theta \times \Gamma$, $m > 2$

	Q^*	αQ^*	IP^*	αIP^*	C^*	αC^*	$IP^*_{<\omega}$	$\alpha IP^*_{<\omega}$	P^*	J^*	B^*	D^*
IP^*_2	+	+	+	+	+	+	+	+	B	B	B	+
αIP^*_2	C	+	C	+	C	+	C	+	B	B	B	+
IP^*_m	G	$Q4$	+	+	+	+	+	+	B	B	B	+
αIP^*_m	C	$Q4$	C	+	C	+	C	+	B	B	B	+
SIP^*_2	+	+	+	+	+	+	+	+	B	B	B	+
αSIP^*_2	C	+	C	+	C	+	C	+	B	B	B	+
SIP^*_m	G	$Q5$	+	+	+	+	+	+	B	B	B	+
αSIP^*_m	C	$Q5$	C	+	C	+	C	+	B	B	B	+

(H) The set $\{\sum_{n\in F} 2^{2n} : F \in \mathcal{P}_f(\mathbb{N})\}$ is an IP set which contains no 3 term arithmetic progression so its complement is P^*, hence also J^* and B^*, and not IP^*, hence not Q^*, not $IP^*_{<\omega}$, and neither IP^*_m nor SIP^*_m for $m \geq 2$.

(I) By [7, Theorem 2.1] there is a subset A of $\mathbb{N}$ which is a C set such that $d(A) = 0$. So $\mathbb{N} \setminus A$ is B^*, hence D^*, and is not C^*, hence not $IP^*_{<\omega}$ and not J^*.

(J) In [12, Theorem 3.1], a subset A of $\mathbb{N}$ was produced which is not a D set in $\mathbb{Z}$, hence not a D set in $\mathbb{N}$, and for each $E \subseteq \mathbb{Z}$ with $d(E) = 0$, $A \setminus E$ is an IP set in $\mathbb{Z}$, hence in $\mathbb{N}$. Then $\mathbb{N} \setminus A$ is a D^* set. Given $E \subseteq \mathbb{N}$ with $d(E) = 0$, $A \setminus E$ is an IP set missing $(\mathbb{N} \setminus A) \cup E$ so by Theorem 1.4(4), $\mathbb{N} \setminus A$ is not an αIP^* set, hence also not $\alpha IP^*_{<\omega}$ and neither αIP^*_m nor αSIP^*_m for any $m \geq 2$.

(K) Let $1 < m < n$. By [10, Corollary 3.8] there is a set $A \subseteq \mathbb{N}$ which is SIP_m and not IP_{m+1} so not IP_n. Then $\mathbb{N} \setminus A$ is IP^*_n, hence SIP^*_n and $IP^*_{<\omega}$, and not SIP^*_m hence not IP^*_m.

(L) Let $1 < m < n$. By Theorem 3.3(a), $(n+1)\mathbb{N}$ is SIP^*_2, hence all of SIP^*_m, SIP^*_n, αSIP^*_m, αSIP^*_n, Q^*, αQ^*, IP^*, αIP^*, C^*, αC^*, $IP^*_{<\omega}$ and $\alpha IP^*_{<\omega}$. By Theorem 4.3 $(n+1)\mathbb{N}$ is not αIP^*_n hence also not αIP^*_m.

(M) Let $1 < m < n$. By Theorem 4.3, $\mathbb{N}n$ is IP^*_n, hence SIP^*_n, αIP^*_n, and αSIP^*_n, but not αIP^*_{n-1}, so not αIP^*_m.

$\Gamma \times \Theta$, $m > 1$

	IP^*_m	αIP^*_m	SIP^*_m	α SIP^*_m
Q^*	A	L	A	$Q6$
αQ^*	C	L	C	$Q6$
IP^*	A	L	A	E
αIP^*	C	L	C	E
C^*	A	L	A	E
αC^*	C	L	C	E
$IP^*_{<\omega}$	K	L	K	$Q7$
α $IP^*_{<\omega}$	C	L	C	$Q7$
P^*	H	$+$	H	$+$
J^*	F	$+$	F	$+$
B^*	F	$+$	F	$+$
D^*	A	J	A	J

(N) Let $1 < m < n$. Then $FS(\langle 2^{2t}\rangle_{t=1}^m)$ is IP_m and not SIP_2 and $FS(\langle 2^{2t}\rangle_{t=1}^n)$ is IP_n and not SIP_2. So so $\mathbb{N} \setminus FS(\langle 2^{2t}\rangle_{t=1}^m)$ is SIP^*_2, hence SIP^*_m, and not IP^*_m. And $\mathbb{N} \setminus FS(\langle 2^{2t}\rangle_{t=1}^n)$ is SIP^*_2, hence SIP^*_m and SIP^*_n, and not IP^*_n.

$\Theta \times \Theta$, $1 < m < n$

	IP^*_m	αIP^*_m	SIP^*_m	α SIP^*_m	IP^*_n	αIP^*_n	SIP^*_n	α SIP^*_n
IP^*_m	+	+	+	+	+	+	+	+
αIP^*_m	C	+	C	+	C	+	C	+
SIP^*_m	N	L	+	+	N	L	+	+
α SIP^*_m	C	L	C	+	C	L	C	+
IP^*_n	K	M	K	$Q8$	+	+	+	+
αIP^*_n	C	M	C	$Q8$	C	+	C	+
SIP^*_n	K	M	K	$Q9$	N	L	+	+
α SIP^*_n	C	M	C	$Q9$	C	L	C	+

□

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