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Bases for Commutative Semigroups and Groups

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Abstract

A *base* for a commutative semigroup $(S,+)$ is an indexed set $\langle x_t\rangle_{t\in A}$ in S such that each element $x\in S$ is uniquely representable as $\sum_{t\in F} x_t$ where F is a finite subset of A and, if S has an identity 0, then $0=\sum_{n\in\emptyset} x_t$. We investigate those commutative semigroups or groups which have a base. We obtain the surprising result that $\mathbb{Q}$ has a base. More generally, we show that an abelian group has a base if and only if it has no elements of odd finite order.

1. *Introduction*

In [**1**] N. G. de Bruijn thoroughly investigated *bases* for the group $\mathbb{Z}$ of integers, that is sequences $\langle x_n\rangle_{n=1}^\infty$ such that every element of $\mathbb{Z}$ is uniquely representable in the form $\sum_{n\in F} x_n$ for some finite subset of $\mathbb{N}$. He showed that there are many such bases for $\mathbb{Z}$ and presented some sufficient conditions and some necessary conditions for a given sequence to be a base for $\mathbb{Z}$. We investigate in this paper bases for other commutative semigroups and groups.

Definition 1·1. Let $(S,+)$ be a commutative semigroup and let $\langle x_t\rangle_{t\in A}$ be an indexed set in S.

(a) $FS(\langle x_t\rangle_{t\in A})=\{\sum_{t\in F} x_t : F \text{ is a finite nonempty subset of } A\}$.
(b) $FS_0(\langle x_t\rangle_{t\in A})=FS(\langle x_t\rangle_{t\in A})\cup\{0\}=\{\sum_{t\in F} x_t : F \text{ is a finite subset of } A\}$.
(c) The indexed set $\langle x_t\rangle_{t\in A}$ *satisfies uniqueness of finite sums* if and only if whenever F and H are finite subsets of A and $F\neq H$, one has that $\sum_{t\in F} x_t\neq\sum_{t\in H} x_t$.
(d) The indexed set $\langle x_t\rangle_{t\in A}$ is a *base* for S if and only if it satisfies uniqueness of finite sums and either $S=FS(\langle x_t\rangle_{t\in A})$ or $S=FS_0(\langle x_t\rangle_{t\in A})$ (depending, of course, on whether or not S has an identity 0).

In Section 2 of this paper we shall show that $(\mathbb{Q},+)$ has a base. We found this surprising. Indeed, we initially believed that $(\mathbb{Q},+)$ could not have a base, and we were not alone in

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this belief. In [**2**] Budak, Işik, and Pym established that members of $\mathbb{Q}$ can be expressed uniquely in the form $\sum_{t\in A} a_t x_t$ for $a_t \in D_t$ where $A = \mathbb{Z}$, $x_t = \dfrac{(-1)^t}{(1-t)!}$ and $D_t = \{0, 1, \ldots, -t\}$ if $t < 0$, and $x_t = (-1)^t(1+t)!$ and $D_t = \{0, 1, \ldots, t+1\}$ if $t \geq 0$. After doing so, they remarked "It does appear unlikely to us that ... $\mathbb{Q}$ can be described ... in such a way that $|D_k| = 2$ for all k."

In Section 3 we shall show that a commutative group has a base if and only if it has no elements of finite odd order.

In Section 4 we shall establish that there are (up to isomorphism) only two countable subsemigroups of $(\mathbb{R}^+, +)$ that are closed under positive differences and have a base, and then only in an essentially unique way. (By $\mathbb{R}^+$ we mean the set of positive real numbers.)

We note that there is a literature on the ways that an abelian group can be expressed as a direct sum of a family of its subsets. For example [**4**, Chapter XV] is devoted to this question.

Our own interest in this subject was motivated by the fact that such expressions have been a useful tool in studying semigroup compactifications. In a subsequent paper [**3**], the authors and S. Ferri were able to use expressions of this type to obtain new results about the Stone-Čech compactification βG and the weakly almost periodic compactification G^{WAP} of of an infinite discrete abelian group G. It is shown there that the smallest ideal of βG contains a free group on $2^{2^{|G|}}$ generators and that G^{WAP} contains a free abelian semigroup on $2^{2^{|G|}}$ generators.

2. $\mathbb{Q}$ *has a base*

In this section we establish some lemmas for later use and prove that $\mathbb{Q}$ has a base. (It is a consequence of Lemma 3·5 below that every nontrivial subgroup of $\mathbb{Q}$ also has a base.)

We start with some special notation which we shall use throughout this section.

Definition 2·1. Let $n \in \mathbb{Z}$.

(a) $D_n = \{2^n \frac{k}{l} : k, l \in 2\mathbb{Z}+1\}$
(b) If $\langle x_t \rangle_{t\in A}$ is an indexed set in $\mathbb{Q}$, then $M_n(\langle x_t \rangle_{t\in A}) = D_n \cap \{x_t : t \in A\}$.

Notice that if $n \in \mathbb{Z}$ and $n \geq 0$, then $D_n \cap \mathbb{Z} = 2^n + 2^{n+1}\mathbb{Z}$, the set of integers congruent to 2^n mod 2^{n+1}, while if $n < 0$, $D_n \cap \mathbb{Z} = \emptyset$.

We omit the routine proof of the following lemma.

LEMMA 2·2. *Let $n, r \in \mathbb{Z}$ with $n \leq r$, let $x \in D_n$, and let $y \in D_r$.*

(a) *If $n < r$, then $x + y \in D_n$.*
(b) *If $n = r$, then $x + y \in D_s$ for some $s > n$.*

LEMMA 2·3. *Let $\langle x_t \rangle_{t\in A}$ be an indexed set in $\mathbb{Q}$. If for each $n \in \mathbb{Z}$, $|M_n(\langle x_t \rangle_{t\in A})| \leq 1$, then $\langle x_t \rangle_{t\in A}$ satisfies uniqueness of finite sums.*

Proof. Suppose we have finite F and H contained in A such that $\sum_{t\in F} x_t = \sum_{t\in H} x_t$ but $F \neq H$. By subtracting any terms in $F \cap H$, we may presume that $F \cap H = \emptyset$. Let $m = \min\{n : \text{there is some } t \in F \cup H \text{ with } x_t \in D_n\}$ and assume without loss of generality that we have $s \in F$ such that $M_m(\langle x_t \rangle_{t\in A}) = \{x_s\}$. For each $t \in F \cup H$ pick $n_t \in \mathbb{Z}$ and k_t and l_t in $2\mathbb{Z}+1$ such that $x_t = 2^{n_t} \frac{k_t}{l_t}$ and let $r = \prod_{t\in F\cup H} l_t$, noting that r is an odd integer. Then $k_s\frac{r}{l_s}+\sum_{t\in F\setminus\{s\}} 2^{n_t-m}k_t\frac{r}{l_t}=\sum_{t\in H} 2^{n_t-m}k_t\frac{r}{l_t}$. This is a contradiction since the number on the left hand side is an odd integer, while the number on the right hand side is an even integer. □

Definition 2·4. Let $\langle x_t\rangle_{t\in A}$ be an indexed set in a semigroup which satisfies uniqueness of finite sums. If $y\in FS(\langle x_t\rangle_{t\in A})$, then $\mathrm{supp}\,(y)$ is the finite subset of A such that $y=\sum_{t\in \mathrm{supp}\,(y)} x_t$.

The following simple observation will be very useful.

LEMMA 2·5. *Let $(G,+)$ be an abelian group and let $\langle x_t\rangle_{t\in A}$ be a base for G. For any $s\in A$ and any $w\in G$,*

(a) *$s\in \mathrm{supp}\,(w)$ if and only if $s\notin \mathrm{supp}\,(w+x_s)$,*
(b) *if r is an odd integer, then $s\in \mathrm{supp}\,(w)$ if and only if $s\notin \mathrm{supp}\,(w+rx_s)$, and*
(c) *if r is an even integer, then $s\in \mathrm{supp}\,(w)$ if and only if $s\in \mathrm{supp}\,(w+rx_s)$.*

Proof. (a) Let $F=\mathrm{supp}\,(w)$ and let $H=\mathrm{supp}\,(w+x_s)$. If $s\notin F$, then $w+x_s=\sum_{t\in F\cup\{s\}} x_t$ so $H=F\cup\{s\}$. If $s\in H$, then $w=\sum_{t\in H\setminus\{s\}} x_t$ so $F=H\setminus\{s\}$.
Conclusions (b) and (c) are immediate consequences of (a). □

The following lemma is proved in [**1**] in the case in which $G=\mathbb{Z}$.

LEMMA 2·6. *Let $(G,+)$ be an abelian group, let $\langle x_t\rangle_{t\in A}$ and $\langle y_t\rangle_{t\in A}$ be indexed sets in G, and assume that $\langle x_t\rangle_{t\in A}$ is a base for G. If for each $t\in A$, y_t is an odd multiple of x_t and $\{t\in A: y_t\neq x_t\}$ is finite, then $\langle y_t\rangle_{t\in A}$ is a base for G.*

Proof. Let $B=\{t\in A: y_t\neq x_t\}$. We proceed by induction on $|B|$, the case $|B|=0$ being trivial. So assume $|B|\geq 1$ and the result is true for smaller sizes. Pick $s\in B$ and pick odd $m\in\mathbb{Z}$ such that $y_t=mx_t$. For each $t\in A$, define

$$z_t=\begin{cases} y_t & \text{if } t\neq s\\ x_t & \text{if } t=s\,.\end{cases}$$

By the induction hypothesis $\langle z_t\rangle_{t\in A}$ satisfies uniqueness of finite sums and $G=FS_0(\langle z_t\rangle_{t\in A})$.

To see that $\langle y_t\rangle_{t\in A}$ satisfies uniqueness of finite sums, let F and H be distinct finite subsets of A and suppose that $\sum_{t\in F} y_t=\sum_{t\in H} y_t$. By subtracting common terms, we may suppose that $F\cap H=\emptyset$. If $s\notin F\cup H$, we have a contradiction, so we may assume that $s\in F$. Then $mz_s+\sum_{t\in F\setminus\{s\}} z_t=\sum_{t\in H} z_t$. Then by Lemma 2·5(b) (with supports computed in terms of $\langle z_t\rangle_{t\in A}$), $s\in H$, a contradiction.

To see that $G=FS_0(\langle y_t\rangle_{t\in A})$, let $w\in G$ and pick finite $F\subseteq A$ such that $w=\sum_{t\in F} z_t$. If $s\notin F$, then $w=\sum_{t\in F} y_t$, so assume that $s\in F$. Pick finite $H\subseteq A$ such that $w-mz_s=\sum_{t\in H} z_t$. By Lemma 2·5(b) we have that $s\notin H$ so $w=\sum_{t\in H\cup\{s\}} y_t$. □

Notice that the restriction that $\{t\in A: y_t\neq x_t\}$ be finite cannot be removed from Lemma 2·6. To see this, consider the sequences $\langle(-2)^t\rangle_{t\in\omega}$ and $\langle 2^t\rangle_{t\in\omega}$. Then $FS_0(\langle(-2)^t\rangle_{t\in\omega})=\mathbb{Z}$ while $FS_0(\langle 2^t\rangle_{t\in\omega})=\omega$.

THEOREM 2·7. *Let G be a subgroup of $(\mathbb{Q},+)$ and assume that $\langle x_t\rangle_{t\in A}$ is a base for G. Then for each $n\in\mathbb{Z}$, $|M_n(\langle x_t\rangle_{t\in A})|\leq 1$.*

Proof. Let $n\in\mathbb{Z}$ and suppose that $|M_n(\langle x_t\rangle_{t\in A})|$ has at least two members, say x_a

and x_b. Pick odd integers k, l, r, and s such that $x_a = 2^n\frac{k}{l}$ and $x_b = 2^n\frac{r}{s}$. Then $lrx_a = 2^n kr = ksx_b$. For $t \in A$, let

$$y_t = \begin{cases} lrx_a & \text{if } t = a \\ ksx_b & \text{if } t = b \\ x_t & \text{otherwise.} \end{cases}$$

By Lemma 2·6, $\langle y_t\rangle_{t\in A}$ satisfies uniqueness of finite sums, while $y_a = y_b$, a contradiction. □

LEMMA 2·8. *Let G be a nontrivial subgroup of $(\mathbb{Q},+)$ and let $W = \{n \in \mathbb{Z} : G \cap D_n \neq \emptyset\}$. Assume that H is a finite subset of W and $w \in G$. If for each $t \in H$, $x_t \in G \cap D_t$, then there are some $F \subseteq H$ and some $r \in W \setminus H$ such that $w - \sum_{t\in F} x_t \in D_r$.*

Proof. Pick $n \in W$ such that $w \in D_n$. If $n \notin H$, let $F = \emptyset$ and $r = n$.

Assume that $n \in H$. We proceed by induction on $|H|$. If $H = \{n\}$, we have by Lemma 2·2 that $w - x_n \in D_r$ for some $r > n$. Now assume that $|H| > 1$ and the lemma is valid for smaller sets. Let $m = \max H$. If $m = n$, then as above we have that $w - x_n \in D_r$ for some $r > n$, so assume that $n \in H \setminus \{m\}$. By the induction hypothesis pick $F \subseteq H \setminus \{m\}$ and $r \in W \setminus (H \setminus \{m\})$ such that $w - \sum_{t\in F} x_t \in D_r$. If $r \neq m$, we are done. If $r = m$, then $w - \sum_{t\in F\cup\{m\}} x_t \in D_s$ for some $s > m$ by Lemma 2·2. □

We now see $(\mathbb{Q},+)$ has a base and that one may make arbitrary assignments of x_t for finitely many t, subject only to the restriction imposed by Theorem 2·7.

THEOREM 2·9. *$\mathbb{Q}$ has a base.*

Proof. Let H be a finite subset of $\mathbb{Z}$ and assume that for all $t \in H$, $x_t \in D_t$. We claim that we may choose x_t for $t \in \mathbb{Z} \setminus H$ such that $x_t \in D_t$ for each t and $\langle x_t\rangle_{t\in W}$ is a base for $\mathbb{Q}$.

Enumerate $\mathbb{Q} \setminus \{0\}$ as $\langle a_t\rangle_{t=1}^\infty$. Assume that we have finite L with $H \subseteq L \subseteq \mathbb{Z}$, and that x_t has been chosen for $t \in L$. Pick the least s such that $a_s \notin FS_0(\langle x_t\rangle_{t\in L})$. Pick $n \in \mathbb{Z}$ such that $a_s \in D_n$. If $n \notin L$, let $x_n = a_s$. If $n \in L$, pick by Lemma 2·8, $F \subseteq L$ and $r \in W \setminus L$ such that $a_s - \sum_{t\in F} x_t \in D_r$ and let $x_r = a_s - \sum_{t\in F} x_t$.

At the completion of this inductive process, let $U = \{r \in \mathbb{Z} : x_r \text{ has been chosen}\}$. We have by Lemma 2·3 that $\langle x_t\rangle_{t\in U}$ satisfies uniqueness of finite sums. Clearly, $FS(\langle x_t\rangle_{t\in\mathbb{Z}}) = \mathbb{Q}$. □

Note that the algorithm implicit in the proofs of Lemma 2·8 and Theorem 2·9 is effectively computable. That is, if an ordering $\langle a_t\rangle_{t=1}^\infty$ of $\mathbb{Q} \setminus \{0\}$ is specified and $y = a_s$, then we know that we will have $y \in FS_0(\langle x_t\rangle_{t=1}^s)$ and we can compute $\langle x_t\rangle_{t=1}^s$ in finite time. Unfortunately, we have not been able to find an explicit definition of a sequence $\langle x_n\rangle_{n=1}^\infty$ which satisfies uniqueness of finite sums such that $\mathbb{Q} = FS_0(\langle x_n\rangle_{n=1}^\infty)$.

3. *Bases for semigroups and groups*

We establish in this section that certain semigroups have bases and that an abelian group has a base if and only if it has no elements of odd finite order.

We omit the obvious proof of the following theorem.

THEOREM 3·1. *Let I be a set and for each $i \in I$ let G_i be either an abelian group or a commutative semigroup with identity 0. If each G_i has a base, then so does $\bigoplus_{i\in I} G_i$.*

As a consequence of Theorems 3·1 and 2·9 we see that $(\mathbb{R},+)$, being the direct sum of $\mathfrak{c}$ copies of $\mathbb{Q}$, has a base and that $(\mathbb{N},\cdot)$ and $(\mathbb{Q}^+,\cdot)$ have bases. (The first is isomorphic to $\bigoplus_{i\in\mathbb{N}}\omega$ and the second is isomorphic to $\bigoplus_{i\in\mathbb{N}}\mathbb{Z}$.) Also $(\mathbb{Q}\setminus\{0\},\cdot)$ has a base because it is isomorphic to the direct sum of $(\mathbb{Q}^+,\cdot)$ and $\mathbb{Z}_2$.

THEOREM 3·2. *Let G be an abelian group and let K be a subgroup of G. If K and G/K have bases, then so does G. In fact, if $\langle x_t\rangle_{t\in A}$ is a base for K, then there exist a set B disjoint from A and an indexed set $\langle x_t\rangle_{t\in B}$ such that $\langle x_t\rangle_{t\in A\cup B}$ is a base for G.*

Proof. Assume that $\langle x_t\rangle_{t\in A}$ and $\langle y_t\rangle_{t\in B}$ are indexed sets satisfying uniqueness of finite sums such that $K = FS_0(\langle x_t\rangle_{t\in A})$ and $G/K = FS_0(\langle y_t\rangle_{t\in B})$. We may assume that $A\cap B=\emptyset$. For $t\in B$ pick $x_t\in G$ such that $x_t+K=y_t$. We claim that $\langle x_t\rangle_{t\in A\cup B}$ satisfies uniqueness of finite sums and $G = FS(\langle x_t\rangle_{t\in A\cup B})$.

To verify the former, let F and H be finite subsets of $A\cup B$ and assume that $\sum_{t\in F}x_t = \sum_{t\in H}x_t$. Then $\sum_{t\in F\cap A}x_t - \sum_{t\in H\cap A}x_t = \sum_{t\in H\cap B}x_t - \sum_{t\in F\cap B}x_t$ so $\sum_{t\in H\cap B}x_t - \sum_{t\in F\cap B}x_t\in K$. Thus

$$\textstyle\sum_{t\in H\cap B}y_t = \sum_{t\in H\cap B}(x_t+K) = \sum_{t\in F\cap B}(x_t+K) = \sum_{t\in F\cap B}y_t\,,$$

and so $H\cap B = F\cap B$. Thus $\sum_{t\in F\cap A}x_t = \sum_{t\in H\cap A}x_t$ so $F=H$ as required.

Now let $w\in G$. Pick finite $F\subseteq B$ such that $w+K=\sum_{t\in F}y_t$. Then $w-\sum_{t\in F}x_t\in K$ so pick finite $H\subseteq A$ such that $w-\sum_{t\in F}x_t = \sum_{t\in H}x_t$. $\square$

THEOREM 3·3. *Let $(G,+)$ be an abelian group which has a base. Then G has no nonzero elements of odd finite order.*

Proof. Pick a base $\langle x_t\rangle_{t\in A}$ for G. Suppose we have $w\in G\setminus\{0\}$ and odd $m\in\mathbb{Z}$ such that $mw=0$. Pick finite $F\subseteq A$ such that $w=\sum_{t\in F}x_t$. Then $0=\sum_{t\in F}mx_t$. But by Lemma 2·6, the sequence $\langle y_t\rangle_{t\in A}$ defined by

$$y_t = \begin{cases} x_t & \text{if } t\notin F\\ mx_t & \text{if } t\in F\end{cases}$$

satisfies uniqueness of finite sums, so $F=\emptyset$, a contradiction. $\square$

By contrast we have the following theorem. In the proof of this theorem, it may be that G_n or G_{n+1} is trivial. In this event note that $\emptyset$ is a base for $\{0\}$.

THEOREM 3·4. *Let $(G,+)$ be an abelian group in which every element has finite order which is some power of 2. Then G has a base.*

Proof. For each $n\in\mathbb{N}$, let $G_n=\{a\in G: 2^n a = 0\}$. Then each G_n is a subgroup of G. We produce inductively a possibly empty set A_n and $\langle x_t\rangle_{t\in A_n}$ such that $A_n\cap A_k=\emptyset$ for $n\neq k$ and if $B_m=\bigcup_{n=1}^m A_n$, then for each $m\in\mathbb{N}$, $\langle x_t\rangle_{t\in B_m}$ satisfies uniqueness of finite sums and $G_m = FS_0(\langle x_t\rangle_{t\in B_m})$.

We have that G_1 is a vector space over $\mathbb{Z}_2$, so is isomorphic to a direct sum of copies of $\mathbb{Z}_2$ so by Theorem 3·1, G_1 has a base. Pick A_1 and $\langle x_t\rangle_{t\in A_1}$ satisfying uniqueness of finite sums such that $G_1 = FS_0(\langle x_t\rangle_{t\in A_1})$.

Inductively, let $m\in\mathbb{N}$ and assume that A_n and $\langle x_t\rangle_{t\in A_n}$ have been chosen for $n\leq m$. Now given $w\in G_{m+1}$, $2w\in G_m$ so G_{m+1}/G_m is a vector space over $\mathbb{Z}_2$ and therefore has a base. By Theorem 3·2 we may choose a set A_{m+1} disjoint from $B_m=\bigcup_{n=1}^m A_n$ and $\langle x_t\rangle_{t\in A_{m+1}}$ such that $\langle x_t\rangle_{t\in B_m\cup A_{m+1}}$ satisfies uniqueness of finite sums and $G_{m+1} = FS_0(\langle x_t\rangle_{t\in B_m\cup A_{m+1}})$.

The induction being complete, let $B = \bigcup_{n=1}^{\infty} A_n$ Then $\langle x_t \rangle_{t \in B}$ satisfies uniqueness of finite sums and $G = FS_0(\langle x_t \rangle_{t \in B})$. $\square$

LEMMA 3·5. *Every subgroup of $\bigoplus_{n=1}^{\infty}(\mathbb{Q}, +)$ has a base.*

Proof. Let G be a subgroup of $\bigoplus_{n=1}^{\infty}(\mathbb{Q}, +)$. If $G = \{\overline{0}\}$, the conclusion is trivial, so assume G is infinite and enumerate $G \setminus \{\overline{0}\}$ as $\langle a_t \rangle_{t=1}^{\infty}$. For $x \in G$ define $\text{supp}\,(x) = \{n \in \mathbb{N} : x(n) \neq 0\}$. Define $f : G \setminus \{\overline{0}\} \to \mathbb{N}$ and $g : G \setminus \{\overline{0}\} \to \mathbb{Z}$ by, for $x \in G \setminus \{\overline{0}\}$, $f(x) = \max \text{supp}\,(x)$ and $x\big(f(x)\big) \in D_{g(x)}$.

Let $x_1 = a_1$. Inductively let $n \in \mathbb{N}$ and assume that we have chosen $\langle x_m \rangle_{m=1}^{n}$ such that if $k \neq m$, then either $f(x_k) \neq f(x_m)$ or $g(x_k) \neq g(x_m)$. Let

$$s = \min\left\{t \in \mathbb{N} : a_t \notin FS(\langle x_m \rangle_{m=1}^{n})\right\}.$$

Let $y_1 = a_s$, let $l \in \{1, 2, \ldots, n\}$, and assume that $\langle y_j \rangle_{j=1}^{l}$ have been chosen. If for all $i \in \{1, 2, \ldots, n\}$, either $f(x_i) \neq f(y_l)$ or $g(x_i) \neq g(y_l)$, let $x_{n+1} = y_l$. Otherwise pick the unique $\alpha(l) \in \{1, 2, \ldots, n\}$ such that $f(x_{\alpha(l)}) = f(y_l)$ and $g(x_{\alpha(l)}) = g(y_l)$ and let $y_{l+1} = y_l - x_{\alpha(l)}$. Notice that either $f(y_{l+1}) < f(y_l)$ or (by virtue of Lemma 2·2) both $f(y_{l+1}) = f(y_l)$ and $g(y_{l+1}) > g(y_l)$. In particular, the function α is injective, so this process must terminate in n steps or fewer. Notice that either $a_s = x_{n+1}$ or $a_s = x_{n+1} + \sum_{i=1}^{l} x_{\alpha(i)}$.

The induction being complete, we have directly that $G = FS_0(\langle x_n \rangle_{n=1}^{\infty})$. We claim that $\langle x_n \rangle_{n=1}^{\infty}$ satisfies uniqueness of finite sums. To this end assume that we have distinct finite subsets F and H of $\mathbb{N}$ such that $\sum_{t \in F} x_t = \sum_{t \in H} x_t$. We may presume that $F \cap H = \emptyset$. Let $k = \max\{f(x_t) : t \in F \cup H\}$, let $F' = \{t \in F : f(x_t) = k\}$, and let $H' = \{t \in H : f(x_t) = k\}$. Since $F' \cup H' \neq \emptyset$ and $F \cap H = \emptyset$, we have that $F' \neq H'$.

Then $\sum_{t \in F'} x_t(k) = \sum_{t \in H'} x_t(k)$ so by Lemma 2·3 there must be some $n \in \mathbb{Z}$ and some distinct t and s in $F' \cup H'$ such that $x_t(k) \in D_n$ and $x_s(k) \in D_n$. But then $f(x_t) = f(x_s)$ and $g(x_t) = g(x_s)$, a contradiction. $\square$

THEOREM 3·6. *Every torsion free abelian group has a base.*

Proof. We proceed by induction on the cardinality of the group G. If G is countable, then as is well known G is isomorphic to a subgroup of $\bigoplus_{n=1}^{\infty}(\mathbb{Q}, +)$. (See for example [**5**, Theorems A14 and A15].) Thus by Lemma 3·5, G has a base.

So assume that $|G| = \kappa > \omega$ and enumerate G as $\langle a_\iota \rangle_{\iota < \kappa}$. For $\sigma < \kappa$, let G_σ be the subgroup of G generated by $\{a_\iota : \iota \leq \sigma\}$ and let $H_\sigma = \{x \in G :$ there is some $n \in \mathbb{Z} \setminus \{0\}$ such that $nx \in G_\sigma\}$. Note that H_σ is a subgroup of G and $|H_\sigma| < \kappa$.

We shall choose inductively A_σ and $\langle x_t \rangle_{t \in A_\sigma}$ for $\sigma < \kappa$ such that $A_\sigma \cap A_\mu = \emptyset$ when $\sigma \neq \mu$ and if $\gamma < \kappa$ and $B_\gamma = \bigcup_{\sigma \leq \gamma} A_\sigma$, then $\langle x_t \rangle_{t \in B_\gamma}$ satisfies uniqueness of finite sums and $H_\gamma = FS_0(\langle x_t \rangle_{t \in B_\gamma})$. To this end, let $\gamma < \kappa$ and assume that we have chosen A_σ and $\langle x_t \rangle_{t \in A_\sigma}$ for $\sigma < \gamma$. Let $C = \bigcup_{\sigma < \gamma} A_\sigma$ and let $H = \bigcup_{\sigma < \gamma} H_\sigma$. (If γ is a successor, say $\gamma = \tau + 1$, then $C = B_\tau$ and $H = H_\tau$.) Then $\langle x_t \rangle_{t \in C}$ satisfies uniqueness of finite sums and $H = FS_0(\langle x_t \rangle_{t \in C})$. Also if $x \in H_\gamma$, $n \in \mathbb{Z} \setminus \{0\}$, and $nx \in H$, then $x \in H$. Thus H_γ / H is torsion free. Since also $|H_\gamma / H| \leq |H_\gamma| < \kappa$, we have by the induction hypothesis that H_γ / H has a base. Thus by Theorem 3·2 we may pick A_γ disjoint from C and $\langle x_t \rangle_{t \in A_\gamma}$ such that $\langle x_t \rangle_{t \in C \cup A_\gamma}$ satisfies uniqueness of finite sums and $H_\gamma = FS_0(\langle x_t \rangle_{t \in C \cup A_\gamma})$.

The construction being complete, let $B = \bigcup_{\gamma < \kappa} A_\gamma$. Then $\langle x_t \rangle_{t \in B}$ satisfies uniqueness of finite sums and $G = FS_0(\langle x_t \rangle_{t \in B})$. $\square$

Corollary 3·7. *Let G be an abelian group. Then G has a base if and only if $G \setminus \{0\}$ has no elements of odd finite order.*

Proof. The necessity is Theorem 3·3. For the sufficiency, let

$$T = \{w \in G : o(w) \text{ is finite}\},$$

the torsion group of G. Then as is well known and very easy to show, G/T is torsion free. Thus, by Theorem 3·6, G/T is has a base. Also by Theorem 3·4, T has a base so by Theorem 3·2, G has a base. $\square$

4. *Subsemigroups of* $(\mathbb{R}^+, +)$

We show in this short section that if S is subsemigroup of $(\mathbb{R}^+, +)$ with the property that whenever $x, y \in S$ and $x < y$ one has $y - x \in S$ and S has a base, then S is either a copy of $\mathbb{N}$ or a copy of the semigroup $\mathbb{D}^+$ of positive dyadic rationals. (That is, $\mathbb{D}^+ = \{k2^n : k \in 2\mathbb{N} - 1 \text{ and } n \in \mathbb{Z}\}$.)

Lemma 4·1. *Let S be a subsemigroup of $(\mathbb{R}^+, +)$ with the property that $y - x \in S$ whenever $x, y \in S$ and $x < y$. Assume that $\langle x_t \rangle_{t \in A}$ is a base for S.*

(a) *If $n, m \in A$ and $x_m < x_n$, then $2x_m \le x_n$.*
(b) *If $n \in A$, then x_n has an immediate successor in $\{x_t : t \in A\}$ and either $x_n = \min\{x_t : t \in A\}$ or x_n has an immediate predecessor in $\{x_t : t \in A\}$.*

Proof. (a) Since $x_n - x_m \in S$, pick a finite subset F of A such that $x_n - x_m = \sum_{t \in F} x_t$. Then $x_n = x_m + \sum_{t \in F} x_t$, so $m \in F$. (If $m \notin F$, then by the uniqueness of finite sums $\{n\} = \{m\} \cup F$.) Consequently $x_n \ge 2x_m$ as required.

(b) By virtue of (a), for each $k \in \mathbb{Z}$ there is at most one $t \in A$ such that $2^k x_n \le x_t < 2^{k+1} x_n$. Consequently we have immediately that if $x_n \ne \min\{x_t : t \in A\}$, then x_n has an immediate predecessor in $\{x_t : t \in A\}$. And, if $x_n \ne \max\{x_t : t \in A\}$, then x_n has an immediate succesor in $\{x_t : t \in A\}$, so suppose that $x_n = \max\{x_t : t \in A\}$. Then for $k \in \mathbb{N}$, $\{t \in A : 2^k x_n \le x_t < 2^{k+1} x_n\} = \emptyset$ so $\sum_{t \in A} x_t < \sum_{k=0}^{\infty} 2^{-k+1} x_n = 4x_n$. This is a contradiction because, since S is a subsemigroup of $\mathbb{R}^+$, S is unbounded above. $\square$

Lemma 4·2. *Let S be a subsemigroup of $(\mathbb{R}^+, +)$ with the property that $y - x \in S$ whenever $x, y \in S$ and $x < y$. Assume that $\langle x_t \rangle_{t \in A}$ is a base for S. If $n, m \in \mathbb{N}$ and x_m is the immediate predecessor of x_n in $\{x_t : t \in A\}$, then $x_n = 2x_m$.*

Proof. By Lemma 4·1(a), $2x_m \le x_n$. Now $2x_m \in S$ so pick a finite subset F of $\mathbb{N}$ such that $2x_m = \sum_{t \in F} x_t$. We cannot have $m \in F$ or uniqueness of finite sums would yield the conclusion that $\{m\} = F \setminus \{m\}$. If we had $x_t < x_m$ for each $t \in F$, then by Lemma 4·1(a) we would have $\sum_{t \in F} x_t \le x_m$. So there is some $l \in F$ such that $x_l > x_m$ and therefore $2x_m \ge x_l \ge x_n$. $\square$

Theorem 4·3. *Let S be a subsemigroup of $(\mathbb{R}^+, +)$ with the property that $y - x \in S$ whenever $x, y \in S$ and $x < y$. Assume that $\langle x_t \rangle_{t \in A}$ is a base for S. If $\{x_t : t \in A\}$ has a smallest member x_k, then $S = x_k \mathbb{N}$ and $\{x_t : t \in A\} = \{x_k 2^t : t \in \omega\}$. If $\{x_t : t \in A\}$ does not have a smallest member and $k \in A$, then $S = x_k \mathbb{D}^+$ and $\{x_t : t \in A\} = \{x_k 2^t : t \in \mathbb{Z}\}$.*

Proof. This is an immediate consequence of Lemma 4·2. $\square$

Notice in particular that $(\mathbb{Q}^+,+)$ and $(\mathbb{R}^+,+)$ do not have bases.

The requirement that $y-x\in S$ whenever $x,y\in S$ and $x<y$ is necessary for the conclusion of Theorem 4·3. We shall now see that there are semigroups of all possible sizes contained in $(\mathbb{R}^+,+)$ which have bases, none of which is a copy of $\mathbb{N}$ or of $\mathbb{D}^+$.

THEOREM 4·4. *Let κ be a cardinal with $\omega\leq\kappa\leq\mathfrak{c}$. There is a subsemigroup S of $(\mathbb{R}^+,+)$ with $|S|=\kappa$ which has a base and there exist $x,y\in S$ with $x<y$ such that $y-x\notin S$ (and in particular S is neither a copy of $\mathbb{N}$ nor a copy of $\mathbb{D}^+$).*

Proof. Pick a subset B of $\mathbb{R}^+$ such that $|B|=\kappa$ and B is linearly independent over $\mathbb{Q}$. Let S be the semigroup generated by B. Then S is isomorphic to $\left(\bigoplus_{\sigma<\kappa}\omega\right)\setminus\{\bar{0}\}$, so has a base by Theorem 3·1. □

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