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Characterization of Simplicity and Cancellativity in βS

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Abstract. We determine precisely when the Stone-Čech compactification βS of a discrete semigroup S is simple and when it is left cancellative or right cancellative. As a consequence we see that βS is cancellative only when it is trivially so. That is, βS is cancellative if and only if S is a finite group.

Given a discrete semigroup S, the operation can be extended to the Stone-Čech compactification βS of S so that βS is a right topological semigroup with S contained in its topological center. (That is, given any $p \in \beta S$, the function $\rho_p : \beta S \to \beta S$ defined by $\rho_p(q) = qp$ is continuous and, given any $x \in S$, the function $\lambda_x : \beta S \to \beta S$ defined by $\lambda_x(q) = xq$ is continuous.) This implies that, for any $x, y \in \beta S$, $xy = \lim_{s \to x} \lim_{t \to y} st$ where s and t denote elements of S.

It has been known for a long time that left or right cancellativity is rare in βS for a typical semigroup S and several papers have been published on the subject. See [3, Chapter 8] and the notes to that chapter for a summary of what was known in 1998. To the best of our knowledge, there have not been more recent results.

A subset T of a semigroup S is called a left ideal if it is non-empty and if $ST \subseteq T$; it is called a right ideal if it is non-empty and if $TS \subseteq T$. It is called an ideal if it is both a left ideal and a right ideal. S is said to be simple if it has no proper subsets which are ideals.

In this note we characterize discrete semigroups S for which βS is simple and those for which βS is left cancellative or right cancellative. We regard βS as being the set of ultrafilters on S, with the points of S identified with the principal ultrafilters. The topology of βS is defined by choosing the sets of the form $\overline{A} = \{p \in \beta S : A \in p\}$ as a base, where A denotes a subset of S. With this topology, $\overline{A} = c\ell_{\beta S}(A)$ and is clopen in βS.

Our characterizations of cancellativity heavily involve the smallest ideal of βS. If a semigroup S has a minimal left ideal or a minimal right ideal which contains an

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idempotent, then it has a smallest two sided ideal $K(S)$. $K(S)$ is the union of all minimal left ideals of S and is the union of all minimal right ideals of S. Given any minimal left ideal L and any minimal right ideal R of S, $L \cap R$ is a group and all groups of this form are isomorphic. The group $L \cap R$ is called the structure group of S. Thus $K(S)$ is the disjoint union of groups, each isomorphic to the structure group.

Every compact right topological semigroup does have a minimal left ideal which contains an indempotent. Thus, if S is an arbitrary discrete semigroup, the preceding remarks hold for βS. See [1] or [3] for a derivation of these facts.

If V is a subset of a semigroup S, $E(V)$ will denote the set of idempotents in V. We shall use the fact that, for any semigroup S, $E(L)$ is a left zero semigroup if L is a minimal left ideal of S which contains an idempotent [3, Lemma 1.30(b)]. Dually, $E(R)$ is a right zero semigroup if R is a minimal right ideal of S which contains an idempotent.

We need the following simple lemmas. When we say that two semigroups with topologies are *topologically isomorphic* we mean that there is a bijection between them which is simultaneously an isomorphism and a homeomorphism.

1 Lemma. *Let S and T be discrete semigroups with S being finite. Then $\beta(S \times T)$ is topologically isomorphic to $S \times \beta T$. If T is a right zero semigroup, so is βT; if T is a left zero semigroup, so is βT.*

Proof. Let π_1 and π_2 denote the projection maps of $S \times T$, and let $\widetilde{\pi}_1 : \beta(S \times T) \to S$ and $\widetilde{\pi}_2 : \beta(S \times T) \to \beta T$ denote their continuous extensions. Define $\theta : \beta(S \times T) \to S \times \beta T$ by $\theta(x) = \big(\widetilde{\pi}_1(x), \widetilde{\pi}_2(x)\big)$.

Since θ is the continuous extension of the inclusion map of $S \times T$ in $S \times \beta T$, which is a homomorphism, θ is a homomorphism [3, Corollary 4.22]. Since $S \times T$ is dense in $S \times \beta T$, θ is surjective. To see that θ is injective, note that, for each $s \in S$, π_2 is injective on $\{s\} \times T$. So $\widetilde{\pi}_2$ is injective on $c\ell_{\beta(S \times T)}(\{s\} \times T)$ [3, Exercise 3.4.1]. Let x and y be distinct elements of $\beta(S \times T)$. If $\widetilde{\pi}_1(x) = \widetilde{\pi}_1(y) = s$, then $x, y \in c\ell_{\beta(S \times T)}(\{s\} \times T)$ and so $\widetilde{\pi}_2(x) \neq \widetilde{\pi}_2(y)$. Thus $\theta(x) \neq \theta(y)$.

If T is a right zero semigroup then, for every $x_1, x_2 \in \beta T$,

$$x_1 x_2 = \lim_{t_1 \to x_1} \lim_{t_2 \to x_2} t_1 t_2 = \lim_{t_2 \to x_2} t_2 = x_2.$$

Thus βT is a right zero semigroup. Similarly, if T is a left zero semigroup, βT is also a left zero semigroup. □

2 Lemma. *Let S be a discrete semigroup and let G be a compact subgroup of βS. Then G is finite.*

Proof. By [2, Theorem 14.25], βS is an F-space, and therefore G is an F-space. Let e be the identity of G. Given any $g, h \in G$, $\rho_{h^{-1}g}$ is a homeomorphism from G to G taking h to g. That is, G is homogeneous. But by [4, Corollary 3.4.2], no infinite compact F-space is homogeneous. □

3 Lemma. *Let S be a discrete semigroup. If $K(\beta S) \cap S \neq \emptyset$, then there is an idempotent $e \in K(\beta S) \cap S$, $G = (e\beta S) \cap (\beta S e)$ is a finite group, and $G \subseteq S$. Furthermore, Se is a minimal left ideal of S.*

Proof. Pick $x \in K(\beta S) \cap S$ and pick a minimal left ideal L of βS and a minimal right ideal R of βS such that $x \in L \cap R$. Then $G = L \cap R$ is a group. Since $L = \beta S x = \rho_x[\beta S]$ and $R = x\beta S = \lambda_x[\beta S]$ we have that G is compact, hence finite by Lemma 2. The powers of x are then a finite group, so the identity e of G is in S. To see that $G \subseteq S$, let $g \in G$. Let g^{-1} be the inverse of g in G. Then $e = \rho_{g^{-1}}(g)$ and $\{e\}$ is a neighborhood of e so pick a neighborhood V of g such that $\rho_{g^{-1}}[V] \subseteq \{e\}$. Since S is dense in βS, pick $y \in V \cap S$. Then $yg^{-1} = e$ so $ye = eg = g$. Since $y \in S$ and $e \in S$, $g \in S$.

To see that Se is a minimal left ideal of S, suppose that T is a left ideal of S for which $T \subseteq Se$. Then $c\ell_{\beta S}(T)$ is a left ideal of βS contained in the minimal left ideal $\beta S e$ of βS [3, Theorem 2.17]. So $c\ell_{\beta S}(T) = \beta S e \supseteq Se$. Since the points of S are isolated in βS, it follows that $T \supseteq Se$. □

We can now characterize the semigroups S for which βS is simple.

4 Theorem. *Let S be any semigroup. The following statements are equivalent:*

(a) βS is simple.

(b) S is a simple semigroup with a minimal left ideal containing an idempotent. Furthermore, the structure group of S is finite and S has only a finite number of minimal left ideals or only a finite number of minimal right ideals.

(c) S contains a finite group G, a left zero semigroup X and a right zero semigroup Y such that S is isomorphic to the semigroup $X \times G \times Y$ with the semigroup operation defined by $(x, g, y)(x', g', y') = (x, gyx'g', y')$ for every $x, x' \in X, g, g' \in G, y, y' \in Y$. Furthermore, either X or Y is finite.

Proof. $(a) \Rightarrow (b)$. By Lemma 3, S has a minimal left ideal which contains an idempotent e. By [3, Theorem 1.65], $K(S) = S \cap K(\beta S) = S \cap \beta S = S$ and so S is simple. Also, by Lemma 3, $G = (e\beta S) \cap (\beta Se) = e\beta Se$ is a finite group. Given $x \in G$, $x \in S$ so $x = exe \in eSe$ so $G = eSe$. Since e is a minimal idempotent, $G = eSe$ is the structure group of S.

Now suppose that S has an infinite number of minimal left ideals and an infinite number of minimal right ideals and choose sequences $\langle L_n \rangle_{n=1}^{\infty}$ and $\langle R_n \rangle_{n=1}^{\infty}$ of distinct minimal left and right ideals respectively. For each n, let x_n be the identity of $Se \cap R_n$ and let y_n be the identity of $eS \cap L_n$. Notice that for each n, $x_n y_n \in R_n \cap L_n$. In particular, $D = \{x_n y_n : n \in \mathbb{N}\}$ is infinite. Pick $q \in \overline{D} \setminus D$. Pick a minimal left ideal I of βS such that $q \in I$. Then $I = \beta S q$ so $q \in \beta S q = c\ell(Sq)$ so pick $s \in S$ such that $D \in sq$. Then $\{t \in S : st \in D\} \in q$, so pick $t \neq v$ in D such that $st \in D$ and $sv \in D$. Pick $k, l, m, n \in \mathbb{N}$ such that $t = x_k y_k$, $v = x_l y_l$, $st = x_m y_m$, and $sv = x_n y_n$. Now $st \in x_m S = R_m$ so $s \in R_m$. Also $sv \in x_n S = R_n$ so $s \in R_n$. Consequently $n = m$ and thus $st = sv \in L_n \cap R_n$. But $t \in L_k$ so $st \in L_k$ and $v \in L_l$ so $sv \in L_l$ so $k = l = n$ and thus $t = v$, a contradiction.

$(b) \Rightarrow (c)$. By [3, Theorem 1.64], there exist a group G, a left zero semigroup X and a right zero semigroup Y such that $K(S)$ is isomorphic to the semigroup $X \times G \times Y$ with the semigroup operation as in the statement of (c). Since S is simple, $S = K(S)$. Since the structure group of $X \times G \times Y$ is G, G is finite. For each $x \in X$, $\{x\} \times G \times Y$ is a minimal right ideal of $X \times G \times Y$ and for each $y \in Y$, $X \times G \times \{y\}$ is a minimal left ideal of $X \times G \times Y$, so either X or Y is finite.

$(c) \Rightarrow (a)$. Assume that X is finite. Notice that $X \times G \times \beta Y$ endowed with the semigroup operation $(x, g, \eta)(x', g', \eta') = (x, g\eta x' g', \eta')$ is a compact right topological semigroup with $X \times G \times Y$ contained in its topological center. For this first note that given $x' \in X$, $g, g' \in G$, and $\eta \in \beta Y$, one has

$$g\eta x' g' = \lambda_g \circ \rho_{x'g'}(\eta) \in \lambda_g \circ \rho_{x'g'}[c\ell_{\beta S}(Y)] = c\ell_{\beta S}(gYx'g') \subseteq c\ell_{\beta S}(G) = G\,.$$

It is then routine to verify that the operation on $X \times G \times \beta Y$ is associative and that $X \times G \times \beta Y$ is a right topological semigroup with $X \times G \times Y$ contained in its topological center.

We claim that βS is isomorphic to $X \times G \times \beta Y$. By Lemma 1, βS can be identified as a topological space with $X \times G \times \beta Y$. (We cannot invoke the algebraic portion of Lemma 1 because the operation on $X \times G \times Y$ is not the direct product.) The semigroup operation of βS is characterized by

$$(x, g, \eta)(x', g', \eta') = \lim_{y \to \eta} \lim_{y' \to \eta'} (x, g, y)(x', g', y') = (x, g\eta x' g', \eta')$$

where the last equality holds because $X \times G \times \beta Y$ is a right topological semigroup with $X \times G \times Y$ contained in its topological center.

We now claim that $X \times G \times \beta Y$ is simple. To see this, for each $x \in X$, let $R_x = \{(x, g, \eta) : g \in G \text{ and } \eta \in \beta Y\}$. We will show that each R_x is a minimal right ideal of $X \times G \times \beta Y$. Since $X \times G \times \beta Y = \bigcup_{x \in X} R_x$, this will suffice. So let $x \in X$ and pick a minimal right ideal $R \subseteq R_x$. Pick $g \in G$ and $\eta \in \beta Y$ such that $(x, g, \eta) \in R$. To see that $R_x \subseteq R$, let $g' \in G$ and $\eta' \in \beta Y$ be given. Now $g\eta xe \in G$ so pick $h \in G$ such that $g\eta xh = g\eta xeh = g'$. Then $(x, g', \eta') = (x, g, \eta)(x, h, \eta') \in R$.

An analogous argument shows that βS is simple if Y is finite. □

We now characterize the semigroups S for which βS is left or right cancellative.

5 Theorem. *Let S be a discrete semigroup. The following statements are equivalent.*

(a) βS is left cancellative.

(b) $K(\beta S)$ contains an element which is left cancelable in βS.

(c) βS has a left cancelable element and $K(\beta S) = \beta S$.

(d) Every idempotent in βS is a left identity for βS.

(e) There exist a finite group $G \subseteq S$ and a compact Hausdorff right zero semigroup $T \subseteq \beta S$ such that $\beta S = GT$ and the function $\varphi : G \times T \to \beta S$ defined by $\varphi(g, \eta) = g\eta$ is both an isomorphism and a homeomorphism.

(f) There exist a finite group G and a right zero semigroup R such that S is isomorphic to $G \times R$.

(g) There exist a finite group G and a discrete right zero semigroup R such that βS is topologically isomorphic to $G \times \beta R$.

(h) S has a left cancelable element and βS is simple.

Proof. $(a) \Rightarrow (b)$. This is obvious.

$(b) \Rightarrow (c)$. Suppose one has $p \in \beta S \setminus K(\beta S)$. Pick a left cancelable $q \in K(\beta S)$ and pick a minimal right ideal R of βS such that $q \in R$. Then $R = q\beta S = q\dot{R}$. So $qp = qr$ for some $r \in R \subseteq K(\beta S)$, a contradiction.

$(c) \Rightarrow (d)$. Suppose that $x \in K(\beta S)$ is left cancelable in βS. We can choose an idempotent $p \in K(\beta S)$ for which $xp = x$. Then $xpy = xy$ and so $py = y$ for every $y \in \beta S$. This shows that $\beta S = p\beta S$ and hence that βS is a minimal right ideal of βS. It follows that $\beta S = q\beta S$ for every idempotent $q \in \beta S$. So, if $y \in \beta S$, $y = qz$ for some $z \in \beta S$ and $qy = qqz = qz = y$.

$(d) \Rightarrow (e)$. We have that βS is simple, because $\beta S = p\beta S \subseteq K(\beta S)$ for any minimal idempotent p in βS. It follows from Lemma 3 that S contains a minimal idempotent e of βS for which $G = (e\beta S) \cap (\beta S e) = e\beta S e$ is a finite subgroup of S. Let $T = E(\beta S)$. Then each $\eta \in T$ is a left identity for βS so $T = \bigcap_{y \in \beta S} {\rho_y}^{-1}[\{y\}]$ so T is compact.

It is routine to verify that φ is a continuous homomorphism. To see that φ is surjective let $q \in \beta S$ and pick $p \in T$ such that $q \in \beta S p$. Then $qp = p$ and $eqe \in G$ so $q = \varphi(eqe, p)$. To see that φ is injective, suppose that $gy = hz$, where $g, h \in G$ and $y, z \in T$. Then $g = ge = gye = hze = he = h$ so $gy = gz$. Then $y = ey = g^{-1}gy = g^{-1}gz = ez = z$.

$(e) \Rightarrow (f)$. The assumption that $\beta S = GT$ implies that e is a left identity for βS. Let $R = T \cap S$. Then $\varphi[G \times R] \subseteq S$. To see that $\varphi[G \times R] = S$ let $x \in S$. Pick $g \in G$ and $y \in T$ such that $x = gy$. Then $g^{-1}x = ey = y$ because e is a left identity for βS.

$(f) \Rightarrow (g)$. This follows from Lemma 1.

$(g) \Rightarrow (a)$. Since βR is a right zero semigroup, this is immediate.

At this stage we have shown that statements (a) through (g) are equivalent. By [3, Lemma 8.1] we have that (h) implies (c). Statements (a) and (c) trivially imply (h). □

6 Theorem. *Let S be a discrete semigroup. The following statements are equivalent.*

(a) βS is right cancellative.

(b) $K(\beta S)$ contains an element which is right cancelable in βS.

(c) βS has a right cancelable element and $K(\beta S) = \beta S$.

(d) Every idempotent in βS is a right identity for βS.

(e) There exist a finite group $G \subseteq S$ and a compact Hausdorff left zero semigroup X such that $\beta S = XG$ and the function $\varphi : X \times G \to \beta S$ defined by $\varphi(x, g) = xg$ is both an isomorphism and a homeomorphism.

(f) There exist a finite group G and a left zero semigroup L such that S is isomorphic to $L \times G$.

(g) There exist a finite group G and a discrete left zero semigroup L such that βS is isomorphic to $\beta L \times G$.

(h) S has a right cancelable element and $K(\beta S) = \beta S$.

Proof. With one exception the proof proceeds by exact left-right switches of the proof of Theorem 5. That exception is the portion of the proof that (d) implies (e) wherein we concluded that T was compact. (It is true that $X = \bigcap_{y \in \beta S} {\lambda_y}^{-1}[\{y\}]$, but we only know that λ_y is continuous for $y \in S$.) Fortunately, (d) states that $X = E(\beta S)$ is the set of right identities of βS. We claim that X is precisely the set of right identities of S. Indeed, given $f \in \beta S$ such that $yf = y$ for all $y \in S$, one has that ρ_f and the identity agree on S and therefore on βS. Consequently $X = \bigcap_{y \in S} {\lambda_y}^{-1}[\{y\}]$, and so X is compact. □

7 Corollary. *If S is a discrete semigroup and if $K(\beta S)$ contains an element left cancelable in βS and an element right cancelable in βS, then S is a finite group.*

Proof. By either of the above theorems we have that $K(\beta S) = \beta S$ and each idempotent of βS is a two sided identity for βS. Thus by Lemma 3 we have an idempotent $e \in S$ such that $G = e\beta S \cap \beta S e$ is a finite group. Since e is both a left identity and a right identity for S we have that $G = \beta S$. □

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