Canonical Partition Relations for (m, p, c)-Systems

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Abstract

By an (m, p, c)-system we mean a choice of an (m, p, c)-set for each $(m, p, c) \in I\!N^3$ together with all finite sums choosing at most one number from each (m, p, c)-set. Here we investigate the canonical situation for arbitrary colorings of an (m, p, c)-system.

1 Introduction

The subject area known as Ramsey Theory apparently began with the 1892 result of Hilbert [6]: If the set $I\!N$ of positive integers is divided into finitely many classes there are arbitrarily large finite sets B and infinitely many translates t with $t + FS(B)$ contained in one class. (For any set of integers, finite or infinite, $FS(B) = \{\sum F : F \text{ is a finite nonempty subset of } B\}$.) There followed many other results about finite partitions including Schur's Theorem, van der Waerden's Theorem, and Ramsey's Theorem, (see [5]).

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A convenient way to partition into finitely many classes is by way of a function. The values of the function are often thought of as "colors". Viewed in this way Ramsey's Theorem says: Given any $k, r \in I\!N$ and any $f : [I\!N]^k \to \{1, 2, \ldots, r\}$ there is an infinite $A \subseteq I\!N$ such that f is constant on $[A]^k$. (Here $[A]^k = \{B : B \subseteq A \text{ and } |B| = k\}$.) If one allows infinitely many colors, the result clearly fails. For example, one may provide each pair with its own color. Alternatively for $x < y$ one could let $f(\{x, y\}) = x$ and $g(\{x, y\}) = y$. The subject of canonical partition relations began with the 1950 result of Erdős and Rado [4] which says, in the case $k = 2$, that these are the only possibilities: Let $f : [I\!N]^2 \to I\!N$. There exists an infinite $A \subseteq I\!N$ such that one of four "canonical" situations holds,

(1) f is constant on $[A]^2$,

(2) f is one-to-one on $[A]^2$,

(3) for $B, C \in [A]^2$, $f(B) = f(C)$ if and only if $\min B = \min C$, or

(4) for $B, C \in [A]^2$, $f(B) = f(C)$ if and only if $\max B = \max C$.

We will also be interested in a canonical version of van der Waerden's theorem (which says, you will recall, that if $I\!N$ is finitely colored, one color class contains arbitrarily long arithmetic progressions). The canonical version, due to Erdős and Graham [3], says that if $f : I\!N \to I\!N$, there are arbitrarily long arithmetic progressions on which f is either one-to-one or constant.

We are interested here in a canonical version for colorings of (m, p, c)-systems. Recall that the (m, p, c)-sets introduced by Deuber [1] characterize the partition regular systems with integer coefficients. Given integers m, p, and c, an (m, p, c)-set is a set of positive integers of the form $\{cx_t + \sum_{i=t+1}^m \lambda_i x_i : t \in \{1, 2, \ldots, m\}$ and for $i \in \{t + 1, \ldots, m\}$, $\lambda_i \in \mathbf{Z}$ and $|\lambda_i| \le p\}$ for some $x_1, x_2, \ldots, x_m$ in $I\!N$. (We will modify the form of this definition in Section 2.) Deuber's theorem says that, given any $r \in I\!N$ and any $m_1, p_1, c_1 \in I\!N$ there exist $m_2, p_2, c_2 \in I\!N$ so that whenever an (m_2, p_2, c_2)-set is r-colored, one color class will contain an (m_1, p_1, c_1)-set. The canonical version of this theorem [8] has, for an arbitrary coloring, the expected two clauses:

(1) f is constant on an (m_1, p_1, c_1)-set;

(2) f is one-to-one on an (m_1, p_1, c_1)-set;

And here, in addition, a third clause:

(3) there is an (m_1, p_1, c_1)-set such that for a, b in this set, $f(a) = f(b)$ if and only if $a = cx_t + \displaystyle\sum_{i=t+1}^{m} \lambda_i x_i$ and $b = cx_t + \displaystyle\sum_{i=t+1}^{m} \gamma_i x_i$ for the same value of t.

By an (m, p, c)-system we mean a choice of some (m, p, c)-set for each $(m, p, c) \in I\!N^3$ together with all finite sums choosing at most one member from each (m, p, c)-set. In [2] it was shown that (m, p, c)-systems are partition regular in the sense that each finite coloring of $I\!N$ has one color class containing an (m, p, c)-system. By [7], we established the stronger partition regularity property: If any (m, p, c)-system is finitely colored, one color class contains an (m, p, c)-system.

We address here a canonical version of this last result. In so doing we follow Taylor's canonical version [9] of the Finite Sum Theorem: Let B be an infinite subset of $I\!N$ and let $f : FS(B) \to I\!N$. There is an infinite set C with $FS(C) \subseteq FS(B)$ so that one of the following five cases holds:

(1) f is constant on $FS(C)$,

(2) f is one-to-one on $FS(C)$,

(3) for finite F, $G \subseteq C$, $f(\sum F) = f(\sum G)$ if and only if $\min F = \min G$,

(4) for finite F, $G \subseteq C$, $f(\sum F) = f(\sum G)$ if and only if $\max F = \max G$,

(5) for finite F, $G \subseteq C$, $f(\sum F) = f(\sum G)$ if and only if $\min F = \min G$ and $\max F = \max G$.

Since we are aiming for a common generalization of [8] and [9], we might (and indeed we did) expect something of the order of 13 clauses (multiplying each of Taylor's conclusions except the "one-to-one" conclusion by 3). We are instead almost able to show that one can get by with six clauses which are effectively Taylor's five clauses plus one additional clause. We do show that none of our clauses can be simply discarded.

Unfortunately we are only "almost" able to establish that six clauses are enough because in one case we need to invoke a conjecture whose validity we cannot establish.

In Section 5 we present our main results. Section 2 consists of a development of some extensive machinery that we will utilize. Section 3 continues this development with an investigation of the ability to write elements of (m, p, c)-systems uniquely. Section 4 introduces the equivalence classes on which our sixth clause is based.

2 Background and Notation

We begin by introducing some notation. The symbols are the same as used in [7], although details of the definition differ. (We will point out the differences.)

2.1 Definition. Let $(m,p,c) \in I\!N^3$ and let $\vec{x} \in I\!N^m$.
$S(m,p,c,\vec{x}) = \{\sum_{i=1}^{t-1} \lambda_i x_i + c x_t : t \in \{1,2,\ldots,m\}$ and for $1 \le i < t$, $\lambda_i \in \{0,1,2,\ldots,p\}\}$. A subset $A \subseteq I\!N$ is an (m,p,c)-set if and only if there is some $\vec{x} \in I\!N^m$ such that $A = S(m,p,c,\vec{x})$.

This definition differs from [7] (and from our introduction) in two ways. First, we reverse the order of summation, putting c last rather than first. This insignificant change allows us to clean up several proofs. More substantively, we restrict the coefficients λ_i to nonnegative integers. This change is very helpful in our discussion of uniqueness of representing an (m,p,c)-system. It is well known that this change causes no harm: If $S'(m,p,c,\vec{x})$ refers to the definition in the introduction then immediately $S(m,p,c,\vec{x}) \subseteq S'(m,p,c,\vec{x})$. On the other hand, given m,p,c and given $\vec{x} \in I\!N^m$, let $p' = (c+1)\cdot(1+p)^{m-1}$. For $i \in \{1,2,\ldots m\}$ let $y_i = \sum_{t=1}^{i}(1+p)^{i-t}x_i$. Then $S'(m,p,c,\vec{y}) \subseteq S(m,p',c,\vec{x})$.

Another significant change from [7] is that we no longer chose (m,p,c)-sets for all triples (m,p,c). We shall show after the definition that this choice does not weaken our results.

2.2 Definition.

(a) For $n \in I\!N$, let $m(n) = n$, $p(n) = n \cdot n!$, and $c(n) = n!$

(b) $V = \times_{n=1}^{\infty} I\!N^{m(n)}$

(c) Given $\vec{x}$ with $\vec{x}\,(n) \in I\!N^{m(n)}$ (such as $\vec{x} \in V$) $S(\vec{x},n) = S(m(n),p(n),c(n),\vec{x}\,(n)) = \{\sum_{i=1}^{t-1} \lambda_i \cdot \vec{x}\,(n)(i) + c(n)\cdot \vec{x}\,(n)(t) : t \in \{1,2,\ldots,m(n)\}$ and for $i \in \{1,2,\ldots,t-1\}$, $\lambda_i \in \{0,1,2,\ldots,p(n)\}\}$

(d) Given $\vec{x} \in V$ and $l \in I\!N$, $FS(< S(\vec{x},n) >_{n=l}^{\infty}) = \{\sum_{n\in F} w_n : F$ is a finite nonempty subset of $I\!N$ and $\min F \ge l$ and for each $n \in F$, $w_n \in S(\vec{x},n)\}$ and $FS(< S(\vec{x},n) >_{n=1}^{l}) = \{\sum_{n\in F} w_n : F$ is a nonempty subset of $\{1,2,\ldots,l\}$ and for each $n \in F$, $w_n \in S(\vec{x},n)\}$.

(e) A set B is an (m,p,c)-system if and only if $B = FS(< S(\vec{x},n) >_{n=1}^{\infty})$ for some $\vec{x} \in V$.

Conclusion (a) of the following lemma shows that we loose nothing by restricting to $(m(n),p(n),c(n))$-sets while conclusion (b) provides our motivation for doing so.

2.3 Lemma. (a) Let $(m,p,c) \in I\!N^3$. There exists $k \in I\!N$ such that for any $n \ge k$ and any $\vec{x} \in I\!N^{m(n)}$, $S(\vec{x},n)$ contains an (m,p,c)-set.

(b) Let $n \in \mathbb{N}$, let $\vec{x} \in \mathbb{N}^{m(n+1)}$. There exists $\vec{y} \in \mathbb{N}^{m(n)}$ with $S(\vec{y}, n) \subseteq S(\vec{x}, n+1)$.

Proof. (a) Let $k \in \mathbb{N}$ with $k \geq \max\{m, p/c\}$ such that c divides $k!$ and let $n \geq k$. Let $\vec{x} \in \mathbb{N}^{m(n)}$ and define $\vec{y} \in \mathbb{N}^m$ by $\vec{y}\,(i) = \frac{c(n)}{c} \cdot \vec{x}\,(i)$. (Since $n \geq k$ and c divides $k!$ we have c divides $c(n)$. Since $m(n) = n \geq m$ we have that $\vec{x}\,(i)$ is defined for all $i \in \{1, 2, \ldots, m\}$.) We claim $S(m, p, c, \vec{y}) \subseteq S(\vec{x}, n)$. To see this let $t \in \{1, 2, \ldots, m\}$ and for $i \in \{1, 2, \ldots, t-1\}$ (if any) let $\lambda_i \in \{0, 1, 2, \ldots, p\}$. Then $\sum_{i=1}^{t-1} \lambda_i \vec{y}\,(i) + c\,\vec{y}\,(t) = \sum_{i=1}^{t-1} (\lambda_i \cdot \frac{c(n)}{c}) \cdot \vec{x}\,(i) + c(n) \cdot \vec{x}\,(t)$ and for each i, $\lambda_i \cdot \frac{c(n)}{c} \in \mathbb{N} \cup \{0\}$ and $\lambda_i \cdot \frac{c(n)}{c} \leq \frac{p}{c} \cdot c(n) \leq n \cdot c(n) = p(n)$.

(b) Observe that $n + 1 > \max\{m(n), \frac{p(n)}{c(n)}\}$ and $c(n)$ divides $(n+1)!$ so the proof of conclusion (a) applies. □

As a consequence of Lemma 2.3 (a) we have the following, whose easy proof is omitted.

2.4 Theorem. Let $\vec{x} \in V$. For each $(m, p, c) \in \mathbb{N}^3$ there exists $\vec{y}\,(m, p, c) \in \mathbb{N}^m$ so that given any nonempty finite $F \subseteq \mathbb{N}^3$ and for each $(m, p, c) \in F$ given $a(m, p, c) \in S(m, p, c, \vec{y}\,(m, p, c))$, one has $\sum_{(m,p,c) \in F} a(m, p, c) \in FS(< S(\vec{x}, n) >_{n=1}^{\infty})$.

Given $\vec{x} \in V$ it may well happen that elements of $FS(< S(\vec{x}, n) >_{n=1}^{\infty})$ can be expressed in more than one way. We find it desirable to be able to look at an element of $FS(< S(\vec{x}, n) >_{n=1}^{\infty})$ and tell how it was obtained. We address this uniqueness in two steps.

2.5 Definition. $V^* = \{\vec{x} \in V :$ for each $n \in \mathbb{N}$, each $a \in S(\vec{x}, n)$, and each $b \in S(\vec{x}, n+1)$, if $t = \max\{i : 2^i \leq a\}$, then 2^{t+1} divides $b\}$.

One can easily see by considering the binary expansion of the members of $FS(< S(\vec{x}, n) >_{n=1}^{\infty})$ that if $\vec{x} \in V^*$, F and G are finite nonempty subsets of $\mathbb{N}$, and for each $n \in F$, $w_n \in S(\vec{x}, n)$ and for each $n \in G$, $v_n \in S(\vec{x}, n)$ and $\sum_{n \in F} w_n = \sum_{n \in G} v_n$ then $F = G$ and each $w_n = v_n$.

2.6 Definition. (a) For $\vec{x} \in V^*$ and $a \in FS(< S(\vec{x}, n) >_{n=1}^{\infty})$, $F(\vec{x}, a)$ is the unique nonempty finite subset of $\mathbb{N}$ and for each $n \in F(\vec{x}, a)$, $w(\vec{x}, a, n)$ is the unique member of $S(\vec{x}, n)$ such that $a = \sum_{n \in F(\vec{x}, a)} w(\vec{x}, a, n)$.

(b) For $\vec{x} \in V^*$ we say $\vec{y}$ refines $\vec{x}$ if and only if $\vec{y} \in V^*$, $FS(< S(\vec{y}, n) >_{n=1}^{\infty}) \subseteq FS(< S(\vec{x}, n) >_{n=1}^{\infty}))$ and for each $n \in \mathbb{N}$, each $a \in S(\vec{x}, n)$, and each $b \in S(\vec{x}, n+1)$ one has $\max F(\vec{x}, a) < \min F(\vec{x}, b)$.

Letting $r = 1$ in the following theorem we see that we can restrict our attention to members of V^*.

2.7 Theorem. Let $r \in \mathbb{N}$, let $\vec{x} \in V$, and let $f : FS(< S(\vec{x}, n) >_{n=1}^{\infty}) \to \{1, 2, \ldots, r\}$. There exists $\vec{y} \in V^*$ such that $FS(< S(\vec{y}, n) >_{n=1}^{\infty}) \subseteq FS(< S(\vec{x}, n) >_{n=1}^{\infty})$ and f is constant on the set $FS(< S(\vec{y}, n) >_{n=1}^{\infty})$. If $\vec{x} \in V^*$, $\vec{y}$ may be chosen so that $\vec{y}$ refines $\vec{x}$.

Proof. [7, Theorem 2.7] □

3 Strong Uniqueness of Representation

Even for $\vec{x} \in V^*$, we may not have complete uniqueness of expression that is, members of $S(\vec{x}, n)$ may be expressible in several different ways in terms of the coefficients λ_i.

We now define a class $W \subseteq V^*$, show that members of W satisfy strong uniqueness of representation, that given any $\vec{x} \in V^*$ there is some $\vec{y} \in W$ such that $\vec{y}$ refines $\vec{x}$, and that if $\vec{x} \in W$ and $\vec{y}$ refines $\vec{x}$ then $\vec{y} \in W$.

3.1 Definition. $W = \{\vec{x} \in V^*$: for each $n \in I\!N$ and each $i \in \{1, 2, \ldots, m(n)\}$, $\vec{x}\ (n)(i) > \sum_{t=1}^{n-1} \sum_{j=1}^{m(t)} t \cdot p(t)^2 \cdot \vec{x}\ (t)(j) + \sum_{j=1}^{i-1} n \cdot p(n)^2 \cdot \vec{x}\ (n)(j)\}$. (If $i = 1$ we take $\sum_{j=1}^{i-1} np(n)^2 \cdot \vec{x}\ (n)(j)$ to be 0.)

3.2 Lemma. Let $\vec{x} \in W$ and let $n \in I\!N$ and for each $t \leq n$ and each $i \in \{1, 2, \ldots, m(t)\}$, let $\lambda_{t,i}$ and $\gamma_{t,i}$ be in $\{0, 1, 2, \ldots, t \cdot p(t)^2\}$. If $\sum_{t=1}^{n} \sum_{i=1}^{m(t)} \lambda_{t,i} \cdot \vec{x}\ (t)(i) = \sum_{t=1}^{n} \sum_{i=1}^{m(t)} \gamma_{t,i} \cdot \vec{x}\ (t)(i)$ then for each $t \in \{1, 2, \ldots, n\}$ and each $i \in \{1, 2, \ldots, m(t)\}$, $\lambda_{t,i} = \gamma_{t,i}$. In particular, the coefficients in the definition of $FS(< S(\vec{x}, n) >_{n=1}^{\infty})$ are all unique.

Proof. Agree that $(t, i) < (s, j)$ if and only if $t < s$ or both $t = s$ and $i < j$. Suppose that the conclusion fails and pick the largest (s, j) such that $\lambda_{s,j} \neq \gamma_{s,j}$, assuming without loss of generality that $\lambda_{s,j} > \gamma_{s,j}$. Then $\vec{x}\ (s)(j) \leq (\lambda_{s,j} - \gamma_{s,j})\ \vec{x}\ (s)(j) = \sum_{(t,i)<(s,j)} (\gamma_{t,i} - \lambda_{t,i})\ \vec{x}\ (t)(i) \leq \sum_{(t,i)<(s,j)} t \cdot p(t)^2 \cdot \vec{x}\ (t)(i) < \vec{x}\ (s)(j)$, a contradiction. □

3.3 Definition. Given $\vec{x} \in W$, $a \in FS(< S(\vec{x}, n) >_{n=1}^{\infty})$, and $n \in F(\vec{x}, a)$, define $r(\vec{x}, a, n) \in \{1, 2, \ldots, m(n)\}$ and $\lambda(\vec{x}, a, n)(i) \in \{0, 1, \ldots, p(n)\}$ for $i \in \{1, 2, \ldots, r(\vec{x}, a, n)\}$, by $w(\vec{x}, a, n) = \sum_{i=1}^{r(\vec{x}, a, n)} \lambda(\vec{x}, a, n)(i) \cdot \vec{x}\ (n)(i)$ and $\lambda(\vec{x}, a, n)(r(\vec{x}, a, n)) = c(n)$.

Definition 3.3 makes sense by virtue of Lemma 3.2

3.4 Theorem. Let $\vec{x} \in V^*$. There exists $\vec{y} \in W$ such that $\vec{y}$ refines $\vec{x}$.

Proof. We inductively construct $\vec{y}\ (n)$. Let $\vec{y}\ (1) = \vec{x}\ (1)$ let $j(1, 1) = 1$. Inductively assume we have chosen $\vec{y}\ (n-1)$ and $j(n-1, i)$ for $i \in \{1, 2, \ldots, m(n-1)\}$.

Pick $j(n, 1) > j(n-1, m(n-1))$ such that $\vec{x}\ (j(n, 1))(1) > \sum_{t=1}^{n-1} \sum_{i=1}^{m(t)} t \cdot p(t)^2 \cdot \vec{y}\ (t)(i)$ and so that if $2^a \leq \sum_{t=1}^{n-1} \sum_{i=1}^{m(t)} p(t) \cdot \vec{y}\ (t)(i)$ then 2^a divides $\vec{x}\ (j(n, 1))(1)$, and let

$$\vec{y}\ (n)(1) = \frac{c(j(n, 1))}{c(n)} \cdot \vec{x}\ (j(n, 1))(1)\ .$$

Given $t \in \{2, 3, \ldots, m(n)\}$ and given $j(n, t-1)$ choose $j(n, t) > j(n, t-1)$ so that

$$\vec{x}\ (j(n, t))(t) > \vec{y}\ (n)(1) + \sum_{i=1}^{t-1} n \cdot p(n)^2 \cdot \vec{y}\ (n)(i)$$

and let $\vec{y}\,(n)(t) = \sum_{i=1}^{t} \frac{c(j(n,i))}{c(n)} \cdot \vec{x}\,(j(n,i))(t)$.

The construction being complete we show that $S(\vec{y}, n) \subseteq FS(< S(\vec{x}, k) >_{k=j(n,1)}^{j(n,m(n))})$ for each $n \in I\!N$. The other required conclusions are then easily established (observing that $j(n,i) \geq n$, so $\frac{c(j(n,i))}{c(n)} \geq 1$). To this end let $t \in \{1, 2, \ldots, m(n)\}$ and for $v \in \{1, 2, \ldots, t-1\}$ let $\lambda_v \in \{0, 1, \ldots, p(n)\}$. Then

$$\begin{aligned}
& \sum_{v=1}^{t-1} \lambda_v \cdot \vec{y}\,(n)(v) + c(n) \cdot \vec{y}\,(n)(t) \\
= & \sum_{v=1}^{t-1} \sum_{i=1}^{v} \frac{c(j(n,i))}{c(n)} \cdot \lambda_v \cdot \vec{x}\,(j(n,i))(v) \\
+ & \sum_{i=1}^{t} c(j(n,i)) \cdot \vec{x}\,(j(n,i))(t) \\
= & \sum_{i=1}^{t-1} (\sum_{v=i}^{t-1} \frac{c(j(n,i))}{c(n)} \cdot \lambda_v \cdot \vec{x}\,(j(n,i))(v) \\
+ & c(j(n,i)) \cdot \vec{x}\,(j(n,i))(t) + c(j(n,t)) \cdot \vec{x}\,(j(n,t))(t).
\end{aligned}$$

Then $c(j(n,t)) \cdot \vec{x}\,(j(n,t))(t) \in S(\vec{x}, j(n,t))$. Also given $i \leq v \leq t-1$ we have

$$\begin{aligned}
& \frac{c(j(n,i))}{c(n)} \cdot \lambda_v \leq \frac{c(j(n,i))}{c(n)} \cdot p(n) \\
= & \; n \cdot c(j(n,i)) \leq j(n,i) \cdot c(j(n,i)) = p(j(n,i)) \\
\text{so} \quad & \sum_{v=i}^{t-1} \frac{c(j(n,i))}{c(n)} \cdot \lambda_v \cdot \vec{x}\,(j(n,i))(v) \\
+ & \; c(j(n,i)) \cdot \vec{x}\,(j(n,i))(t) \in S(\vec{x}, j(n,i))
\end{aligned}$$

as required. □

We see in the next lemma that if $\vec{x} \in W$ and $\vec{y}$ refines $\vec{x}$, a construction similar to that used in the proof of Theorem 3.4 is in fact forced on us.

3.5 Lemma. Let $\vec{x} \in W$ and let $\vec{y}$ refine $\vec{x}$. Let $n > 1$ in $I\!N$ be given. Let $v \in \{2, 3, \ldots, m(n)\}$ be given and let H be a nonempty subset of $\{1, 2, \ldots, m(n) - 1\}$ with $\max H < v$.

Let $a = \sum_{u \in H} c(n) \cdot \vec{y}\,(n)(u)$ and let $b = c(n) \cdot \vec{y}\,(n)(v)$

(a) For all $t \in F(\vec{x}, a)$ and all $i \in \{1, 2, \ldots, r(\vec{x}, a, t)\}$, $c(n)$ divides $\lambda(\vec{x}, a, t)(i)$.

(b) $F(\vec{x}, a) \subseteq F(\vec{x}, b)$

(c) For all $t \in F(\vec{x}, a)$, $r(\vec{x}, a, t) < r(\vec{x}, b, t)$.

Proof. Let $d = \sum_{u \in H} \vec{y}\,(n)(u) + c(n) \cdot \vec{y}\,(n)(v)$. Then $a, b, d \in S(\vec{y}, n) \subseteq FS(< S(\vec{x}, t) >_{t=1}^{\infty})$. Now

$$
\begin{aligned}
& \sum_{t \in F(\vec{x},d)} \sum_{i=1}^{r(\vec{x},d,t)} \lambda(\vec{x}, d, t)(i) \cdot \vec{x}\,(t)(i) = d \\
= \quad & \frac{a}{c(n)} + b = \sum_{t \in F(\vec{x},a)} \sum_{i=1}^{r(\vec{x},a,t)} \frac{\lambda(\vec{x}, a, t)(i)}{c(n)} \cdot \vec{x}\,(t)(i) \\
+ \quad & \sum_{t \in F(\vec{x},b)} \sum_{i=1}^{r(\vec{x},b,t)} \lambda(\vec{x}, b, t)(i) \cdot \vec{x}\,(t)(i) \\
= \quad & \sum_{t \in G} \sum_{i=1}^{s(t)} \left(\frac{\lambda(\vec{x}, a, t)(i)}{c(n)} + \lambda(\vec{x}, b, t)(i) \right) \cdot \vec{x}\,(t)(i),
\end{aligned}
$$

where $G = F(\vec{x}, a) \cup F(\vec{x}, b)$, $s(t) = \max\{r(\vec{x}, a, t), r(\vec{x}, b, t)\}$, and if say $t \notin F(\vec{x}, a)$, then we have $r(\vec{x}, a, t) = 0$ and $\lambda(\vec{x}, a, t)(i) = 0$ for $i > r(\vec{x}, a, t)$. Multiplying both extremes of the above equation by $c(n)$ we get

$$
\begin{aligned}
& \textstyle\sum_{t \in F(\vec{x},d)} \sum_{i=1}^{r(\vec{x},d,t)} c(n) \cdot \lambda(\vec{x}, d, t)(i) \cdot \vec{x}\,(t)(i) \\
= \quad & \textstyle\sum_{t \in G} \sum_{i=1}^{s(t)} (\lambda(\vec{x}, a, t)(i) + c(n) \cdot \lambda(\vec{x}, b, t)(i)) \cdot \vec{x}\,(t)(i).
\end{aligned}
$$

Therefore by Lemma 3.2 we have $F(\vec{x}, d) = G$ (since for each $t \in F(\vec{x}, d) \cup G$ some coefficient is nonzero) and for each $t \in G$, $s(t) = r(\vec{x}, d, t)$ and for each $t \in G$ and $i \in \{1, 2, \ldots, s(t)\}$, $c(n) \cdot \lambda(\vec{x}, d, t)(i) = \lambda(\vec{x}, a, t)(i) + c(n) \cdot \lambda(x, b, t)(i)$. Conclusion (a) then follows immediately.

For conclusion (b), let $t \in F(\vec{x}, a)$ and suppose $t \notin F(\vec{x}, b)$. Then $t \in G$ and $r(\vec{x}, b, t) = 0$ so $r(\vec{x}, d, t) = s(t) = r(\vec{x}, a, t)$ and $\lambda(\vec{x}, b, t)(r(\vec{x}, d, t)) = 0$ so $c(n) \cdot c(t) = c(n) \cdot \lambda(\vec{x}, d, t)(r(\vec{x}, d, t)) = \lambda(\vec{x}, a, t)(r(\vec{x}, d, t)) = \lambda(\vec{x}, a, t)(r(\vec{x}, a, t)) = c(t)$, a contradiction.

To see conclusion (c), let $t \in F(\vec{x}, a)$ and let $i = r(\vec{x}, d, t)$. Then $c(n) \cdot c(t) = c(n) \cdot \lambda(\vec{x}, d, t)(i) = \lambda(\vec{x}, a, t)(i) + c(n) \cdot \lambda(\vec{x}, b, t)(i)$. Now $i = r(\vec{x}, d, t) = s(t) = \max\{r(\vec{x}, a, t),\ r(\vec{x}, b, t)\}$. Suppose first $r(\vec{x},a,t) > r(\vec{x},b,t)$. Then $\lambda(\vec{x},b,t)(i) = 0$ so $c(n) \cdot c(t) = c(t)$, which is a contradiction. Next, if $r(\vec{x},a,t) = r(\vec{x},b,t)$ then $c(n) \cdot c(t) = c(t) + c(n) \cdot c(t)$, again a contradiction. Thus $r(\vec{x},a,t) < r(\vec{x},b,t)$. $\square$

Given that $\vec{x} \in W$ and $\vec{y}$ refines $\vec{x}$ it is immediate that for any $n \in \mathbb{N}$ and any $a \in S(\vec{y},n)$, $\min F(\vec{x},a) \geq n$. We see now that equality can hold only under very restricted circumstances.

3.6 Lemma. Let $\vec{x} \in W$, let $\vec{y}$ refine $\vec{x}$, let $n \in \mathbb{N}$, let $l \in \{1,2,\ldots,m(n)\}$, and let $b = c(n) \cdot \vec{y}\,(n)(l)$. Assume that $\min F(\vec{x},b) = n$.

(a) For all $t \in \{1,2,\ldots,n-1\}$, $\vec{y}\,(t) = \vec{x}\,(t)$.

(b) If $l > 1$ and $a = \sum_{j=1}^{l-1} c(n) \cdot \vec{y}\,(n)(j)$, then either $n \notin F(\vec{x},a)$ or we have for each $i \in \{1,2,\ldots,r(\vec{x},a,n)\}$, $\lambda(\vec{x},a,n)(i) = 0$ or $\lambda(\vec{x},a,n)(i) = c(n)$.

Proof. (a) For $t < n$, since $\min F(\vec{x},b) = n$, one must have for each $i \in \{1,2,\ldots,m(t)\}$ that $F(\vec{x},c(t) \cdot \vec{y}\,(t)(i)) = \{t\}$.

By Lemma 3.5 (c) we have for each $i \in \{1,2,\ldots,m(t)-1\}$ the relation $r(\vec{x},c(t) \cdot \vec{y}\,(t)(i),t) < r(\vec{x},c(t) \cdot \vec{y}\,(t)(i+1),t)$ so, since $r(\vec{x},c(t) \cdot \vec{y}\,(t)(m(t)),t) \leq m(t)$ we must have for each $i \in \{1,2,\ldots,m(t)\}$ that $r(\vec{x},c(t) \cdot \vec{y}\,(t)(i),t) = i$. Hence, for each $i \in \{1,2,\ldots,m(t)\}$, $c(t) \cdot \vec{y}\,(t)(i) = \sum_{j=1}^{i-1} \lambda_{i,j} \cdot \vec{x}\,(t)(j) + c(t) \cdot \vec{x}\,(i)$ (where $\lambda_{i,j} = \lambda(\vec{x},c(t) \cdot \vec{y}\,(t)(i),t)(j)$). We claim each $\lambda_{i,j} = 0$, so that $\vec{y}\,(t)(i) = \vec{x}\,(t)(i)$ as required. Suppose not and pick the least i such that some $j < i$ has $\lambda_{i,j} \neq 0$ and pick such j. Then $\vec{y}\,(t)(j) = \vec{x}\,(t)(j)$. Let $d = p(t) \cdot \vec{y}\,(t)(j) + c(t) \cdot \vec{y}\,(t)(i)$.

Then

$$\begin{aligned} & \sum_{v=1}^{i-1} \lambda_{i,v} \cdot \vec{x}\,(t)(v) + c(t) \cdot \vec{x}\,(t)(i) + p(t) \cdot \vec{x}\,(t)(j) = d \\ = & \sum_{v=1}^{r(\vec{x},d,t)} \lambda(\vec{x},d,t)(v) \cdot \vec{x}\,(t)(v). \end{aligned}$$

By Lemma 3.2 (since $\lambda_{i,j} + p(t) \leq p(t) + p(t) < t \cdot p(t)^2$) the coefficients are equal and in particular $\lambda(\vec{x},d,t)(j) = \lambda_{i,j} + p(t) > p(t)$, a contradiction.

(b) Assume $n \in F(\vec{x},a)$. We have by Lemma 3.5 (a) that for each $i \in \{1,2,\ldots,r(\vec{x},a,n)\}$, $c(n)$ divides $\lambda(\vec{x},a,n)(i)$. For each $i \in \{1,2,\ldots,r(\vec{x},a,n)\}$ pick $h(i) \in \mathbb{N} \cup \{0\}$ such that $\lambda(\vec{x},a,n)(i) = h(i) \cdot c(n)$. We show that each $h(i) \in \{0,1\}$. Let $d = \sum_{j=1}^{l-1} p(n) \cdot \vec{y}\,(n)(j) + c(n) \cdot \vec{y}\,(n)(l)$. Then $d = \frac{p(n)}{c(n)} \cdot a + b$.

Then

$$\sum_{t\in F(\vec{x},d)} \sum_{i=1}^{r(\vec{x},d,t)} \lambda(\vec{x},d,t)(i)\cdot \vec{x}\,(t)(i) = d = \frac{p(n)}{c(n)}\cdot a + b$$

$$= \sum_{t\in F(\vec{x},a)} \sum_{i=1}^{r(\vec{x},a,t)} \frac{p(n)}{c(n)}\cdot \lambda(\vec{x},a,t)(i)\cdot \vec{x}\,(t)(i)$$

$$+ \sum_{t\in F(\vec{x},b)} \sum_{i=1}^{r(\vec{x},a,t)} \lambda(\vec{x},b,t)(i)\cdot \vec{x}\,(t)(i).$$

Again by Lemma 3.2 we have all corresponding coefficents are equal. (The coefficeints on the right hand side are at most $\frac{p(n)}{c(n)}\cdot p(t) + p(t) = (n+1)\cdot p(t) < t\cdot p(t)^2$). In particular, since $n \in F(\vec{x},a)$ we have for each $i \in \{1,2,\ldots,r(\vec{x},a,n)\}$ that $\lambda(\vec{x},d,t)(i) = \frac{p(n)}{c(n)}\cdot\lambda(\vec{x},a,t)(i) + \lambda(\vec{x},b,t)(i) = p(n)\cdot h(i) + \lambda(\vec{x},b,t)(i)$. Therefore, $h(i) \le 1$. $\square$

We are now ready to establish that the process of passing to refinements keeps us in the class W.

3.7 Theorem. Let $\vec{x}\in W$ and let $\vec{y}$ refine $\vec{x}$. Then $\vec{y}\in W$.

Proof. Let $n \in I\!N$ and let $l \in \{1,2,\ldots,m(n)\}$. Let $b = c(n)\cdot\,\vec{y}\,(n)(l)$. We show first that if $k \in F(\vec{x},b)$ then

$$(*) \qquad \frac{c(k)}{c(n)}\cdot \sum_{t=1}^{k-1}\sum_{i=1}^{m(t)} t\cdot p(t)^2\cdot \vec{x}\,(t)(i)$$

$$\ge \sum_{s=1}^{n-1}\sum_{j=1}^{m(s)} s\cdot p(s)^2\cdot \vec{y}\,(s)(j).$$

If $k = n$ this is immediate from Lemma 3.6 (a), so we assume $k > n$. For each $s \in \{1,2,\ldots,n-1\}$, let $a_s = \sum_{j=1}^{m(s)} c(s)\cdot\,\vec{y}\,(s)(j)$, and let $\alpha(s) = \max F(\vec{x},a_s)$ and let $\alpha(0) = 0$. Then since $\vec{y}$ refines $\vec{x}$ we have for each s that $F(\vec{x},a_s) \subseteq \{\alpha(s-1)+1,\alpha(s-1)+2,\ldots,\alpha(s)\}$. Further $\alpha(n-1) \le k-1$. Now given $s \in \{1,2,\ldots,n-1\}$ and $t \in \{\alpha(s-1)+1,\alpha(s-1)+2,\ldots,\alpha(s)\}$ we have $s \le t$ and hence

$$\frac{s\cdot p(s)^2\cdot p(t)}{c(s)} \le \frac{t\cdot p(t)^2\cdot c(k)}{c(n)} \qquad \text{since} \qquad \frac{p(s)}{c(s)} = s < n < k \le \frac{c(k)}{c(n)}.$$

Then we have

$$\begin{aligned}
& \sum_{s=1}^{n-1}\sum_{j=1}^{m(s)} s\cdot p(s)^2\cdot \vec{y}\,(s)(j) \\
= & \sum_{s=1}^{n-1}\frac{s\cdot p(s)^2}{c(s)}\cdot\sum_{j=1}^{m(s)} c(s)\cdot \vec{y}\,(s)(j) \\
= & \sum_{s=1}^{n-1}\frac{s\cdot p(s)^2}{c(s)}\cdot a_s \\
= & \sum_{s=1}^{n-1}\frac{s\cdot p(s)^2}{c(s)}\cdot\sum_{t\in F(\vec{x},a_s)}\sum_{i=1}^{r(\vec{x},a_s,t)}\lambda(\vec{x},a_s,t)(i)\cdot \vec{x}\,(t)(i) \\
\leq & \sum_{s=1}^{n-1}\sum_{t\in F(\vec{x},a_s)}\frac{s\cdot p(s)^2\cdot p(t)}{c(s)}\cdot\sum_{i=1}^{m(t)}\vec{x}\,(t)(i) \\
\leq & \sum_{s=1}^{n-1}\sum_{t=\alpha(s-1)+1}^{\alpha(s)}\frac{t\cdot p(t)^2\cdot c(k)}{c(n)}\cdot\sum_{i=1}^{m(t)}\vec{x}\,(t)(i) \\
\leq & \sum_{t=1}^{k-1}\frac{t\cdot p(t)^2\cdot c(k)}{c(n)}\cdot\sum_{i=1}^{m(t)}\vec{x}\,(t)(i)
\end{aligned}$$

and so (*) is established.

Assume now $l = 1$ and let $k = \min F(\vec{x}, b)$. Then

$$\begin{aligned}
\vec{y}\,(n)(1) &\geq \frac{c(k)}{c(n)}\cdot \vec{x}\,(k)(r(\vec{x},b,k)) \\
&\geq \frac{c(k)}{c(n)}\cdot \vec{x}\,(k)(1) \\
&> \frac{c(k)}{c(n)}\cdot\sum_{t=1}^{k-1}\sum_{i=1}^{m(t)} t\cdot p(t)^2\cdot \vec{x}\,(t)(i) \\
&\geq \sum_{s=1}^{n-1}\sum_{j=1}^{m(s)} s\cdot p(s)^2\cdot \vec{y}\,(s)(j)
\end{aligned}$$

by (*).

Now assume $l > 1$. Then

$$
\begin{aligned}
\vec{y}\,(n)(l) &= \frac{1}{c(n)} \cdot \sum_{k\in F(\vec{x},b)} \sum_{i=1}^{r(\vec{x},b,k)} \lambda(\vec{x},b,k)(i)\cdot \vec{x}\,(k)(i). \\
&\geq \frac{1}{c(n)} \cdot \sum_{k\in F(\vec{x},b)} c(k)\cdot \vec{x}\,(k)(r(\vec{x},b,k)) \\
&> \sum_{k\in F(\vec{x},b)} \frac{c(k)}{c(n)} \cdot \sum_{t=1}^{k-1}\sum_{i=1}^{m(t)} t\cdot p(t)^2\cdot \vec{x}\,(t)(i) \\
&\quad + \sum_{k\in F(\vec{x},b)} \frac{c(k)}{c(n)} \cdot \sum_{i=1}^{r(\vec{x},b,k)-1} k\cdot p(k)^2\cdot \vec{x}\,(k)(i).
\end{aligned}
$$

Now

$$
\begin{aligned}
&\sum_{k\in F(\vec{x},b)} \frac{c(k)}{c(n)} \sum_{t=1}^{k-1}\sum_{i=1}^{m(t)} t\cdot p(t)^2\cdot \vec{x}\,(t)(i) \\
\geq\ & \sum_{k\in F(\vec{x},b)} \sum_{s=1}^{n-1}\sum_{j=1}^{m(s)} s\cdot p(s)^2\cdot \vec{y}\,(s)(j) \\
\geq\ & \sum_{s=1}^{n-1}\sum_{j=1}^{m(s)} s\cdot p(s)^2\cdot \vec{y}\,(s)(j) \qquad\qquad \text{by} \quad (*).
\end{aligned}
$$

Thus it suffices to show that

$$
\sum_{k\in F(\vec{x},b)} \frac{c(k)}{c(n)} \cdot \sum_{i=1}^{r(\vec{x},b,k)-1} k\cdot p(k)^2\cdot \vec{x}\,(k)(i) \geq \sum_{j=1}^{l-1} n\cdot p(n)^2\cdot \vec{y}\,(n)(j).
$$

To this end let $a = \sum_{j=1}^{l-1} c(n)\cdot \vec{y}\,(n)(j)$. Then

$$
\begin{aligned}
&\sum_{j=1}^{l-1} n\cdot p(n)^2\cdot \vec{y}\,(n)(j) = \frac{n\cdot p(n)^2}{c(n)}\cdot a \\
=\ & \sum_{k\in F(\vec{x},a)} \frac{n\cdot p(n)^2}{c(n)} \cdot \sum_{i=1}^{r(\vec{x},a,k)} \lambda(\vec{x},a,k)(i)\cdot \vec{x}\,(k)(i).
\end{aligned}
$$

Since by Lemma 3.5 $F(\vec{x}, a) \subseteq F(\vec{x}, b)$ and for each

$$k \in F(\vec{x}, a), \quad r(\vec{x}, a, k) \leq r(\vec{x}, b, k) - 1$$

it in turn suffices to show that for each $k \in F(\vec{x}, a)$,

$$\frac{n \cdot p(n)^2}{c(n)} \cdot \sum_{i=1}^{r(\vec{x},a,k)} \lambda(\vec{x}, a, k)(i) \cdot \vec{x}\,(k)(i)$$

$$\leq \frac{c(k)}{c(n)} \cdot \sum_{i=1}^{r(\vec{x},a,k)} k \cdot p(k)^2 \cdot \vec{x}\,(k)(i).$$

If $k = n$, by Lemma 3.6 (b) we have each $\lambda(\vec{x}, a, k)(i)$ is 0 or $c(n)$ so this inequality is immediate. Then assume $k > n$. Then given $i \in \{1, 2, \ldots, r(\vec{x}, a, k)\}$, $\lambda(\vec{x}, a, k)(i) \leq p(k)$ and $p(n) = n \cdot c(n) < c(n+1) \leq c(k)$ so $\frac{n \cdot p(n)^2 \cdot p(k)}{c(n)} \leq \frac{k \cdot c(k)^2 \cdot p(k)}{c(n)}$ so the inequality follows. □

4 Equivalence with respect to members of W

Given $\vec{x} \in W$ we define an equivalence relation on $FS(< S(\vec{x}, n) >_{n=1}^{\infty})$ which provides one of the clauses of our canonization theorem.

4.1 Definition. Let $\vec{x} \in W$ and let $a, b \in FS(< S(\vec{x}, n) >_{n=1}^{\infty})$. Then $a \approx_{\vec{x}} b$ if and only if $F(\vec{x}, a) = F(\vec{x}, b)$ and for each $n \in F(\vec{x}, a), r(\vec{x}, a, n) = r(\vec{x}, b, n)$.

Thus to say that $a \approx_{\vec{x}} b$ is to say that a and b are constructed from the same parts of the same (m, p, c)-sets.

4.2 Lemma. Let $\vec{x} \in W$, let $\vec{y}$ refine $\vec{x}$, and let $a \in FS(< S(\vec{y}, n) >_{n=1}^{\infty})$.

(a) For $n \in F(\vec{y}, a), t \in F(\vec{x}, c(n) \cdot \vec{y}\,(n)(r(\vec{y}, a, n))$, and $j \in \{1, 2, \ldots, r(\vec{x}, c(n) \cdot \vec{y}\,(n)(r(\vec{y}, a, n)), t)\}$ let $\alpha(n, t, j) = \min\{i : t \in F(\vec{x}, c(n) \cdot \vec{y}\,(n)(i))$ and $j \leq r(\vec{x}, c(n) \cdot (n)(i), t)\}$. Then for each $n \in F(\vec{y}, a)$, $\{(i, t, j) : 1 \leq i \leq r(\vec{y}, a, n)$ and $t \in F(\vec{x}, c(n) \cdot \vec{y}\,(n)(i))$ and $1 \leq j \leq r(\vec{x}, c(n) \cdot \vec{y}\,(n)(i), t)\} = \{(i, t, j) : t \in F(\vec{x}, c(n) \cdot y(n)(r(\vec{y}, a, n)))$ and $1 \leq j \leq r(\vec{x}, c(n) \cdot \vec{y}\,(n)(r(\vec{y}, a, n)), t)$ and $\alpha(n, t, j) \leq i \leq r(\vec{y}, a, n)\}$.

(b) $F(\vec{x}, a) = \cup_{n \in F(\vec{y}, a)} F(\vec{x}, c(n) \cdot \vec{y}\,(n)(r(\vec{y}, a, n)))$.

(c) Given $t \in F(\vec{x}, a)$ and $n \in F(\vec{y}, a)$, if $t \in F(\vec{x}, c(n) \cdot \vec{y}\,(n)(r(\vec{y}, a, n)))$, then $r(\vec{x}, a, t) = r(\vec{x}, c(n) \cdot \vec{y}\,(n)(r(\vec{y}, a, n)), t)$.

Proof. (a) Observe that for each t and j, $\alpha(n,t,j) \leq r(\vec{y},a,n)$. Now let (i,t,j) be given with $1 \leq i \leq r(\vec{y},a,n)$ and $t \in F(\vec{x},c(n)\cdot \vec{y}\ (n)(i))$ and $1 \leq j \leq r(\vec{x},c(n)\cdot \vec{y}\ (n)(i),t)$. As $i \leq r(\vec{y},a,n)$ we have by Lemma 3.5 (b) that $t \in F(\vec{x},c(n)\cdot \vec{y}\ (n)(r(\vec{y},a,n)))$ and by Lemma 3.5 (c) that $j \leq r(\vec{x},c(n)\cdot \vec{y}\ (n)(r(\vec{y},a,n)),t)$. That $\alpha(n,t,j) \leq i$ follows immediately from the definition of $\alpha(n,t,j)$.

For the reverse inclusion, let a triple (i,t,j) be given with $t \in F(\vec{x},c(n)\cdot \vec{y}\ (n)(r(\vec{y},a,n)),$ $1 \leq j \leq r(\vec{x},c(n)\cdot \vec{y}\ (n)(r(\vec{y},a,n)),t)$ and $\alpha(n,t,j) \leq i \leq r(\vec{y},a,n)$. From Lemma 3.5 and the definition of $\alpha(n,t,j)$ we have $t \in F(\vec{x},c(n)\cdot \vec{y}\ (n)(i))$ and $j \leq r(\vec{x},c(n)\cdot \vec{y}\ (n)(i),t)$.

We now establish (b) and (c). Observe that for $n \in I\!N$ and $i \in \{1,2,\ldots,m(n)\}$, $\vec{y}\ (n)(i) = \sum_{t\in F(\vec{x},c(n)\cdot\vec{y}(n)(i))} \sum_{j=1}^{r(\vec{x},c(n),\vec{y}(n)(i),t)} \left(\frac{\lambda(\vec{x},c(n)\cdot\vec{y}(n)(i),t)(j)}{c(n)}\right) \cdot \vec{x}\ (t)(j)$. Then we have

$$
\begin{aligned}
& \sum_{t\in F(\vec{x},a)} \sum_{j=1}^{r(\vec{x},a,t)} \lambda(\vec{x},a,t)(j)\cdot \vec{x}\ (t)(j) = a \\
= & \sum_{n\in F(\vec{y},a)} \sum_{i=1}^{r(\vec{y},a,n)} \lambda(\vec{y},a,n)(i)\cdot \vec{y}\ (n)(i) \\
= & \sum_{n\in F(\vec{y},a)} \sum_{i=1}^{r(\vec{y},a,n)} \sum_{t\in F(\vec{x},c(n)\cdot\vec{y}(n)(i))} \\
& \sum_{j=1}^{r(\vec{x},c(n)\cdot\vec{y}(n)(i),t)} \left(\frac{\lambda(\vec{y},a,n)(i)\cdot\lambda(\vec{x},c(n)\cdot \vec{y}\ (n)(i),t)(j)}{c(n)}\right)\cdot \vec{x}\ (t)(j) \\
= & \sum_{n\in F(\vec{y},a);\, t\in F(\vec{x},c(n)\cdot\vec{y}(n)(r(\vec{y},a,n)))} \sum_{j=1}^{r(\vec{x},c(n)\cdot\vec{y}(n)(r(\vec{y},a,n),t)} \\
& \sum_{i=\alpha(n,t,j)}^{r(\vec{y},a,n)} \left(\frac{\lambda(\vec{y},a,n)(i)\cdot\lambda(\vec{x},c(n)\cdot \vec{y}\ (n)(i),t)(j)}{c(n)}\right)\cdot \vec{x}\ (t)(j),
\end{aligned}
$$

where the last equality follows from part (a). Multiplying both extremes by $c(n)$ we get

$$
\begin{aligned}
& \sum_{t\in F(\vec{x},a)} \sum_{j=1}^{r(\vec{x},a,t)} c(n)\cdot\lambda(\vec{x},a,t)(j)\cdot \vec{x}\ (t)(j) \\
= & \sum_{n\in F(\vec{y},a);\, t\in F(\vec{x},c(n)\cdot\vec{y}(n)(r(\vec{y},a,n)))} \sum_{j=1}^{r(\vec{x},c(n)\cdot\vec{y}(n)(r(\vec{y},a,n),t)}
\end{aligned}
$$

$$\sum_{i=\alpha(n,t,j)}^{r(\vec{y},a,n)} \lambda(\vec{y},a,n)(i)\cdot\lambda(\vec{x},c(n)\cdot\vec{y}\,(n)(i),t)(j)\cdot\vec{x}\,(t)(j).$$

The coefficients of $\vec{x}\,(t)(j)$ on the left hand side are all at most $c(n)\cdot p(t)\leq c(t)\cdot p(t)$. On the right hand side the coefficient of $\vec{x}\,(t)(j)$ is

$$\sum_{i=\alpha(n,t,i)}^{r(\vec{y},a,n)} \lambda(\vec{y},a,n)(i)\cdot\lambda(\vec{x},c(n)\cdot\vec{y}\,(n)(i),t)(j)$$

$$\leq\sum_{i=1}^{m(a)} p(n)\cdot p(t)=n\cdot p(n)\cdot p(t)\leq t\cdot p(t)^2.$$

Observe also that for $n\neq s$ in $F(\vec{y},a)$,

$$F(\vec{x},c(n)\cdot\vec{y}\,(n)(r(\vec{y},a,n)))\cap F(\vec{x},c(s)\cdot\vec{y}\,(s)(r(\vec{y},a,s)))=\emptyset.$$

Thus by Lemma 3.2 we have

$$F(\vec{x},a)=\cup_{n\in F(\vec{y},a)}F(\vec{x},c(n)\cdot\vec{y}\,(n)(r(\vec{y},a,n)))$$

(since each t on either side has some nonzero coefficient associated with it). Given $t\in F(\vec{x},a)$ and $j\in\{1,2,\ldots,r(\vec{x},a,t)\}$, if n is the member of $F(\vec{y},a)$ with $t\in F(\vec{x},c(n)\cdot\vec{y}\,(n)(r(\vec{y},a,n)))$, then

$$\lambda(x,a,t)(j)=\sum_{i=\alpha(n,t,j)}^{r(y,a,n)}\frac{\lambda(\vec{y},a,n)(i)\cdot\lambda(\vec{x},c(n)\cdot\vec{y}\,(n)(i),t)(j)}{c(n)}$$

and if $\lambda(\vec{x},a,t)(j)\neq 0$ then necessarily

$$j\in\{1,2,\ldots,r(\vec{x},c(n)\cdot\vec{y}\,(n)(r(\vec{y},a,n)),t)\}.$$

This again is from Lemma 3.2. Since $\lambda(\vec{x},a,t)(r(\vec{x},a,t))=c(t)\neq 0$ one has

$$r(\vec{x},a,t)\leq r(\vec{x},c(n)\cdot\vec{y}\,(n)(r(\vec{y},a,n)),t)\;.$$

Likewise, if $j=r(\vec{x},c(n)\cdot\vec{y}\,(n)(r(\vec{y},a,n)),t)$ then

$$\sum_{i=\alpha(n,t,j)}^{r(y,a,n)}\frac{\lambda(\vec{y},a,n)(i)\cdot\lambda(\vec{x},c(n)\cdot\vec{y}\,(n)(i),t)(j)}{c(n)}$$

$$\geq\frac{\lambda(\vec{y},a,n)(r,(\vec{y},a,n))\cdot\lambda(\vec{x},c(n)\cdot\vec{y}\,(n)(r(\vec{y},a,n)),t)(j)}{c(n)}=\frac{c(n)\cdot c(t)}{c(n)}=c(t)$$

so $r(\vec{x}, c(n)\cdot \vec{y}\ (n)(r(\vec{y}, a, n)), t) \leq r(\vec{x}, a, t)$. $\square$

The following theorem is important to us because we are, in our canonization arguments always passing to refinements.

4.3 Theorem. Let $\vec{x} \in W$, let $\vec{y}$ refine $\vec{x}$, and let $a, b \in FS(< S(\vec{y}, n) >_{n=1}^{\infty})$. Then $a \approx_{\vec{y}} b$ if and only if $a \approx_{\vec{x}} b$.

Proof. *Necessity*. By Lemma 4.2 (b) and (c) we have

$$\begin{aligned} F(\vec{x}, a) &= \cup_{n \in F(\vec{y}, a)} F(\vec{x}, c(n)\cdot \vec{y}\ (n)(r(\vec{y}, a, n))) \\ &= \cup_{n \in F(\vec{y}, b)} F(\vec{x}, c(n)\cdot \vec{y}\ (n)(r(\vec{y}, b, n))) = F(\vec{x}, b). \end{aligned}$$

Now let $t \in F(\vec{x}, a)$ and pick $n \in F(\vec{y}, a)$ such that $t \in F(\vec{x}, c(n)\cdot \vec{y}\ (n)(r(\vec{y}, a, n)))$. Then we have $t \in F(\vec{x}, c(n)\cdot \vec{y}\ (n)(r(\vec{y}, a, n)))$ and also

$$r(\vec{x}, a, t) = r(\vec{x}, c(n)\cdot \vec{y}\ (n)(r(\vec{y}, a, n)), t) = r(\vec{x}, c(n)\cdot \vec{y}\ (n)(r(\vec{y}, b, n)), t) = r(\vec{x}, b, t)\ .$$

.

Sufficiency. We show first that $F(\vec{x}, a) \subseteq F(\vec{y}, b)$. Let $n \in F(\vec{y}, a)$ and pick any element $t \in F(\vec{x}, c(n)\cdot \vec{y}\ (n)(r(\vec{y}, a, n)))$. Then $t \in F(\vec{x}, a)$ so $t \in F(\vec{x}, b)$ so (by Lemma 4.2) for some $s \in F(\vec{y}, b), t \in F(\vec{x}, c(s)\cdot \vec{y}\ (s)(r(\vec{y}, b, s)))$. Now we have $c(s)\cdot \vec{y}\ (s)(r(\vec{y}, b, s)) \in S(\vec{y}, s)$ and $c(n)\cdot \vec{y}\ (n)(r(\vec{y}, b, n)) \in S(\vec{y}, n)$ and $t \in F(\vec{x}, c(s)\cdot \vec{y}\ (s)(r(\vec{y}, b, s))) \cap F(\vec{x}, c(n)\cdot \vec{y}\ (n)(r(\vec{y}, b, n)))$ so, since $\vec{y}$ refines $\vec{x}$, $s = n$. Similarly $F(\vec{y}, b) \subseteq F(\vec{y}, a)$ so $F(\vec{y}, b) = F(\vec{y}, a)$.

Now let $n \in F(\vec{y}, a)$ and pick $t \in F(\vec{x}, c(n)\cdot \vec{y}\ (n)(r(\vec{y}, a, n))$. Then again using Lemma 4.2 we have $r(\vec{x}, c(n)\cdot \vec{y}\ (n)(r(\vec{y}, a, n)), t) = r(\vec{x}, a, t) = r(\vec{x}, b, t) = r(\vec{x}, c(n)\cdot \vec{y}\ (n)(r(\vec{y}, b, n)), t)$. If we had say $r(\vec{y}, b, n) < r(\vec{y}, a, n)$ then by Lemma 3.5 (c) we would have $r(\vec{x}, c(n)\cdot \vec{y}\ (n)(r(\vec{y}, b, n)), t) < r(\vec{x}, c(n)\cdot \vec{y}\ (n)(r(\vec{y}, a, n)), t)$. Thus $r(\vec{y}, b, n) = r(\vec{y}, a, n)$. $\square$

We conclude this section by establishing the basis for our ability to dispense with the three separate classes from [8] being applied to each (m, p, c)-set.

4.4 Theorem. Let $\vec{x} \in W$. There exists $\vec{y}$ refining $\vec{x}$ so that for any a and b in $FS(< S(\vec{y}, n) >_{n=1}^{\infty})$ statements (a), (b), (c) and (d) are equivalent and statements (e), (f), (g), and (h) are equivalent.

(a) If $t = \min F(\vec{x}, a)$, then $t = \min F(\vec{x}, b)$ and $w(\vec{x}, a, t) = w(\vec{x}, b, t)$.

(b) If $t = \min F(\vec{x}, a)$, then $t = \min F(\vec{x}, b)$ and $r(\vec{x}, a, t) = r(\vec{x}, b, t)$.

(c) $\min F(\vec{x},a) = \min F(\vec{x},b)$.

(d) If $n = \min F(\vec{y},a)$, then $n = \min F(\vec{y},b)$ and $r(\vec{y},a,n) = r(\vec{y},b,n)$.

(e) If $t = \max F(\vec{x},a)$, then $t = \max F(\vec{x},b)$ and $w(\vec{x},a,t) = w(\vec{x},b,t)$.

(f) If $t = \max F(\vec{x},a)$, then $t = \max F(\vec{x},b)$ and $r(\vec{x},a,t) = r(\vec{x},b,t)$.

(g) $\max F(\vec{x},a) = \max F(\vec{x},b)$

(h) If $n = \max F(\vec{y},a)$, then $n = \max F(\vec{y},b)$ and $r(\vec{y},a,n) = r(\vec{y},b,n)$.

Proof. Define a sequence $< \mu(n) >_{n=1}^{\infty}$, by $\mu(1) = 1$ and $\mu(n+1) = \mu(n)+2m(n)-1 = \mu(n)+2n-1$.

For each $n \in I\!N$ and each $j \in \{1,2,\ldots,m(n)\}$ define $\vec{y}\ (n)(j) = \sum_{t=\mu(n)+m(n)-j}^{\mu(n)+m(n)+j-2} \frac{c(t)}{c(n)} \cdot \vec{x}\ (t)(j)$. We show first that $\vec{y}$ refines $\vec{x}$. Since $F(\vec{x}, c(n) \cdot \vec{y}\ (n)(m(n))) = \{\mu(n), \mu(n)+1, \ldots, \mu(n+1)-1\}$ and for $j < m(n)$ we have $F(\vec{x}, c(n) \cdot \vec{y}\ (n)(j)) \subseteq F(\vec{x}, c(n) \cdot \vec{y}\ (n)(m(n)))$, it suffices to show that $FS(< S(\vec{y},n) >_{n=1}^{\infty}) \subseteq FS(< S(\vec{x},t) >_{t=1}^{\infty})$. To see this it suffices to let $n \in I\!N$ and show that $S(\vec{y},n) \subseteq FS(< S(\vec{x},t) >_{t=\mu(n)}^{\mu(n+1)-1})$.

Let $n \in I\!N$ and $a \in S(\vec{y},n)$ be given. Then $\{(j,t) : 1 \le j \le r(\vec{y},a,n)$ and $\mu(n)+m(n)-j \le t \le u(n)+m(n)+j-2\} = \{(j,t) : \mu(n)+m(n)-r(\vec{y},a,n) \le t \le \mu(n)+m(n)+r(\vec{y},a,n)-2$ and $1+|\mu(n)+m(n)-t-1| \le j \le r(\vec{y},a,n)\}$ as can be routinely established. Therefore

$$\begin{aligned}
a &= \sum_{j=1}^{r(\vec{y},a,n)} \lambda(\vec{y},a,n)(j) \cdot \vec{y}\ (n)(j) \\
&= \sum_{j=1}^{r(\vec{y},a,n)} \sum_{t=\mu(n)+m(n)-j}^{\mu(n)+m(n)+j-2} \lambda(\vec{y},a,n)(j) \cdot \frac{c(t)}{c(n)} \cdot \vec{x}\ (t)(j) \\
&= \sum_{t=\mu(n)+m(n)-r(\vec{y},a,n)}^{\mu(n)+m(n)+r(\vec{y},a,n)-2} \quad \sum_{j=1+|\mu(n)+m(n)-t-1|}^{r(\vec{y},a,n)} \lambda(\vec{y},a,n)(j) \cdot \frac{c(t)}{c(n)} \cdot \vec{x}\ (t)(j).
\end{aligned}$$

For each t and j we have $\lambda(\vec{y},a,n)(j) \cdot \frac{c(t)}{c(n)} \le p(n) \cdot \frac{c(t)}{c(n)} = n \cdot c(t) \le p(t)$. Further if $j = r(\vec{y},a,n)$, then $\lambda(\vec{y},a,n)(j) \cdot \frac{c(t)}{c(n)} = c(n) \cdot \frac{c(t)}{c(n)} = c(t)$ so $a \in FS(< S(\vec{x},t) >_{t=\mu(n)}^{\mu(n+1)-1})$ as required.

We now show that the statements (a), (b), (c) and (d) are equivalent, omitting the similar verification of the equivalence of (e), (f), (g) and (h).

Since $w(\vec{x},a,t) = \sum_{i=1}^{r(\vec{x},a,t)} \lambda(\vec{x},a,t)(i) \cdot \vec{x}\ (t)(i)$ that (a) implies (b) is trivial as is the fact that (b) implies (c). To see that (c) implies (d), let $t = \min F(\vec{x},a) = \min F(\vec{x},b)$. Pick n such that $\mu(n) \le t < \mu(n+1)$. Then $n = \min F(\vec{y},a) = \min F(\vec{y},b)$. Now as above we see that

$$w(\vec{y},a,n) = \sum_{j=1}^{r(\vec{y},a,n)} \lambda(\vec{y},a,n)(j)\cdot \vec{y}\,(n)(j)$$
$$= \sum_{t=\mu(n)+m(n)-r(\vec{y},a,n)}^{\mu(n)+m(n)+r(\vec{y},a,n)-2} \sum_{j=1+|\mu(n)+m(n)-t-1|}^{r(\vec{y},a,n)} \lambda(\vec{y},a,n)(j)\cdot \frac{c(t)}{c(n)}\cdot \vec{x}\,(t)(j).$$

Then $\min F(\vec{x},a) = \mu(n)+m(n)-r(\vec{y},a,n)$. Similarly $\min F(\vec{x},b) = \mu(n)+m(n)-r(\vec{y},b,n)$ and hence $r(\vec{y},a,n) = r(\vec{y},b,n)$.

Finally to see that (d) implies (a), let $n = \min F(\vec{y},a) = \min F(\vec{y},b)$ and assume $r(\vec{y},a,n) = r(\vec{y},b,n)$. Then

$$w(\vec{y},a,n) = \sum_{t=\mu(n)+m(n)-r(\vec{y},a,n)}^{\mu(n)+m(n)+r(\vec{y},a,n)-2} \sum_{j=1+|\mu(n)+m(n)-t-1|}^{r(\vec{y},a,n)} \lambda(\vec{y},a,n)(j)\cdot \frac{c(t)}{c(n)}\cdot \vec{x}\,(t)(j).$$

Let $t = \mu(n)+m(n)-r(\vec{y},a,n)$. Then $t = \min F(\vec{x},a)$ and

$$w(\vec{x},a,t) = \sum_{j=1+|\mu(n)+m(n)-t-1|}^{r(\vec{y},a,n)} \lambda(\vec{y},a,n)(j)\cdot \frac{c(t)}{c(n)}\cdot \vec{x}\,(t)(j).$$

Likewise $t = \min F(\vec{x},b)$ and

$$w(\vec{x},b,t) = \sum_{j=1+|\mu(n)+m(n)-t-1|}^{r(\vec{y},a,n)} \lambda(\vec{y},b,n)(j)\cdot \frac{c(t)}{c(n)}\cdot \vec{x}\,(t)(j).$$

But now $1+|\mu(n)+m(n)-t-1| = 1+r(\vec{y},a,n)-1 = r(\vec{y},a,n)$ so

$$w(\vec{x},a,t) = c(n)\cdot \frac{c(t)}{c(n)}\cdot \vec{x}\,(t)(r(\vec{y},a,n)) = c(t)\cdot \vec{x}\,(t)(r(\vec{y},a,n)).$$

Likewise

$$w(\vec{x},b,t) = c(n)\cdot \frac{c(t)}{c(n)}\cdot \vec{x}\,(t)(r(\vec{y},b,n)) = c(t)\cdot \vec{x}\,(t)(r(\vec{y},a,n)) = w(\vec{x},a,t).$$

$\square$

5 The Canonical Partition Relations

We address here the following problem. Find certain canonical colorings so that given any $\vec{x}\in V^*$ and any coloring f of $FS(< S(\vec{x},n) >_{n=1}^{\infty})$, there is some refinement $\vec{y}$ of $\vec{x}$ such that f agrees with one of the canonical colorings on $FS(< S(\vec{y}, n) >_{n=1}^{\infty})$. We produce here six canonical colorings, none of which can be avoided in the strong sense that for each of them there exist $\vec{x} \in V^*$ and a coloring f of $FS(< S(\vec{x}, n) >_{n=1}^{\infty})$ so that whenever $\vec{y}$ refines $\vec{x}$ one has f agrees with the given coloring on $FS(< S(\vec{y}, n) >_{n=1}^{\infty})$. (By way of contrast, in [8] the third clause cannot be proved to hold in this strong sense, that is, there is no coloring such that every solution is colored according to this third clause.)

We also know that if a certain conjecture, which is inspired by the Erdős-Graham canonical theorem for arithmetic progressions, holds then some of these six colorings must apply for some refinement $\vec{y}$ of $\vec{x}$.

We will have need of a multidimensional version of Theorem 2.7.

5.1 Definition. Let $\vec{x} \in V^*$ and let $k \in \mathbb{N}$. $[FS(< S(\vec{x}, n) >_{n=1}^{\infty})]^k_< = \{\{a_1, a_2, \ldots, a_k\}$: each $a_i \in FS(< S(\vec{x}, n) >_{n=1}^{\infty})$ and if $i < k$ then $\max F(\vec{x}, a_i) < \min F(\vec{x}, a_{i+1})\}$.

5.2 Theorem. Let $\vec{x} \in V^*$ and let $f : [FS(< S(\vec{x}, n) >_{n=l}^{\infty})]^k_< \to \{1, 2, \ldots, l\}$. There exsits $\vec{y}$ refining $\vec{x}$ such that f is constant on $[FS(< S(\vec{y}, n) >_{n=1}^{\infty})]^k_<$.

Proof. [7, Theorem 3.3].

Our approach follows closely that of Taylor in [9]. We assume throughout this section, until after the proof of Theorem 5.14, that we have $\vec{x} \in W$ and $f : FS(< S(\vec{x}, n) >_{n=1}^{\infty}) \to \mathbb{N}$. (By Theorem 3.4 we loose nothing by assuming that $\vec{x} \in W$.)

5.3 Definition. Let $\mathcal{H} = \{h : \{1, 2, 3\} \to \{1, 2\}\}$. Define $g : [FS(< S(\vec{x}, n) >_{n=1}^{\infty})]^3_< \to \mathcal{H}$ as follows: Given a_1, a_2, $a_3 \in FS(< S(\vec{x}, n) >_{n=1}^{\infty})$ with

$$\max F(\vec{x}, a_1) < \min F(\vec{x}, a_2) \leq \max F(\vec{x}, a_2) < \min F(\vec{x}, a_3),$$

(a) $g(\{a_1, a_2, a_3\})(1) = 1$ if and only if $f(a_1 + a_2 + a_3) = f(a_1)$.

(b) $g(\{a_1, a_2, a_3\})(2) = 1$ if and only if $f(a_1 + a_2 + a_3) = f(a_3)$.

(c) $g(\{a_1, a_2, a_3\})(3) = 1$ if and only if $f(a_1 + a_2 + a_3) = f(a_1 + a_3)$.

By Theorem 5.2 we may, and do, assume that g is constant on $[FS(< S(\vec{x}, n) >_{n=1}^{\infty})]^3_<$. We let $h \in \mathcal{H}$ be the constant value of g on $[FS(< S(\vec{x}, n) >_{n=1}^{\infty})]^3_<$.

5.4 Lemma. If $h(1) = h(2) = 1$, then there exists $\vec{y}$ refining $\vec{x}$ such that f is constant on $FS(< S(\vec{y}, n) >_{n=1}^{\infty})$.

Proof. For $n \in \mathbb{N}$ and $i \in \{1, 2, \ldots, m(n)\}$, let $\vec{y}\,(n)(i) = \frac{c(n+2)}{c(n)} \cdot \vec{x}\,(n+2)(i)$. Clearly $\vec{y}$ refines $\vec{x}$. Let $a_1 = \vec{x}\,(1)(1)$ and $a_2 = 2\,\vec{x}\,(2)(1)$. Then given any $b \in FS(< S(\vec{y}, n) >_{n=1}^{\infty})$

we have $\{a_1, a_2, b\} \in [FS(< S(\overrightarrow{x}, n) >_{n=1}^{\infty})]^3_<$ so $g(\{a_1, a_2, b\})(1) = 1$ and $g(\{a_1, a_2, b\})(2) = 1$ so $f(b) = f(a_1 + a_2 + b) = f(a_1)$. $\square$

5.5 Lemma. If $h(1) = 1$ and $h(2) = 2$, then there exists $\overrightarrow{y}$ refining $\overrightarrow{x}$ such that for all elements $a, b \in FS(< S(\overrightarrow{y}, n) >_{n=1}^{\infty})$ it is $f(a) = f(b)$ if and only if $\min F(\overrightarrow{y}, a) = \min F(\overrightarrow{y}, b) = n$ and $r(\overrightarrow{y}, a, n) = f(\overrightarrow{y}, b, n)$.

Proof. For each $n \in \mathbb{N}$ and $i \in \{1, 2, \ldots, m(n)\}$ let
$z(n)(i) = \frac{c(3n)}{c(n)} \cdot \overrightarrow{x}\ (3n)(i) + \frac{c(3n+1)}{c(n)} \cdot \overrightarrow{x}\ (3n+1)(i) + \frac{c(3n+2)}{c(n)} \cdot \overrightarrow{x}\ (3n+2)(i)$. Then $\overrightarrow{z}$ refines $\overrightarrow{x}$. Pick $\overrightarrow{y}$ refining $\overrightarrow{z}$ as guaranteed by Theorem 4.4.

Let $a, b \in FS(< S(\overrightarrow{y}, n) >_{n=1}^{\infty})$ and assume first that $\min F(\overrightarrow{y}, a) = \min F(\overrightarrow{y}, b) = n$ and $r(\overrightarrow{y}, a, n) = r(\overrightarrow{y}, b, n)$. Then by "(d) implies (a)" of Theorem 4.4, $\min F(\overrightarrow{z}, a) = \min F(\overrightarrow{z}, b) = t$ and $w(\overrightarrow{z}, a, t) = w(\overrightarrow{z}, b, t)$. Let $d = w(\overrightarrow{z}, a, t)$. Then $d = e_1 + e_2 + e_3$ where $e_1 \in S(\overrightarrow{x}, 3t)$, $e_2 \in S(\overrightarrow{x}, 3t+1)$, and $e_3 \in S(\overrightarrow{x}, 3t+2)$. Also we have $s, q \in FS(< S(\overrightarrow{x}, l) >_{l=3t+3}^{\infty}) \cup \{0\}$ such that $a = e_1 + e_2 + e_3 + s$ and $b = e_1 + e_2 + e_3 + q$. Then $\{e_1, e_2, e_3 + s\}$ and $\{e_1, e_2, e_3 + q\}$ are in $[FS(< S(\overrightarrow{x}, l) >_{l=1}^{\infty})]^3_<$ so

$$g(\{e_1, e_2, e_3 + s\})(1) = 1 = g(\{e_1, e_2, e_3 + q\})(1)$$

so

$$f(a) = f(e_1 + e_2 + e_3 + s) = f(e_1) = f(e_1 + e_2 + e_3 + q) = f(b).$$

Now assume $f(a) = f(b)$ and suppose the conclusion fails. Then by "(c) implies (d)" of Theorem 4.4 we have $\min F(\overrightarrow{z}, a) \neq F(\overrightarrow{z}, b)$. Therefore, without loss of generality we can assume that $t = \min F(\overrightarrow{z}, a) < \min F(\overrightarrow{z}, b)$.

Let $d = w(\overrightarrow{z}, a, t)$ and pick $e_1 \in S(\overrightarrow{x}, 3t)$, $e_2 \in S(\overrightarrow{x}, 3t+1)$, and $e_3 \in S(\overrightarrow{x}, 3t+s)$ with $d = e_1 + e_2 + e_3$. Also pick $s \in FS(< S(\overrightarrow{x}, l) >_{l=3t+3}^{\infty}) \cup \{0\}$ with $a = e_1 + e_2 + e_3 + s$. Then $\{e_1, e_2, e_3 + s\}$, $\{e_1, e_2, e_3 + b\}$, and $\{e_1 + e_2, e_3, b\}$ are all in $[FS(< S(\overrightarrow{x}, n) >_{n=1}^{\infty})]^3_<$ so $g(\{e_1, e_2, e_3 + s\})(1) = 1 = g(\{e_1, e_2, e_3 + b\})(1)$ and $g(\{e_1 + e_2, e_3, b\})(2) = 2$ so

$$f(a) = f(e_1 + e_2 + e_3 + s) = f(e_1) = f(e_1 + e_2 + e_3 + b) \neq f(b),$$

a contradiction. $\square$

The proof of the following lemma is very similar to the proof of Lemma 5.5 so we omit it.

5.6 Lemma. If $h(1) = 2$ and $h(2) = 1$, then there exists $\overrightarrow{y}$ refining $\overrightarrow{x}$ so that for all elements $a, b \in FS(< S(\overrightarrow{y}, n) >_{n=1}^{\infty})$ it is $f(a) = f(b)$ if and only if $\max F(\overrightarrow{y}, a) = \max F(\overrightarrow{y}, b) = n$ and $r(\overrightarrow{y}, a, n) = r(\overrightarrow{y}, b, n)$.

The next case we will consider is $h(1) = h(2) = 2$ and $h(3) = 1$. For this we need a preliminary lemma.

5.7 Lemma. If $h(1) = 2$, then there exists $\vec{y}$ refining $\vec{x}$ such that for all $a, b \in FS(< S(\vec{y}, n) >_{n=1}^{\infty})$, if $\max F(\vec{y}, a) < \max F(\vec{y}, b)$, then $f(a) < f(b)$.

Proof. We construct $\vec{y}$ inductively. Let $\vec{y}\ (1) = \vec{x}\ (1)$, let $n \in \mathbb{N}$, and assume we have chosen $\vec{y}\ (1), \vec{y}\ (2), \ldots, \vec{y}\ (n)$. Let $B = FS(< S(\vec{y}, j)_{s=1}^{n})$, let $t = \max\{f(a) : a \in B\}$, and let $l = \max\{\max F(\vec{x}, a) : a \in S(\vec{y}, n)\}$. Define $\varphi : FS(< S(\vec{x}, j) >_{j=1}^{\infty}) \to \{1, 2, 3\}$ as follows for each $b \in FS(< S(\vec{x}, j) >_{j=1}^{\infty})$.

(1) If $\min F(\vec{x}, b) \leq l$, then $\varphi(b) = 1$.

(2) If $\min F(\vec{x}, b) > l$ and there exists $a \in B \cup \{0\}$ with $f(a + b) \leq t$, then $\varphi(b) = 2$.

(3) If $\min F(\vec{x}, b) > l$ and for all $a \in B \cup \{0\}$, $f(a + b) > t$, then $\varphi(b) = 3$.

By Theorem 2.7 choose $\vec{z}$ refining $\vec{x}$ such that φ is constant on $FS(< S(\vec{z}, j) >_{j=1}^{\infty})$. We claim the constant value of φ on $FS(< S(\vec{z}, j) >_{j=1}^{\infty})$ is 3. Since $\vec{z}$ refines $\vec{x}$ it clearly cannot be 1 (because $\min F(\vec{x}, c(l + 1)\cdot \vec{z}\ (l + 1)(1)) \geq l + 1$). Suppose that the constant value is 2. Define a mapping $\tau : FS(< S(\vec{z}, j) >_{j=1}^{\infty}) \to B \cup \{0\}$ so that for each $b \in FS(< S(\vec{z}, j) >_{j=1}^{\infty})$ one has $f(\tau(b) + b) \leq t$ and pick by Theorem 2.7 some $\vec{u}$ refining $\vec{z}$ so that τ is constant on $FS(< S(\vec{u}, j) >_{j=1}^{\infty})$ and let $a \in B \cup \{0\}$ be this constant value. Now define $\gamma : FS(< S(\vec{u}, j)_{j=1}^{\infty}) \to \{1, 2, \ldots, t\}$ by $\gamma(b) = f(a + b)$. Applying Theorem 2.7 one more time we get some $\vec{v}$ refining $\vec{u}$ with γ constant on $FS(< S(\vec{v}, j) >_{j=1}^{\infty})$. Pick $d_1 \in S(\vec{v}, 1)$, $d_2 \in S(\vec{v}, 2)$, and $d_3 \in S(\vec{v}, 3)$. Then $\{a + d_1, d_2, d_3\} \in [FS(< S(\vec{x}, j) >_{j=1}^{\infty})]^3_<$ so $g(\{a + d_1, d_2, d_3\})(1) = 2$, so $f(a + d_1 + d_2 + d_3) \neq f(a + d_1)$. But d_1 and $d_1 + d_2 + d_3$ are in $FS(< S(\vec{v}, j) >_{j=1}^{\infty})$ so $f(a + d_1 + d_2 + d_3) = \gamma(d_1 + d_2 + d_3) = \gamma(d_1) = f(a + d_1)$, a contradiction.

Thus the constant value of φ on $FS(< S(\vec{z}, j) >_{j=1}^{\infty})$ is 3. Let $\vec{y}\ (n) = \vec{z}\ (n)$. □

5.8 Lemma. If $h(1) = h(2) = 2$ and $h(3) = 1$, then there is some $\vec{y}$ refining $\vec{x}$ such that for all $a, b \in FS(< S(\vec{y}, n) >_{n=1}^{\infty})$ it is $f(a) = f(b)$ if and only if $\min F(\vec{y}, a) = \min F(\vec{y}, b)$ and $r(\vec{y}, a, \min F(\vec{y}, a)) = r(\vec{y}, b, \min F(\vec{y}, a))$ and $\max F(\vec{y}, a) = \max F(\vec{y}, b)$ and $r(\vec{y}, a, \max F(\vec{y}, a)) = r(\vec{y}, b, \max F(\vec{y}, b))$.

Proof. By Lemma 5.7 we may presume that for all $a, b \in FS(< S(\vec{x}, n) >_{n=1}^{\infty})$, if $\max F(\vec{x}, a) < \max F(\vec{x}, b)$, then $f(a) < f(b)$. Define $\vec{u}\ (n)(i) = \frac{c(2n)}{c(n)} \cdot \vec{x}\ (2n)(i) + \frac{c(2n+1)}{c(n)} \cdot \vec{x}\ (2n + 1)(i)$ for $n \in \mathbb{N}$ and $i \in \{1, 2, \ldots, m(n)\}$. Then $\vec{u}$ refines $\vec{x}$. Choose $\vec{y}$ refining $\vec{u}$ as guaranteed by Theorem 4.4. We claim $\vec{y}$ is as required.

Let $a, b \in FS(< S(\vec{y}, n) >_{n=1}^{\infty})$ and assume first that $\min F(\vec{y}, a) = \min F(\vec{y}, b)$ and $r(\vec{y}, a, \min F(\vec{y}, a)) = r(\vec{y}, b, \min F(\vec{y}, a))$ and $\max F(\vec{y}, a) = \max F(\vec{y}, b)$ and $r(\vec{y}, a, \max F(\vec{y}, a)) = r(\vec{y}, b, \max F(\vec{y}, a))$. Then using "(d) implies (a)" and "(h) implies (e)" of Theorem 4.4 we have $\min F(\vec{u}, a) = \min F(\vec{u}, b)$ and $w(\vec{u}, a, \min F(\vec{u}, a)) = w(\vec{u}, b, \min F(\vec{u}, a))$ and $\max F(\vec{u}, a) = \max F(\vec{u}, b)$ and $w(\vec{u}, a, \min F(\vec{u}, a)) = w(\vec{u}, b, \min F(\vec{u}, a))$ and $\max F(\vec{u}, a) = \max F(\vec{u}, b)$ and $w(\vec{u}, a, \max F(\vec{u}, a)) = w(\vec{u}, b, \max F(\vec{u}, a))$. Next let $d = w(\vec{u}, a, \min F(\vec{u}, a))$ and let $e = w(\vec{u}, a, \max F(\vec{u}, a))$. If $d = e$, then $a = d = e = h$, so $f(a) = f(b)$. Thus we may assume $d \neq e$ and hence $\min F(\vec{u}, a) < \max F(\vec{u}, a)$. Let $l = \min F(\vec{u}, a)$ and $n = \max F(\vec{u}, a)$. Then pick q, $s \in FS(< S(\vec{u}, i) >_{i=l+1}^{n-1}) \cup \{0\}$ so that $a = d + q + e$ and $b = d + s + e$. Pick $v_1 \in S(\vec{x}, 2l), v_2 \in S(\vec{x}, 2l + 1)$, $v_3 \in S(\vec{x}, 2n)$ and $v_4 \in S(\vec{x}, 2n + 1)$ with $d = v_1 + v_2$ and $e = v_3 + v_4$. Then $\{v_1, v_2 + q + v_3, v_4\}$ and $\{v_1, v_2 + s + v_3, v_4\}$ are in $[FS(< S(\vec{x}, j) >_{j=1}^{\infty})]^3_<$ so $g(\{v_1, v_2 + q + v_3, v_4\}(2) = 1$ and $g\{v_1, v_2 + s + v_3, v_4\})(2) = 1$ so $f(a) = f(v_1 + v_2 + q + v_3 + v_4) = f(v_1 + v_4) = f(v_1 + v_2 + s + v_3 + v_4) = f(b)$.

For the other implication assume we have $f(a) = f(b)$. Then $\max F(\vec{x}, a) = \max F(\vec{x}, b)$ (since if we had, say $\max F(\vec{x}, a) < \max F(\vec{x}, b)$ we would have $f(a) < f(b)$). Consequently, we have $\max F(\vec{u}, a) = 2 \max F(\vec{x}, a) + 1 = \max F(\vec{u}, b)$. Then using "(g) implies (h)" of Theorem 4.4 we have $\max F(\vec{y}, a) = \max F(\vec{y}, b)$ and $r(\vec{y}, a, \max F(\vec{y}, a)) = r(\vec{y}, b, \max F(\vec{y}, b))$. Also using "(g) implies (e)" of Theorem 4.4 we have $w(\vec{u}, a, \max F(\vec{u}, a)) = w(\vec{u}, b, \max F(\vec{u}, a))$. Let $n = \max F(\vec{u}, a)$ and let $d = w(\vec{u}, a, n)$.

To complete the proof we use Theorem 4.4 "(c) implies (d)". It suffices to show $\min F(\vec{u}, a) = \min F(\vec{u}, b)$. Suppose not, and let without loss of generality $l = \min F(\vec{u}, a) < \min F(\vec{u}, b)$. Pick $s \in FS(< S(\vec{u}, j) >_{j=l+1}^{n-1}) \cup \{0\}$ such that $b = s + d$. Let $v = w(\vec{u}, a, l)$ and pick $q \in FS(< S(\vec{u}, j) >_{j=l+1}^{n-1}) \cup \{0\}$ such that $a = v + q + d$. Pick $e_1 \in FS(\vec{x}, 2l)$, $e_2 \in F(\vec{x}, 2l+1)$, $e_3 \in F(\vec{x}, 2n)$, and $e_4 \in F(\vec{x}, 2n+1)$ with $v = e_1 + e_2$ and $d = e_3 + e_4$. Then $\{e_1 + e_2, q + e_3, e_4\}$, $\{e_1 + e_2, s + e_3, e_4\}$, and $\{e_1, e_2, b\}$ are in $[FS(< S(\vec{x}, j) >_{j=1}^{\infty})]^3_<$ so $g(\{e_1 + e_2, s + e_3, e_4\})(3) = 1$, $g(\{e_1 + e_2, s + e_3, e_4\})(3) = 1$, and $g(\{e_1, e_2, b\})(2) = 2$. Then $f(b) \neq f(e_1 + e_2 + b) = f(e_1 + e_2 + s + e_3 + e_4) = f(e_1 + e_2 + e_4) = f(e_1 + e_2 + q + e_3 + e_4) = f(a)$, a contradiction. $\square$

We need two more preliminary lemmas before we can attack the case $h(1) = h(2) = h(3) = 2$.

5.9 Definition. Let $\vec{y} \in W$, let $k \in I\!N$, and let $a \in FS(< S(\vec{y}, n) >_{n=t}^{\infty})$.
$F(\vec{y}, a, k) = \{n \in F(\vec{y}, a) : n > k\}$.

5.10 Lemma. Assume $h(1) = h(2) = h(3) = 2$. There exists $\vec{y}$ refining $\vec{x}$ such that for all $a, b \in FS(< S(\vec{y}, n) >_{n=1}^{\infty})$, if there exists $k \in F(\vec{y}, a) \Delta F(\vec{y}, b)$ such that $\sum_{n \in F(\vec{y}, a, k)} w(\vec{y}, a, n) = \sum_{n \in F(\vec{y}, b, k)} w(\vec{y}, b, n)$, then $f(a) \neq f(b)$.

Proof. By Lemma 5.7 we may presume that for all $a, b \in FS(< S(\vec{x}, n) >_{n=1}^{\infty})$ with $\max F(\vec{x}, a) <$

$\max F(\vec{x}, b)$ one has $f(a) < f(b)$. We construct inductively $<\vec{y}\,(k)>_{k=1}^{\infty}$ and $<\vec{z}_k>_{k=0}^{\infty}$ so that

(1) $\vec{z}_0 = \vec{x}$

(2) $\vec{z}_k \in W$

(3) for $k > 0$, $\vec{z}_k$ refines $\vec{z}_{k-1}$

(4) for $k > 0$, $\min F(\vec{z}_{k-1}, \vec{z}_k\,(1)(1)) \geq k$

(5) for $k > 0$, $\vec{y}\,(k) = \vec{z}_k\,(k)$

(6) for $k > 0$, for all $v \in FS(< S(\vec{z}_{k,n}) >_{n=k+1}^{\infty})$ and all q, $s \in FS(< S(\vec{y}, n) >_{n=1}^{k}) \cup \{0\}$, if $\max F(\vec{y}, q) \neq \max F(\vec{y}, s)$, then $f(q+v) \neq f(s+v)$ (where "$\max F(\vec{y}, q) \neq \max F(\vec{y}, s)$" includes the possibility that say $q \neq 0$ and $s = 0$).

Observe that (3) and (4) combined tell us that $FS(< S(\vec{z}_k, n) >_{n=1}^{\infty}) \subseteq FS(< S(\vec{z}_{k-1}, n)_{n=k}^{\infty})$.

To ground the induction, let $\vec{z}_0 = \vec{x}$, let $\vec{y}\,(1)(1) = \vec{z}_1\,(1)(1) = \vec{z}_0\,(1)(1) + 2\,\vec{z}_0\,(2)(1)$ and for $n > 1$ and for $i \in \{1, 2, \ldots, m(n)\}$ let $\vec{z}_1\,(n)(i) = \frac{c(n+1)}{c(n)} \cdot \vec{z}_0\,(n+1)(i)$. Then hypotheses (1) through (5) follow immediately. To verify hypothesis (6), let $v \in FS(< S(\vec{z}_1, n) >_{n=2}^{\infty})$ and let $q, s \in FS(< S(\vec{y}, n) >_{n=1}^{1}) \cup \{0\}$ with $\max F(\vec{y}, q) \neq \max F(\vec{y}, s)$. Since $FS(< S(\vec{y}, n) >_{n=1}^{1}) = \{\vec{y}\,(1)(1)\}$, this means (without loss of generality) that $q = \vec{y}\,(1)(1)$ and $s = 0$. Now $q + v = \vec{x}\,(1)(1) + 2\,\vec{x}\,(2)(1) + v$ and $\{\vec{x}\,(1)(1), 2\,\vec{x}\,(2)(1), v\} \in [FS(< S(\vec{x}, n) >_{n=1}^{\infty}]_<^3$ so $g(\{\vec{x}\,(1)(1), 2\,\vec{x}\,(2)(1), v\})(2) = 2$ so $f(q+v) = f(\vec{x}\,(1)(1) + 2\,\vec{x}\,(2)(1) + v) \neq f(v) = f(s+v)$.

Assume the construction has continued through $k-1$. For the next step we define a coloring $\varphi : [FS(< S(\vec{z}_{k-1}, n) >_{n=k}^{\infty})]_<^2 \to \{1, 2\}$ as follows: for v_1, $v_2 \in FS(< S(\vec{z}_{k-1}, n) >_{n=k}^{\infty}$ with $\max F(\vec{z}_{k-1}, v_1) < \min F(\vec{z}_{k-1}, v_2) : \varphi(\{v_1, v_2\}) = 1$ if and only if there exist elements $s, q \in FS(< S(\vec{y}, n) >_{n=1}^{k-1}) \cup \{0\}$ with $f(s + v_2) = f(q + v_1 + v_2)$. By Theorem 5.2, pick $\vec{z}_k$ refining $\vec{z}_{k-1}$ so that φ is constant on $[FS(< S(\vec{z}_k, n) >_{n=1}^{\infty})]_<^2$. We claim that the constant value is 2.

Suppose instead the constant value is 1. Let $B = FS(<\vec{y}, n >_{n=1}^{k-1}) \cup \{0\}$ and define $\tau : [FS(< S(\vec{z}_k, n) >_{n=1}^{\infty})]_<^2 \to B \times B$ so that if $\tau(\{v_1, v_2\}) = (s, q)$ then $f(s+v_2) = f(q+v_1+v_2)$ (where of course $\max F(\vec{z}_k, v_1) < \min F(\vec{z}_k, v_2)$). Now pick $\vec{u}$ refining $\vec{z}_k$ so that τ is constant on the set $[FS(< S(\vec{u}, n) >_{n=1}^{\infty})]_<^2$ and let (s, q) be the constant value of τ. Pick $v_1, v_2, v_3 \in FS(< S(\vec{u}, n) >_{n=1}^{\infty})$ with $\max F(\vec{u}, v_1) < \min F(\vec{u}, v_2)$ and $\max F(\vec{u}, v_2) < \min F(\vec{u}, v_3)$. Then $\{v_1, v_3\}$ and $\{v_1 + v_2, v_3\}$ are in $[FS(< S(\vec{u}, n)) >_{n=1}^{\infty})]_<^2$ so $f(s + v_3) = f(q + v_1 + v_3)$ and $f(s + v_3) = f(q + v_1 + v_2 + v_3)$. But $\{q + v_1, v_2, v_3\} \in [FS(< S(\vec{x}, n) >_{n=1}^{\infty})]_<^3$ (which follows from hypotheses (1), (4) and (5)) so $g(\{q + v_1, v_2, v_3\})(3) = 2$ so $f(q + v_1 + v_2 + v_3) \neq f(q + v_1 + v_3)$, a contradiction.

Let $\overrightarrow{y}\ (k) = \overrightarrow{z}_k\ (k)$. Then hypotheses (1) through (5) are satisfied directly. For hypothesis (4) note only that φ was only defined on $[FS(< S(\overrightarrow{z}_{k-1}, n) >_{n=k}^{\infty}]^2_<$. To verify hypothesis (6) let $v \in FS(< S(\overrightarrow{z}_k, n) >_{n=k+1}^{\infty})$ and let $q, s \in FS(< S(\overrightarrow{y}, n) >_{n=1}^{k}) \cup \{0\}$ with $\max F(\overrightarrow{y}, s) \neq \max F(\overrightarrow{y}, q)$. Assume without loss of generality that $\max F(\overrightarrow{y}, s) < \max F(\overrightarrow{y}, q)$ (including the possibility that $s = 0$). If $\max F(\overrightarrow{y}, q) < k$, the result follows from hypothesis (6) at $k-1$, so we assume $\max F(\overrightarrow{y}, q) = k$. Let $q_2 = w(\overrightarrow{y}, q, k)$ and let $q_1 = q - q_2$. Then $q_1, s \in FS(< S(\overrightarrow{y}, n) >_{n=1}^{k-1}) \cup \{0\}$ and $\{q_2, v\} \in [FS(< S(\overrightarrow{z}_k, n) >_{n=k}^{\infty})]^2_<$ so $\varphi(\{q_2, v\}) = 2$ so $f(s+v) \neq f(q_1+q_2+v) = f(q+v) = f(q+v)$ as required.

The construction being complete, let $a, b \in FS(< S(\overrightarrow{y}, n) >_{n=1}^{\infty}$ and assume that we have $k \in F(\overrightarrow{y}, a)\Delta F(\overrightarrow{y}, b)$ with

$$\sum_{n \in F(\overrightarrow{y}, a, k)} w(\overrightarrow{y}, a, n) = \sum_{n \in F(\overrightarrow{y}, b, k)} w(\overrightarrow{y}, b, n).$$

Assume without loss of generality that $k \in F(\overrightarrow{y}, a) \backslash F(\overrightarrow{y}, b)$.

Let $v \in \sum_{n \in F(\overrightarrow{y}, a, k)} w(\overrightarrow{y}, a, n)$. If $v = 0$, then we have $k = \max F(\overrightarrow{y}, a) > \max F(\overrightarrow{y}, b)$ and so $\max F(\overrightarrow{x}, a) > \max F(\overrightarrow{x}, b)$ so $f(a) > f(b)$. Assume $v \neq 0$, let $q = a - v$ and let $s = b - v$. Then $v \in FS(< S(\overrightarrow{z}_k, n) >_{n=k+1}^{\infty})$, $q, s \in FS(< S(\overrightarrow{y}, n) >_{n=1}^{k}) \cup \{0\}$, and $k = \max F(\overrightarrow{y}, q) > \max F(\overrightarrow{y}, s)$ so $f(a) = f(q+v) \neq f(s+v) = f(b)$. □

5.11 Lemma.

Assume $h(1) = h(2) = h(3) = 2$. Then there exists $\overrightarrow{y}$ refining $\overrightarrow{x}$ such that for all elements $a, b \in FS(< S(\overrightarrow{y}, n) >_{n=1}^{\infty})$ if there exists $k \in I\!N$ with $\sum_{n \in F(\overrightarrow{y}, a, k)} w(\overrightarrow{y}, a, n) = \sum_{n \in F(\overrightarrow{y}, b, k)} w(\overrightarrow{y}, b, n)$ and either $k \in F(\overrightarrow{y}, a)\Delta F(\overrightarrow{y}, b)$ or $k \in F(\overrightarrow{y}, a) \cap F(\overrightarrow{y}, b)$ and $r(\overrightarrow{y}, a, k) \neq r(\overrightarrow{y}, b, k)$, then $f(a) \neq f(b)$.

Proof. By Lemma 5.10 we may presume that if there is some $k \in F(\overrightarrow{x}, a)\Delta F(\overrightarrow{x}, b)$ such that $\sum_{n \in F(\overrightarrow{x}, a, k)} w(\overrightarrow{x}, a, n) = \sum_{n \in F(\overrightarrow{x}, b, k)} w(\overrightarrow{y}, b, n)$ then $f(a) \neq f(b)$. We define $\overrightarrow{y}$ in a fashion similar to that done in the proof of Theorem 4.4 (except that we are not concerned here with minimums). Define a sequence $< \mu(n) >_{n=1}^{\infty}$ by $u(1) = 1$ and $\mu(n+1) = \mu(n) + m(n)$. For $a \in I\!N$ and $j \in \{1, 2, \ldots, m(n)\}$, let $\overrightarrow{y}\ (n)(j) = \sum_{t=\mu(n)}^{\mu(n)+j-1} \frac{c(t)}{c(n)} \cdot \overrightarrow{x}\ (t)(j)$. Thus as in the proof of Theorem 4.4 we have that $\overrightarrow{y}$ refines $\overrightarrow{x}$. Observe that for any $a \in FS(< S(\overrightarrow{y}, n) >_{n=1}^{\infty})$ and any $n \in F(\overrightarrow{y}, a)$ we have

$$w(\overrightarrow{y}, a, n) \quad = \quad \sum_{j=1}^{r(\overrightarrow{y}, a, n)} \lambda(\overrightarrow{y}, a, n)(j) \cdot \overrightarrow{y}\ (n)(j)$$

$$= \sum_{j=1}^{r(\vec{y},a,n)} \sum_{t=\mu(n)}^{\mu(n)+j-1} \lambda(\vec{y},a,n)(j) \cdot \frac{c(t)}{c(n)} \cdot \vec{x}\,(t)(j)$$

$$= \sum_{t=\mu(n)}^{\mu(n)+r(\vec{y},a,n)-1} \sum_{j=t-\mu(n)+1}^{r(\vec{y},a,n)} \lambda(\vec{y},a,n)(j) \cdot \frac{c(t)}{c(n)} \cdot \vec{x}\,(t)(j).$$

Thus given $t \in F(\vec{x},a)$, if $n \in I\!N$ with $\mu(n) \leq t < \mu(n+1)$, then it follows $w(\vec{x},a,t) = \sum_{j=t-\mu(n)+1}^{r(\vec{y},a,n)} \lambda(\vec{y},a,n)(j) \cdot \frac{c(t)}{c(n)} \cdot \vec{x}\,(t)(j)$.

Also

$$F(\vec{x},a) = \cup_{n \in F(\vec{y},a)} \{\mu(n), \mu(n)+1, \ldots, r(\vec{y},a,n)-1\}.$$

Now let a, b, and k be given as in the statement of the lemma. Assume without loss of generality that either $k \in F(\vec{y},a) \backslash F(\vec{y},b)$ or $k \in F(\vec{y},a) \cap F(\vec{y},b)$ and $r(\vec{y},a,k) > r(\vec{y},b,k)$. In either case, let $l = \mu(k) + r(\vec{y},a,k) - 1$. Then $l \in F(\vec{x},a) \backslash F(\vec{x},b)$. (If $k \notin F(\vec{y},b)$ then $F(\vec{x},b) \cap \{\mu(k), \mu(k)+1, \ldots, \mu(k+1)-1\} = \emptyset$. If $k \in F(\vec{y},b)$ and $r(\vec{y},b,k) < r(\vec{y},a,k)$, then $F(\vec{x},b) \cap \{\mu(k), \mu(k)+1, \ldots, \mu(k+1)-1\} = \{\mu(k), \mu(k)+1, \ldots, \mu(k)+r(\vec{y},b,k)-1\}$.)

Also

$$f(\vec{x},a,l) = \cup_{n \in F(\vec{y},a,k)} \{\mu(n), \mu(n)+1, \ldots, \mu(n)+r(\vec{y},a,n)-1\}$$

$$= \cup_{n \in F(\vec{y},b,k)} \{\mu(n), \mu(n)+1, \ldots, \mu(n)+r(\vec{y},b,n)-1\} = F(\vec{x},b,l)$$

so

$$\sum_{t \in F(\vec{x},a,l)} w(\vec{x},a,t) = \sum_{n \in F(\vec{y},a,k)} \sum_{t=\mu(n)}^{\mu(n)+r(\vec{y},b,n)-1} \sum_{j=t-\mu(n)+1}^{r(\vec{y},a,n)} \lambda(\vec{y},a,n)(j) \cdot \frac{c(t)}{c(n)} \cdot \vec{x}\,(t)(j)$$

$$= \sum_{n \in F(\vec{y},a,k)} \sum_{j=1}^{r(\vec{y},a,n)} \sum_{t=\mu(n)}^{\mu(n)+j-1} \lambda(\vec{y},a,n)(j) \cdot \frac{c(t)}{c(n)} \cdot \vec{x}\,(t)(j)$$

$$= \sum_{n \in F(\vec{y},n,k)} w(\vec{y},a,n) = \sum_{n \in F(\vec{y},b,k)} w(\vec{y},b,n)$$

$$= \sum_{t \in F(\vec{x},b,l)} w(\vec{x},b,t).$$

Hence $f(a) \neq f(b)$. $\square$

As our final preliminary we have:

5.12 Lemma. Assume $h(1) = h(2) = h(3) = 3$ and that for all $a, b \in FS(< S(\vec{x}, n) >_{n=1}^{\infty})$ if $a \approx_{\vec{x}} b$ then $f(a) = f(b)$. Then there exists $\vec{y}$ refining $\vec{x}$ such that for all $a, b \in FS(< S(\vec{y}, n) >_{n=1}^{\infty})$ $f(a) = f(b)$ if and only if $a \approx_{\vec{y}} b$.

Proof. Pick $\vec{y}$ as guaranteed by Lemma 5.11. One implication follows immediately from Theorem 4.3 so assume $a, b \in FS(< S(\vec{y}, n) >_{n=1}^{\infty})$ and $a \not\approx_{\vec{y}} b$. Pick k so that $F(\vec{y}, a, k) = F(\vec{y}, b, k)$ and for each $n \in F(\vec{y}, a, k)$, $r(\vec{y}, a, n) = r(\vec{y}, b, n)$ and either $k \in F(\vec{y}, a) \Delta F(\vec{y}, b)$ or $k \in F(\vec{y}, a) \cap F(\vec{y}, b)$ and $r(\vec{y}, a, k) \neq r(\vec{y}, b, k)$. If $F(\vec{y}, a, k) = \emptyset$, the conditions of Lemma 5.11 are satisfied and hence $f(a) \neq f(b)$. Thus we assume $F(\vec{y}, a, k) \neq \emptyset$.

Let $d = \sum_{n \in F(\vec{y}, a) \setminus F(\vec{y}, a, k)} w(\vec{y}, a, n) + \sum_{n \in F(\vec{y}, a, k)} w(\vec{y}, b, n)$. Then b and d satisfy the condition of Lemma 5.11 so $f(b) \neq f(d)$. Also $d \approx_{\vec{y}} a$ so $f(d) = f(a)$. Thus $f(b) \neq f(a)$ as required. $\square$

For the main result of this section we need to assume the validity of the following conjecture. The rationale for the conjecture comes from the theorem of Erdős and Graham [3]: If $f : \mathbb{N} \to \mathbb{N}$ there are arbitrarily long arithmetic progressions on which f is either one-to-one or constant.

5.13 Conjecture.

Given any $\vec{y} \in W$ and any coloring $\varphi : FS(< S(\vec{y}, n) >_{n=1}^{\infty}) \to \mathbb{N}$ there exists $\vec{z}$ refining $\vec{y}$ such that φ is either one-to-one on $FS(< S(\vec{z}, n) >_{n=1}^{\infty})$ or constant on $\approx_{\vec{z}}$-equivalence classes of $FS(< S(\vec{z}, n) >_{n=1}^{\infty})$.

5.14 Claim.

If Conjecture 5.13 is valid, then given $\vec{x} \in W$ and $f : FS(< S(\vec{x}, n) >_{n=1}^{\infty}) \to \mathbb{N}$, there exists $\vec{y}$ refining $\vec{x}$ such that one of the following six statements holds.

(a) For all $a, b \in FS(< S(\vec{y}, n) >_{n=1}^{\infty})$, $f(a) = f(b)$.

(b) For all $a, b \in FS(< S(\vec{y}, n) >_{n=1}^{\infty})$, $f(a) = f(b)$ if and only if $\min F(\vec{y}, a) = \min F(\vec{y}, b)$ and $r(\vec{y}, a, \min F(\vec{y}, a)) = r(\vec{y}, b, \min F(\vec{y}, b))$.

(c) For all $a, b \in FS(< S(\vec{y}, n) >_{n=1}^{\infty})$, $f(a) = f(b)$ if and only if $\max F(\vec{y}, a) = \max F(\vec{y}, b)$ and $r(\vec{y}, a, \max F(\vec{y}, a)) = r(\vec{y}, b, \max F(\vec{y}, b))$.

(d) For all $a, b \in FS(< S(\vec{y}, n) >_{n=1}^{\infty})$, $f(a) = f(b)$ if and only if $\min F(\vec{y}, a) = \min F(\vec{y}, b)$ and $r(\vec{y}, a, \min F(\vec{y}, a)) = r(\vec{y}, b, \min F(\vec{y}, b))$ and $\max F(\vec{y}, a) = \max F(\vec{y}, b)$ and also $r(\vec{y}, a, \max F(\vec{y}, a)) = r(\vec{y}, b, \max F(\vec{y}, b))$

(e) For all $a, b \in FS(< S(\vec{y}, n) >_{n=1}^{\infty})$, $f(a) = f(b)$ if and only if $a \approx_{\vec{y}} b$.

(f) For all $a, b \in FS(< S(\vec{y}, n) >_{n=1}^{\infty})$, $f(a) = f(b)$ if and only if $a = b$.

Proof. We are assuming g is constantly equal to h on $[FS(< S(\vec{x}, n) >_{n=1}^{\infty})]^3_<$. If $h(1) = h(2) = 1$, clause (a) applies by Lemma 5.4. If $h(1) = 1$ and $h(2) = 2$, clause (b) applies by Lemma 5.5. If $h(1) = 2$ and $h(2) = 1$, clause (c) applies by Lemma 5.6. If $h(1) = h(2) = 2$ and $h(3) = 1$, clause (d) applies by Lemma 5.8. Assume then that $h(1) = h(2) = h(3) = 2$. By the assumed validity of conjecture 5.13 we can assume that either f is one-to-one on $FS(< S(\vec{x}, n) >_{n=1}^{\infty})$, in which case clause (f) applies, or f is constant on the $\approx_{\vec{x}}$-equivalence classes in which case clause (e) applies by Lemma 5.12. □

5.15 Theorem. Regardless of the validity of Conjecture 5.13, none of the clauses of Claim 5.14 can be dispensed with.

Proof. If $\vec{x} \in W$ and f is defined on $FS(< S(\vec{x}, n) >_{n-1}^{\infty})$ in accordance with any one of the six clauses of Claim 5.14 and $\vec{y}$ is any refinement of $\vec{x}$, then the same clause applies to the set $FS(< S(\vec{y}, n) >_{n=1}^{\infty})$.

Indeed this is immediate for clauses (a) and (f) and Theorem 4.3 yields the result for clause (e). The other three are similar. We shall consider clause (c). We let $a, b \in FS(< S(\vec{y}, n) >_{n=1}^{\infty})$ and show that $\min F(\vec{y}, a) = \min F(\vec{y}, b)$ and $r(\vec{y}, a, \min F(\vec{y}, a)) = r(\vec{y}, b, \min F(\vec{y}, b))$ if and only if $\min F(\vec{x}, a) = \min F(\vec{x}, b)$ and $r(\vec{x}, a, \min F(\vec{x}, a)) = r(\vec{x}, b, \min F(\vec{x}, b))$. Assume first $n = \min F(\vec{y}, a) = \min F(\vec{y}, b)$ and $r(\vec{y}, a, n) = r(\vec{y}, b, n)$. By Theorem 4.2 (b) we have

$$\min F(\vec{x}, a) = \min F(\vec{x}, c(n) \cdot \vec{y}\ (n)(r(\vec{y}, a, n)))$$

$$= \min F(\vec{x}, c(n) \cdot \vec{y}\ (n)(r(\vec{y}, b, n))) = \min F(\vec{x}, b).$$

By Theorem 4.2 (c), if $t = \min F(\vec{x}, a)$, we have

$$r(\vec{x}, a, t) = r(\vec{x}, c(n) \cdot \vec{y}\ (n)(r(\vec{y}, a, n)), t) = r(\vec{x}, c(n) \cdot \vec{y}\ (n)(r(\vec{y}, b, n)), t) = r(\vec{x}, b, t).$$

Now assume $t = \min F(\vec{x}, a) = \min F(\vec{x}, b)$ and $r(\vec{x}, a, t) = r(\vec{x}, b, t)$. Pick $n \in F(\vec{y}, a)$ and $s \in F(\vec{y}, b)$ with

$$t \in F(\vec{x}, c(n) \cdot \vec{y}\ (n)(r(\vec{y}, a, n))) \cap F(\vec{x}, c(s) \cdot \vec{y}\ (s)(r(\vec{y}, a, s))).$$

Then $n = s$ since the intersection above is nonempty and in fact $n = \min F(\vec{y}, a) = \min F(\vec{y}, b)$. If one had say $r(\vec{y}, a, n) < r(\vec{y}, b, n)$ then Theorem 4.2 (c) and Lemma 3.5 (c) would yield

$$r(\vec{x}, a, t) = r(\vec{x}, c(n) \cdot \vec{y}\ (n)(r(\vec{y}, a, n)), t)$$

$$< r(\overrightarrow{x}, c(n) \cdot \overrightarrow{y}(n)(r(\overrightarrow{y}, b, n)), t) = r(\overrightarrow{x}, b, t)).$$

□

We close with an observation regarding our Conjecture 5.13. Assume as in Section 5 that we have g constantly equal to h on $[FS(< S(\overrightarrow{x}, n) >_{n=1}^{\infty})]^3_<$. By Lemmas 5.4, 5.5, 5.6, and 5.8 we have that f is constant on the $\approx_{\overrightarrow{x}}$-equivalence classes except possibly if $h(1) = h(2) = h(3) = 2$. In fact, Claim 5.14 is valid without the assumption of Conjecture 5.13 if and only if Conjecture 5.13 holds. The sufficiency was proved in the proof of Claim 5.14. On the other hand, if any of the first five clauses holds, then f is constant on the $\approx_{\overrightarrow{y}}$-equivalence classes.

6 References

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