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THE GRAHAM-ROTHSCHILD THEOREM AND THE ALGEBRA OF βW

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Abstract. In a previous paper we established an infinitary extension of the Graham-Rothschild Theorem by producing an infinite decreasing chain of idempotents in the Stone-Čech compactification of the set of variable words over a nonempty alphabet. In this paper we investigate further the algebraic structure of that compactification and determine which finite chains of idempotents are extendable to an infinite chain as above.

1. Introduction

Throught this paper A will denote a nonempty set (the *alphabet*). We write ω for the set $\{0, 1, 2, \ldots\}$ of finite ordinals and $\mathbb{N} = \omega \setminus \{0\}$. We choose a set $V = \{v_n : n \in \omega\}$ (of *variables*) such that $A \cap V = \emptyset$ and define W to be the semigroup of words over the alphabet $A \cup V$, including the empty word, with concatenation as the semigroup operation. (Formally a *word* w is a function from an initial segment $\{0, 1, \ldots, k-1\}$ of ω to the alphabet and the length $\ell(w)$ of w is k. We shall occasionally need to resort to this formal meaning, so

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that if $i \in \{0, 1, \ldots, \ell(w) - 1\}$, then $w(i)$ denotes the $(i+1)^{\text{st}}$ letter of w.)

For each $n \in \mathbb{N}$, we define W_n to be the set of words over the alphabet $A \cup \{v_0, v_1, \ldots, v_{n-1}\})$ and we define W_0 to be the set of words over A. We note that each W_n is a subsemigroup of W.

Definition 1.1. Let $n \in \mathbb{N}$, let $k \in \omega$ with $k \leq n$, and let $\emptyset \neq B \subseteq A$. Then $[B]\binom{n}{k}$ is the set of all words w over the alphabet $B \cup \{v_0, v_1, \ldots, v_{k-1}\}$ of length n such that

(1) for each $i \in \{0, 1, \ldots, k-1\}$, if any, v_i occurs in w and
(2) for each $i \in \{0, 1, \ldots, k-2\}$, if any, the first occurrence of v_i in w precedes the first occurrence of v_{i+1}.

Definition 1.2. Let $k \in \mathbb{N}$. Then the *set of k-variable words* is $S_k = \bigcup_{n=k}^{\infty} [A]\binom{n}{k}$. Also $S_0 = W_0$.

Given $w \in S_n$ and $u \in W$ with $\ell(u) = n$, we define $w\langle u \rangle$ to be the word with length $\ell(w)$ such that for $i \in \{0, 1, \ldots, \ell(w) - 1\}$

$$w\langle u \rangle(i) = \begin{cases} w(i) & \text{if } w(i) \in A \\ u(j) & \text{if } w(i) = v_j\,. \end{cases}$$

That is, $w\langle u \rangle$ is the result of substituting $u(j)$ for each occurrence of v_j in w.

The following theorem is commonly known as the Graham-Rothschild Theorem. The original theorem [4] (or see [7]) is stated in a significantly stronger fashion. However this stronger version is derivable from Theorem 1.3 in a reasonably straightforward manner. (See [3, Theorem 5.1].)

Theorem 1.3 (Graham-Rothschild)**.** *Assume that the alphabet A is finite, let $m, n \in \omega$ with $m < n$, and let S_m be finitely colored. There exists $w \in S_n$ such that $\left\{w\langle u \rangle : u \in [A]\binom{n}{m}\right\}$ is monochrome.*

In [3] we established a strong infinitary extension of the Graham-Rothschild Theorem by producing an infinite sequence of idempotents in βW, the Stone-Čech compactification of W. In order to discuss this, let us briefly review some facts about the Stone-Čech compactification βT of a (discrete) semigroup $(T, \cdot)$. We take the points of βT to be the ultrafilters on T, the principal ultrafilters being identified with the points of T. Given a set $A \subseteq T$, $\overline{A} = \{p \in \beta T : A \in p\}$. The set $\{\overline{A} : A \subseteq T\}$ is a basis for the open sets (as well as a basis for the closed sets) of βT. If $R \subseteq T$ we shall identify an ultrafilter p on R with the ultrafilter $\{A \subseteq T : A \cap R \in p\}$ and thereby pretend that $\beta R \subseteq \beta T$. We let $T^* = \beta T \setminus T$.

There is a natural extension of the operation $\cdot$ of T to βT making βT a compact right topological semigroup with T contained in its topological center. This says that for each $p \in \beta T$ the function $\rho_p : \beta T \to \beta T$ is continuous and for each $x \in T$, the function $\lambda_x : \beta T \to \beta T$ is continuous, where $\rho_p(q) = q \cdot p$ and $\lambda_x(q) = x \cdot q$. Given $B \subseteq T$ and $x \in T$, let $x^{-1}B = \{y \in T : x \cdot y \in B\}$. Then for any $p, q \in \beta T$ and any $B \subseteq T$, one has that $B \in p \cdot q$ if and only if $\{x \in T : x^{-1}B \in q\} \in p$. In particular, if $B \in p$ and $C \in q$, then $B \cdot C \in p \cdot q$. See [6] for an elementary introduction to the semigroup βT and for any unfamiliar algebraic facts encountered in this paper.

A subset U of a semigroup T is called a left ideal if it is nonempty and $TU \subseteq U$. It is called a right ideal if it is nonempty and $UT \subseteq U$. It is called a two-sided ideal, or simply an ideal, if it is both a left ideal and a right ideal. Any compact Hausdorff right topological semigroup T has a smallest two sided ideal $K(T)$ which is the union of all of the minimal left ideals of T and is also the union of all of the minimal right ideals of T. If $x \in K(T)$, then xT is the minimal right ideal with x as a member and Tx is the minimal left ideal with x as a member. The intersection of any minimal left ideal and any minimal right ideal is a group. Thus if p is a minimal idempotent in T, then p is the unique idempotent of T in $pT \cap Tp$. There is a partial ordering of the idempotents of T determined by $p \leq q$ if and only if $p = p \cdot q = q \cdot p$. An idempotent p is minimal with respect to this order if and only if $p \in K(T)$ [6, Theorem 1.59]. Such an idempotent is called simply "minimal". The intersection of any right ideal and any left ideal of T contains a minimal idempotent. We shall also frequently use the following fact [6, Theorem 1.65]: If T is a compact right topological semigroup, D is a compact subsemigroup of T, and $D \cap K(T) \neq \emptyset$, then $K(D) = D \cap K(T)$.

If $(T, \cdot)$ is a discrete semigroup, there is also a natural extension $*$ of the operation $\cdot$ to βT, for which $(\beta T, *)$ is a compact left topological semigroup. This means that, for each $x \in \beta T$, λ_x is continuous. The algebraic facts stated in the preceding paragraph are valid for compact left topological semigroups as well as compact right topological semigroups. For this reason, many of the results obtained in [3], as well as the present paper, are valid for $(\beta W, *)$, as well as $(\beta W, \cdot)$. This remark applies to [3, Theorem 2.12] and to Theorem 1.5, Theorem 1.14 and Theorem 2.3 in the present paper.

Definition 1.4. Let $u \in W$ with length n. Then $h_u : W \to W$ is the homomorphism such that, for all $w \in A \cup V$,

$$h_u(w) = \begin{cases} w & \text{if } w \in A \\ u(j) & \text{if } w = v_j \text{ and } j < n \\ w & \text{if } w = v_j \text{ and } j \geq n\,. \end{cases}$$

Notice that if $w \in S_n$, $u \in W$, and the length of u is n, then $h_u(w) = w\langle u \rangle$. Given $u \in W$, the function h_u has a continuous extension from βW to βW. We shall also denote this extension by h_u, and observe that $h_u : \beta W \to \beta W$ is a homomorphism. (See [6, Corollary 4.22].) We shall refer to the mappings h_u as *reductions*. If $u, w \in W$, we may call $h_u(w)$ a *reduction* of w.

The following theorem is a special case of the main algebraic result of [3]. It is this result that we used to establish infinitary extensions of Theorem 1.3.

Theorem 1.5. *Let p be a minimal idempotent in βS_0. There is a sequence $\langle p_n \rangle_{n=0}^{\infty}$ such that*

(1) *$p_0 = p$;*
(2) *for each $n \in \mathbb{N}$, p_n is a minimal idempotent of βS_n;*
(3) *for each $n \in \mathbb{N}$, $p_n \leq p_{n-1}$;*
(4) *for each $n \in \mathbb{N}$ and each $u \in [A]\binom{n}{n-1}$, $h_u(p_n) = p_{n-1}$.*

Further, p_1 can be any minimal idempotent of βS_1 such that $p_1 \leq p_0$ and p_2 can be any minimal idempotent of βW_2 such that $p_2 \in p_1 h_{v_1}(p_1)\beta W_2 \cap \beta W_2 h_{v_1}(p_1)p_1$.

Proof. This is [3, Theorem 2.12] in the case where $D = \{e\}$ and T_e is the identity. (The conclusion about p_2 is proved there, but not stated.) Or see the appendix to this paper for the proof of a stronger result. $\square$

The results of [3] suggest the importance of the relation $\prec$ which we now define.

Definition 1.6. The binary relation $\prec$ on $\bigcup_{n<\omega} \beta S_n$ is defined by $q \prec p$ if and only if there exist $m < n < \omega$ such that $q \in \beta S_m$, $p \in \beta S_n$, and $h_u(p) = q$ for all $u \in [A]\binom{n}{m}$.

One fairly easily establishes (using Lemma 1.11 below) that $\prec$ is transitive. In fact, ones sees (using Lemma 1.12) that $\prec$ is a tree (i.e., the set of predecessors of any element is linearly ordered). In [3], strong combinatorial consequences are drawn from the existence of certain kinds of infinite branches through $\prec$. In Section 3 of this paper we will characterize which ultrafilters lie on such branches and do the same for other kinds of branches. In addition we will consider other structural properties of $\prec$ such as the existence of maximal elements and branching degree.

Recall that the ordinal sum $1+\omega=\omega$.

Definition 1.7. Let $\alpha \in \omega \cup \{\omega\}$. Then $\langle p_i \rangle_{i<\alpha}$ is a *reductive sequence of length* α if and only if $p_i \in \beta S_i$ for each $i < \alpha$ and whenever $i < j < \alpha$ and $u \in [A]\binom{j}{i}$, $h_u(p_j) = p_i$. If in addition p_i is a minimal idempotent in βS_i for each $i < \alpha$ and $p_{i+1} < p_i$ whenever $i+1 < \alpha$, then $\langle p_i \rangle_{i<\alpha}$ is a *special reductive sequence*.

If $n < \omega$, $q \in \beta S_n$, $p \in \beta S_{n+1}$, and $h_u(p) = q$ for all $u \in [A]\binom{n+1}{n}$, then q is the *unique reduction* of p in βS_n.

Theorem 1.5 tells us that any minimal idempotent in βS_0 is a term of an infinite special reductive sequence, and that any minimal idempotent in βS_1 which is less than some minimal idempotent in βS_0 is also a term of an infinite special reductive sequence. It was shown in [3, Theorem 3.6] that there exist minimal idempotents in βS_1 that are not part of any reductive sequence of length greater than 2. As we have remarked above, we shall be concerned in Section 3 of this paper with the order relation $\prec$. In particular, we shall be concerned with determining which idempotents are terms of special infinite reductive sequences. The characterizations that we obtain are in terms of certain special subsemigroups of βS_n. We study those semigroups in Section 2.

We are working in this paper in a more restrictive setting than in [3]. (In the terminology of that paper, $D = E = \{e\}$, T_e is the identity, and for each $n < \omega$, $v_n = (e, \nu_n)$.) We do this primarily because the maps h_u as defined here are much easier to comprehend than their more general version as defined in [3].

We conclude this introduction with some preliminary results which will be used later.

Theorem 1.8. *Assume that the alphabet A is finite, let $m, n \in \omega$ with $m < n$, and let $r \in \mathbb{N}$. There exists $k \in \mathbb{N}$ such that $k > n$*

and whenever $[A]\binom{k}{m}$ *is* r*-colored, there exists* $w \in [A]\binom{k}{n}$ *such that* $\bigl\{w\langle u\rangle : u \in [A]\binom{n}{m}\bigr\}$ *is monochrome.*

Proof. This is a consequence of Theorem 1.3 by a standard compactness argument. (See [5, Section 1.5] or [6, Section 5.5].) □

Lemma 1.9. *Let* $m < n < \omega$ *and let* $u \in [A]\binom{n}{m}$*. Then* $h_u[K(\beta S_n)] \subseteq K(\beta S_m)$*.*

Proof. By Theorem 1.5, $h_u[\beta S_n] \cap K(\beta S_m) \neq \emptyset$ and thus

$$K(h_u[\beta S_n]) \subseteq h_u[\beta S_n] \cap K(\beta S_m)\,.$$

By [6, Exercise 1.7.3] $K(h_u[\beta S_n]) = h_u[K(\beta S_n)]$. □

We remark that if $m > 0$, the inclusion of Lemma 1.9 may be proper. To see this, pick $a \in A$ and let $u = av_0v_1\cdots v_{m-1}aa\cdots a$. Then $h_u[\beta S_n]$ misses the right ideal $v_0\beta S_m$ of βS_m.

Lemma 1.10. *Let* $r, s \in W$*, let* $k = \ell(r)$*, and let* $m = \ell(s)$*. If* $k \leq m$*, let* $u = h_r(s)$*. If* $k > m$*, let* $u = h_r(s)^\frown r(m)r(m+1)\cdots r(k-1)$*. Then* $h_u = h_r \circ h_s$*.*

Proof. It suffices to verify that $h_r\bigl(h_s(w)\bigr) = h_u(w)$ for every $w \in A \cup V$. Assume first that $k \leq m$. If $w \in A \cup \{v_j : j \geq m\}$, then $h_u(w) = w = h_r(w) = h_r\bigl(h_s(w)\bigr)$. If $w = v_j$ for some $j < m$, then $h_u(w) = u(j) = h_r\bigl(s(j)\bigr) = h_r\bigl(h_s(w)\bigr)$.

Now assume that $k > m$. If $w \in A \cup \{v_j : j \geq k\}$, then $h_u(w) = w = h_r(w) = h_r\bigl(h_s(w)\bigr)$. If $w = v_j$ for some j with $m \leq j < k$, then $h_u(w) = u(j) = r(j) = h_r(w) = h_r\bigl(h_s(w)\bigr)$. If $w = v_j$ for some $j < m$, then $h_u(w) = u(j) = h_r\bigl(s(j)\bigr) = h_r\bigl(h_s(w)\bigr)$. □

Lemma 1.11. *Let* $k < m < n < \omega$*, let* $r \in [A]\binom{m}{k}$*,* $s \in [A]\binom{n}{m}$*, and* $u \in [A]\binom{n}{k}$*. Then* $h_r \circ h_s = h_u$ *if and only if* $u = s\langle r\rangle$*.*

Proof. The sufficiency is a special case of Lemma 1.10. For the necessity, let $x = v_0v_1\cdots v_{n-1}$. Then $u = h_u(x) = h_r\bigl(h_s(x)\bigr) = h_r(s) = s\langle r\rangle$. □

Lemma 1.12. *Let* $k \leq m \leq n < \omega$ *and let* $u \in [A]\binom{n}{k}$*. Then there exist* $r \in [A]\binom{n}{m}$ *and* $s \in [A]\binom{m}{k}$ *such that* $u = r\langle s\rangle$*.*

Proof. If $m = k$ or $m = n$, the result is trivial, so we assume that $k < m < n$. We note that it suffices to establish the result under the additional assumption that $m = n-1$. (For then, using Lemma 1.11, one establishes the general result by induction on $n-m$.)

Either $u(j) \in A$ for some $j \in \{0,1,\ldots,n-1\}$ or else there exists $t \in \{0,1,\ldots,k-1\}$ such that v_t occurs more than once in u. In the second case, let t be the smallest index for which this happens. Then $u(t) = v_t$ and one may choose $j > t$ such that $u(j) = v_t$. In either case, we define r and s as follows for $i \in \{0,1,\ldots,n-1\}$ and $l \in \{0,1,\ldots,n-2\}$:

$$r(i) = \begin{cases} v_i & \text{if } i < j \\ u(j) & \text{if } i = j \\ v_{i-1} & \text{if } j < i \end{cases} \quad \text{and } s(l) = \begin{cases} u(l) & \text{if } l < j \\ u(l+1) & \text{if } j \le l\,. \end{cases}$$

It is routine to verify that $u = r\langle s\rangle$. □

Lemma 1.13. *Let $0 < m < n < \omega$ and let $u, u' \in [A]\binom{m}{m-1}$. There exist $w, w' \in [A]\binom{n}{m}$ such that $w\langle u\rangle = w'\langle u'\rangle$.*

Proof. There exist $i, j \in \{0,1,\ldots,m-1\}$, $t \in A \cup \{v_\delta : \delta < i\}$, and $s \in A \cup \{v_\delta : \delta < j\}$ such that for $l \in \{0,1,\ldots,m-1\}$,

$$u(l) = \begin{cases} v_l & \text{if } l < i \\ t & \text{if } l = i \\ v_{l-1} & \text{if } i < l \end{cases} \quad \text{and } u'(l) = \begin{cases} v_l & \text{if } l < j \\ s & \text{if } l = j \\ v_{l-1} & \text{if } j < l\,. \end{cases}$$

We may assume that $j \le i$. Pick $a \in A$ and for $l \in \{0,1,\ldots,n-1\}$, let

$$w(l) \quad = \quad \begin{cases} v_l & \text{if } l < j \\ s & \text{if } l = j \\ v_{l-1} & \text{if } j < l < m \\ a & \text{if } m \le l < n \end{cases} \quad \text{and}$$

$$w'(l) \quad = \quad \begin{cases} v_l & \text{if } l \le i \\ t & \text{if } l = i+1 \text{ and } t \in A \cup \{v_\delta : \delta < j\} \\ v_{\delta+1} & \text{if } l = i+1\,,\ t = v_\delta, \text{ and } j \le \delta < i \\ v_{l-1} & \text{if } i+1 < l < m \\ a & \text{if } m \le l < n\,. \end{cases}$$

It is routine to verify that w and w' are as required. □

We now state a theorem which is a significant extension of [3, Theorem 2.12]. The proof of this theorem, which we give in an appendix, is valid under the hypotheses used in [3], without the restrictions that $D = \{e\}$ or that T_e is the identity, which we introduced in the present paper.

Theorem 1.14. *Let X be a subsemigroup of βW such that $h_u[X] \subseteq X$ for every $u \in W$, $X \cap \beta W_n$ is compact and $X \cap \beta S_n$ is non-empty for every $n \in \omega$. Let p_0 be a minimal idempotent of $X \cap \beta W_0$ and $p_1 < p_0$ a minimal idempotent of $X \cap \beta W_1$. Then there is an infinite reductive sequence $(p_0, p_1, p_2, p_3, \ldots)$ such that p_n is a minimal idempotent of $X \cap \beta S_n$ and $p_{n+1} < p_n$ for every $n \in \omega$.*

Proof. The proof of [3, Theorem 2.12] provides a proof of this theorem, provided that βW is replaced by X, βW_n by $X \cap \beta W_n$ and βS_n by $X \cap \beta S_n$ for every $n \in \omega$. This includes defining $x \leq_R y$ and $x \leq_L y$ for $x, y \in X$ to mean that $x \in yX$ and $x \in Xy$ respectively, rather than $x \in y\beta W$ and $x \in \beta W y$. See the appendix to this paper for the details. $\square$

We observe that the algebraic results of the present paper have Ramsey theoretic applications, which will be the subject of a subsequent paper.

We should mention that Lemma 2.10 and Theorem 3.1 were proved in [2]. (See Lemma 7.1 and Claim 6 in §7 of [2].) We provide the proofs, however, because the terminology of [2] is significantly different from the terminology used in this paper.

2. Some subsemigroups of βS_n

Definition 2.1. Let $n \in \omega$.

$$\begin{aligned} C_n &= \big\{x \in \beta S_n : \ h_u(x) = h_{u'}(x) \text{ whenever } m < n \\ &\qquad\qquad \text{and } u, u' \in [A]\tbinom{n}{m}\big\} \\ GR_n &= \bigcap_{r>n} \bigcap \big\{h_u[C_r] : u \in [A]\tbinom{r}{n}\big\} \\ T_n &= \big\{x \in \beta S_n : \ (\forall r > n)(\exists y \in \beta S_r)\big(\forall u \in [A]\tbinom{r}{n}\big) \\ &\qquad\qquad (h_u(y) = x)\big\}\,. \end{aligned}$$

We shall see in Theorem 2.3 that the objects defined in Definition 2.1 are all subsemigroups of βS_n.

Lemma 2.2. *Let $m < n < \omega$ and let $u \in [A]\binom{n}{m}$. Then $h_u[C_n] \subseteq C_m$ and $h_u[GR_n] \subseteq GR_m$.*

Proof. The first assertion is an immediate consequence of Lemma 1.11. To verify the second assertion, let $y \in GR_n$ and let $x = h_u(y)$. Let $k \in \mathbb{N}$ with $k > m$ be given. We need to show that for each $w \in [A]\binom{k}{m}$, $x \in h_w[C_k]$.

Assume first that $k > n$. Choose any $q \in [A]\binom{k}{n}$ and pick $z \in C_k$ such that $y = h_q(z)$. Then $x = h_u\big(h_q(z)\big) = h_{q\langle u\rangle}(z)$ by Lemma 1.11. Given $w \in [A]\binom{k}{m}$, $h_w(z) = h_{q\langle u\rangle}(z)$ because $z \in C_k$.

Now assume that $k \leq n$. Pick by Lemma 1.12, $r \in [A]\binom{n}{k}$ and $s \in [A]\binom{k}{m}$ such that $h_u = h_s \circ h_r$. Then $x = h_s\big(h_r(y)\big)$ and $h_r(y) \in C_k$ by the first assertion in the current lemma, so for any $w \in [A]\binom{k}{m}$, $h_w\big(h_r(y)\big) = h_s\big(h_r(y)\big) = x$. $\square$

Theorem 2.3. *Let $n \in \omega$. Then GR_n, T_n, and C_n are subsemigroups of βS_n that meet the smallest ideal $K(\beta S_n)$ and $GR_n \subseteq T_n \subseteq C_n$.*

Proof. Pick by Theorem 1.5 an infinite special reductive sequence $\langle p_m\rangle_{m<\omega}$. For each $m < \omega$, $p_m \in GR_m \cap T_m \cap C_m \cap K(\beta S_m)$, so in particular each is nonempty. Also, for each $m < r < \omega$, and each $u \in [A]\binom{r}{m}$, $h_u[S_r] \subseteq S_m$, so $GR_n \subseteq \beta S_n$. Using the fact that h_u is a homomorphism for each $u \in W$, it is routine to verify that each of GR_n, T_n, and C_n is algebraically closed.

To see that $GR_n \subseteq T_n$, let $x \in GR_n$ and let $r > n$. Pick any $w \in [A]\binom{r}{n}$ and any $y \in C_r$ such that $x = h_w(y)$. Let $u \in [A]\binom{r}{n}$. Since $y \in C_r$, $h_u(y) = h_w(y) = x$.

Finally assume that $x \in T_n$ and suppose that $x \notin C_n$. Pick $m < n$ and $u, u' \in [A]\binom{n}{m}$ such that $h_u(x) \neq h_{u'}(x)$. Pick disjoint subsets Y and Y' of S_m such that $Y \in h_u(x)$ and $Y' \in h_{u'}(x)$. Let $X = {h_u}^{-1}[Y] \cap {h_{u'}}^{-1}[Y']$. Then $X \in x$.

Pick $z \in h_u[S_n] \cap h_{u'}[S_n]$. (We know this intersection is nonempty because it is a member of any member of T_m.) Pick w and w' in S_n such that $z = h_u(w) = h_{u'}(w')$. That is, $z = w\langle u\rangle = w'\langle u'\rangle$. This implies that w and w' have the same length, say k. Then $w, w' \in [A]\binom{k}{n}$. Since $x \in T_n$, pick $y \in \beta S_k$ such that $x = h_w(y) = h_{w'}(y)$. Then ${h_w}^{-1}[X] \cap {h_{w'}}^{-1}[X] \cap S_k \in y$ so pick $t \in S_k$ such that $h_w(t) \in X$ and $h_{w'}(t) \in X$. Then by Lemma 1.11,

$$h_u\big(h_w(t)\big) = h_{w\langle u\rangle}(t) = h_{w'\langle u'\rangle}(t) = h_{u'}\big(h_{w'}(t)\big)$$

so $Y \cap Y' \neq \emptyset$, a contradiction. $\square$

The fact that GR_n meets $K(\beta S_n)$ shows, surprisingly, that every element q of βS_n is a factor of an element in GR_n. More precisely, for every $p \in K(GR_n)$, p is a member of a minimal right ideal R and a minimal left ideal L of βS_n. Then $R = pq\beta S_n$ and $L = \beta S_n qp$ so $p = pqx = yqp$ for some $x, y \in \beta S_n$.

We shall see in Corollary 2.5 that the semigroups C_n have a simpler description than that given by their definition.

Theorem 2.4. *Let $m < n < \omega$, let $p \in \beta S_n$, and let $q \in \beta S_m$. If $\{h_u(p) : u \in [A]\binom{n}{m}\} = \{q\}$, then $q \in C_m$. In particular, if $k < m$, then $\{h_u(p) : u \in [A]\binom{n}{k}\}$ is also a singleton.*

Proof. We show by induction on $m - k$ that if $k < m$ and $u, u' \in [A]\binom{m}{k}$, then $h_u(q) = h_{u'}(q)$. So assume first that $k = m-1$ and let $u, u' \in [A]\binom{m}{m-1}$. By Lemma 1.13 we may choose $w, w' \in [A]\binom{n}{m}$ such that $w\langle u\rangle = w'\langle u'\rangle$. Then, using Lemma 1.11,

$$h_u(q) = h_u\big(h_w(p)\big) = h_{w\langle u\rangle}(p) = h_{w'\langle u'\rangle}(p) = h_{u'}\big(h'_w(p)\big) = h_{u'}(q)\,.$$

Now assume that $k < m-1$ and for all $u, u' \in [A]\binom{m}{k+1}$, $h_u(q) = h_{u'}(q)$. Let $u, u' \in [A]\binom{m}{k}$. Pick by Lemma 1.12 some $s, s' \in [A]\binom{m}{k+1}$ and $r, r' \in [A]\binom{k+1}{k}$ such that $u = s\langle r\rangle$ and $u' = s'\langle r'\rangle$. By Lemma 1.13 choose $w, w' \in [A]\binom{m}{k+1}$ such that $w\langle r\rangle = w'\langle r'\rangle$. Then, using Lemma 1.11, we have

$$\begin{aligned} h_u(q) &= h_r\big(h_s(q)\big) = h_r\big(h_w(q)\big) = h_{w\langle r\rangle}(q) \\ &= h_{w'\langle r'\rangle}(q) = h_{r'}\big(h_{w'}(q)\big) = h_{r'}\big(h_{s'}(q)\big) = h_{u'}(q)\,. \end{aligned}$$

The "in particular" conclusion now follows by Lemma 1.12. $\square$

Corollary 2.5. *Let $n \in \mathbb{N}$. Then*

$$\begin{aligned} C_n &= \{q \in \beta S_n : \ \text{there exists a reductive sequence} \\ &\qquad\qquad \langle p_m\rangle_{m<n+1} \text{ with } p_n = q\} \\ &= \{q \in \beta S_n : \ h_u(q) = h_{u'}(q) \text{ whenever } u, u' \in [A]\tbinom{n}{n-1}\}\,. \end{aligned}$$

Proof. It is an immediate consequence of Theorem 2.4 that $C_n = \{q \in \beta S_n : h_u(q) = h'_u(q)$ whenever $u, u' \in [A]\binom{n}{n-1}\}$. It is also immediate that $\{q \in \beta S_n :$ there exists a reductive sequence $\langle p_m\rangle_{m<n+1}$ with $p_n = q\} \subseteq C_n$. To establish the reverse inclusion, let $q \in C_n$. For each $m < n$ choose any $u_m \in [A]\binom{n}{m}$. Let $p_n = q$ and for $m < n$, let $p_m = h_{u_m}(q)$. To see that $\langle p_m\rangle_{m<n+1}$ is a reductive sequence, assume that $n > 1$, let $k < m < n$, and let $w \in [A]\binom{m}{k}$. Then by Lemma 1.11 $h_w(p_m) = h_w\big(h_{u_m}(q)\big) = h_{u_m\langle w\rangle}(q) = h_{u_k}(q) = p_k$. $\square$

We saw in Theorem 2.4 that if $\{h_u(p) : u \in [A]\binom{n}{m}\}$ is a singleton and $k < m$, then $\{h_u(p) : u \in [A]\binom{n}{k}\}$ is also a singleton. In terms of the relation $\prec$ of Definition 1.6, if p has a predecessor in βS_m, then it has a predecessor in βS_k for all $k < m$. We see now that this conclusion need not hold if $m < k < n$.

Theorem 2.6. *There exists an idempotent $p \in \beta S_3$ such that $\{h_u(p) : p \in [A]\binom{3}{1}\}$ is a singleton but $\{h_u(p) : p \in [A]\binom{3}{2}\}$ is not a singleton. So p has a predecessor with respect to the relation $\prec$ in βS_1, but not in βS_2.*

Proof. Let p_0 be a minimal idempotent in βS_0 and pick a minimal idempotent p_1 in βS_1 such that $p_1 \leq p_0$. Let $q_2 = h_{v_1}(p_1)$ and let $q_3 = h_{v_2}(p_1)$. Let B be the set of words over $A \cup \{v_1\}$ and let C be the set of words over $A \cup \{v_2\}$ and note that $B \in q_2$ and $C \in q_3$. Then $S_1BC \in p_1q_2q_3$ and $S_1BC \subseteq S_3$ so $p_1q_2q_3\beta S_3$ is a right ideal of βS_3. Similarly $v_0v_1CBS_1 \in v_0v_1q_3q_2p_1$ and $v_0v_1CBS_1 \subseteq S_3$ so $\beta S_1v_0v_1CBS_1$ is a left ideal of βS_1. Pick an idempotent $p_3 \in p_1q_2q_3\beta S_3 \cap \beta S_1v_0v_1CBS_1$. Pick $r, s \in \beta S_3$ such that $p_3 = p_1q_2q_3r = sv_0v_1q_3q_2p_1$.

Pick a letter $a \in A$. Then $v_0v_1a, v_0av_1 \in [A]\binom{3}{2}$. We show first that $h_{v_0v_1a}(p_3) \neq h_{v_0av_1}(p_3)$, using the fact that $p_3 = sv_0v_1q_3q_2p_1$. Now $h_{v_0v_1a}[S_3] \subseteq S_2$, $h_{v_0v_1a}(v_0) = v_0$, $h_{v_0v_1a}(v_1) = v_1$, $h_{v_0v_1a}[C] \subseteq S_0$, $h_{v_0v_1a}[B] \subseteq B$, and $h_{v_0v_1a}[S_1] \subseteq S_1$. Thus $S_2v_0v_1S_0BS_1 \in h_{v_0v_1a}(p_3)$. Also $h_{v_0av_1}[S_3] \subseteq S_2$, $h_{v_0av_1}(v_0) = v_0$, $h_{v_0av_1}(v_1) = a$, $h_{v_0av_1}[C] \subseteq B$, $h_{v_0av_1}[B] \subseteq S_0$, and $h_{v_0av_1}[S_1] \subseteq S_1$. Thus $S_2v_0aBS_0S_1 \in h_{v_0av_1}(p_3)$. Since $S_2v_0v_1S_0BS_1 \cap S_2v_0aBS_0S_1 = \emptyset$ we have that $h_{v_0v_1a}(p_3) \neq h_{v_0av_1}(p_3)$. (The displayed v_0 is the rightmost v_0 which has a later v_1. In one of these sets it is followed by v_1 while in the other it is followed by a.)

Now let $u \in [A]\binom{3}{1}$. If $u = v_0w$ for some $w \in S_0 \cup S_1$, then h_u is the identity on S_1 so $h_u(p_1) = p_1$ and therefore $h_u(p_3) = h_u(sv_0v_1q_3q_2)p_1 = p_1h_u(q_2q_3r)$ so $h_u(p_3) \leq p_1$ and thus $h_u(p_3) = p_1$.

Next assume that $u = bv_0t$ where $b \in A$ and $t \in A \cup \{v_0\}$. Then $h_u(p_1) \leq h_u(p_0) = p_0$ so $h_u(p_1) = p_0$. Also, using Lemma 1.11, $h_u(q_2) = h_{bv_0t}\big(h_{v_1}(p_1)\big) = h_{v_1\langle bv_0t\rangle}(p_1) = h_{v_0}(p_1) = p_1$. Thus

$h_u(p_3) = h_u(sv_0v_1q_3)p_1p_0 = p_0p_1h_u(q_3r) = h_u(sv_0v_1q_3)p_1 = p_1h_u(q_3r)$ so $h_u(p_3) = p_1$.

Finally assume that $u = bcv_0$ where $b, c \in A$. Then $h_u(p_1) \leq h_u(p_0) = p_0$ so $h_u(p_1) = p_0$. Also $h_u(q_2) = h_{bcv_0}\big(h_{v_1}(p_1)\big) = h_{v_1\langle bcv_0\rangle}(p_1) = h_c(p_1) \leq h_c(p_0) = p_0$ so $h_u(q_2) = p_0$. And $h_u(q_3) = h_{bcv_0}\big(h_{v_2}(p_1)\big) = h_{v_2\langle bcv_0\rangle}(p_1) = h_{v_0}(p_1) = p_1$ Thus $h_u(p_3) = h_u(sv_0v_1)p_1p_0p_0 = p_0p_0p_1h_u(r) = h_u(sv_0v_1)p_1 = p_1h_u(r)$ so $h_u(p_3) = p_1$. □

We now introduce a family which will help us establish that $GR_n = T_n$ for all $n \in \omega$. Given a set X, we write $\mathcal{P}_f(X) = \{B \subseteq X : B$ is finite and nonempty$\}$.

Definition 2.7. Let $n \in \omega$. Then

$$\begin{aligned}\mathcal{R}_n = \{X \subseteq S_n :\ & (\forall r > n)\big(\forall B \in \mathcal{P}_f(A)\big)(\exists w \in S_r) \\ & \big(\forall u \in [B]\tbinom{r}{n}\big)(h_u(w) \in X)\}\,.\end{aligned}$$

Lemma 2.8. *Let $n \in \omega$ and let $p \in \beta S_n$. Then $p \in T_n$ if and only if $p \subseteq \mathcal{R}_n$.*

Proof. Assume $p \in T_n$. To see that $p \subseteq \mathcal{R}_n$, let $X \in p$. Let $r > n$ and let $B \in \mathcal{P}_f(A)$. Pick $y \in \beta S_r$ such that $h_u(y) = p$ for all $u \in [A]\binom{r}{n}$. Then $\bigcap\{{h_u}^{-1}[X] : u \in [B]\binom{r}{n}\} \in y$ so pick $w \in \bigcap\{{h_u}^{-1}[X] : u \in [B]\binom{r}{n}\}$.

Conversely, suppose that $p \subseteq \mathcal{R}_n$ and let $r > n$. Let $\mathcal{Q} = \{(P, B) : P \in p$ and $B \in \mathcal{P}_f(A)\}$ and direct $\mathcal{Q}$ by agreeing that $(P, B) \leq (P', B')$ if and only if $P' \subseteq P$ and $B \subseteq B'$. Pick for each $(P, B) \in \mathcal{Q}$ some $w_{P,B} \in S_r$ such that $\big\{h_u(w_{P,B}) : u \in [B]\binom{r}{n}\big\} \subseteq P$. Let y be a limit point of the net $\langle w_{P,B}\rangle_{(P,B)\in\mathcal{Q}}$ in βS_r. Let $u \in [A]\binom{r}{n}$. We claim that $h_u(y) = p$. Suppose instead that we have some $P \in p \backslash h_u(y)$ and pick $B \in \mathcal{P}_f(A)$ such that $u \in [B]\binom{r}{n}$. Then ${h_u}^{-1}[S_n \setminus P] \in y$ so pick $(P', B') \in \mathcal{Q}$ such that $(P', B') \geq (P, B)$ and $w_{P',B'} \in {h_u}^{-1}[S_n \setminus P]$. Then $u \in [B']\binom{r}{n}$ and $h_u(w_{P',B}) \in P' \subseteq P$, a contradiction. So $p \in T_n$. □

Lemma 2.9. *Let $n \in \omega$ and let $X \in \mathcal{P}(S_n) \setminus \mathcal{R}_n$. Then*

$$\begin{aligned}&(\exists k > n)\big(\exists B \in \mathcal{P}_f(A)\big)(\forall r \geq k) \\ &(\forall w \in S_r)\big(\exists u \in [B]\tbinom{r}{n}\big)(h_u(w) \notin X)\,.\end{aligned}$$

Proof. By the definition of $\mathcal{R}_n$, pick $B \in \mathcal{P}_f(A)$ and $k > n$ such that $(\forall w \in S_k)\big(\exists u \in [B]\binom{k}{n}\big)(h_u(w) \notin X)$. Let $r \geq k$ and let $w \in S_r$.

Pick $a \in B$ and define $s \in [B]\binom{r}{k}$ by $s = v_0 v_1 \cdots v_{k-1} aa \cdots a$. Then $w\langle s\rangle \in S_k$ so pick $u \in [B]\binom{k}{n}$ such that $h_u(w\langle s\rangle) \notin X$. Then $s\langle u\rangle \in [B]\binom{r}{n}$ and, by Lemma 1.11, $h_{s\langle u\rangle}(w) = h_u\big(h_s(w)\big) = h_u(w\langle s\rangle) \notin X$. □

Lemma 2.10. *Let $X, Y \in \mathcal{P}(S_n)$. If $X \notin \mathcal{R}_n$ and $Y \notin \mathcal{R}_n$, then $X \cup Y \notin \mathcal{R}_n$.*

Proof. Pick by Lemma 2.9 some $B \in \mathcal{P}_f(A)$ and some $r > n$ such that

(1) $(\forall w \in S_r)\big(\exists u \in [B]\binom{r}{n}\big)(h_u(w) \notin X)$ and
(2) $(\forall w \in S_r)\big(\exists u \in [B]\binom{r}{n}\big)(h_u(w) \notin Y)$.

Pick by Theorem 1.8 some $k \in \mathbb{N}$ such that $k > r$ and whenever $[B]\binom{k}{n}$ is 2-colored, there exists $w \in [B]\binom{k}{r}$ such that $\big\{w\langle u\rangle : u \in [B]\binom{r}{n}\big\}$ is monochrome.

Suppose that $X \cup Y \in \mathcal{R}_n$ and pick $s \in S_k$ such that

$$\big(\forall t \in [B]\tbinom{k}{n}\big)(h_t(s) \in X \cup Y)\,.$$

That is, $\big\{s\langle t\rangle : t \in [B]\binom{k}{n}\big\} \subseteq X \cup Y$. Then the members t of $[B]\binom{k}{n}$ are 2-colored according to whether $s\langle t\rangle$ is in X or not, and if not, $s\langle t\rangle \in Y$. Pick $w \in [B]\binom{k}{r}$ such that either

$$\big\{s\big\langle w\langle u\rangle\big\rangle : u \in [B]\tbinom{r}{n}\big\} \subseteq X \text{ or}$$
$$\big\{s\big\langle w\langle u\rangle\big\rangle : u \in [B]\tbinom{r}{n}\big\} \subseteq Y.$$

We may assume without loss of generality that the former holds.

Now $s\langle w\rangle \in S_r$ so pick $u \in [B]\binom{r}{n}$ such that $h_u(s\langle w\rangle) \notin X$. But by Lemma 1.11,

$$h_u(s\langle w\rangle) = h_u\big(h_w(s)\big) = h_{w\langle u\rangle}(s) = s\big\langle w\langle u\rangle\big\rangle\,,$$

a contradiction. □

Lemma 2.11. *Let $n < r < \omega$ and let $u \in [A]\binom{r}{n}$. Then $T_n \subseteq h_u[T_r]$.*

Proof. Let $p \in T_n$ and let $\mathcal{F}$ be the filter generated by $\{{h_u}^{-1}[P] \cap S_r : P \in p\}$. We claim that $\mathcal{F} \subseteq \mathcal{R}_r$. To see this, let $P \in p$, let $k > r$, and let $B \in \mathcal{P}_f(A)$. We need to produce $x \in S_k$ such that $\big\{h_w(x) : w \in [B]\binom{k}{r}\big\} \subseteq {h_u}^{-1}[P]$.

Since $p \in T_n$, pick $z \in \beta S_k$ such that for all $l \in [A]\binom{k}{n}$, $h_l(z) = p$. Then

$$\bigcap\big\{{h_{w\langle u\rangle}}^{-1}[P] : w \in [B]\tbinom{k}{r}\big\} \in z$$

so pick $x \in S_k \cap \bigcap \{{h_{w\langle u\rangle}}^{-1}[P] : w \in [B]\binom{k}{r}\}$. Then given $w \in [B]\binom{k}{r}$, $h_u\big(h_w(x)\big) = h_{w\langle u\rangle}(x) \in P$.

Let $\mathcal{A} = \{\mathcal{H} \subseteq \mathcal{P}(S_r) : \mathcal{F} \subseteq \mathcal{H} \subseteq \mathcal{R}_r$ and $\mathcal{H}$ is a filter$\}$. Pick a maximal member q of $\mathcal{A}$. We claim that q is an ultrafilter. Suppose instead that we have some $X \subseteq S_r$ such that $X \notin q$ and $S_r \setminus X \notin q$. Then the filter generated by $q \cup \{X\}$ is not contained in $\mathcal{R}_r$ and the filter generated by $q \cup \{S_r \setminus X\}$ is not contained in $\mathcal{R}_r$. So pick $Q, R \in q$ such that $X \cap Q \notin \mathcal{R}_r$ and $R \setminus X \notin \mathcal{R}_r$. Then by Lemma 2.10 $(X \cap Q) \cup (R \setminus X) \notin \mathcal{R}_r$. But $Q \cap R \subseteq (X \cap Q) \cup (R \setminus X)$ and $Q \cap R \in \mathcal{R}_r$, a contradiction.

Since $\mathcal{F} \subseteq q$ we have that $h_u(q) = p$. By Lemma 2.8, $q \in T_r$. □

Theorem 2.12. *Let $n \in \omega$. Then $GR_n = T_n$.*

Proof. By Theorem 2.3 we have that $GR_n \subseteq T_n$. To establish the other inclusion, let $p \in T_n$. Let $r > n$ and let $u \in [A]\binom{r}{n}$. By Lemma 2.11 $p \in h_u[T_r]$, so by Theorem 2.3, $p \in h_u[C_r]$. □

In light of Theorems 2.3 and 2.12 it is natural to ask about the relationship between the semigroups C_n and T_n. Since $C_0 = \beta S_0$ it is not hard to show that $T_0 \neq C_0$. And we shall see in Corollary 3.17 and Theorem 3.18 that for each $n \geq 1$, $T_n \neq C_n$.

We see now that T_n has a rich algebraic structure.

Theorem 2.13. *Let $\kappa = |S_0|$. (So $\kappa = \max\{\omega, |A|\}$.) For each $n \in \omega$, T_n has 2^{2^κ} minimal left ideals and 2^{2^κ} minimal right ideals. Each minimal right ideal has 2^{2^κ} idempotents and each minimal left ideal has 2^{2^κ} idempotents.*

Proof. We have that βS_0 has 2^{2^κ} minimal left ideals and at least $2^{\mathfrak{c}}$ minimal right ideals by [6, Theorem 6.42 and Corollary 6.41]. We claim that in fact βS_0 has 2^{2^κ} minimal right ideals. If $|A| \leq \omega$, then $\mathfrak{c} = 2^\kappa$, so we may assume that $|A| > \omega$. Pick $a \in A$ and let S' be the set of words over $A \setminus \{a\}$. Then as is well known, $|\beta S'| = 2^{2^\kappa}$. (See, for example, [6, Theorem 3.58].) Given $x \neq y$ in $\beta S'$, one has that $xa\beta S_0$ and $ya\beta S_0$ are disjoint right ideals, each of which contains a minimal right ideal. (If $X \in x$, $Y \in y$, and $X \cap Y = \emptyset$, then $xa\beta S_0 \subseteq \overline{XaS_0}$, $ya\beta S_0 \subseteq \overline{YaS_0}$, and $XaS_0 \cap YaS_0 = \emptyset$.)

Note that if p is a minimal idempotent in βS_0 and $n \in \omega$, then $T_n \cap \beta S_n p \neq \emptyset$ and $T_n \cap p\beta S_n \neq \emptyset$. To see this, pick by Theorem 1.5 an infinite special reductive sequence $\langle p_m \rangle_{m<\omega}$ with $p_0 = p$. Then $p_n \in T_n \cap p\beta S_n \cap \beta S_n p$.

Now let p and q be members of distinct minimal left ideals of βS_0. We claim that $\beta S_n p \cap \beta S_n q = \emptyset$ (so that $T_n \cap \beta S_n p$ and $T_n \cap \beta S_n q$ are disjoint left ideals of T_n). Suppose instead one has some $x \in \beta S_n p \cap \beta S_n q$. Pick any $u \in [A]\binom{n}{0}$. Then h_u is the identity on βS_0 so $h_u(x) \in h_u[\beta S_n p] \cap h_u[\beta S_n q] \subseteq \beta S_0 p \cap \beta S_0 q$, a contradiction.

Since each left ideal contains a minimal left ideal, the first assertion is thus established. A similar argument establishes the assertion about the number of minimal right ideals. The conclusions about idempotents follow from the fact that the intersection of any minimal left ideal and any minimal right ideal has an idempotent. □

We now develop a method for establishing inequalities in βW by considering patterns of segments within words. This will be used in this section to establish the existence of large free groups in βW and in the next section to establish large branching degree in the $\prec$ tree.

Definition 2.14. Assume B is an alphabet and $c \in B$. Let S be the semigroup of words in B. For $w \in S$, a segment s of w is a *c-gap* of w if c does not occur in s and $w = w_1 csc w_2$ for some $w_1, w_2 \in S$. Suppose G is a group and let S' be the collection of words in which c does not occur. For any function $\mu : S' \to G$, define $\mu^+ : S \to G$ so that $\mu^+(w) = \mu(s_1) + \cdots + \mu(s_n)$ where $s_1, \ldots, s_n$ enumerates the c-gaps of w in the order they occur and $+$ denotes the group operation of G (if there are no c-gaps of w in X, $\mu^+(w)$ is the identity of G).

In the case where G is the set of integers mod n and μ is the characteristic function of some subset X of S', $\mu^+(w)$ counts the number of c-gaps of w which are in X mod n.

As usual, μ and μ^+ extend naturally to a function mapping $\beta S'$ and βS respectively into βG. In the cases that will interest us, G will be finite so that βG is the same as G. Notice that μ^+ will not generally be a homormorphism since multiplying two words together often creates a new c-gap which isn't in either of the individual words.

Definition 2.15. Assume B is some alphabet, $c \in B$ and S is the semigroup of words over B. For $w \in S$, define $\tau_c(w)$, the *tail of w*

with respect to c, to be the longest end segment of w which does not contain c and define $\eta_c(w)$, the *head of* w *with respect to* c, to be the longest initial segment of w which does not contain c. For $p \in \beta S$, c *persists in* p if the set of words containing c is in p.

Notice that for $p, q \in \beta S$, if c persists in q then $\eta_c(qp) = \eta_c(q)$ and $\tau_c(pq) = \tau_c(q)$. On the other hand, if c does not persist in q then $\eta_c(qp) = q\eta_c(p)$ and $\tau_c(pq) = \tau_c(p)q$.

Lemma 2.16. *Assume B is an alphabet, $c \in B$ and S is the semigroup of words over B. Also suppose G is a finite group with identity 0 and $\mu : S' \to G$ where S' is the set of words in which c does not occur. If $p, p', q, q' \in \beta S$ and c persists in p, p', q and q' then*

(a) $\mu^+(pq) = \mu^+(p) + \mu\big(\tau_c(p)\eta_c(q)\big) + \mu^+(q)$,
(b) *if p is an idempotent then* $\mu^+(p) = -\mu\big(\tau_c(p)\eta_c(p)\big)$,
(c) *if c does not persist in $x \in \beta S$ then* $\mu^+(px) = \mu^+(p) = \mu^+(xp)$.
(d) *if* $\eta(q) = \eta(q')$ *then* $\mu^+(pq) = \mu^+(pq')$ *iff* $\mu^+(q) = \mu^+(q')$.
(e) *if* $\tau(p) = \tau(p')$ *then* $\mu^+(pq) = \mu^+(p'q)$ *iff* $\mu^+(p) = \mu^+(p')$.

Proof. Parts (a) and (c) are straightforward. Part (b) follows from part (a) and parts (d) and (e) can each be derived using (a) and (c). □

Theorem 2.17. *Assume B is a nonempty alphabet and S is the semigroup of words over B. If $p, q \in \beta S$ then $p\beta Sq$ contains a free group on 2^{2^κ} generators where $\kappa = \max\{\omega, |B|\}$.*

Proof. Without loss of generality, p is a minimal idempotent and $p = q$. Note that c persists in p. If B has only one element, S is isomorphic to $\mathbb{N}$ and the lemma follows from Corollary 7.37 of [6]. Suppose B has more than one element and fix an element c of B. Let S' be the elements of S which have no occurence of c. S' has size κ, so there are 2^{2^κ} elements of $\beta S'$. We will show that the collection of $pcxcp$ where x is an element of $\beta S'$ and not equal to either $\tau_c(p)\eta_c(p)$, $\tau_c(p)$ or $\eta_c(p)$ generates a free group. For this, it suffices to show that any finite subcollection generates a free group.

Suppose $x_1, \ldots, x_n$ are distinct elements of $\beta S'$ which are distinct from $\tau_c(p)\eta_c(p)$, $\tau_c(p)$ and $\eta_c(p)$. Let F denote the free group on generators $a_1, a_2, \ldots, a_n$. Suppose that $x \in p\beta Sp$ can be written as $x = r_1 r_2 \cdots r_m$, where for each i, r_i is either pcx_jcp or the inverse of

pcx_jcp in $p\beta Sp$ for some j. Define $b \in F$ by $b = b_1b_2\cdots b_m$, where $b_i = a_j$ if $r_i = pcx_jcp$ and $b_i = a_j^{-1}$ if r_i is the inverse of pcx_jp in $p\beta Sp$. We shall show that $x \neq p$ if b is not the identity of F. In this case, there is a homorphism f mapping F to a finite group G for which $f(b)$ is not equal to the identity by [6, Theorem 1.23].

Define $\mu : S' \to G$ by $\mu(s) = f(a_i)$ if $s \in X_i$ and $\mu(s)$ is the identity if $s \notin \bigcup_{i=1}^n X_i$. Then μ^+ is a homomorphism on $p\beta Sp$ by Lemma 2.16(a). Since $\mu^+(pcx_icp) = f(a_i)$ for each $i \in \{1, 2, \cdots, n\}$, $\mu^+(x) = f(b)$. So $x \neq p$. □

Theorem 2.18. *For each $n \in \omega$, every maximal group in $K(T_n)$ contains a free group on 2^{2^κ} generators where $\kappa = \max\{|A|, \omega\}$.*

Proof. Let p_0 be a minimal idempotent in βW_0. By Theorem 2.17 we may let $\{p_0x_\iota p_0 : \iota < 2^{2^\kappa}\}$ be a set of elements in $p_0\beta W_0p_0$ which generate a free group in $p_0\beta W_0p_0$. We can choose by Theorem 1.5 a minimal idempotent p_n in T_n satisfying $p_0 \prec p_n$. Then $\{p_nx_\iota p_n : \iota < 2^{2^\kappa}\} \subseteq T_n$ generates a free group in $p_nT_np_n$, because any reduction h_u for which $u \in [A]\binom{n}{0}$ is a homomorphism mapping each $p_nx_\iota p_n$ to $p_0x_\iota p_0$. It follows from [6, Theorem 2.11] that every maximal group in $K(\beta T_n)$ contains a free group on 2^{2^κ} generators. □

We now set out to characterize the members of T_m in terms of their members.

Definition 2.19. Let $m < n < \omega$, let φ be a finite coloring of S_n, and let $B \in \mathcal{P}_f(A)$. Then

$$E_{m,n,\varphi,B} = \Big\{s \in S_m : \begin{array}{l}(\exists\tau : [B]\tbinom{n}{m} \to S_n)\big(\varphi\circ\tau \text{ is constant and}\\ \big(\forall u \in [B]\tbinom{n}{m}\big)\big(h_u\big(\tau(u)\big) = s\big)\big)\Big\}.\end{array}$$

Theorem 2.20. *Let $m \in \omega$ and let $p \in \beta S_m$. Given $n > m$, there exists $q \in \beta S_n$ such that $h_u(q) = p$ for all $u \in [A]\binom{n}{m}$ if and only if for every finite coloring φ of S_n and every $B \in \mathcal{P}_f(A)$, $E_{m,n,\varphi,B} \in p$. In particular, $p \in T_m$ if and only if for every $n > m$, every finite coloring φ of S_n and every $B \in \mathcal{P}_f(A)$, $E_{m,n,\varphi,B} \in p$.*

Proof. It suffices to establish the first conclusion. So let $n > m$.

Necessity. Let φ be a finite coloring of S_n and let $B \in \mathcal{P}_f(A)$. Pick $q \in \beta S_n$ such that $h_u(q) = p$ for all $u \in [A]\binom{n}{m}$. Pick $Q \in q$ on which φ is constant. Then $\bigcap\big\{h_u[Q] : u \in [B]\binom{n}{m}\big\} \in p$ and $\bigcap\big\{h_u[Q] : u \in [B]\binom{n}{m}\big\} \subseteq E_{m,n,\varphi,B}$.

Sufficiency. For each $B \in \mathcal{P}_f(A)$, let $D_B = \bigcap \big\{\beta S_n \cap {h_u}^{-1}[\{p\}] : u \in [B]\binom{n}{m}\big\}$. We claim that each $D + B \neq \emptyset$. So suppose instead that we have $B \in \mathcal{P}_f(A)$ such that $D_B = \emptyset$. For each $x \in \beta S_n$ choose $u_x \in [B]\binom{n}{m}$ such that $h_{u_x}(x) \neq p$ and pick $X_x \in x$ such that $h_{u_x}[X_x] \notin p$. Then $\{\overline{X_x} : x \in \beta S_n\}$ is an open cover of βS_n so pick finite $F \subseteq \beta S_n$ such that $\beta S_n = \bigcup_{x \in F} \overline{X_x}$. For each $y \in S_n$ choose $\varphi(y) \in F$ such that $y \in X_{\varphi(y)}$. Then φ is a finite coloring of S_n so $E_{m,n,\varphi,B} \in p$. Pick $s \in E_{m,n,\varphi,B} \setminus \bigcup_{x \in F} h_{u_x}[X_x]$ and pick $\tau : [B]\binom{n}{m} \to S_n$ such that $\varphi \circ \tau$ is constant and for all $u \in [B]\binom{n}{m}$, $h_u\big(\tau(u)\big) = s$. Let $x \in F$ be the constant value of $\varphi \circ \tau$. Then $h_{u_x}\big(\tau(u_x)\big) = s$ and $\tau(u_x) = X_{\varphi\left(\tau(u_x)\right)} = X_x$, so $s \in h_{u_x}[X_x]$, a contradiction.

If $B \subseteq C$, then $D_C \subseteq D_B$ so $\{D_B : B \in \mathcal{P}_f(A)\}$ is a set of closed subsets of βS_n with the finite intersection property so choose $q \in \bigcap_{B \in \mathcal{P}_f(A)} D_B$. Then for each $u \in [A]\binom{n}{m}$, $h_u(q) = p$. □

The reductions h_u are also continuous homomorphisms from $(\beta W, *)$ to itself, where $*$ denotes the natural extension of the semigroup operation from W to βW for which βW is left topological. The subsets C_n, GR_n and T_n of βW do not depend on which semigroup operation on βW is being used. These sets are compact subsemigroups of $(\beta W, *)$ as well as $(\beta W, \cdot)$.

As we remarked in the introduction, Theorem 1.14 is valid for $(\beta W, *)$ as well as $(\beta W, \cdot)$, because it depends only on algebraic propties which hold in compact left topological semigroups as well as compact right topological semigroups. Thus infinite special reductive sequences also exist in $(\beta W, *)$. These are reductive sequences in $(\beta W, \cdot)$ as well, but are are far from being special reductive sequences in $(\beta W, \cdot)$. It was shown in the proof of [1, Theorem 3.13] that, if S denotes the free semigroup over an alphabet with two letters and if p is a minimal idempotent in $(\beta S, *)$, then $p \notin \beta S \cdot p$. This statement can be extended to the free semigroup over any alphabet with more than one letter, by applying a homomorphism which reduces the number of letters to two. So, if $n \in \mathbb{N}$, a minimal idempotent in $(\beta W_n, *)$ is not an idempotent in $(\beta W_n, \cdot)$ and is not in $K(\beta W_n, \cdot)$. In fact, it can be shown that it is right cancelable in $(\beta W_n, \cdot)$ if A is countable.

3. Extending reductive sequences

Our first objective is to determine those elements of βS_n which are part of infinite reductive sequences.

Lemma 3.1. *Let $0 < n < \omega$, let p_n be an idempotent in T_n, and let p_{n-1} be the unique reduction of p_n in βS_{n-1}. If $p_n < p_{n-1}$, then there is an idempotent $p_{n+1} \in T_{n+1}$ such that $p_{n+1} < p_n$ and p_n is the unique reduction of p_{n+1} in βS_n.*

Proof. We show first that

(∗) if s is an idempotent in T_{n+1} such that $h_u(s) = p_n$ for all $u \in [A]\binom{n+1}{n}$, then $h_u(sp_n) = p_n = h_u(p_n s)$ for all $u \in [A]\binom{n+1}{n}$ and for every $k \geq n+1$ there exist $x_k, y_k \in \beta S_k$ such that $h_u(x_k) = sp_n$ and $h_u(y_k) = p_n s$ for all $u \in [A]\binom{k}{n+1}$.

To establish the first assertion, let $u \in [A]\binom{n+1}{n}$ and let $w = u_{|n}$. Then $h_u(p_n) = h_w(p_n)$. If $w \in [A]\binom{n}{n}$, then $h_w(p_n) = p_n$ so $h_u(sp_n) = p_n p_n = p_n = h_u(p_n s)$. If $w \in [A]\binom{n}{n-1}$, then $h_w(p_n) = p_{n-1}$ so $h_u(sp_n) = p_n p_{n-1} = p_n = p_{n-1} p_n = h_u(p_n s)$.

We establish the second assertion by induction on k. If $k = n+1$, let $x_k = sp_n$ and let $y_k = p_n s$. If $u \in [A]\binom{n+1}{n+1}$, then h_u is the identity on βS_{n+1} so $h_u(x_k) = sp_n$ and $h_u(y_k) = p_n s$.

Now assume that $k > n+1$ and the statement is true for $k-1$. Since $s \in T_{n+1}$ pick $z \in \beta S_k$ such that $h_u(z) = s$ for all $u \in [A]\binom{k}{n+1}$. By the induction hypothesis pick $x_{k-1}, y_{k-1} \in \beta S_{k-1}$ such that $h_u(x_{k-1}) = sp_n$ and $h_u(y_{k-1}) = p_n s$ for all $u \in [A]\binom{k-1}{n+1}$. Let $u \in [A]\binom{k}{n+1}$ and let $w = u_{|k-1}$. Then $h_u(x_{k-1}) = h_w(x_{k-1})$. If $w \in [A]\binom{k-1}{n+1}$, then $h_w(x_{k-1}) = sp_n$ and $h_w(y_{k-1}) = p_n s$ so $h_u(x_k) = ssp_n = sp_n$ and $h_u(y_k) = p_n ss = p_n s$. So assume that $w \in [A]\binom{k-1}{n}$. Pick by Lemma 1.12 some $u_1 \in [A]\binom{k-1}{n+1}$ and $u_2 \in [A]\binom{n+1}{n}$ such that $w = u_1\langle u_2\rangle$. Then $h_w(x_{k-1}) = h_{u_2}\big(h_{u_1}(x_{k-1})\big) = h_{u_2}(sp_n) = p_n$ and $h_w(y_{k-1}) = h_{u_2}\big(h_{u_1}(y_{k-1})\big) = h_{u_2}(p_n s) = p_n$. Thus $h_u(x_k) = sp_n$ and $h_u(y_k) = p_n s$. Thus (∗) is established.

Now by Lemma 2.11 we have that

$$\big\{s \in T_{n+1} : \big(\forall u \in [A]\tbinom{n+1}{n}\big)(h_u(s) = p_n)\big\} \neq \emptyset\,.$$

(If $s \in T_{n+1}$, then $s \in C_{n+1}$.) So this set is a compact subsemigroup of βS_{n+1} so we may pick an idempotent $s \in T_{n+1}$ such that $h_u(s) = p_n$ for all $u \in [A]\binom{n+1}{n}$. Then by $(*)$, $sp_n \in T_{n+1}$. So

$$sp_n \in T_{n+1} \cap \bigcap \left\{{h_u}^{-1}[\{p_n\}] : u \in [A]\tbinom{n+1}{n}\right\} \cap \beta S_{n+1}p_n$$

and thus this set is a compact subsemigroup of βS_{n+1}. Pick an idempotent

$$q \in T_{n+1} \cap \bigcap \left\{{h_u}^{-1}[\{p_n\}] : u \in [A]\tbinom{n+1}{n}\right\} \cap \beta S_{n+1}p_n$$

and note that $qp_n = q$ because $q \in \beta S_{n+1}p_n$.

Then by $(*)$, $p_nq \in T_{n+1}$ and $h_u(p_nq) = p_n$ for all $u \in [A]\binom{n+1}{n}$. Let $p_{n+1} = p_nq$. Then $p_{n+1}p_{n+1} = p_nqp_nq = p_nqq = p_nq = p_{n+1}$, $p_{n+1}p_n = p_nqp_n = p_nq = p_{n+1}$, and $p_np_{n+1} = p_np_nq = p_nq = p_{n+1}$. □

Theorem 3.2. *Let $n < \omega$ and let p_n be an idempotent in βS_n. Then p_n is a term of an infinite reductive sequence consisting of idempotents for which $p_{k+1} < p_k$ for each $k < \omega$ if and only if $p_n \in T_n$ and either $n = 0$ or $p_n < p_{n-1}$, where p_{n-1} is the unique reduction of p_n in βS_{n-1}.*

Proof. The necessity is trivial. For the sufficiency, assume first that $n > 0$, $p_n \in T_n$, and $p_n < p_{n-1}$. Pick $a \in A$. Inductively, for $k \in \{0, 1, \ldots, n-2\}$, if any, assume that p_{k+1} is an idempotent in T_{k+1} with $p_{k+2} < p_{k+1}$. Let $u = av_0v_1 \cdots v_{k-1} \in [A]\binom{k+1}{k}$ and let $p_k = h_u(p_{k+1})$. Then p_k is an idempotent which is the unique reduction of p_{k+1} in βS_k. To see that $p_k \in T_k$, let $r > k+1$ and choose $q \in \beta S_r$ such that $h_w(q) = p_{k+1}$ for all $w \in [A]\binom{r}{k+1}$. Let $x \in [A]\binom{r}{k}$ and pick by Lemma 1.12 $w \in [A]\binom{r}{k+1}$ and $s \in [A]\binom{k+1}{k}$ such that $x = w\langle s\rangle$. Then by Lemma 1.11 $h_x(q) = h_s\big(h_w(q)\big) = h_s(p_{k+1}) = p_k$.

Let $w = av_0v_1 \cdots v_k$ and note that $h_w(p_{k+1}) = h_u(p_{k+1})$ so $p_k = h_u(p_{k+1}) = h_w(p_{k+1}) > h_w(p_{k+2}) = p_{k+1}$. Thus we have $\langle p_k\rangle_{k=0}^n$ is a reductive sequence consisting of idempotents such that $p_k \in T_k$ for all $k \leq n$ and $p_k < p_{k+1}$ for all $k < n$.

Now let $m \geq n$ and assume that $\langle p_k\rangle_{k=0}^m$ is a reductive sequence consisting of idempotents such that $p_k \in T_k$ for all $k \leq n$ and $p_k < p_{k+1}$ for all $k < m$. By Lemma 3.1, pick an idempotent $p_{m+1} \in T_{m+1}$ such that $p_{m+1} < p_m$ and p_m is the unique reduction of p_{m+1} in βS_m.

Now assume that $n = 0$ and $p_0 \in T_0$. We claim that $T_1 p_0 \subseteq T_1$. Certainly $\beta S_1 p_0 \subseteq \beta S_1$. Let $q \in T_1$ and let $r > 1$. Pick $y \in \beta S_r$ such that for all $u \in [A]\binom{r}{1}$, $h_u(y) = q$. Then for all $u \in [A]\binom{r}{1}$, $h_u(p_0) = p_0$ and so $h_u(yp_0) = qp_0$ as required.

Pick $a \in A$ and pick by Lemma 2.11 $q \in T_1$ such that $h_a(q) = p_0$. Then $q \in C_1$ by Theorem 2.3, so $h_c(q) = p_0$ for all $c \in A$ and thus

$$qp_0 \in T_1 p_0 \cap \bigcap_{c \in A} {h_c}^{-1}[\{p_0\}] .$$

Pick an idempotent $r \in T_1 p_0 \cap \bigcap_{c \in A} {h_c}^{-1}[\{p_0\}]$ and let $p_1 = p_0 r$. Then p_1 is an idempotent in T_1 and $p_1 < p_0$ so the already established case where $n = 1$ applies. $\square$

We now see that the requirement of Theorem 3.2 that p_n be a member of T_n can be weakened in the case in which $n = 1$.

Theorem 3.3. *Let p_1 be an idempotent in C_1. If $p_1 < p_0$, where p_0 denotes the unique reduction of p_1 in βW_0, then there is an infinite reductive sequence $\langle p_0, p_1, p_2, p_3, \ldots \rangle$ consisting of idempotents, such that $p_{n+1} < p_n$ for every $n \in \omega$.*

Proof. By Theorem 3.2 it is enough to show that $p_1 \in T_1$. Given $n > 1$, put $q_n = h_{v_0}(p_1) h_{v_1}(p_1) \cdots h_{v_{n-1}}(p_1)$. Then $q_n \in \beta S_n$. Let $u \in [A]\binom{n}{1}$. We claim that $h_u(q_n) = p_1$. To see this note that if $a \in A$ and $m \in \{0, 1, \ldots, n-1\}$, then $h_u\big(h_{v_m}(a)\big) = a$ while $h_u\big(h_{v_m}(v_0)\big) = u(m)$. Thus if $m \in \{0, 1, \ldots, n-1\}$ and $w \in S_1$, then $h_u\big(h_{v_m}(w)\big) = h_{u(m)}(w)$. Therefore, if $u(m) \in A$, then $h_u\big(h_{v_m}(p_1)\big) = p_0$, while if $u(m) = v_0$, then $h_u\big(h_{v_m}(p_1)\big) = p_1$. Since there is at least one $m \in \{0, 1, \ldots, n-1\}$ for which $u(m) = v_0$, we have $h_u(q_n) = p_1$. So $p_1 \in T_1$. $\square$

Theorem 3.4. *Let $n < \omega$ and let p_n be a minimal idempotent in βS_n. Then p_n is a term of an infinite special reductive sequence if and only if either $n = 0$ or $p_n \in T_n$ and $p_n < p_{n-1}$, where p_{n-1} is the unique reduction of p_n in βS_{n-1}.*

Proof. Again the necessity is trivial. If $n = 0$, Theorem 1.5 applies, so assume that $n > 0$. Pick $a \in A$. For $k \in \{0, 1, \ldots, n-2\}$, if any, let $u = av_0 v_1 \cdots v_{k-1} \in [A]\binom{k+1}{k}$ and let $p_k = h_u(p_{k+1})$. Exactly as in the proof of Theorem 3.2 we have that $p_k \in T_k$ and $p_{k+1} < p_k$. By Lemma 1.9, p_k is minimal in βS_k. Thus we have that $\langle p_k \rangle_{k=0}^{n}$ is a special reductive sequence. Let $m \geq n$ and assume that $\langle p_k \rangle_{k=0}^{m}$ is a special reductive sequence. By Lemma 3.1 we can choose an idempotent $q_{n+1} \in T_{n+1}$ such that $q_{n+1} < p_n$ and p_n is the unique reduction of q_{n+1} in βS_n. Pick by [6, Theorem 1.60] a minimal idempotent p_{n+1} of T_{n+1} such that $p_{n+1} \leq q_{n+1}$. Given $u \in [A]\binom{n+1}{n}$, $h_u(p_{n+1}) \leq h_u(q_{n+1}) = p_n$ so $h_u(p_{n+1}) = p_n$. $\square$

It is natural to ask whether the requirement that $p_n < p_{n-1}$, where p_{n-1} is the unique reduction of p_n in βS_{n-1}, is needed. We see that it is.

Theorem 3.5. *Let $n \in \mathbb{N}$. There is a minimal idempotent q of T_n such that there is no minimal idempotent r of βS_{n-1} with $q < r$. In particular, if r is the unique reduction of q in βS_{n-1}, then it is not the case that $q < r$.*

Proof. The length function $\ell : W \to \mathbb{N}$ is a surjective homomorphism, hence so is its continuous extension from βW to $\beta\mathbb{N}$ which we also denote by ℓ. Notice that for any $u \in W$, $\ell \circ h_u = \ell$. Pick any nonminimal idempotent x of $\beta\mathbb{N}$ and let $X = \ell^{-1}[\{x\}]$. Notice that for each $k < \omega$, $\ell[S_k] = \{t \in \mathbb{N} : t \geq k\}$ and so $X \cap \beta S_k \neq \emptyset$.

Pick a minimal idempotent p_0 of $X \cap S_0$. We claim that $p_0 \in T_0$. So let $k > 0$ be given and pick an idempotent y of $X \cap \beta S_k$ such that $y < p_0$. Then for all $u \in [A]\binom{k}{0}$, $h_u(y) \leq h_u(p_0) = p_0$. Since $\ell\big(h_u(y)\big) = \ell(y) = x$ we have $h_u(y) \in X \cap \beta S_0$ and so $h_u(y) = p_0$.

By Theorem 3.2 we may pick $p_1, p_2, \ldots$ such that $\langle p_k \rangle_{k<\omega}$ is a reductive sequence and for each $k \in \omega$, $p_{k+1} < p_k$ and $p_k \in T_k$.

Recall that we have fixed $n \in \mathbb{N}$. Given any $u \in [A]\binom{n}{0}$, $h_u(p_n) = p_0$ and so $\ell(p_n) = \ell\big(h_u(p_n)\big) = \ell(p_0) = x$ and thus $p_n \in X$. Pick a minimal idempotent q of T_n such that $q \leq p_n$. Suppose that we have a minimal idempotent r of βS_{n-1} such that $q < r$.

Pick $a \in A$ and let G be the free group over $\{a\} \cup V$. Define a homomorphism $f : W \to G$ by agreeing for $W \in A \cup V$, that

$$f(w) = \begin{cases} w & \text{if } w \in V \\ a & \text{if } w \in A\,. \end{cases}$$

Denote also by f its continuous extension from βW to βG.

Now $f(q) \leq f(p_n)$ and $f(q) \leq f(r)$ so $\beta G f(p_n) \cap \beta G f(r) \neq \emptyset$. Since G is countable we have by [6, Corollary 6.20] that either $f(p_n) \in \beta G f(r)$ or $f(r) \in \beta G f(p_n)$. Since $f(r)$ and $f(p_n)$ are idempotents, this says that $f(p_n) = f(p_n)f(r) = f(p_n r)$ or $f(r) = f(r)f(p_n) = f(r p_n)$. Let $B = \{w \in W : v_{n-1} \text{ occurs in } w\}$. Then

$B \in rp_n$ so $f[B] \in f(rp_n)$. Since $f[S_{n-1}] \in f(r)$ and $f[S_{n-1}] \cap f[B] = \emptyset$, we have that $f(r) \neq f(rp_n)$ and so $f(p_n) = f(p_n r)$.

Let $\ell' : G \to \mathbb{N}$ be the length function on G and denote also by ℓ' its continuous extension from βG to $\beta\mathbb{N}$. Then $\ell'\big(f(p_n)\big) = \ell'\big(f(p_n r)\big)$ and for $w \in W$, $\ell'\big(f(w)\big) = \ell(w)$ so $\ell(p_n) = \ell(p_n r) = \ell(p_n) + \ell(r)$. Since $\ell[S_{n-1}] = \{t \in \mathbb{N} : t \geq n-1\}$ and $r \in K(\beta S_{n-1})$, $\ell(r) \in K(\beta\mathbb{N})$ and so $x = \ell(p_n) \in K(\beta\mathbb{N})$, a contradiction. $\square$

We observed in the introduction that the relation $\prec$ defined in Definition 1.6 has the property that the set of predecessors (if any) of an element of βS_n is linearly ordered. We shall show in Theorem 3.7 that elements of βS_n may have many successors.

We begin with a lemma which allows us to propagate branching upwards along special reductive sequences in the $\prec$ tree.

Lemma 3.6. *Assume $(p_0, \ldots, p_{n+1})$ is a special reductive sequence. If $(p_0, \ldots, p_{n-1}, x)$ is a reductive sequence (equivalently, either $n = 0$ or $p_{n-1} \prec x$) then $(p_0, \ldots, p_n, \overline{x})$ is a special reductive sequence where, letting $\tilde{x}$ be the inverse of $p_n x p_n$ in the group $p_n \beta S_n p_n$, $\overline{x} = \tilde{x} p_{n+1} x p_n$.*

Proof. Noting that $\overline{x} = \tilde{x} p_{n+1} p_n x p_n$, a straightforward calculation shows that $\overline{x}$ is an idempotent. Since $p_{n+1} \in K(\beta S_{n+1})$ and p_{n+1} is a factor of $\overline{x}$, $\overline{x}$ is a minimal idempotent. Clearly, $\overline{x} < p_n$.

Suppose $u \in [A]\binom{n+1}{n}$. We wish to show $h_u(\overline{x}) = p_n$. Of course, $h_u(\overline{x})$ is an idempotent since h_u is a homomorphism. So, showing that $h_u(\overline{x}) \leq p_n$ will suffice. This is immediate if the restriction of u to n is in $[A]\binom{n}{n}$ i.e. is $v_0 \ldots v_{n-1}$. So suppose otherwise. Notice that this implies that $n \neq 0$. We have $h_u(p_n x p_n) = p_{n-1} p_{n-1} p_{n-1} = p_{n-1}$. Since $\tilde{x}(p_n x p_n) = p_n$, $h_u(\tilde{x}) p_{n-1} = p_{n-1}$. Since $h_u(\tilde{x})$ is in the group $p_{n-1} \beta S_{n-1} p_{n-1}$, this implies that $h_u(\tilde{x}) = p_{n-1}$. A simple calculation now shows that $h_u(\overline{x}) = p_n$. $\square$

Theorem 3.7. *Let $\kappa = \max\{\omega, |A|\}$. If $(p_0, \ldots, p_n)$ is a special reductive sequence which can be extended to a special reductive sequence $(p_0, \ldots, p_{n+1})$ then there are 2^{2^κ} elements x of βS_{n+1} such that $(p_0, \ldots, p_n, x)$ is a special reductive sequence. Moreover, if $p_{n+1} \in T_{n+1}$ then there are as many such x in T_{n+1}.*

Proof. For convenience, whenever $z \in \beta S_{k+1}$ and v_k persists in z, we will write $\tau(z)$ and $\eta(z)$ for $\tau_{v_k}(z)$ and $\eta_{v_k}(z)$ respectively.

By Theorem 2.17, there is a subset U of βW_0 of size 2^{2^κ} such that $\tau(p_1)x\eta(p_1)$ are distinct as x ranges over U. By shrinking U if necessary, we may also assume all are distinct from $\tau(p_1)\eta(p_1)$. Let $\tilde{x}$ be the inverse of p_0xp_0 in $p_0\beta W_0p_0$ for $x \in U$. By shrinking U again, we may assume that whenever x and y are distinct elements of U, $\tau(p_1)x\eta(p_1) \neq \tau(p_1)\tilde{y}\eta(p_1)$. (If the collection of $\tau(p_1)\tilde{y}\eta(p_1)$ has size less than 2^{2^κ} this is clear, otherwise the desired subcollection can be constructed inductively.)

For $x \in U$ define $x_k \in \beta S_k$ for $k = 0, \ldots, n+1$ by induction according to Lemma 3.6 so that $x_0 = x$ and whenever $k \leq n$, $x_{k+1} = \tilde{x}_k p_{k+1} x_k p_k$ where $\tilde{x}_k$ is the inverse of $p_k x_k p_k$ in the group $p_k \beta S_k p_k$. Lemma 3.6 implies that if $x \in U$ and $0 < k \leq n+1$ then $(p_0, \ldots, p_{k-1}, x_k)$ is a special reductive sequence.

We first show that if x and y are distinct elements of U then $x_{n+1} \neq y_{n+1}$. Fix such x and y. Let $P(k)$ denote the following: $\tau(p_{k+1})x_k\eta(p_{k+1})$ is not equal to

$\tau(p_{k+1})y_k\eta(p_{k+1})$,
$\tau(p_{k+1})\tilde{y}_k\eta(p_{k+1})$ or
$\tau(p_{k+1})\eta(p_{k+1})$.

We will show by induction on $k = 0, \ldots, n$ that $P(k)$ holds, but first notice that this will imply that $x_{n+1} \neq y_{n+1}$ as follows. Since

$$\tau(p_{n+1})x_n\eta(p_{n+1}) \neq \tau(p_{n+1})y_n\eta(p_{n+1})$$

and $\eta(p_{n+1}) = p_n\eta(p_{n+1})$, we must also have

$$\tau(p_{n+1})x_np_n \neq \tau(p_{n+1})y_np_n\,.$$

Since $\tau(p_{n+1})x_np_n = \tau(x_{n+1})$ and $\tau(p_{n+1})y_np_n = \tau(y_{n+1})$, we conclude $x_{n+1} \neq y_{n+1}$.

To begin the proof by induction that $P(k)$ holds for $k = 0, \ldots, n$, notice that $P(0)$ is true by choice of U.

Assume $k < n$ and $P(k)$ holds. $\tau(p_{k+1})x_k\eta(p_{k+1})$ contains an element X which is not in $\tau(p_{k+1})y_k\eta(p_{k+1})$, $\tau(p_{k+1})\tilde{y}_k\eta(p_{k+1})$ or $\tau(p_{k+1})\eta(p_{k+1})$. Let μ be the characteristic function of X as a subset of S_k modulo 3 so that μ^+ counts the number of v_k-gaps from X modulo 3 in elments of S_{k+1}. We see that

$\mu\big(\tau(p_{k+1})y_k\eta(p_{k+1})\big) = 0$,
$\mu\big(\tau(p_{k+1})\tilde{y}_k\eta(p_{k+1})\big) = 0$,
$\mu\big(\tau(p_{k+1})\eta(p_{k+1})\big) = 0$ and

$$\mu\big(\tau(p_{k+1})x_k\eta(p_{k+1})\big) = 1\,.$$

We will show that $\mu^+\big(\tau(p_{k+2})x_{k+1}\eta(p_{k+2})\big)$ is not equal to

$$\mu^+\big(\tau(p_{k+2})y_{k+1}\eta(p_{k+2})\big),$$
$$\mu^+\big(\tau(p_{k+2})\tilde{y}_{k+1}\eta(p_{k+2})\big) \text{ or}$$
$$\mu^+\big(\tau(p_{k+2})\eta(p_{k+2})\big)$$

thus completing the inductive argument.

Using parts (a), (d) and (e) of Lemma 2.16 and the fact that $\tau(p_{k+2}) = \tau(p_{k+2})p_{k+1}$ and $\eta(p_{k+2}) = p_{k+1}\eta(p_{k+2})$, it will suffice to show that $\mu^+(p_{k+1}x_{k+1}p_{k+1})$ is not equal to

$$\mu^+(p_{k+1}y_{k+1}p_{k+1}),$$
$$\mu^+(p_{k+1}\tilde{y}_{k+1}p_{k+1}) \text{ or}$$
$$\mu^+(p_{k+1}).$$

Part (b) of Lemma 2.16 implies that $\mu^+(p_{k+1}) = 0$. Using the definitions of y_{k+1} and x_{k+1} we can use Lemma 2.16 again to compute that $\mu^+(p_{k+1}y_{k+1}p_{k+1}) = 0$ and $\mu^+(p_{k+1}x_{k+1}p_{k+1})$ is either 1 or 2 depending on whether X is in $\tau(p_{k+1})\tilde{x}_k\eta(p_{k+1})$ or not. Using the fact that $(p_{k+1}\tilde{y}_{k+1}p_{k+1})(p_{k+1}y_{k+1}p_{k+1}) = \tilde{y}_{k+1}(p_{k+1}y_{k+1}p_{k+1}) = p_{k+1}$ and Lemma 2.16 yet again, we see that $\mu^+(p_{k+1}\tilde{y}_{k+1}p_{k+1})$ is also 0.

Now assume that $p_{n+1} \in T_{n+1}$. In order to complete the proof of the theorem, it will suffice to show that $x_{n+1} \in T_{n+1}$ for all $x \in U$.

Since $p_{n+1} \in T_{n+1}$, $p_k \in T_k$ for $k \le n$ by Lemma 2.2 and Theorem 2.12. By the definition of T_{n+1}, for $r > n+1$ choose p_r such that $p_{n+1} \prec p_r$ (implying $p_k \prec p_r$ for $k \le n$ also). Notice that by Theorem 3.4 we could have chosen the p_r so that $< p_i >_{i<\omega}$ would be a special reductive sequence, but we won't need that assumption.

Fix $x \in U$. We will show by induction on $k = 1, \ldots, n+1$ that $x_k \in T_k$.

Since (p_0, x_1) is a special reductive sequence, $x_1 \in T_1$ by Theorem 3.3.

Assume $1 \le k < n+1$ and $x_k \in T_k$. By Theorem 2.3, $p_k x_k p_k \in T_k$. Moreover, $\tilde{x}_k \in T_k$ since $p_k T_k p_k$ is a subgroup of $p_k \beta S_k p_k$. We will now show by induction on $r \ge k+1$ that there is some $x_r^{k+1} \in \beta S_r$ such that $x_{k+1} \preceq x_r^{k+1}$, thus verifying that $x_{k+1} \in T_{k+1}$.

For $r = k+1$, simply take $x_r^{k+1} = x_{k+1}$.

Suppose $k+1 \le r$ and $x_{k+1} \preceq x_r^{k+1}$ where $x_r^{k+1} \in \beta S_r$. Choose $x_r^k, \tilde{x}_r^k \in \beta S_r$ such that $x_k \prec x_r^k$ and $\tilde{x}_k \prec \tilde{x}_r^k$. Let x_{r+1}^{k+1} be an idempotent in βS_{r+1} such that $x_{r+1}^{k+1} \in \beta W(\tilde{x}_r^k p_{r+1} x_r^k p_r x_r^{k+1})$ and $x_{r+1}^{k+1} \in (x_r^{k+1} \tilde{x}_r^k p_{r+1} x_r^k p_r)\beta W$.

In order to show $x_{k+1} \prec x_{r+1}^{k+1}$, suppose $u \in [A]\binom{r+1}{k+1}$. Notice that $h_u(x_{r+1}^{k+1})$ is an idempotent in βS_{k+1}, so to complete our proof it will suffice to show that $h_u(x_{r+1}^{k+1}) \leq x_{k+1}$. This is immediate when considering the two possible cases: $u|r \in [A]\binom{r}{k+1}$ or $u|r \in [A]\binom{r}{k}$. In the first case, use the fact that $h_u(x_r^{k+1}) = x_{k+1}$. In the second case, notice that $h_u(x_r^{k+1}) = p_k$ since $p_k \prec x_{k+1} \prec x_r^{k+1}$. □

We have seen that there are finite special reductive sequences which have 2^{2^κ} continuations, where $\kappa = \max(\omega, |A|)$, and shall see that there are others which cannot be continued.

By Theorem 1.5, if p_0 is any minimal idempotent in βS_0 and p_1 is any minimal idempotent in βS_1 such that $p_1 < p_0$, then in fact $p_0 \prec p_1$. We see now that such a statement cannot be extended to $n = 2$.

Theorem 3.8. *Let p_0 be a minimal idempotent in βS_0 and let p_1 be a minimal idempotent in βS_1 such that $p_1 < p_0$. There exists a minimal idempotent p_2 in βS_2 such that $p_2 < p_1$ but it is not the case that $p_1 \prec p_2$.*

Proof. Pick a minimal idempotent q of βW_1 such that $q \in p_1 \beta W_1 \cap \beta W_1 p_0$ and $q \neq p_1$. (Let $a \in A$. Then the left ideals $\beta W_1 v_0 v_0 p_0$ and $\beta W_1 a v_0 p_0$ are disjoint subsets of $\beta W_1 p_0$. The intersection of each of them with $p_1 \beta W_1$ contains an idempotent minimal in βW_1.) Notice that the minimal left ideals $\beta W_1 p_1$ and $\beta W_1 q$ are disjoint, since p_1 is the unique idempotent in $p_1 \beta W_1 \cap \beta W_1 p_1$. (We know that p_1 is minimal in βW_1 because S_1 is an ideal of W_1.)

Pick a minimal idempotent p_2 of βS_2 such that $p_2 \in p_1 \beta S_2 \cap \beta S_2 h_{v_1}(q) p_1$. Pick $r \in \beta S_2$ such that $p_2 = r h_{v_1}(q) p_1$ and pick $a \in A$. Then

$$\begin{aligned}
h_{av_0}(p_2) &= h_{av_0}(r) h_{v_1\langle av_0\rangle}(q) h_{av_0}(p_1) \\
&= h_{av_0}(r) h_{v_0}(q) h_a(p_1) \\
&= h_{av_0}(r) q p_0 = h_{av_0}(r) q \text{ and} \\
h_{v_0 a}(p_2) &= h_{v_0 a}\big(r h_{v_1}(q)\big) h_{v_0 a}(p_1) \\
&= h_{v_0 a}\big(r h_{v_1}(q)\big) p_1
\end{aligned}$$

and so $h_{av_0}(p_2)$ and $h_{v_0 a}(p_2)$ are in disjoint left ideals of βW_2. □

It is natural to ask whether every finite special reductive sequence $\langle p_i\rangle_{i=0}^n$ can be extended to a special reductive sequence with $n+2$ terms. The answer is "yes" if $n=0$ or $n=1$, by Theorem 1.5. We shall show in Theorem 3.16 that the answer is "no" if $n>1$. We shall use some special notation.

Definition 3.9. Let $n\in\omega$. Then $[A]^*\binom{n}{0}=[A]\binom{n}{0}$ and if $0<m\leq n$, then $[A]^*\binom{n}{m}=\big\{u\in[A]\binom{n}{m}:u(n-1)=v_{m-1}$ and $u_{|n-1}\in[A]\binom{n-1}{m-1}\big\}$.

Also $D_n=\big\{x\in\beta W_n:(\forall m<n)\big(\forall u,u'\in[A]^*\binom{n}{m}\big)\big(h_u(x)=h_{u'}(x)\big)\big\}$.

Lemma 3.10. *Let $m<n<\omega$ and let $u\in[A]^*\binom{n}{m}$. Then $h_u[D_n]\subseteq D_m$.*

Proof. We know that $h_u[\beta W_n]\subseteq\beta W_m$. If $m=0$, then $D_m=\beta W_m$, so assume that $m>0$, let $k<m$, let $x\in D_n$, and let $w,w'\in[A]^*\binom{m}{k}$. Then $u\langle w\rangle$ and $u\langle w'\rangle$ are in $[A]^*\binom{n}{k}$ and so $h_w\big(h_u(x)\big)=h_{u\langle w\rangle}(x)=h_{u\langle w'\rangle}(x)=h_{w'}\big(h_u(x)\big)$. □

Lemma 3.11. *Let $n\in\mathbb{N}$ and let p_0 be a minimal idempotent in βS_0. Let $x_n\in D_n$ and for each $m<n$ let x_m be the unique value of $h_u(x_n)$ for $u\in[A]^*\binom{n}{m}$. There is a special reductive sequence $\langle q_0,q_1,\ldots,q_n\rangle$ such that $q_0=p_0$ and for each $i\in\{1,2,\ldots,n\}$,*

$$q_i\in q_{i-1}x_i\beta W_i\cap\beta W_ix_iq_{i-1}\,.$$

Proof. We can assume that $x_n\in K(D_n)$ because we can pick $y_n\in K(D_n)\cap x_nD_n\cap D_nx_n$ and, given $m<n$, if y_m is the unique value of $h_u(y_n)$ for $u\in[A]^*\binom{n}{m}$, then $y_m\beta W_m\subseteq x_m\beta W_m$ and $\beta W_my_m\subseteq\beta W_mx_m$.

Assume first that $n=1$, let $q_0=p_0$, and let q_1 be a minimal idempotent of βW_1 with $q_1\in q_0x_1\beta W_1\cap\beta W_1x_1q_0$. Then $q_1<q_0$ and so for any $u\in[A]\binom{1}{0}$, $h_u(q_1)<h_u(q_0)=q_0$ and thus $h_u(q_1)=q_0$. Also $q_1\in K(\beta W_1)=K(\beta S_1)$ because S_1 is an ideal of W_1.

Now assume that $n>1$ and the lemma is valid for $n-1$. Note that by Theorem 1.5 $D_n\cap K(\beta W_n)\neq\emptyset$ and thus $x_n\in K(D_n)=D_n\cap K(\beta W_n)$. By Lemma 3.10 $x_{n-1}\in D_{n-1}$. We also observe that $x_{n-1}\in K(\beta W_{n-1})$. To see this, let $u=v_0v_0v_1\cdots v_{n-2}$. Then $u\in[A]^*\binom{n}{n-1}$ so $h_u(x_n)=x_{n-1}$. Also $h_u[W_n]=W_{n-1}$ so $h_u[\beta W_n]=\beta W_{n-1}$ and thus by [6, Exercise 1.7.3], $h_u[K(\beta W_n)]=K(\beta W_{n-1})$. Thus $x_{n-1}=h_u(x_n)\in K(\beta W_{n-1})$.

By the induction hypothesis we may pick a special reductive sequence $\langle q_0, q_1, \ldots, q_{n-1}\rangle$ such that $q_0 = p_0$ and for each $i \in \{1, 2, \ldots, n-1\}$ $q_i \in q_{i-1}x_i\beta W_i \cap \beta W_i x_i q_{i-1}$. Pick a minimal idempotent q_n of βW_n such that $q_n \in q_{n-1}x_n\beta W_n \cap \beta W_n x_n q_{n-1}$. Then $q_n < q_{n-1}$. Also, since $S_{n-1} \in q_{n-1}$ and $\{w \in W_n : v_{n-1}$ occurs in $w\} \in x_n$, we have that $q_n \in \beta S_n$ and so q_n is minimal in βS_n.

Let $u \in [A]\binom{n}{n-1}$. It remains to show that $h_u(q_n) = q_{n-1}$. If $u = v_0 v_1 \cdots v_{n-2} t$ for some $t \in A \cup \{v_0, v_1, \ldots, v_{n-2}\}$, then $h_u(q_{n-1}) = q_{n-1}$ so $h_u(q_n) \leq q_{n-1}$ and thus $h_u(q_n) = q_{n-1}$.

So assume that $u = u' v_{n-2}$ for some $u' \in [A]\binom{n-1}{n-2}$. Then $u \in [A]^*\binom{n}{n-1}$ and so $h_u(x_n) = x_{n-1}$. Also $h_u(q_{n-1}) = h_{u'}(q_{n-1}) = q_{n-2}$. Thus $h_u(q_{n-1}x_n) = q_{n-2}x_{n-1}$ and $h_u(x_n q_{n-1}) = x_{n-1}q_{n-2}$. Since $x_{n-1} \in K(\beta W_{n-1})$, we may pick a minimal right ideal R of βW_{n-1} and a minimal left ideal L of βW_{N-1} such that $q_{n-2}x_{n-1} \in R$ and $x_{n-1}q_{n-2} \in L$. Then $q_{n-1} \in R \cap L$ so $q_{n-2}x_{n-1} \in R = q_{n-1}\beta W_{n-1}$ and $x_{n-1}q_{n-2} \in L = \beta W_{n-1}q_{n-1}$ so $h_u(q_n) \leq q_{n-1}$ and thus $h_u(q_n) = q_{n-1}$. $\square$

Notice that if in Lemma 3.11, x_1 is a minimal idempotent in βS_1 and $x_1 < p_0$, then $q_1 \in q_0 x_1 \beta W_1 \cap \beta W_1 x_1 q_0 = x_1 \beta W_1 \cap \beta W_1 x_1$ and so $q_1 = x_1$.

Definition 3.12. We choose any $c \in A$ and define E to be the set of words in W_0 in which c does not occur.

We now give an inductive definition of a subset R_n of W_n for each $n \geq 2$.

$$R_2 = W_1 v_1 E v_0 W_2 \text{ and if } n > 2\,,$$
$$R_n = W_{n-1} h_{v_{n-1}}[W_1] v_{n-1} R_{n-1} W_n\,.$$

We observe that, for every $n \geq 2$, R_n is a right ideal of W_n and $W_{n-1}R_n \subseteq R_n$.

Lemma 3.13. *If p_0 is any minimal idempotent in βS_0, there is a special reductive sequence $\langle p_0, p_1, p_2\rangle$ for which $R_2 \in p_2$.*

Proof. Let c be the element of A used to define R_2 and let $B = A \setminus \{c\}$. We first deal with the case in which $B = \emptyset$. Then $E = \emptyset$ and so $R_2 = W_1 v_1 v_0 W_2$. Let q be a minimal idempotent of βW_2 satisfying $q \in p_0 v_1 v_0 \beta W_2 \cap \beta W_2 p_0$. Then $W_1 \in p_0$ so $W_1 v_1 v_0 W_2 \in q$ and thus $R_2 \in q$. Note also that $h_{cc}(q) \leq h_{cc}(p_0) = p_0$ and so $h_{cc}(q) = p_0$. Let $p_1 = h_{cv_0}(q)$. Now q is minimal in the subsemigroup $c\beta W_2$ of

βW_2 by [6, Theorem 1.65] and so p_1 is minimal in $h_{cv_0}[c\beta W_2] = c\beta W_1$, hence in βW_1. Since also $p_1 \in \beta S_1$ we have that $p_1 \in \beta S_1 \cap K(\beta W_1) = K(\beta S_1)$.

Now choose p_2 to be a minimal idempotent in βS_2 such that $p_2 \in p_1 q\beta S_2 \cap \beta S_2 q p_1$. Then $W_1 R_2 W_2 \in p_2$ and $W_1 R_2 W_2 \subseteq R_2$ so $R_2 \in p_2$. Now let $u \in [A]\binom{2}{1}$. If $u = cv_0$, then $h_u(cv_0) = cc$ so by Lemma 1.10 $h_u(p_1) = h_{cc}(q) = p_0$ and thus $h_u(p_2) \in p_0 p_1 h_u[\beta S_2] \cap h_u[\beta S_2] p_1 p_0 \subseteq p_1 \beta S_1 \cap \beta S_1 p_1$ so $h_u(p_2) = p_1$. If $u = v_0 t$ for some $t \in \{c, v_0\}$, then $h_u(cv_0) = cv_0$ so by Lemma 1.10 $h_u(p_1) = h_{cv_0}(q) = p_1$ and thus $h_u(p_2) \in p_1 h_u[q\beta S_2] \cap h_u[\beta S_2 q] p_1$ so $h_u(p_2) = p_1$. So $\langle p_0, p_1, p_2 \rangle$ is a special reductive sequence.

We now assume that $B \neq \emptyset$. Recall that E is the semigroup of words over B. Let W_1' be the semigroup of words over $A \cup \{v_0\}$. Pick a minimal idempotent q_0 of βE and a minimal idempotent q_1 of $\beta W_1'$ such that $q_1 < q_0$. Note that for any $b \in B$, $h_b(q_1) = q_0$.

Let $y = h_c(q_1)$. Then $y \leq h_c(q_0) = q_0$. Let $z = v_1 q_1 y q_1 y$. Note that $Ev_0 W_1 \in q_1$ so $v_1 E v_0 W_1 W_2 \in z$ and thus $R_2 \in z$.

Let $H = \{x \in \beta W_2 : \text{for all } a, b \in A,\ h_{av_o}(x) = h_{bv_0}(x)\}$. Note that by Theorem 1.5, $H \cap K(\beta W_2) \neq \emptyset$. In particular, H is a compact subsemigroup of βW_2 and $K(H) = H \cap K(\beta W_2)$. If $b \in B$, then $h_{bv_0}(q_1) = h_b(q_1) = q_0$ and so $h_{bv_0}(z) = v_0 q_0 y q_0 y = v_0 y$ and $h_{bv_0}(p_0) = p_0$. Also $h_{cv_0}(z) = v_0 yyyy = v_0 y$ and $h_{cv_0}(p_0) = p_0$. Thus $p_0 z \in H$ and $z p_0 \in H$. We can choose a minimal idempotent x of H with $x \in p_0 z H \cap H z p_0$. Then $x \in K(\beta W_2)$, $x \leq p_0$, and $R_2 \in x$.

Let $p_1 = h_{cv_0}(x)$ and note that, given any $a \in A$, $h_{av_0}(x) = h_{cv_0}(x) = p_1$. Let I be the ideal of W_1 consisting of words in which c occurs. Then $K(\beta W_1) \subseteq \overline{I} \subseteq h_{cv_0}[\beta W_2]$ and so

$$p_1 \in h_{cv_0}[K(\beta W_2)] = K(h_{cv_0}[\beta W_2]) = h_{cv_0}[\beta W_2] \cap K(\beta W_1)$$

and so $p_1 \in K(\beta W_1) = K(\beta S_1)$. Also $p_1 \leq h_{cv_0}(p_0) = p_0$ and therefore $h_a(p_1) = p_0$ for all $a \in A$.

Now choose p_2 to be a minimal idempotent of βS_2 with $p_2 \in p_1 x \beta S_2 \cap \beta S_2 x p_1$. Then $p_2 < p_1$ and since $R_2 \in x$, $R_2 \in p_2$. Finally, let $u \in [A]\binom{2}{1}$. We show that $h_u(p_2) = p_1$. If $u = av_0$ for some $a \in A$, then $h_u(p_2) \in p_0 p_1 h_u[\beta S_2] \cap h_u[\beta S_2] p_1 p_0 \subseteq p_1 \beta S_1 \cap \beta S_1 p_1$. If $u = v_0 t$ for some $t \in A \cup \{v_0\}$, then $h_u(p_2) \in p_1 h_u[x\beta S_2] \cap h_u[\beta S_2 x] p_1 \subseteq p_1 \beta S_1 \cap \beta S_1 p_1$. Thus, in either case, $h_u(p_1) = p_2$. $\square$

Lemma 3.14. *Let $n > 1$, let p_0 be a minimal idempotent in βS_0 and let $p_1 < p_0$ be a minimal idempotent in βS_1. Then there exists a special reductive sequence $\langle q_0, q_1, \ldots, q_n \rangle$ such that $q_0 = p_0$ and $R_n \in q_n$. Furthermore, if $n > 2$, then $q_1 = p_1$ and if $n > 3$, then $q_2 \in T_2$.*

Proof. By Lemma 3.13, this holds if $n = 2$. So assume that $n \geq 3$ and that the statement of the lemma is true for $n-1$. Pick a special reductive sequence $\langle r_0, r_1, \ldots, r_{n-1} \rangle$ with $r_0 = p_0$, $R_{n-1} \in r_{n-1}$, and with $r_1 = p_1$ if $n > 3$. Let $x_n = h_{v_{n-1}}(p_1)r_{n-1}$. Then $R_n \in x_n$, because $W_1 v_0 W_0 = S_1 \in p_1$ so $h_{v_{n-1}}[W_1]v_{n-1}W_0R_{n-1} \in x_n$ and $W_0 R_{n-1} \subseteq R_{n-1}$.

We claim that $x_n \in D_n$. So let $m < n$ and let $u \in [A]^*\binom{n}{m}$. Then $h_u(r_{n-1}) = h_{u_{|n-1}}(r_{n-1}) = r_{m-1}$ and $h_u\big(h_{v_{n-1}}(p_1)\big) = h_{v_{n-1}\langle u \rangle}(p_1)$ $= h_{v_{m-1}}(p_1)$. Thus $h_u(x_n) = h_{v_{m-1}}(p_1)r_{m-1}$, which is independent of the choice of u.

Pick by Lemma 3.11 a special reductive sequence $\langle q_0, q_1, \ldots, q_n \rangle$ such that $q_0 = p_0$ and, for each $m \in \{1, 2, \ldots, n\}$,

$$q_m \in q_{m-1}x_m\beta W_m \cap \beta W_m x_m q_{m-1}\,,$$

where for each $m \in \{1, 2, \ldots, n-1\}$, $x_m = h_{v_{m-1}}(p_1)r_{m-1}$. Notice in particular that $x_1 = h_{v_0}(p_1)r_0 = p_1 p_0 = p_1$ and thus, since $q_1 \in q_0 x_1 \beta W \cap \beta W x_1 q_0$, we have $q_1 = p_1$.

Now $q_n \in q_{n-1}x_n\beta W_n$ and $R_n \in x_n$. Since $W_{n-1}R_nW_n \subseteq R_n$, it follows that $R_n \in q_n$.

Finally assume that $n > 3$. Then $r_1 = p_1$ so $x_2 = h_{v_1}(p_1)p_1$. Therefore

$$q_2 \in p_1 x_2 \beta W_2 \cap \beta W_2 x_2 p_1 \subseteq p_1 h_{v_1}(p_1)\beta W_2 \cap \beta W_2 h_{v_1}(p_1)p_1\,.$$

By Theorem 2.9, $q_2 \in T_2$. $\square$

Lemma 3.15. *Let $n \geq 2$ and let $c \in A$ be the letter used in the definition of E and R_2. Define u_n and w_n in $[A]\binom{n+1}{n}$ by $u_n = cv_0v_1\cdots v_{n-1}$ and $w_n = v_0cv_1v_2\cdots v_{n-1}$. Then ${h_{u_n}}^{-1}[R_n] \cap {h_{w_n}}^{-1}[R_n] = \emptyset$.*

Proof. Notice that ${h_{u_n}}^{-1}[W_n] \subseteq W_{n+1}$ and ${h_{w_n}}^{-1}[W_n] \subseteq W_{n+1}$.

Assume first that $n = 2$ and suppose we have $x \in W_3$ such that $h_{u_2}(x) \in R_2$ and $h_{w_2}(x) \in R_2$. Then $h_{u_2}(x) \in W_1v_1zv_0W_2$ for some $z \in E$ so $x \in W_2v_2zv_1W_3$ and thus the first variable after the first occurrence of v_2 in x is v_1. Similarly $h_{w_2}(x) \in W_1v_1yv_0W_2$ for some

$y \in E$ so $x \in W_2 v_2 y v_0 W_3$ and thus the first variable after the first occurrence of v_2 in x is v_0, a contradiction.

Now assume that $n > 2$ and ${h_{u_{n-1}}}^{-1}[R_{n-1}] \cap {h_{w_{n-1}}}^{-1}[R_{n-1}] = \emptyset$. For each $k \in \omega$ and $x \in W$, define a v_k *block* in x as a segment of x in which all the letters are in $A \cup \{v_k\}$ with the first and last letters being v_k and which is maximal with respect to this condition. Also, if $k > 0$, let $W_k^{\lhd} = \{x \in W_k : v_{k-1} \text{ opccurs in } x\}$. Define $\varphi_k : W_k^{\lhd} \to W_{k-1}$ as follows. Let $x \in W_k^{\lhd}$. If there is only one v_{k-1} block in x, let $\varphi_k(x)$ be the word which begins after the v_{k-1} block and continues to the end of x. Otherwise let $\varphi_k(x)$ be the word which begins after the first v_{k-1} block and ends immediately before the next occurrence of v_{k-1}. For example, if $a \in A$, then $\varphi_3(v_0 v_2 a v_2) = \emptyset$ and $\varphi_3(v_0 v_2 a v_1) = \varphi_3(v_0 v_2 a v_2 a v_1 v_2 v_1) = a v_1$.

We claim that if $x \in R_n$, then $\varphi_n(x) \in R_{n-1}$. Indeed, from the definition of R_n, we have that $x = y v_{n-1} z \alpha$ for some $y \in W_{n-1} h_{v_{n-1}}[W_1]$, $z \in R_{n-1}$, and $\alpha \in W_n$. If $\alpha \in W_{n-1}$, then $\varphi_n(x) = z\alpha$. Otherwise, $\alpha = \delta v_{n-1} \gamma$ where $\delta \in W_{n-1}$ and $\gamma \in W_n$ so that $\varphi_n(x) = z\delta$. In either case, $\varphi_n(x) \in R_{n-1} W_{n-1} \subseteq R_{n-1}$.

Next observe that if $x \in W_{n+1}^{\lhd}$ has the property that $h_{u_n}(x) \in R_n$ and $h_{w_n}(x) \in R_n$, then h_{u_n} and h_{w_n} map the first v_n-block of x to the first v_{n-1}-block of $h_{u_n}(x)$ and $h_{w_n}(x)$ respectively. Indeed, if this statement does not hold for h_{u_n}, v_0 must occur in x between the first v_n-block of x and the next occurrence of v_n in x, and v_0 must be the only variable which does. However, v_0 is then the only variable which occurs in $h_{w_n}(x)$ between the first v_{n-1}-block of $h_{w_n}(x)$ and the next occurrence of v_{n-1} in $h_{w_n}(x)$. Since $n > 2$, this contradicts the assumption that $h_{w_n}(x) \in R_n$. The assumption that h_{w_n} does not map the first v_n-block of x to the first v_{n-1}-block of $h_{w_n}(x)$, leads to a contradiction in a similar way.

It follows that

$$\begin{aligned} h_{u_{n-1}}\big(\varphi_{n+1}(x)\big) &= \varphi_n\big(h_{u_n}(x)\big) \text{ and} \\ h_{w_{n-1}}\big(\varphi_{n+1}(x)\big) &= \varphi_n\big(h_{w_n}(x)\big) \end{aligned}$$

because, for $y \in W_n$, $h_{u_n}(y) = h_{u_{n-1}}(y)$ and $h_{w_n}(y) = h_{w_{n-1}}(y)$.

Now suppose we have some $x \in {h_{u_n}}^{-1}[R_n] \cap {h_{w_n}}^{-1}[R_n]$. Then $x \in W_{n+1}^{\lhd}$ because v_{n-1} occurs in any member of R_n and $\varphi_n\big(h_{u_n}(x)\big) \in R_{n-1}$ and $\varphi_n\big(h_{w_n}(x)\big) \in R_{n-1}$ so

$$\varphi_{n+1}(x) \in {h_{u_{n-1}}}^{-1}[R_{n-1}] \cap {h_{w_{n-1}}}^{-1}[R_{n-1}]\,,$$

a contradiction. □

Theorem 3.16. *Let $n > 1$, let p_0 be a minimal idempotent in βS_0, and let p_1 be a minimal idempotent in βS_1 such that $p_1 < p_0$. Then there exists a special reductive sequence $\langle q_0, q_1, \ldots, q_n\rangle$ such that $q_0 = p_0$ and there is no $r \in \beta W_{n+1}$ for which $q_n \prec r$. If $n > 2$, then $q_1 = p_1$.*

Proof. Pick $\langle q_0, q_1, \ldots, q_n\rangle$ as guaranteed by Lemma 3.14 and suppose we have some $r \in \beta W_{n+1}$ for which $q_n \prec r$. Let u_n and w_n be as in Lemma 3.15. Then $h_{u_n}(r) = h_{w_n}(r) = q_n$ and so ${h_{u_n}}^{-1}[R_n] \in r$ and ${h_{w_n}}^{-1}[R_n] \in r$, a contradiction. □

Corollary 3.17. *Let $n > 1$. There is a minimal idempotent of βS_n in $C_n \setminus T_n$.*

Proof. Let $\langle q_0, q_1, \ldots, q_n\rangle$ be as guaranteed by Theorem 3.16. Then $q_n \in C_n \setminus T_n$. □

We need a different argument to show that $C_1 \neq T_1$.

Theorem 3.18. *There is a minimal idempotent of βS_1 in $C_1 \setminus T_1$.*

Proof. Choose any $c \in A$. Let X denote the set of elements of S_1 in which there is no occurrence of c before the first occurrence of v_0. We observe that $cl_{\beta S_1}(X) \cap T_1 = \emptyset$, because $X \cap h_{cv_0}[S_2] = \emptyset$. We shall show that $cl_{\beta S_1}(X) \cap C_1 \neq \emptyset$.

If $A = \{c\}$, then $\beta S_1 = C_1$ and so $cl_{\beta S_1}(X) \cap C_1 \neq \emptyset$.

Assume that $|A| > 1$. Let $S_0' = \{w \in S_0 : c$ does not occur in $w\}$ and let Let $S_1' = \{w \in S_1 : c$ does not occur in $w\}$. Let q_0 be a minimal idempotent in $\beta S_0'$ and let q_1 be a minimal idempotent in $\beta S_1'$ such that $q_1 \leq q_0$. Then $h_a(q_1) = q_0$ for all $a \in A \setminus \{c\}$. Let $x_1 = q_0 q_1 h_c(q_1)$. Then, for any $a \in A \setminus \{c\}$, $h_a(x_1) = h_c(x_1) = q_0 h_c(q_1)$. So $x_1 \in C_1$. Since $x_1 \in c\ell_{\beta S_1}(X)$, we again have $cl_{\beta S_1}(X) \cap C_1 \neq \emptyset$.

Now X is a right ideal of S_1 and so $cl_{\beta S_1}(X)$ is a right ideal of βS_1 by [6, Theorem 2.15]. Thus $cl_{\beta S_1}(X) \cap C_1$ contains a minimal idempotent of C_1, and any minimal idempotent of C_1 is also a minimal idempotent of βS_1. □

4. Appendix – Proof of Theorem 1.14

We provide here the necessary adaptations of the proof of [3, Theorem 2.12] to establish Theorem 1.14. As we have previously remarked, this theorem holds in the more general setting of [3], in which it is not assumed that $D = \{e\}$ or that T_e is the identity. The reader is referred to [3] for the definition of the more general parameter system used there.

Definition 4.1. Let $n \in \mathbb{N}$ with $n \geq 2$.

(a) For $i \in \{0, 1, \ldots, n-1\}$, $w_{n,i}$ is the word obtained from $v_0 v_1 \cdots v_{n-1}$ by deleting v_i.

(b) For $i \in \{0, 1, \ldots, n-1\}$,

$$\begin{aligned} U_{n,i} = \{w \in W : \ & \ell(w) = n\,, w(i) \in A \cup \{v_l : l < i\}\,, \\ & \text{and for all } j \in \{0,1,\ldots,n-1\}, \text{ if } j < i, \text{ then} \\ & w(j) = v_j \text{ and if } j > i, \text{ then } w(j) = v_{j-1}\}\,. \end{aligned}$$

Thus if $0 < i < n-1$, a member of $U_{n,i}$ is of the form $v_0 \cdots v_{i-1} t v_i \cdots v_{n-2}$ where $t \in A \cup \{v_0, v_1, \ldots, v_{i-1}\}$.

Notice that for any $n \in \mathbb{N}$ with $n \geq 2$, $[A]\binom{n}{n-1} = \bigcup_{i=0}^{n-1} U_{n,i}$.

Theorem 1.14. *Let X be a subsemigroup of βW such that $h_u[X] \subseteq X$ for every $u \in W$, $X \cap \beta W_n$ is compact and $X \cap \beta S_n$ is non-empty for every $n \in \omega$. Let p_0 be a minimal idempotent of $X \cap \beta W_0$ and let p_1 be a minimal idempotent of $X \cap \beta W_1$ such that $p_1 < p_0$. Then there is an infinite reductive sequence $(p_0, p_1, p_2, p_3, \ldots)$ such that p_n is a minimal idempotent of $X \cap \beta S_n$ and $p_{n+1} < p_n$ for every $n \in \omega$.*

Proof. Note that $h_u(p_1) = p_0$ for all $u \in [A]\binom{1}{0}$. We first show how p_2 can be defined. Let $\alpha = h_{v_1}(p_1)$. Then $\alpha \in X \cap \beta W_2$ so we may pick an idempotent $p_2 \in p_1\alpha(X \cap \beta W_2) \cap (X \cap \beta W_2)\alpha p_1$ which is minimal in $X \cap \beta W_2$. Since $p_1\alpha \in \beta S_2$, $p_2 \in \beta S_2$ so p_2 is minimal in $X \cap \beta S_2$.

Now let $u \in [A]\binom{2}{1}$. Then $h_u[S_2] \subseteq S_1$ so $h_u(p_2) \in X \cap \beta S_1$. It thus suffices to show that $h_u(p_2) \leq p_1$. If $u \in U_{2,1}$, then h_u is the identity on S_1, so $h_u(p_2) \leq h_u(p_1) = p_1$. Now assume that $u \in U_{2,0}$ and pick $t \in A$ such that $u = tv_0$. For $w \in S_1$, $h_u(w) = h_t(w)$, and so $h_u(p_1) = h_t(p_1) = p_0$. Also, by Lemma 1.10, $h_{tv_0} \circ h_{v_1}$ is the identity on W_1. So $h_u(\alpha) = h_{tv_0}\big(h_{v_1}(p_1)\big) = p_1$. Therefore $h_u(p_2) \in p_0 p_1 h_u[X \cap \beta W_2] \cap h_u[X \cap \beta W_2] p_1 p_0 \subseteq p_1(X \cap \beta W_1) \cap (X \cap \beta W_1)p_1$ so $h_u(p_2) \leq p_1$.

We now proceed to an inductive construction. Let $n \in \mathbb{N}$ with $n \geq 2$.

We shall introduce elements, (such as η_i or γ_i) which depend on n as well as on i. However, in an effort to reduce the number of subscripts used, we shall not indicate the dependence on n in the notation.

We make the inductive assumption that we have chosen p_i for $i \in \{0, 1, 2, \ldots n\}$, η_i, η'_i, δ_i, and δ'_i for $i \in \{1, 2, 3, \ldots, n-1\}$, and γ_i and γ'_i for $i \in \{2, 3, \ldots, n-2\}$, if any, so that the following hypotheses are satisfied.

(a) For each $i \in \{0, 1, \ldots, n\}$, p_i is a minimal idempotent of $X \cap \beta S_i$.
(b) For each $i \in \{1, 2, \ldots, n\}$, $p_i \leq p_{i-1}$ and $h_u(p_i) = p_{i-1}$ for every $u \in [A]\binom{i}{i-1}$.
(c) For every $i \in \{1, 2, \ldots, n-1\}$, η_i and η'_i are minimal idempotents in $X \cap \beta W_{n-1}$.
(d) For every $i \in \{1, 2, \ldots, n-1\}$, $\eta_i \in X p_{n-1}$ and $\eta'_i \in p_{n-1} X$.
(e) For $i \in \{1, 2, \ldots, n-1\}$, $\delta_i = h_{w_{n,n-i-1}}(\eta_i)$, $\delta'_i = h_{w_{n,n-i-1}}(\eta'_i)$,

$$p_n \in p_{n-1}\delta_1 \cdots \delta_{n-1} X, \text{ and}$$
$$p_n \in X\delta'_{n-1} \cdots \delta'_1 p_{n-1}\,.$$

(f) For every $i \in \{1, 2, \ldots, n-2\}$, if any,

$$\eta_i \in \gamma_i \cdots \gamma_{n-2}\eta_{n-1} X \text{ and}$$
$$\eta'_i \in X\eta'_{n-1}\gamma'_{n-2} \cdots \gamma'_i\,.$$

(g) For every choice of $u_{n,i} \in U_{n,i}$ for $i \in \{0, 1, \ldots, n-1\}$, the entry in the row labeled by u and the column labeled by x in the following tables is $h_u(x)$.

u $\backslash x:$	p_{n-1}	δ_1	δ_2	δ_3	$\ldots$	δ_{n-2}	δ_{n-1}
$u_{n,n-1}$	p_{n-1}						
$u_{n,n-2}$	p_{n-2}	η_1					
$u_{n,n-3}$	p_{n-2}	γ_1	η_2				
$u_{n,n-4}$	p_{n-2}	γ_1	γ_2	η_3			
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$		
$u_{n,1}$	p_{n-2}	γ_1	γ_2	γ_3	$\ldots$	η_{n-2}	
$u_{n,0}$	p_{n-2}	γ_1	γ_2	γ_3	$\ldots$	γ_{n-2}	η_{n-1}

Table 1

$u \setminus x:$	δ'_{n-1}	δ'_{n-2}	$\dots$	δ'_3	δ'_2	δ'_1	p_{n-1}
$u_{n,n-1}$							p_{n-1}
$u_{n,n-2}$						η'_1	p_{n-2}
$u_{n,n-3}$					η'_2	γ'_1	p_{n-2}
$u_{n,n-4}$				η'_3	γ'_2	γ'_1	p_{n-2}
$\vdots$			$\iddots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$u_{n,1}$		η'_{n-2}	$\dots$	γ'_3	γ'_2	γ'_1	p_{n-2}
$u_{n,0}$	η'_{n-1}	γ'_{n-2}	$\dots$	γ'_3	γ'_2	γ'_1	p_{n-2}

Table 2

We observe that these assumptions do hold if $n = 2$, with $\eta_1 = \eta'_1 = p_1$. For hypothesis (e), note that $\delta_1 = \delta'_1 = \alpha$. Hypothesis (f) is vacuous, and we have already verified the table entries of hypothesis (g).

Notice that since $h_{w_{n,n-i-1}}[W_{n-1}] \subseteq W_n$ one has that each $\delta_i \in X \cap \beta W_n$. Also, since $h_u[W_n] \subseteq W_{n-1}$ for each $u \in [A]\binom{n}{n-1}$, we have that each $\gamma_i \in X \cap \beta W_{n-1}$.

By assumption (e), $p_n \in p_{n-1}\delta_1 \cdots \delta_{n-1}X$. So there is some $x \in X$ such that $p_{n-1}\delta_1 \cdots \delta_{n-1}x = p_n = p_n p_n \in p_n X$. Such x is necessarily in βW_n because $p_n \in \beta W_n$. So

$$\{x \in X \cap \beta W_n : p_{n-1}\delta_1 \cdots \delta_{n-1}x \in p_n X\}$$

is nonempty and is therefore a right ideal of $X \cap \beta W_n$. So we can choose a minimal idempotent μ_n of $X \cap \beta W_n$ which is in this right ideal and in the left ideal $(X \cap \beta W_n)p_n$ of $X \cap \beta W_n$.

Now let $i \in \{2, 3, \dots, n-1\}$. Note that $\delta_i \cdots \delta_{n-1}\mu_n = \delta_i \cdots \delta_{n-1}\mu_n\mu_n$, so

$$\{x \in X \cap \beta W_n : p_{n-1}\delta_1\delta_2 \cdots \delta_{i-1}x \in p_n X \text{ and } x \in \delta_i \cdots \delta_{n-1}\mu_n X\}$$

is nonempty, because it contains $\delta_i \cdots \delta_{n-1}\mu_n$. It is therefore a right ideal of $X \cap \beta W_n$, and we can choose a minimal idempotent μ_i of $X \cap \beta W_n$ which is in this right ideal and is also in the left ideal $(X \cap \beta W_n)p_n$ of $X \cap \beta W_n$.

Similarly, $\{x \in X \cap \beta W_n : p_{n-1}x \in p_n X \text{ and } x \in \delta_1 \cdots \delta_{n-1}\mu_n X\}$ is nonempty because $\delta_1 \cdots \delta_{n-1}\mu_n$ is a member, and thus we may choose a minimal idempotent μ_1 of $X \cap \beta W_n$ which is in this right ideal of βW_n and also in the left ideal $(X \cap \beta W_n)p_n$.

Thus we have chosen minimal idempotents $\mu_1, \mu_2, \ldots, \mu_n$ in βW_n which satisfy the following conditions:

$$(*)\qquad \begin{array}{l} \mu_i \in X p_n \text{ for all } i \in \{1,2,\ldots,n\}; \\ p_{n-1}\delta_1 \cdots \delta_{i-1}\mu_i \in p_n X \text{ for all } i \in \{2,3,\ldots,n\}; \\ p_{n-1}\mu_1 \in p_n X; \text{ and} \\ \mu_i \in \delta_i \cdots \delta_{n-1}\mu_n X \text{ for all } i \in \{1,2,3,\ldots,n-1\}\,. \end{array}$$

By a left-right switch of these arguments, we can chose minimal idempotents $\mu_1', \mu_2', \ldots, \mu_n'$ in βW_n which satisfy the following conditions:

$$(**)\qquad \begin{array}{l} \mu_i' \in p_n X \text{ for all } i \in \{1,2,\ldots,n\}; \\ \mu_i'\delta_{i-1}' \cdots \delta_1' p_{n-1} \in X p_n \text{ for all } i \in \{2,3,\ldots,n\}; \\ \mu_1' p_{n-1} \in X p_n; \text{ and} \\ \mu_i' \in X \mu_n' \delta_{n-1}' \cdots \delta_i' \text{ for all } i \in \{1,2,3,\ldots,n-1\}\,. \end{array}$$

(While βW is right topological and not left topological, all of the algebraic facts that we are using in this proof are valid from both sides.)

For $i \in \{1,2,\ldots,n\}$, let $\epsilon_i = h_{w_{n+1,n-i}}(\mu_i)$, let $\epsilon_i' = h_{w_{n+1,n-i}}(\mu_i')$, and note that $\epsilon_i, \epsilon_i' \in X \cap \beta W_{n+1}$. Then $p_n\epsilon_1 \cdots \epsilon_n(X \cap \beta W_{n+1})$ and $(X \cap \beta W_{n+1})\epsilon_n' \cdots \epsilon_1' p_n$ are respectively right and left ideals of $(X \cap \beta W_{n+1})$. Pick a minimal idempotent p_{n+1} of $(X \cap \beta W_{n+1})$ such that

$$p_{n+1} \in p_n\epsilon_1 \cdots \epsilon_n(X \cap \beta W_{n+1}) \cap (X \cap \beta W_{n+1})\epsilon_n' \cdots \epsilon_1' p_n\,.$$

Since $\{w \in W_{n+1} : v_n \text{ occurs in } w\} \in \epsilon_1$, $p_{n+1} \in \beta S_{n+1}$. Consequently, p_{n+1} is minimal in $X \cap \beta S_{n+1}$.

We now claim that the induction hypotheses are satisfied for $n+1$ with η_i, η_i', δ_i, δ_i', γ_i, and γ_i' replaced by μ_i, μ_i', ϵ_i, ϵ_i', δ_i, and δ_i' respectively. That is, we claim that

(a) For each $i \in \{0,1,\ldots,n+1\}$, p_i is a minimal idempotent of $X \cap \beta S_i$.
(b) For each $i \in \{1,2,\ldots,n+1\}$, $p_i \leq p_{i-1}$ and $h_u(p_i) = p_{i-1}$ for every $u \in [A]\binom{i}{i-1}$.
(c) For every $i \in \{1,2,\ldots,n\}$, μ_i and μ_i' are minimal idempotents in $X \cap \beta W_n$.
(d) For every $i \in \{1,2,\ldots,n\}$, $\mu_i \in X p_n$ and $\mu_i' \in p_n X$.

(e) For $i \in \{1, 2, \ldots, n\}$, $\epsilon_i = h_{w_{n+1,n-i}}(\mu_i)$, $\epsilon'_i = h_{w_{n+1,n-i}}(\mu'_i)$,

$$p_{n+1} \in p_n \epsilon_1 \cdots \epsilon_n X, \text{ and}$$
$$p_{n+1} \in X \epsilon'_n \cdots \epsilon'_1 p_n\,.$$

(f) For every $i \in \{1, 2, \ldots, n-1\}$,

$$\mu_i \in \delta_i \cdots \delta_{n-1} \mu_n X \text{ and}$$
$$\mu'_i \in X \mu'_n \delta'_{n-1} \cdots \delta'_i\,.$$

(g) For every choice of $u_{n+1,i} \in U_{n+1,i}$ for $i \in \{0, 1, \ldots, n\}$, the entry in the row labeled by u and the column labeled by x in the following tables is $h_u(x)$.

u \ x:	p_n	ϵ_1	ϵ_2	ϵ_3	$\ldots$	ϵ_{n-1}	ϵ_n
$u_{n+1,n}$	p_n						
$u_{n+1,n-1}$	p_{n-1}	μ_1					
$u_{n+1,n-2}$	p_{n-1}	δ_1	μ_2				
$u_{n+1,n-3}$	p_{n-1}	δ_1	δ_2	μ_3			
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$		
$u_{n+1,1}$	p_{n-1}	δ_1	δ_2	δ_3	$\ldots$	μ_{n-1}	
$u_{n+1,0}$	p_{n-1}	δ_1	δ_2	δ_3	$\ldots$	δ_{n-1}	μ_n

Table 3

u \ x:	ϵ'_n	ϵ'_{n-1}	$\ldots$	ϵ'_3	ϵ'_2	ϵ'_1	p_n
$u_{n+1,n}$							p_n
$u_{n+1,n-1}$						μ'_1	p_{n-1}
$u_{n+1,n-2}$					μ'_2	δ'_1	p_{n-1}
$u_{n+1,n-3}$				μ'_3	δ'_2	δ'_1	p_{n-1}
$\vdots$			$\iddots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$u_{n+1,1}$		μ'_{n-1}	$\ldots$	δ'_3	δ'_2	δ'_1	p_{n-1}
$u_{n+1,0}$	μ'_n	δ'_{n-1}	$\ldots$	δ'_3	δ'_2	δ'_1	p_{n-1}

Table 4

All of these conclusions can be easily verified except (g) and the assertion in (b) that $h_u(p_{n+1}) = h_u(p_n)$ for all $u \in [A]\binom{n}{n-1}$. We show first that this latter assertion follows from statement (g).

For any $i \in \{0, 1, \ldots, n\}$, $h_{u_{n+1,i}}(p_{n+1}) \in X \cap \beta S_n$ and p_n is minimal in $X \cap \beta S_n$, so it suffices to show that $h_{u_{n+1,i}}(p_{n+1}) \leq p_n$.

Since $p_{n+1} \leq p_n$ and $h_{u_{n+1,n}}$ is the identity on W_n, we have that $h_{u_{n+1,n}}(p_{n+1}) \leq h_{u_{n+1,n}}(p_n) = p_n$.

Now let $i \in \{0, 1, \ldots, n-1\}$ and let $u = u_{n+1,i}$. We have $p_{n+1} \in p_n\epsilon_1 \cdots \epsilon_{n-i}X$ and so $h_u(p_{n+1}) \in h_u(p_n\epsilon_1 \cdots \epsilon_{n-i})X$ and by $(*)$ and Table 3, $h_u(p_n\epsilon_1 \cdots \epsilon_{n-i}) \in p_nX$. Also $p_{n+1} \in X\epsilon'_{n-i} \cdots \epsilon'_1 p_n$ so $h_u(p_{n+1}) \in Xh_u(\epsilon'_{n-i} \cdots \epsilon'_1 p_n)$ and by $(**)$ and Table 4,

$$h_u(\epsilon'_{n-i} \cdots \epsilon'_1 p_n) \in Xp_n \,.$$

It thus suffices to verify the entries of Table 3 and Table 4. We shall write out the verification for Table 3. The verification for Table 4 follows by a left-right switch of the arguments. To this end, let a choice of $u_{n+1,i} \in U_{n+1,i}$ for $i \in \{0, 1, \ldots, n\}$ be given.

We have that $h_{u_{n+1,n}}$ is the identity on S_n so $h_{u_{n+1,n}}(p_n) = p_n$. For $i \in \{0, 1, \ldots, n-1\}$, $h_{u_{n+1,i}} = h_{u_{n,i}}$ on S_n so $h_{u_{n+1,i}}(p_n) = h_{u_{n,i}}(p_n) = p_{n-1}$ by hypothesis (b).

The diagonal entries are correct because $\epsilon_i = h_{w_{n+1,n-i}}(\mu_i)$ for $i \in \{1, 2, \ldots, n\}$ and $h_{u_{n+1,n-i}} \circ h_{w_{n+1,n-i}}$ is the identity on W_n.

Let $k \in \{1, 2, \ldots, n-1\}$, let $i \in \{0, 1, \ldots, n-k-1\}$, and let $u \in U_{n+1,i}$. To finish the proof we need to show that $h_u(\epsilon_k) = \delta_k$. Now $\epsilon_k = h_{w_{n+1,n-k}}(\mu_k)$ so we are showing that $h_u\big(h_{w_{n+1,n-k}}(\mu_k)\big) = \delta_k$. Since $i < n-k$, we have that

$$h_u\big(h_{w_{n+1,n-k}}(\mu_k)\big) = h_{w_{n,n-k-1}}\big(h_u(\mu_k)\big) \,.$$

So it suffices to show that

$$h_{w_{n,n-k-1}}\big(h_u(\mu_k)\big) = \delta_k \,.$$

Now $h_{w_{n,n-k-1}}(\eta_k) = \delta_k$ by hypothesis (e), so it suffices to show that $h_u(\mu_k) = \eta_k$. And since $h_u(\mu_k)$ and η_k are idempotents in $X \cap \beta W_{n-1}$ and η_k is minimal in $X \cap \beta W_{n-1}$ it suffices to show that $h_u(\mu_k) \leq \eta_k$.

Now $\mu_k \in Xp_n$ by $(*)$ so that $h_u(\mu_k) \in Xh_u(p_n) = Xp_{n-1}$, the equality holding by hypothesis (b). Since $\eta_k \in Xp_{n-1}$ by hypothesis (d), $\eta_k = \eta_k p_{n-1} \in (X \cap \beta W_{n-1})p_{n-1}$. Since $(X \cap \beta W_{n-1})p_{n-1}$ is a minimal left ideal of $X \cap \beta W_{n-1}$, $(X \cap \beta W_{n-1})\eta_k = (X \cap \beta W_{n-1})p_{n-1}$. Thus we have that $h_u(\mu_k) = h_u(\mu_k)p_{n-1} \in (X \cap \beta W_{n-1})p_{n-1} = (X \cap \beta W_{n-1})\eta_k$.

It remains to show that $h_u(\mu_k) \in \eta_k X$. We have by $(*)$ that $\mu_k \in \delta_k \cdots \delta_{n-1}\mu_n X$. If $i = n-k-1$, we have that $h_u(\mu_k) \in h_u(\delta_k)X = \eta_k X$ by hypothesis (g), so assume that $i < n-k-1$. Then $h_u(\mu_k) \in$

$h_u(\delta_k)\cdots h_u(\delta_{n-i-1})X = \gamma_k\cdots\gamma_{n-i-2}\eta_{n-i-1}X$, the equality holding by hypothesis (g). If $i = 0$, we have directly that $h_u(\mu_k) \in \gamma_k\cdots\gamma_{n-2}\eta_{n-1}X$. Otherwise $\eta_{n-i-1} \in \gamma_{n-i-1}\cdots\gamma_{n-2}\eta_{n-1}X$ by hypothesis (f) so again $h_u(\mu_k) \in \gamma_k\cdots\gamma_{n-2}\eta_{n-1}X$. Also $\eta_k \in \gamma_k\cdots\gamma_{n-2}\eta_{n-1}X$ by hypothesis (f). Now $\eta_{n-1} \in K\big(X \cap \beta W_{n-1}\big)$ and $\gamma_k\cdots\gamma_{n-2} \in X \cap \beta W_{n-1}$ so $\gamma_k\cdots\gamma_{n-2}\eta_{n-1} \in K\big(X \cap \beta W_{n-1}\big)$ and thus as in the previous paragraph, $h_u(\mu_k) \in \eta_k X$. □

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