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SOME PROPERTIES OF CARTESIAN PRODUCTS AND STONE-ČECH COMPACTIFICATIONS

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ABSTRACT. Given a discrete space S , the Stone-Čech compactification βS of S consists of all of the ultrafilters on S . If $p \in \beta S$ and $q \in \beta T$, then the *tensor product*, $p \otimes q \in \beta(S \times T)$. If (S, \cdot) is a semigroup and $p, q \in \beta S$, then $p \otimes q$ is intimately related to the algebraic product $p \cdot q$. We investigate tensor products in this paper, showing among other things, that tensor products are topologically rare. For example, $S^* \otimes T^*$ is nowhere dense in $\beta(S \times T)$, where $S^* = \beta S \setminus S$.

We also investigate Cartesian products of Stone-Čech compactifications, considering the question of whether, given semigroups (S, \cdot) and (T, \cdot) , $(\beta S)^u$ and $(\beta T)^v$ can be isomorphic for distinct positive integers u and v . We obtain conditions guaranteeing that the answer is “no” as well as some examples where the answer is “yes”.

1. INTRODUCTION

The tensor product of two ultrafilters is a special case of the notion of the *sum of ultrafilters* introduced by Frolík in paragraph 1.2 of [7].

Definition 1.1. Let S and T be discrete spaces, let $p \in \beta S$, and let $q \in \beta T$. Then the *tensor product of p and q* is defined by

$$p \otimes q = \{A \subseteq S \times T : \{x \in S : \{y \in T : (x, y) \in A\} \in q\} \in p\}.$$

It is an easy exercise to show that $p \otimes q$ is an ultrafilter on $S \times T$.

More generally, we have the following. (We take \mathbb{N} to be the set of positive integers; ω , the first infinite cardinal, we take to be $\mathbb{N} \cup \{0\}$.)

Definition 1.2. Let $k \in \mathbb{N} \setminus \{1\}$. For each $i \in \{1, 2, \dots, k\}$, let S_i be a discrete space and let $p_i \in \beta S_i$. The *tensor product of p_1, p_2, \dots, p_k* is

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defined by

$$p_1 \otimes p_2 \otimes \cdots \otimes p_k = \left\{ A \subseteq \times_{i=1}^k S_i : \begin{aligned} & \{x_1 \in S_1 : \{x_2 \in S_2 : \dots \{x_k \in S_k : \\ & (x_1, x_2, \dots, x_k) \in A\} \in p_k\} \in p_{k-1}\} \dots \} \in p_1 \right\}.$$

The tensor product is, in a certain sense, associative. For example, let discrete spaces S_1 , S_2 , and S_3 , and $p_1 \in \beta S_1$, $p_2 \in \beta S_2$, and $p_3 \in \beta S_3$ be given. Then

$$(p_1 \otimes p_2) \otimes p_3 \in \beta((S_1 \times S_2) \times S_3) \text{ and } p_1 \otimes (p_2 \otimes p_3) \in \beta(S_1 \times (S_2 \times S_3)).$$

If $f : \beta((S_1 \times S_2) \times S_3) \rightarrow \beta(S_1 \times S_2 \times S_3)$ is the continuous extension of the function $((x_1, x_2), x_3) \mapsto (x_1, x_2, x_3)$ and $g : \beta(S_1 \times (S_2 \times S_3)) \rightarrow \beta(S_1 \times S_2 \times S_3)$ is the continuous extension of the function $(x_1, (x_2, x_3)) \mapsto (x_1, x_2, x_3)$, then $f((p_1 \otimes p_2) \otimes p_3) = g(p_1 \otimes (p_2 \otimes p_3)) = p_1 \otimes p_2 \otimes p_3$.

Some of our results about tensor products deal with the algebraic structure of βS , as do all of our results about Cartesian products of Stone-Ćech compactifications. We give a brief introduction to that structure now. For a detailed elementary development see [14, Part I].

Let S be a discrete space. As we mentioned in the abstract, we take the points of βS to be the ultrafilters on S , identifying the points of S with the principal ultrafilters. If C is a compact Hausdorff space, and $f : S \rightarrow C$, there is a unique continuous extension from $\beta S \rightarrow C$ which we will denote by \tilde{f} . If $f : S \rightarrow C$ and C is not compact, we will write $f : S \rightarrow C \subseteq D$, where D is compact to indicate that $\tilde{f} : \beta S \rightarrow D$. For example, in Definition 2.18 we write $g : \mathbb{R}_d \times \mathbb{R}_d \rightarrow \mathbb{Z} \times \mathbb{Z} \subseteq \beta(\mathbb{Z} \times \mathbb{Z})$ to indicate that $\tilde{g} : \beta(\mathbb{R}_d \times \mathbb{R}_d) \rightarrow \beta(\mathbb{Z} \times \mathbb{Z})$, while in Theorem 2.20 we write $\iota : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N} \times \mathbb{N} \subseteq \beta\mathbb{N} \times \beta\mathbb{N}$ to indicate that $\tilde{\iota} : \beta(\mathbb{N} \times \mathbb{N}) \rightarrow \beta\mathbb{N} \times \beta\mathbb{N}$.

Let S be a discrete space and let $*$ be a binary operation on S . The operation on S extends to an operation on βS , also denoted by $*$, so that for each $p \in \beta S$, the function ρ_p is continuous and for each $x \in S$, λ_x is continuous, where for $q \in \beta S$, $\rho_p(q) = q * p$ and $\lambda_x(q) = x * q$. If the operation on S is associative, so is its extension. That is, if $(S, *)$ is a semigroup, then $(\beta S, *)$ is a compact Hausdorff right topological semigroup with S contained in its topological center. The operation may be characterized in terms of limits by, for $p, q \in \beta S$, $p * q = \lim_{s \rightarrow p} \lim_{t \rightarrow q} s * t$, where s and t denote members of S and the limits are computed in βS . The operation may also be characterized by the fact that for $A \subseteq S$, $A \in p * q$ if and only if $\{s \in S : s^{-1}A \in q\} \in p$, where $s^{-1}A = \{t \in S : s * t \in A\}$.

Now let (T, \cdot) be a compact Hausdorff right topological semigroup. Then T has a smallest two sided ideal $K(T)$ which is the union of all

minimal left ideals of T and is the union of all of the minimal right ideals of T . If L is a minimal left ideal of T and R is a minimal right ideal of T , then $L \cap R$ is a group, and any two such groups are isomorphic. In particular, T has idempotents.

The set $E(T)$ of idempotents of T is partially ordered by the relation \leq defined by $p \leq q$ if and only if $p \cdot q = q \cdot p = p$. An idempotent p is minimal with respect to this order if and only if $p \in K(T)$.

Given a set X , we let $\mathcal{P}_f(X)$ be the set of finite nonempty subsets of X . If $\langle x_n \rangle_{n=1}^\infty$ is a sequence in a semigroup (S, \cdot) we let $FP(\langle x_n \rangle_{n=1}^\infty) = \{\prod_{t \in F} x_t : F \in \mathcal{P}_f(\mathbb{N})\}$ where $\prod_{t \in F} x_t$ is computed in increasing order of indices.

In Section 2 we present our results about tensor products, including the fact that the set of tensor products is topologically small. We mentioned in the abstract that if S and T are infinite, then $S^* \otimes T^*$ is nowhere dense in $(S \times T)^*$. Further, if S and T are countably infinite, then $\beta S \otimes \beta T$ is not a Borel subset of $\beta(S \times T)$.

In Section 3 we present our results about Cartesian products of βS . It turns out to be critical whether or not βS has nonminimal idempotents. In Theorem 3.2 we show that if S and T are countable left cancellative semigroups and either βS or βT has a nonminimal idempotent, then for any distinct $u, v \in \mathbb{N}$, $(\beta S)^u$ is not isomorphic to $(\beta T)^v$. On the other hand, there are easy examples showing that such a result may fail if all idempotents of βS are minimal. We characterize those left cancellative semigroups S for which all idempotents of βS are minimal.

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2. TENSOR PRODUCTS

If S and T are arbitrary discrete spaces, $p \in \beta S$, and $q \in \beta T$, the tensor product $p \otimes q$ can be characterized in terms of limits as follows. We have that $p \otimes q = \lim_{s \rightarrow p} \lim_{t \rightarrow q} (s, t)$, where s denotes a member of S and t denotes a member of T and the limits are in the space $\beta(S \times T)$.

It follows at once from Definition 1.1 or from the above characterization that the map $p \mapsto p \otimes q$ from $\beta S \rightarrow \beta(S \times T)$ is continuous for every $q \in \beta T$, and the map $q \mapsto s \otimes q$ from βT to $\beta(S \times T)$ is continuous for every $s \in S$.

We note a fundamental property of tensor products, which follows immediately from their characterization in terms of limits. Let S and T be arbitrary discrete spaces and let $f : S \rightarrow S \subseteq \beta S$ and $g : T \rightarrow T \subseteq \beta T$ be arbitrary maps. Define $h : S \times T \rightarrow S \times T \subseteq \beta(S \times T)$ by $h(s, t) = (f(s), g(t))$. Then for every $p \in \beta S$ and $q \in \beta T$, $\tilde{h}(p \otimes q) = \tilde{f}(p) \otimes \tilde{g}(q)$.

Furthermore, \tilde{h} is injective if f and g are injective, and \tilde{h} is surjective if f and g are surjective, by [14, Exercise 3.4.1]. If S and T are semigroups and f and g are homomorphisms, then \tilde{h} is also a homomorphism, by [14, Theorem 4.8].

Of course, if $p \in \beta S$ and $q \in \beta T$, we can also define $p \odot q \in \beta(S \times T)$ by

$$p \odot q = \{A \subseteq S \times T : \{y \in T : \{x \in S : (x, y) \in A\} \in p\} \in q\}.$$

In terms of limits, this operation is characterized by reversing the order of the limits so that $p \odot q = \lim_{t \rightarrow q} \lim_{s \rightarrow p} (s, t)$, where s denotes a member of S and t denotes a member of T .

To simplify the presentation, in our results we shall assume that $S = T$, the corresponding more general statement usually being obvious.

If S is an arbitrary set and $*$ is a binary operation on S , we have observed that $*$ can be extended to a binary operation on βS . If $\sigma : S \times S \rightarrow S \subseteq \beta S$ is defined by $\sigma(s, t) = s * t$, then, for every $p, q \in \beta S$, $\tilde{\sigma}(p \otimes q) = p * q$. We shall now see that this property characterizes the tensor product. For $i \in \{1, 2\}$ we shall let $\pi_i : S \times S \rightarrow S \subseteq \beta S$ be the projection map. Note that for any $p, q \in \beta S$, $\tilde{\pi}_1(p \otimes q) = p$ and $\tilde{\pi}_2(p \otimes q) = q$.

Theorem 2.1. *Let S be an infinite set and let $r \in \beta(S \times S)$. The following statements are equivalent.*

- (a) $r = \tilde{\pi}_1(r) \otimes \tilde{\pi}_2(r)$.
- (b) *Whenever $*$ is a binary operation on S and $\sigma : S \times S \rightarrow S \subseteq \beta S$ satisfies $\sigma(s, t) = s * t$ for s and t in S , then $\tilde{\sigma}(r) = \tilde{\pi}_1(r) * \tilde{\pi}_2(r)$.*

Proof. Let $p = \tilde{\pi}_1(r)$ and let $q = \tilde{\pi}_2(r)$.

(a) \Rightarrow (b). Assume that $r = p \otimes q$.

$$\tilde{\sigma}(p \otimes q) = \tilde{\sigma}\left(\lim_{s \rightarrow p} \lim_{t \rightarrow q} (s, t)\right) = \lim_{s \rightarrow p} \lim_{t \rightarrow q} \sigma(s, t) = \lim_{s \rightarrow p} \lim_{t \rightarrow q} s * t = p * q.$$

(b) \Rightarrow (a). Pick an injection $\sigma : S \times S \rightarrow S \subseteq \beta S$ and define a binary operation $*$ on S by $s * t = \sigma(s, t)$. Assume that $\tilde{\sigma}(r) = p * q$. Then

$$\tilde{\sigma}(r) = \lim_{s \rightarrow p} \lim_{t \rightarrow q} s * t = \lim_{s \rightarrow p} \lim_{t \rightarrow q} \sigma(s, t) = \tilde{\sigma}\left(\lim_{s \rightarrow p} \lim_{t \rightarrow q} (s, t)\right) = \tilde{\sigma}(p \otimes q),$$

where s and t denote elements of S . By [14, Exercise 3.4.1], $\tilde{\sigma}$ is injective, so $r = p \otimes q$. \square

If S is an infinite discrete space of cardinality at least κ , we say that an ultrafilter on S is κ -uniform if all its members have cardinality at

least κ . We let $U_\kappa(S)$ denote the space of κ -uniform ultrafilters on S . In particular, $U_\omega(S) = S^*$. If $p \in S^*$, $\|p\|$ will denote $\min(\{|P| : P \in p\})$.

Theorem 2.2. *Let S denote an infinite discrete space of cardinality κ .*

- (1) *If $p, q \in S^*$ and $\|p\| = \|q\|$, then $p \otimes q \neq p \odot q$.*
- (2) *If $p \in S^*$, $p \otimes p \neq p \odot p$.*
- (3) *$\mathcal{cl}(U_\kappa(S) \otimes U_\kappa(S))$ does not meet $\mathcal{cl}(U_\kappa(S) \odot U_\kappa(S))$.*

Proof. (1) Suppose that $\|p\| = \|q\| = \lambda$. We can choose members P and Q of p and q respectively for which $|P| = |Q| = \lambda$. We enumerate P as $\langle s_i \rangle_{i < \lambda}$ and Q as $\langle t_i \rangle_{i < \lambda}$. Then $\{(s_i, t_j) : i < j\} \in p \otimes q$ and $\{(s_i, t_j) : i > j\} \in p \odot q$.

(2) This is immediate from (1).

(3) We enumerate S as $\langle u_i \rangle_{i < \kappa}$. Then $U_\kappa(S) \otimes U_\kappa(S) \subseteq \mathcal{cl}(\{(u_i, u_j) : i < j\})$ and $U_\kappa(S) \odot U_\kappa(S) \subseteq \mathcal{cl}(\{(u_i, u_j) : i > j\})$. \square

We turn our attention to two results showing that the set of tensor products is topologically small. Notice that if $\omega \leq \kappa \leq |S|$, then $\beta S \otimes U_\kappa(S) \subseteq U_\kappa(S \times S)$ and $U_\kappa(S) \otimes \beta S \subseteq U_\kappa(S \times S)$. Given a set X and a cardinal κ , we let $[X]^\kappa = \{A \subseteq X : |A| = \kappa\}$.

Theorem 2.3. *If S is an infinite discrete space, and $\omega \leq \kappa \leq |S|$, then $U_\kappa(S) \otimes U_\kappa(S)$ is nowhere dense in $U_\kappa(S \times S)$. In particular $S^* \otimes S^*$ is nowhere dense in $(S \times S)^*$.*

Proof. Let $X = U_\kappa(S) \otimes U_\kappa(S)$. It suffices to show that whenever $A \in [S \times S]^\kappa$, there exists $D \in [A]^\kappa$ such that $\overline{D} \cap X = \emptyset$, so let $A \in [S \times S]^\kappa$ be given. If $\overline{A} \cap X = \emptyset$, we may let $D = A$, so assume $\overline{A} \cap X \neq \emptyset$ and pick $p, q \in U_\kappa(S)$ such that $A \in p \otimes q$. Let $B = \{s \in S : \{t \in S : (s, t) \in A\} \in q\}$. Then $B \in p$ so $|B| \geq \kappa$. For $b \in B$, let $C_b = \{t \in S : (b, t) \in A\}$. Then $C_b \in q$ so $|C_b| = \kappa$.

Enumerate B as $\langle b(\sigma) \rangle_{\sigma < \kappa}$. Choose $c(0) \in C_{b(0)}$. Let $0 < \tau < \kappa$ and assume we have chosen $\langle c(\sigma) \rangle_{\sigma < \tau}$. Pick $c(\tau) \in C_{b(\tau)} \setminus \{c(\sigma) : \sigma < \tau\}$. Let $D = \{(b(\sigma), c(\sigma)) : \sigma < \kappa\}$. Then $\overline{D} \cap X = \emptyset$. \square

Theorem 2.4. *Let S be a countably infinite discrete space. Then $\beta S \otimes \beta S$ is not a Borel subset of $\beta(S \times S)$.*

Proof. Pick a bijection $f : \mathbb{N} \rightarrow S$, define $g : \mathbb{N} \times \mathbb{N} \rightarrow S \times S \subseteq \beta(S \times S)$ by $g(n, m) = (f(n), f(m))$. Then \tilde{g} is a homeomorphism, $\beta S \otimes \beta S = \tilde{g}[\beta \mathbb{N} \otimes \beta \mathbb{N}]$, and $\beta(S \times S) = \tilde{g}[\beta(\mathbb{N} \times \mathbb{N})]$. So it suffices to prove that $\beta \mathbb{N} \otimes \beta \mathbb{N}$ is not a Borel subset of $\beta(\mathbb{N} \times \mathbb{N})$.

We show first that $\mathbb{N}^* \otimes \mathbb{N}^*$ is not a Borel subset of $\beta(\mathbb{N} \times \mathbb{N})$.

In [15, Lemma 3.1] it was shown that every Borel subset of $\beta \mathbb{N}$ is the union of at most \mathfrak{c} compact sets. Since $\beta(\mathbb{N} \times \mathbb{N})$ is homeomorphic to $\beta \mathbb{N}$,

the same assertion applies to $\beta(\mathbb{N} \times \mathbb{N})$. Suppose that $\mathbb{N}^* \otimes \mathbb{N}^*$ is a Borel subset of $\beta(\mathbb{N} \times \mathbb{N})$ and is thus the union of at most \mathfrak{c} compact sets. We have seen that if $\sigma : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N} \subseteq \beta\mathbb{N}$ is defined by $\sigma(n, m) = n + m$, then for any $p, q \in \beta\mathbb{N}$, $\tilde{\sigma}(p \otimes q) = p + q$, so $\mathbb{N}^* + \mathbb{N}^*$ is the union of at most \mathfrak{c} compact sets. It was shown in [15, Theorem 3.5] that $\mathbb{N}^* + \mathbb{N}^*$ is not the union of at most \mathfrak{c} compact sets, so this is a contradiction.

To see that $\beta\mathbb{N} \otimes \beta\mathbb{N}$ is not a Borel subset of $\beta(\mathbb{N} \times \mathbb{N})$, note that $\mathbb{N}^* \otimes \mathbb{N}^* = \beta\mathbb{N} \otimes \beta\mathbb{N} \setminus (\{(n, m) : n, m \in \mathbb{N}\} \cup \bigcup_{n \in \mathbb{N}} (n \otimes \mathbb{N}^*) \cup \bigcup_{n \in \mathbb{N}} (\mathbb{N}^* \otimes n))$ and $\{(n, m) : n, m \in \mathbb{N}\} \cup \bigcup_{n \in \mathbb{N}} (n \otimes \mathbb{N}^*) \cup \bigcup_{n \in \mathbb{N}} (\mathbb{N}^* \otimes n)$ is a countable union of compact sets, so is Borel. Thus if $\beta\mathbb{N} \otimes \beta\mathbb{N}$ were Borel, $\mathbb{N}^* \otimes \mathbb{N}^*$ would also be Borel. \square

Corollary 2.5. *Let S be an infinite set. Then $\beta S \otimes \beta S$ is not a Borel subset of $\beta(S \times S)$.*

Proof. Choose an infinite countable subset A of S , and let $B = (\beta S \otimes \beta S) \cap \text{cl}_{\beta(S \times S)}(A \times A)$. If $\beta S \otimes \beta S$ were Borel, then B would also be Borel. We claim that $B = \text{cl}_{\beta S}(A) \otimes \text{cl}_{\beta S}(A)$. To see this, observe that, if $p, q \in \beta S$ and $p \otimes q \in B$, then $\tilde{\pi}_1(p \otimes q) = p \in \text{cl}_{\beta S}(A)$ and $\tilde{\pi}_2(p \otimes q) = q \in \text{cl}_{\beta S}(A)$. So $p \otimes q \in \text{cl}_{\beta S}(A) \otimes \text{cl}_{\beta S}(A)$. Thus $B \subseteq \text{cl}_{\beta S}(A) \otimes \text{cl}_{\beta S}(A)$, and the reverse inclusion is straightforward. It follows from Theorem 2.4 that B is not Borel. \square

We now explore the algebraic properties of $\beta S \otimes \beta S$. We will in particular be interested in idempotents in $S^* \otimes S^*$. We leave the easy proof of the following remark to the reader.

Remark 2.6. Let G be an infinite group with identity e and let r be an idempotent in $\beta(G \times G)$. Then one of the following statements holds.

- (a) $r = (e, e)$;
- (b) $\tilde{\pi}_1(r) = e$ and $\tilde{\pi}_2(r)$ is an idempotent in G^* , in which case $r = e \otimes \tilde{\pi}_2(r)$;
- (c) $\tilde{\pi}_2(r) = e$ and $\tilde{\pi}_1(r)$ is an idempotent in G^* , in which case $r = \tilde{\pi}_1(r) \otimes e$; or
- (d) $\tilde{\pi}_1(r)$ and $\tilde{\pi}_2(r)$ are idempotents in G^* .

Recall that a semigroup S is *weakly left cancellative* provided $\{x \in S : a \cdot x = b\}$ is finite for all $a, b \in S$; it is *weakly right cancellative* provided $\{x \in S : x \cdot a = b\}$ is finite for all $a, b \in S$.

Theorem 2.7. *Let S be a countably infinite semigroup which is weakly left cancellative, right cancellative, and has only a finite number of right identities. Let E denote the set of right identities of S , let $p, q \in S^*$, and let $r = p \otimes q$. If u is a left identity for r in $\beta(S \times S)$, then $\tilde{\pi}_2(u) \in E$.*

Proof. Assume we have $u \in \beta(S \times S)$ such that $r = u \cdot r$. It suffices to show that $S \times E \in u$, so suppose that $S \times (S \setminus E) \in u$.

We first show that $s \cdot q \neq q$ if $s \in S \setminus E$. To see this, suppose that $s \cdot q = q$. Then $\{t \in S : s \cdot t = t\} \in q$ by [14, Theorem 3.35]. Pick $t \in S$ such that $s \cdot t = t$. Then for every $v \in S$, $v \cdot s \cdot t = v \cdot t$ and so $v \cdot s = v$. Thus $s \in E$.

Let $S' = S \setminus E$, enumerate the elements of S' as a sequence, and write $s \prec t$ if s precedes t in this sequence. Since E is finite, $S' \in q$. Given $d \in S'$, pick $A_d \in (d \cdot q) \setminus q$. For $s \in S'$, let $Q_s = S' \cap \bigcap \{(d^{-1}A_d) \cap (S \setminus A_d) : d \in S' \text{ and } d \prec s\}$. Then $Q_s \in q$ and if $d \in S'$ and $d \prec s$, then $d \cdot Q_s \cap Q_s = \emptyset$.

Let $A = \bigcup_{s \in S'} (\{s\} \times Q_s)$. Since E is finite and $p \in S^*$, $A \in p \otimes q = u \cdot r$ so that $\{(c, d) \in S \times S : (c, d)^{-1}A \in r\} \in u$. Since also $S \times S' \in u$ we may pick $(c, d) \in S \times S'$ such that $(c, d)^{-1}A \in r$.

Let $D = \{(s, t) \in S' \times S' : c \cdot s \in S', d \prec c \cdot s, \text{ and } t \in Q_{c \cdot s}\}$. We claim that $D \in r$. To see this, let $H = \{s \in S : \{t \in S : (s, t) \in D\} \in q\}$. We want to show that $H \in p$. Let $I = \{s \in S' : c \cdot s \in S' \text{ and } d \prec c \cdot s\}$. To show that $H \in p$, it suffices that I is cofinite and $I \subseteq H$. Since E is finite and for $a \in E$, $\{s \in S : c \cdot s = a\}$ is finite we have that $\{s \in S : c \cdot s \in E\}$ is finite. Likewise, $\{a \in S' : a \prec d\}$ is finite so $\{s \in S : c \cdot s \in S' \text{ and } c \cdot s \preceq d\}$ is finite so I is cofinite as claimed. To see that $I \subseteq H$, let $s \in I$. Then $Q_{c \cdot s} \in q$ and $Q_{c \cdot s} \subseteq \{t \in S : (s, t) \in D\}$ so $s \in H$ as required.

Pick $(s, t) \in (c, d)^{-1}A \cap D$. Then $(c, d) \cdot (s, t) \in A$ so pick $s' \in S'$ and $t' \in Q_{s'}$ such that $(c, d) \cdot (s, t) = (s', t')$. Now $(s, t) \in D$ so $d \prec c \cdot s$, $c \cdot s \in S'$, and $t \in Q_{c \cdot s}$. Then $t' = d \cdot t \in d \cdot Q_{c \cdot s}$ while $t' \in Q_{s'} = Q_{c \cdot s}$ so $d \cdot Q_{c \cdot s} \cap Q_{c \cdot s} \neq \emptyset$, a contradiction. \square

Corollary 2.8. *If S is a countably infinite semigroup which is weakly left cancellative, right cancellative, and has only a finite number of right identities, then $S^* \otimes S^*$ contains no idempotents.*

Proof. Let $p, q \in S^*$ and let $r = p \otimes q$. If r is an idempotent, then by Theorem 2.7, $\tilde{\pi}_2(r)$ is a right identity for S , while $\tilde{\pi}_2(r) = q \in S^*$. \square

Corollary 2.9. *Let S be a countably infinite semigroup which is weakly left cancellative, right cancellative, and has only a finite number of right identities. Then $\beta S \otimes \beta S$ does not meet $K(\beta(S \times S))$.*

Proof. Let $p, q \in \beta S$ and let $r = p \otimes q$. By [14, Exercise 1.7.3],

$$\tilde{\pi}_1[K(\beta(S \times S))] = K(\beta S) \text{ and } \tilde{\pi}_2[K(\beta(S \times S))] = K(\beta S).$$

Assume that $r \in K(\beta(S \times S))$. Then $p = \tilde{\pi}_1(r)$ and $q = \tilde{\pi}_2(r)$ are in $K(\beta S)$ so that that $p, q \in S^*$ by [14, Theorem 4.36]. Since $r \in K(\beta(S \times S))$ which is a union of groups, there exists $u \in K(\beta(S \times S))$ such that

$r = u \cdot r$. But then by Theorem 2.7, $\tilde{\pi}_2(u) \in S$ while $\tilde{\pi}_2(u) \in K(\beta S) \subseteq S^*$. \square

We observe that we have an algebraic identity in the case in which (S, \cdot) is a semigroup, in which case $S \times S$ is also a semigroup.

Lemma 2.10. *Let (S, \cdot) be a semigroup, let $p, x, y \in \beta S$, and let $t \in S$. Then $(p \otimes t) \cdot (x \otimes y) = (p \cdot x) \otimes (t \cdot y)$.*

Proof. Since $\lim_{s \rightarrow p} \lim_{u \rightarrow x} \lim_{v \rightarrow y} (s, t) \cdot (u, v) = \lim_{s \rightarrow p} \lim_{u \rightarrow x} \lim_{v \rightarrow y} (s \cdot u, t \cdot v)$, where s, u, v denote elements of S , the conclusion follows. \square

Note that the above proof uses the fact that λ_t is continuous. The corresponding identity does not hold if $t \in S^*$. For example, if (S, \cdot) is a group and p is an idempotent in S^* , then by Corollary 2.8 one cannot have $(p \otimes p) \cdot (p \otimes p) = (p \cdot p) \otimes (p \cdot p)$.

Lemma 2.11. *Let S be an arbitrary semigroup and let $r \in \beta(S \times S)$. If $\tilde{\pi}_2(r) = t \in S$, then $r = \tilde{\pi}_1(r) \otimes t$.*

Proof. For every $R \in r$, we can choose $s_R \in \pi_1[R]$ for which $(s_R, t) \in R$. Direct r by reverse inclusion. Taking limits as the net $\langle s_R, t \rangle_{R \in r}$ converges to r , we obtain $r = \tilde{\pi}_1(r) \otimes t$. \square

Corollary 2.12. *Let S be a countable group and let $p, q \in S^*$. Then $p \otimes q$ is right cancelable in $\beta(S \times S)$ if and only if p is right cancelable in βS .*

Proof. Let e denote the identity of S . Let $r = p \otimes q$ and assume that r is not right cancellable in $\beta(S \times S)$. By the equivalence of (1) and (3) in [14, Theorem 8.18], there is an idempotent u in $(S \times S)^*$ for which $u \cdot r = r$. By Theorem 2.7, $\tilde{\pi}_2(u) = e$. It follows from Lemma 2.11 that $u = v \otimes e$ where $v = \tilde{\pi}_1(u) \in S^*$. Then by Lemma 2.10 $r = (v \otimes e) \cdot (p \otimes q) = (v \cdot p) \otimes q$, and so $p = \tilde{\pi}_1(r) = v \cdot p$. Thus p is not right cancelable in βS by the equivalence of (1) and (2) in [14, Theorem 8.18].

Conversely, if p is not right cancelable in βS , $p = w \cdot p$ for some $w \in S^*$. Then $r = (w \otimes e) \cdot (p \otimes q)$, and so r is not right cancellable in $\beta(S \times S)$. \square

Corollary 2.12 also holds if $S = \mathbb{N}$; but this requires a different proof since $(\mathbb{N} \times \mathbb{N})$ is not a group. (In [14, Theorem 8.18] it is assumed that S is either $(\mathbb{N}, +)$ or a countable group.)

In the proof of the following corollary, we shall assume that $\beta\mathbb{N}$ is embedded in $\beta\mathbb{Z}$ and that $\beta(\mathbb{N} \times \mathbb{N})$ is embedded in $\beta(\mathbb{Z} \times \mathbb{Z})$.

Corollary 2.13. *Let $p, q \in \mathbb{N}^*$. Then $p \otimes q$ is right cancelable in $\beta(\mathbb{N} \times \mathbb{N})$ if and only if p is right cancelable in $\beta\mathbb{N}$.*

Proof. Let $r = p \otimes q$, and assume that r is not right cancelable in $\beta(\mathbb{N} \times \mathbb{N})$. Then there exist distinct elements $x, y \in \beta(\mathbb{N} \times \mathbb{N})$ such that $x + r = y + r$. We can choose disjoint subsets X and Y of $\mathbb{N} \times \mathbb{N}$ which are members of x and y respectively. Then $x + r$ is in $cl(X + r) \cap cl(Y + r)$. By [14, Theorem 3.40], there exists $(a, b) \in X$ such that $(a, b) + r \in y' + r$ for some $y' \in cl(Y)$, or else there exists $(c, d) \in Y$ such that $(c, d) + r = x' + r$ for some $x' \in cl(X)$. Without loss of generality, we may assume the former. Note that it follows from [14, Lemma 6.28] that $y' \in (\mathbb{N} \times \mathbb{N})^*$. Now the equation $r = -(a, b) + y' + r$ holds in $\beta\mathbb{Z}$. By applying Theorem 2.7 with $S = \mathbb{Z}$, we see that $\tilde{\pi}_2(-(a, b) + y') = 0$ so $\tilde{\pi}_2(y') = b$. So $\tilde{\pi}_1(y') \in \mathbb{N}^*$ and hence $-a + \tilde{\pi}_1(y') \in \mathbb{N}^*$, because \mathbb{N}^* is a left ideal of $\beta\mathbb{Z}$ by [14, Exercise 4.3.5]. Let $v = -a + \tilde{\pi}_1(y')$. Then by Lemma 2.11, $-(a, b) + y' = v \otimes 0$. So $(v \otimes 0) + (p \otimes q) = p \otimes q$. Applying $\tilde{\pi}_1$ to this equation, we see that $v + p = p$, and hence that p is not right cancelable in $\beta\mathbb{N}$ by the equivalence of (1) and (2) in [14, Theorem 8.18].

Conversely, assume that p is not right cancelable in $\beta\mathbb{N}$. Then there exist distinct $w, z \in \beta\mathbb{N}$ such that $w + p = z + p$. So, using Lemma 2.10 twice, $(w \otimes 1) + r = (w + p) \otimes (1 + q) = (z + p) \otimes (1 + q) = (z \otimes 1) + r$, and r is not right cancelable in $\beta(\mathbb{N} \times \mathbb{N})$. \square

In [1] Argabright and Wilde showed that a left cancellative semigroup S is left amenable if and only if it satisfies the *strong Følner condition* (SFC):

$$(\forall H \in \mathcal{P}_f(S))(\forall \epsilon > 0)(\exists F \in \mathcal{P}_f(S))(\forall s \in H)(|F \Delta s \cdot F| < \epsilon \cdot |F|).$$

Semigroups satisfying SFC have a natural notion of density.

Definition 2.14. Let S be a semigroup which satisfies SFC. Then

$$d(A) = \sup \left\{ \alpha : (\forall H \in \mathcal{P}_f(S))(\forall \epsilon > 0)(\exists K \in \mathcal{P}_f(S))(|A \cap K| \geq \alpha \cdot |K| \text{ and } (\forall s \in H)(|K \Delta s \cdot K| < \epsilon \cdot |K|)) \right\}.$$

Definition 2.15. If S is a semigroup which satisfies SFC, $\Delta(S) = \{p \in \beta S : (\forall A \in \mathcal{P}) (d(A) > 0)\}$.

In the proof of the following theorem, we shall use the well-known fact that, if S is left cancellative and left amenable, a subset A of $S \times S$ has positive density if and only if there is a probability measure μ on βS , which is invariant under translations by elements of S and has the property that $\mu(\overline{A}) > 0$. (See, for example, [12, Theorems 4.7 and 4.16].)

Theorem 2.16. *Let S be a countably infinite, cancellative, left amenable semigroup. Then $\beta S \otimes \beta S$ does not meet $\Delta(S \times S)$. Consequently, $\beta S \otimes \beta S \cap cl(K(\beta(S \times S))) = \emptyset$.*

Proof. Let $p, q \in \beta S$ and let $r = p \otimes q$. We shall show that there is a member A of r such that $\mu(\overline{A}) = 0$ for every probability measure μ on $\beta(S \times S)$ which is invariant under translations by elements of $S \times S$.

First suppose that $p, q \in S^*$. We enumerate S as a sequence and write $s \prec s'$ if s occurs before s' in this sequence. By [14, Lemma 6.28], $b \cdot q \neq c \cdot q$ if b and c are distinct elements of S . Hence, for each $s \in S$, we can choose $Q_s \in q$ such that $b \cdot Q_s \cap c \cdot Q_s = \emptyset$ whenever $b \prec s, c \prec s$ and $b \neq c$. Let $A = \bigcup_{s \in S} (\{s\} \times Q_s)$. Then $A \in r$.

Let μ be a probability measure on $\beta(S \times S)$ which is invariant under translations by elements of $S \times S$. Suppose that $\mu(\overline{A}) = \alpha > 0$.

We claim that if $a, b, c \in S$ and $b \neq c$, then $\mu((a, b)\overline{A} \cap (a, c)\overline{A}) = 0$. So let a, b , and c be given and assume without loss of generality that $b \prec c$. Let $F = \{s \in S : s \preceq c\}$. We claim that $(a, b)A \cap (a, c)A \subseteq \bigcup_{s \in F} (\{s\} \times S)$. To see this, let $(u, v) \in (a, b)A \cap (a, c)A$ and pick $(s, t), (s', t') \in A$ such that $(u, v) = (a, b) \cdot (s, t) = (a, c) \cdot (s', t')$. Then $s = s'$ and $b \cdot t = c \cdot t'$. Since t and t' are both in Q_s , we can't have $c \prec s$ because then one would have $b \cdot Q_s \cap c \cdot Q_s \neq \emptyset$. Therefore $s \in F$. It is obvious that, for every given $s \in S$, $\{s\} \times S$ has an infinite number of disjoint translations by elements of $S \times S$, and so $\mu(\text{cl}(\{s\} \times S)) = 0$. Since $(a, b)\overline{A} \cap (a, c)\overline{A} \subseteq \text{cl}(\bigcup_{s \in F} (\{s\} \times S)) = \bigcup_{s \in F} \text{cl}(\{s\} \times S)$, it follows that $\mu((a, b)\overline{A} \cap (a, c)\overline{A}) = 0$ as claimed.

Now pick $n \in \mathbb{N}$ such that $\frac{1}{n} < \alpha$ and pick $a \in S$ and distinct b_1, b_2, \dots, b_n in S . Then

$$\begin{aligned} \mu\left(\bigcup_{i=1}^n (a, b_i)\overline{A}\right) &\geq \\ \sum_{i=1}^n \mu((a, b_i)\overline{A}) - \sum_{i=1}^{n-1} \sum_{j=i+1}^n \mu((a, b_i)\overline{A} \cap (a, b_j)\overline{A}) &= \\ \sum_{i=1}^n \mu(\overline{A}) = n \cdot \alpha &> 1, \end{aligned}$$

a contradiction.

If $p = s \in S$, we put $A = \{s\} \times S$. Then $A \in r$ and $\mu(\overline{A}) = 0$. Similarly, if $q = s \in S$, we put $A = S \times \{s\}$. Then, again, $A \in r$ and $\mu(\overline{A}) = 0$.

For the second conclusion, by [13, Theorems 2.12, 2.14, and 5.9], $\Delta(S \times S)$ is a closed two sided ideal of $\beta(S \times S)$. \square

Several of the preceding results have only applied to countable semigroups. However, some of them can be extended to a class of uncountable semigroups as indicated in the following theorem. An example of semigroups satisfying the hypotheses of statement (1) of this theorem include any free semigroup. A free semigroup T can be mapped onto $(\mathbb{N}, +)$ by the homomorphism which sends a word in T to its length. Another example is the direct sum of any number of copies of ω , where an element is mapped to the sum of its coordinates in ω . Recall that in a cancellative semigroup any right or left identity is a two sided identity.

Theorem 2.17. *Let T be a semigroup which can be mapped into a countable cancellative semigroup S by a homomorphism f .*

- (1) *If f has the property that whenever $q \in T^*$, one has $\tilde{f}(q)$ is not an element of S which is a (right or left) identity for S , then no element in $T^* \otimes T^*$ can have a left identity in $T^* \otimes T^*$. So no element of $T^* \otimes T^*$ can be an idempotent.*
- (2) *If f is surjective, then no element of $\beta T \otimes \beta T$ can be in $K(\beta(T \times T))$.*
- (3) *If f is surjective and S is left amenable, then no element of $\beta T \otimes \beta T$ can be in $\text{cl}(K(\beta(T \times T)))$.*

Proof. Define $g : T \times T \rightarrow S \times S \subseteq \beta(S \times S)$ by $g(x, y) = (f(x), f(y))$. Then for $p, q \in \beta T$, $\tilde{g}(p \otimes q) = \tilde{f}(p) \otimes \tilde{f}(q)$.

(1). Let $p, q \in T^*$, let $u = p \otimes q$, and suppose that u is a left identity for r in $\beta(T \times T)$. Then $\tilde{g}(u)$ is a left identity for $\tilde{f}(p) \otimes \tilde{f}(q)$ and so, by Theorem 2.7, $\tilde{\pi}_2(\tilde{g}(u))$ is a right identity for S , and hence a two sided identity. But $\tilde{\pi}_2(\tilde{g}(u)) = \tilde{f}(q)$, contradicting our assumption about f .

(2). Since f is surjective, so is g and therefore so is \tilde{g} . By [14, Exercise 1.7.3], $K(\beta(S \times S)) = \tilde{g}[K(\beta(T \times T))]$. By Corollary 2.9, $\beta S \otimes \beta S$ misses $K(\beta(S \times S))$ and $\tilde{g}[\beta T \otimes \beta T] \subseteq \beta S \otimes \beta S$ so $\beta T \otimes \beta T$ misses $K(\beta(T \times T))$.

(3). By Theorem 2.16, $(\beta S \otimes \beta S) \cap \text{cl}(K(\beta(S \times S))) = \emptyset$. Since $\tilde{g}[\text{cl}(K(\beta(T \times T)))] = \text{cl}(g[(K(\beta(T \times T))])$, the conclusion follows as in (2). \square

In a similar way, we can show that some of the preceding theorems apply to the semigroup $(\mathbb{R}_d, +)$, where \mathbb{R}_d denotes the real line with the discrete topology.

Definition 2.18. We define the nearest integer function $f : \mathbb{R} \rightarrow \mathbb{Z} \subseteq \beta\mathbb{Z}$ by $f(x) = \lfloor x + \frac{1}{2} \rfloor$. We define $g : \mathbb{R}_d \times \mathbb{R}_d \rightarrow \mathbb{Z} \times \mathbb{Z} \subseteq \beta(\mathbb{Z} \times \mathbb{Z})$ by $g(s, t) = (f(s), f(t))$.

As we noted at the beginning of this section $\tilde{g}(p \otimes q) = \tilde{f}(p) \otimes \tilde{f}(q)$ for every $p, q \in \beta S$.

Theorem 2.19. *Let $S = (\mathbb{R}_d, +)$. Then $S^* \times S^*$ does not meet $\Delta(S \times S)$. In particular, $S^* \otimes S^*$ does not meet $\text{cl}K(\beta(S \times S))$. If $p, q \in S^*$ and $\tilde{f}(p) \neq 0$, then $p \otimes q$ is not an idempotent.*

Proof. We shall prove the last assertion first. We observe that, if $s, t \in S$ and $|s - f(s)| < 1/4$ and $|t - f(t)| < 1/4$, then $f(s + t) = f(s) + f(t)$. Let $h : \mathbb{R} \rightarrow \mathbb{T}$ denote the canonical homomorphism, where \mathbb{T} is the unit circle regarded as the quotient \mathbb{R}/\mathbb{Z} . We shall use real numbers in $[-\frac{1}{2}, \frac{1}{2})$ to denote the corresponding points of \mathbb{T} . We define $j : S \times S \rightarrow \mathbb{T} \times \mathbb{T}$

by $j(s, t) = (h(s), h(t))$. If r is an idempotent in $\beta(S \times S)$, since \tilde{j} is a homomorphism, $\tilde{j}(r) = (0, 0)$. So $\{(s, t) \in S \times S : |s - f(s)| < 1/4 \text{ and } |t - f(t)| < 1/4\} \in r$. It follows easily that, for any idempotents $r_1, r_2 \in \beta(S \times S)$, $\tilde{g}(r_1 + r_2) = \tilde{g}(r_1) + \tilde{g}(r_2)$.

Assume that $p, q \in S^*$, $\tilde{f}(p) \neq 0$, and $r = p \otimes q$ is an idempotent. Then $\tilde{g}(r)$ is an idempotent in $\beta(\mathbb{Z} \times \mathbb{Z})$ and $\tilde{g}(r) = \tilde{f}(p) \otimes \tilde{f}(q)$. Since $\tilde{\pi}_1(\tilde{g}(r)) = \tilde{f}(p) \neq 0$, by Remark 2.6 we must have $\tilde{f}(p) \in \mathbb{Z}^*$ and either $\tilde{f}(q) \in \mathbb{Z}^*$ or $\tilde{f}(q) = 0$. The first of these possibilities is ruled out by Corollary 2.8, so we assume that $\tilde{f}(p) \in \mathbb{Z}^*$ and $\tilde{f}(q) = 0$.

The functions $\tau, \psi : \mathbb{R}_d \times \mathbb{R}_d \rightarrow \mathbb{R}_d \times \mathbb{R}_d \subseteq \beta(\mathbb{R}_d \times \mathbb{R}_d)$ defined by $\tau(x, y) = (-x, -y)$ and $\psi(x, y) = (-x, y)$ are both isomorphisms so we may assume without loss of generality that $(0, \infty) \in p$ and $(0, \infty) \in q$. We then have that for any $r > 0$, $(0, r) \in q$ and $(r, \infty) \in p$. Let $A = \{(s, t) \in S \times S : s > 0 \text{ and } 0 < t < \frac{1}{s}\} \in r$. Since $(0, \infty) \subseteq \{s \in S : \{t \in S : (s, t) \in A\} \in q\}$, $A \in r$. Since r is an idempotent, $B = \{(s, t) \in S \times S : -(s, t) + A \in r\} \in r$ so pick $(s, t) \in A$ such that $-(s, t) + A \in r$. Let $B' = \{(s', t') \in S \times S : \frac{1}{t} < s + s' \text{ and } t' > 0\}$. Then $(\frac{1}{t}, \infty) \subseteq \{s' \in S : \{t' \in S : (s', t') \in B'\} \in q\}$, $B' \in r$. Pick $(s', t') \in B' \cap (-(s, t) + A)$. Then $t < t + t' < \frac{1}{s + s'} < t$, a contradiction.

Now suppose that $w \in \Delta(S \times S) \cap (S^* \otimes S^*)$. Let $w = u \otimes v$, where $u, v \in S^*$. We shall show that $\tilde{f}(u) \otimes \tilde{f}(v) \in \Delta(\mathbb{Z} \times \mathbb{Z})$, contradicting Theorem 2.16. To see this, let $A \in \tilde{f}(u) \otimes \tilde{f}(v) = \tilde{g}(w)$. Then $g^{-1}[A] \in w$. Since $S \times S$ is commutative and cancellative, it is amenable, and so there is a probability measure ν on $\beta(S \times S)$, which is invariant under translations by elements of $S \times S$, for which $\nu(\text{cl}_{\beta(S \times S)}(g^{-1}[A])) > 0$. We can define a probability measure μ on $\beta(\mathbb{Z} \times \mathbb{Z})$ by putting $\mu(B) = \nu(\tilde{g}^{-1}[B])$ for every Borel subset B of $\beta(\mathbb{Z} \times \mathbb{Z})$. Then $\mu(\text{cl}_{\beta(\mathbb{Z} \times \mathbb{Z})}(A)) > 0$. We claim that μ is invariant under translations by elements of $\mathbb{Z} \times \mathbb{Z}$. To see this observe that, for every $m, n \in \mathbb{Z}$ and every $s, t \in S$, $g((m, n) + (s, t)) = (f(m + s), f(n + t)) = (m + f(s), n + f(t)) = (m, n) + g(s, t)$. By taking limits as (s, t) tends to x , it follows that $\tilde{g}((m, n) + x) = (m, n) + \tilde{g}(x)$ for every $x \in \beta(S \times S)$. So, for every subset D of $\beta(\mathbb{Z} \times \mathbb{Z})$ and every $x \in \beta(S \times S)$, $\tilde{g}((m, n) + x) \in D \Leftrightarrow (m, n) + \tilde{g}(x) \in D$. Thus

$$\tilde{g}^{-1}[-(m, n) + D] = -(m, n) + \tilde{g}^{-1}[D].$$

It follows that μ is invariant under translations by elements of $\mathbb{Z} \times \mathbb{Z}$. Hence, since $\mu(\text{cl}_{\beta(\mathbb{Z}^2)}(A)) > 0$, A has positive density. So $\tilde{f}(u) \otimes \tilde{f}(v) \in \Delta(\mathbb{Z} \times \mathbb{Z})$, as claimed. \square

Let $\iota : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N} \times \mathbb{N} \subseteq \beta\mathbb{N} \times \beta\mathbb{N}$ be the identity map. We now consider the way that tensor products occur in studying the map $\tilde{\iota}$. We

first note that if $p, q \in \beta\mathbb{N}$, then, where all limits are in the space $\beta\mathbb{N} \times \beta\mathbb{N}$ and s and t denote members of \mathbb{N} ,

$$\tilde{\iota}(p \otimes q) = \tilde{\iota}\left(\lim_{s \rightarrow p} \lim_{t \rightarrow q}(s, t)\right) = \lim_{s \rightarrow p} \lim_{t \rightarrow q} \tilde{\iota}(s, t) = \lim_{s \rightarrow p} \lim_{t \rightarrow q}(s, t) = (p, q)$$

and

$$\tilde{\iota}(p \odot q) = \tilde{\iota}\left(\lim_{t \rightarrow q} \lim_{s \rightarrow p}(s, t)\right) = \lim_{t \rightarrow q} \lim_{s \rightarrow p} \tilde{\iota}(s, t) = \lim_{t \rightarrow q} \lim_{s \rightarrow p}(s, t) = (p, q).$$

So $p \otimes q$ is the unique element of $\beta\mathbb{N} \otimes \beta\mathbb{N}$ in $\tilde{\iota}^{-1}[\{(p, q)\}]$. Similarly, $p \odot q$ is the unique element of $\beta\mathbb{N} \odot \beta\mathbb{N}$ in $\tilde{\iota}^{-1}[\{(p, q)\}]$. We also know by Theorem 2.2(1) that if $p, q \in \mathbb{N}^*$, then $p \otimes q \neq p \odot q$.

Recall that $p \in \mathbb{N}^*$ is a *P-point* of \mathbb{N}^* if and only if whenever $\langle A_n \rangle_{n=1}^\infty$ is a sequence of members of p , there is some $B \in p$ such that $B \setminus A_n$ is finite for each $n \in \mathbb{N}$; p is *selective* if and only if whenever $f : \mathbb{N} \rightarrow \mathbb{N}$ there is some $B \in p$ such that the restriction of f to B is either injective or constant. Given $p, q \in \mathbb{N}^*$, $p \leq_{RK} q$ if and only if there exists $f : \mathbb{N} \rightarrow \mathbb{N} \subseteq \beta\mathbb{N}$ such that $f(q) = p$. Given $p, q \in \mathbb{N}^*$, p and q are isomorphic if and only if there exists a bijection $f : \mathbb{N} \rightarrow \mathbb{N} \subseteq \beta\mathbb{N}$ such that $\tilde{f}(q) = p$.

Theorem 2.20. *It is consistent with ZFC that there exist $p, q \in \mathbb{N}^*$ such that $\tilde{\iota}^{-1}[\{(p, q)\}] = \{p \otimes q, p \odot q\}$.*

Proof. M. Duguet proved [6, Théoreme 1] that for $p, q \in \mathbb{N}^*$, $|\tilde{\iota}^{-1}[\{(p, q)\}]| = 2$ if and only if p and q are P-points and there does not exist $r \in \mathbb{N}^*$ such that $r \leq_{RK} p$ and $r \leq_{RK} q$. An elementary proof of this fact is provided in [3, Theorem 17]. The proof in [3] in fact shows that if p and q are as specified, then $\tilde{\iota}^{-1}[\{(p, q)\}] = \{p \otimes q, p \odot q\}$. In [3, Corollary 19] it was shown that if p and q are non-isomorphic selective ultrafilters, then $|\tilde{\iota}^{-1}[\{(p, q)\}]| = 2$.

In [2, Corollary 9, page 53], Blass showed that Martin's Axiom (in fact the weaker cardinal number hypothesis $\mathfrak{p} = \mathfrak{c}$) implies that there are $2^{\mathfrak{c}}$ selective ultrafilters on \mathbb{N} , and since there are only \mathfrak{c} functions from \mathbb{N} to \mathbb{N} , given any selective p , there must be a non-isomorphic selective q . \square

In [3, Theorem 14] Blass and Moche showed that if $p, q \in \mathbb{N}^*$ and $\tilde{\iota}^{-1}[\{(p, q)\}]$ is finite, then there exist P-points in \mathbb{N}^* . And it was shown by Shelah [18] that the existence of P-points cannot be proved in ZFC. (We thank the referee for pointing out that there is now a simpler proof of this fact in [4].) It is therefore consistent that $\tilde{\iota}^{-1}[\{(p, q)\}]$ is infinite for all $p, q \in \mathbb{N}^*$, and as an infinite compact subspace of $\beta(\mathbb{N} \times \mathbb{N})$, $|\tilde{\iota}^{-1}[\{(p, q)\}]| = 2^{\mathfrak{c}}$ by [14, Theorem 3.59]. This is another indication that $\mathbb{N}^* \otimes \mathbb{N}^*$ is a small subset of $\beta(\mathbb{N} \times \mathbb{N})$. We see now, that if p (or q) is close to the smallest ideal of $\beta\mathbb{N}$, then it must be that $|\tilde{\iota}^{-1}[\{(p, q)\}]| = 2^{\mathfrak{c}}$.

Theorem 2.21. *Let $p \in \text{cl}K(\beta\mathbb{N})$. Then for every $q \in \mathbb{N}^*$, $|\tilde{v}^{-1}[\{(p, q)\}]| = 2^c$.*

Proof. It was shown in [11, Theorem 3.1] that if $p \in \mathbb{N}^*$ and none of the members of p have “property S ”, then for every $q \in \mathbb{N}^*$, $|\tilde{v}^{-1}[\{(p, q)\}]| = 2^c$. It is a routine exercise to show that a subset A of \mathbb{N} has property S if and only if A is not piecewise syndetic. So the assertion that none of the members of p have property S is the same as the assertion that all of the members of p are piecewise syndetic, which is equivalent by [14, Corollary 4.41] to the assertion that $p \in \text{cl}K(\beta\mathbb{N})$. \square

Although $\mathbb{N}^* \otimes \mathbb{N}^*$ contains no idempotents and does not meet $\Delta(\mathbb{N} \times \mathbb{N})$, members of $\mathbb{N}^* \otimes \mathbb{N}^*$ may have rich combinatorial properties.

Theorem 2.22. *($S, +$) be an infinite commutative semigroup, let $u, v \in \mathbb{N}$, let A be a $u \times v$ matrix with entries from ω , and let $p, q \in S^*$. Assume that whenever $B \in p$ or $B \in q$, there exists $\vec{x} \in S^v$ such that $A\vec{x} \in B^u$. Then whenever $D \in p \otimes q$, there exists $\vec{z} \in (S \times S)^v$ such that $A\vec{z} \in D^u$.*

Proof. Let $D \in p \otimes q$. Let $B = \{x \in S : \{y \in S : (x, y) \in D\} \in q\}$. Pick $\vec{x} \in S^v$ such that $\vec{w} = A\vec{x} \in B^u$. Let $C = \bigcap_{i=1}^u \{y \in S : (w_i, y) \in D\}$. Then $C \in q$ so pick $\vec{y} \in S^v$ such that $A\vec{y} \in C^u$. Define $\vec{z} \in (S \times S)^v$ by, for $i \in \{1, 2, \dots, v\}$, $z_i = (x_i, y_i)$. \square

In particular, for $k \in \mathbb{N}$, applying Theorem 2.22 to the semigroups $(\mathbb{N}, +)$ and (\mathbb{N}, \cdot) respectively and the matrix

$$A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ \vdots & \vdots \\ 1 & k \end{pmatrix}$$

one has for any $p, q \in \mathbb{N}^*$

- (1) if every member of p and every member of q contains a length $k + 1$ arithmetic progression so does every member of $p \otimes q$ and
- (2) if every member of p and every member of q contains a length $k + 1$ geometric progression so does every member of $p \otimes q$.

3. CARTESIAN PRODUCTS

In [16, Theorem 5.1] we showed that if $u, v \in \mathbb{N}$ and $u > 1$, then $\beta(\mathbb{N}^v)$ and $(\beta\mathbb{N})^u$ are not isomorphic and $\beta(\mathbb{Z}^v)$ and $(\beta\mathbb{Z})^u$ are not isomorphic. (It has been known since 1959 [10] that they are not homeomorphic. An easy way to see that $\beta(\mathbb{Z}^v)$ and $(\beta\mathbb{Z})^u$ are not homeomorphic is to note

that by [9, Theorem 14.25], $\beta(\mathbb{Z}^v)$ is an F-space while by [9, Exercise 14Q(1)], $(\beta\mathbb{Z})^u$ is not.)

In this section we investigate the question of when or whether $(\beta S)^u$ and $(\beta T)^v$ can be isomorphic, as well as the corresponding questions for $(S^*)^u$, for $\beta(S^u)$, and for $(S^u)^*$.

The following easy lemma strengthens [14, Corollary 6.23].

Lemma 3.1. *Let S be a countable left cancellative semigroup and let e and f be idempotents in βS such that $\beta S \cdot e \cap \beta S \cdot f \neq \emptyset$. Then $e \cdot f = e$ or $f \cdot e = f$.*

Proof. By [14, Theorem 6.19] we may assume without loss of generality that there are some $s \in S$ and some $x \in \beta S$ such that $s \cdot e = x \cdot f$. Then $s \cdot e \cdot f = x \cdot f \cdot f = x \cdot f = s \cdot e$. By [14, Lemma 8.1], s is left cancelable in βS so $e \cdot f = e$. \square

Theorem 3.2. *Let S and T be left cancellative semigroups. Let $u, v \in \mathbb{N}$ with $u \neq v$.*

- (1) *If $(\beta S)^u$ and $(\beta T)^v$ are isomorphic and if βT has a nonminimal idempotent, then βS also has a nonminimal idempotent.*
- (2) *If S is countable and either βS or βT has a nonminimal idempotent, then $(\beta S)^u$ is not isomorphic to $(\beta T)^v$ if $v > u$.*
- (3) *If S and T are both countable and either βS or βT has a nonminimal idempotent, then $(\beta S)^u$ is not isomorphic to $(\beta T)^v$.*

Proof. (1) Suppose that $(\beta S)^u$ and $(\beta T)^v$ are isomorphic. Let $\varphi : (\beta T)^v \rightarrow (\beta S)^u$ be an isomorphism.

Assume that βT has a nonminimal idempotent p . By [14, Theorem 2.23], $K((\beta T)^v) = (K(\beta T))^v$ so $\vec{p} = (p, p, \dots, p)$ is a nonminimal idempotent in $(\beta T)^v$ and thus $\varphi(\vec{p}) = \vec{q} = (q_1, q_2, \dots, q_u)$ is a nonminimal idempotent in $(\beta S)^u$. Therefore (again using [14, Theorem 2.23]), some $q_i \notin K(\beta S)$. So βS also has a nonminimal idempotent.

(2) Assume that S is countable, that $v > u$ and that $(\beta S)^u$ and $(\beta T)^v$ are isomorphic. By (1) we may assume that βT has a nonminimal idempotent p .

By [14, Theorem 1.60], pick a minimal idempotent $q \in \beta T$ such that $q < p$. For $i \in \{1, 2, \dots, v\}$, define $\vec{r}_i \in (\beta T)^v$ by, for $t \in \{1, 2, \dots, v\}$,

$$\vec{r}_i(t) = \begin{cases} q & \text{if } t \neq i \\ p & \text{if } t = i. \end{cases}$$

Then each \vec{r}_i is a nonminimal idempotent in $(\beta T)^v$ and, if $i \neq j$ in $\{1, 2, \dots, v\}$, then $\vec{r}_i \cdot \vec{r}_j = \vec{q} = (q, q, \dots, q)$, which is minimal in $(\beta T)^v$.

Since $(\beta T)^v$ is isomorphic to $(\beta S)^u$, we can pick nonminimal idempotents $\vec{p}_1, \vec{p}_2, \dots, \vec{p}_v$ in $(\beta S)^u$ and a minimal idempotent \vec{w} in $(\beta S)^u$ such that $\vec{p}_i \cdot \vec{p}_j = \vec{w}$ whenever i and j are distinct members of $\{1, 2, \dots, v\}$.

We claim that, given $k \in \{1, 2, \dots, u\}$, there is at most one $i \in \{1, 2, \dots, v\}$ such that $\vec{p}_i(k) \neq \vec{w}(k)$. Suppose instead we have $i \neq j$ in $\{1, 2, \dots, v\}$ such that $\vec{p}_i(k) \neq \vec{w}(k)$ and $\vec{p}_j(k) \neq \vec{w}(k)$. We have that $\vec{p}_i(k) \cdot \vec{p}_j(k) = \vec{p}_j(k) \cdot \vec{p}_i(k) = \vec{w}(k)$ so $\beta S \cdot \vec{p}_i(k) \cap \beta S \cdot \vec{p}_j(k) \neq \emptyset$. Therefore by Lemma 3.1 without loss of generality, $\vec{p}_i(k) = \vec{p}_i(k) \cdot \vec{p}_j(k) = \vec{w}(k)$, a contradiction.

For $k \in \{1, 2, \dots, u\}$, pick $f(k) \in \{1, 2, \dots, v\}$ such that, if $i \in \{1, 2, \dots, v\} \setminus \{f(k)\}$, then $\vec{p}_i(k) = \vec{w}(k)$. Pick $i \in \{1, 2, \dots, v\} \setminus \{f(k) : k \in \{1, 2, \dots, u\}\}$. Then $\vec{p}_i = \vec{w}$, a contradiction.

(3) This follows immediately from (2). \square

Corollary 3.3. *Theorem 3.2 remains valid if all occurrences of βS are replaced by S^* and all occurrences of βT are replaced by T^* .*

Proof. By [14, Corollary 4.29], S^* is a compact right topological semigroup and so [14, Theorem 2.23] applies to S^* . We note that if $S^* \cdot p \cap S^* \cdot q \neq \emptyset$, then $\beta S \cdot p \cap \beta S \cdot q \neq \emptyset$. So this proof may be taken verbatim from the proof of Theorem 3.2. \square

We shall see in Theorem 3.19 that (for left cancellative S), βS has a nonminimal idempotent if and only if S^* has a nonminimal idempotent.

We shall now consider isomorphisms between spaces of the form $\beta(S^u)$ and $\beta(T^v)$. We say that topological spaces with algebraic structure are *topologically isomorphic* provided there is a function from one to the other which is simultaneously a homeomorphism and an isomorphism.

We point out that, for arbitrary semigroups S and T and arbitrary positive integers u and v , the assumption that $(\beta S)^u$ and $(\beta T)^v$ are topologically isomorphic, implies that $\beta(S^u)$ and $\beta(T^v)$ are topologically isomorphic. To see this, observe that a topological isomorphism from $(\beta S)^u$ onto $(\beta T)^v$ maps S^u onto T^v , because it maps the isolated points of $(\beta S)^u$ onto the isolated points of $(\beta T)^v$. Our claim then follows from [14, Corollary 4.22] and [14, Exercise 3.4.1]. In the case in which S and T are groups, an algebraic isomorphism from $(\beta S)^u$ onto $(\beta T)^v$, or from $\beta(S^u)$ onto $\beta(T^v)$, defines an isomorphism from S^u onto T^v and hence a topological isomorphism from $\beta(S^u)$ onto $\beta(T^v)$. This follows from the fact that the points of S^u can be characterized algebraically in $(\beta S)^u$ and in $\beta(S^u)$ as the set of invertible elements. To see this note that by [14, Theorem 4.36], $(S^u)^*$ is an ideal of $\beta(S^u)$ and S^* is an ideal of βS . So elements of $(S^u)^*$ are not invertible in $\beta(S^u)$ and elements of $(\beta S)^u \setminus S^u$ are not invertible in $(\beta S)^u$.

For the question of whether $\beta(S^u)$ and $\beta(S^v)$ are isomorphic, no results nearly as strong as Theorem 3.2 are possible.

If T is any semigroup with identity, and $S = \bigoplus_{n=1}^{\infty} T$, then for any $u, v \in \mathbb{N}$, S^u and S^v are isomorphic and so $\beta(S^u)$ and $\beta(S^v)$ are isomorphic

and $(S^u)^*$ and $(S^v)^*$ are isomorphic. We do have results for \mathbb{N} and for finitely generated abelian groups.

Recall that a sequence $\langle x_n \rangle_{n=1}^\infty$ has distinct finite products if and only if whenever F and H are distinct finite nonempty subsets of \mathbb{N} , one has $\prod_{t \in F} x_t \neq \prod_{t \in H} x_t$.

Lemma 3.4. *Let S be an infinite semigroup. If S contains an infinite right cancellative and weakly left cancellative semigroup T , then S^* has a nonminimal idempotent.*

Proof. Assume that T is a countably infinite right cancellative and weakly left cancellative semigroup contained in S . By [14, Lemma 6.31] there is a sequence $\langle x_n \rangle_{n=1}^\infty$ in T which has distinct finite products. Let $M = \bigcap_{m=1}^\infty FP(\langle x_n \rangle_{n=m}^\infty)$. By [14, Lemma 5.11], M is a subsemigroup of βT which therefore contains an idempotent by [14, Theorem 2.5]. Note that if $a = \prod_{t \in F} x_t$ and $m > \min F$, then $a \notin FP(\langle x_n \rangle_{n=m}^\infty)$ because $\langle x_n \rangle_{n=1}^\infty$ has distinct finite products. Hence M is a G_δ subset of T^* and therefore by [14, Corollary 6.33], M contains 2^c nonminimal idempotents, and thus so does S^* . \square

The following lemma is well known. We cannot find a reference for it so we provide a sketch of a proof.

Lemma 3.5. *Let $u, v \in \mathbb{N}$ with $u \neq v$.*

- (1) \mathbb{N}^u and \mathbb{N}^v are not isomorphic.
- (2) \mathbb{Z}^u and \mathbb{Z}^v are not isomorphic.

Proof. Assume that $v < u$. The u unit vectors in \mathbb{Z}^u are linearly independent over \mathbb{Q} . If \mathbb{Z}^u and \mathbb{Z}^v were isomorphic, we would have u nonzero elements of \mathbb{Q}^v which are linearly independent over \mathbb{Q} .

It is easy to check that an isomorphism h from \mathbb{N}^u to \mathbb{N}^v can be extended to an isomorphism from \mathbb{Z}^u to \mathbb{Z}^v by putting $h(\vec{x} - \vec{y}) = h(\vec{x}) - h(\vec{y})$ for every $\vec{x}, \vec{y} \in \mathbb{N}^u$. \square

Lemma 3.6. *Let $u \in \mathbb{N}$ and let p be an idempotent in $\beta(\mathbb{N}^u)$. Then $\beta(\mathbb{Z}^u) + p \subseteq \beta(\mathbb{N}^u)$.*

Proof. Let $\Theta = \{q \in \beta(\mathbb{N}^u) : \text{for all } i \in \{1, 2, \dots, u\}, \tilde{\pi}_i(q) \in \mathbb{N}^*\}$. Since p is an idempotent, $p \in \Theta$. By [16, Lemma 3.2], Θ is a left ideal of $\beta(\mathbb{Z}^u)$ so $\beta(\mathbb{Z}^u) + p \subseteq \Theta \subseteq \beta(\mathbb{N}^u)$. \square

Lemma 3.7. *Let $u \in \mathbb{N}$ and let p be a nonminimal idempotent in $\beta(\mathbb{N}^u)$. The center of $p + \beta(\mathbb{N}^u) + p$ is $\mathbb{Z}^u + p$ which is isomorphic to \mathbb{Z}^u .*

Proof. Note that \mathbb{Z}^u is the group of quotients of \mathbb{N}^u so by [14, Theorem 6.63], the center of $p + \beta(\mathbb{N}^u) + p$ is $(\mathbb{Z}^u + p) \cap (p + \beta(\mathbb{N}^u) + p)$. By [14,

Theorem 6.54] \mathbb{Z}^u is the center of $\beta(\mathbb{Z}^u)$. Using Lemma 3.6 we have that $\mathbb{Z}^u + p = \mathbb{Z}^u + p + p = p + \mathbb{Z}^u + p + p \subseteq p + \beta(\mathbb{N}^u) + p$ so the center of $p + \beta(\mathbb{N}^u) + p$ is $\mathbb{Z}^u + p$. Define $\varphi : \mathbb{Z}^u \rightarrow \mathbb{Z}^u + p$ by $\varphi(\vec{x}) = \vec{x} + p$. Then φ is a surjective homomorphism which is injective by [14, Lemma 6.28(1)]. \square

Theorem 3.8. *Let $u, v \in \mathbb{N}$ with $u \neq v$.*

- (1) $\beta(\mathbb{N}^u)$ and $\beta(\mathbb{N}^v)$ are not isomorphic.
- (2) $(\mathbb{N}^u)^*$ and $(\mathbb{N}^v)^*$ are not isomorphic.

Proof. (1) By [14, Theorem 6.54], the centers of $\beta(\mathbb{N}^u)$ and $\beta(\mathbb{N}^v)$ are \mathbb{N}^u and \mathbb{N}^v respectively and these are not isomorphic by Lemma 3.5(1).

(2) Suppose that $(\mathbb{N}^u)^*$ and $(\mathbb{N}^v)^*$ are isomorphic and let $\varphi : (\mathbb{N}^u)^* \rightarrow (\mathbb{N}^v)^*$ be an isomorphism. Pick by Lemma 3.4 a nonminimal idempotent $p \in (\mathbb{N}^u)^*$. Then $\varphi(p)$ is a nonminimal idempotent in $(\mathbb{N}^v)^*$.

By [14, Theorem 4.36], $(\mathbb{N}^u)^*$ is an ideal of $\beta(\mathbb{N}^u)$ so $p + \beta(\mathbb{N}^u) + p = p + \beta(\mathbb{N}^u) + p + p \subseteq p + (\mathbb{N}^u)^* + p \subseteq p + \beta(\mathbb{N}^u) + p$. So by Lemma 3.7, the center of $p + (\mathbb{N}^u)^* + p$ is isomorphic to \mathbb{Z}^u while the center of $\varphi(p) + (\mathbb{N}^v)^* + \varphi(p)$ is isomorphic to \mathbb{Z}^v and \mathbb{Z}^u is not isomorphic to \mathbb{Z}^v by Lemma 3.5(2). \square

The statements in Theorem 3.8 with \mathbb{Z} replacing \mathbb{N} are consequences of Theorem 3.11.

Lemma 3.9. *Let $(G, +)$ be a nontrivial finitely generated abelian group and let $u, v \in \mathbb{N}$ with $u \neq v$. Then G^u and G^v are not isomorphic.*

Proof. By [8, Theorem 10.4], G is the direct sum of finitely many cyclic groups, so there exist a finite group H and a group $K = \mathbb{Z}^z$ for some $z \in \omega$ such that G is isomorphic to $H \oplus K$.

Suppose that G^u and G^v are isomorphic. Then $H^u \oplus K^u$ and $H^v \oplus K^v$ are isomorphic. Let $\varphi : H^u \oplus K^u \rightarrow H^v \oplus K^v$ be an isomorphism. Then φ takes elements of finite order to elements of finite order and elements of infinite order to elements of infinite order, so $\varphi[H^u \times \{\vec{0}\}] = H^v \times \{\vec{0}\}$. So $|H|^u = |H|^v$, and thus we must have $H = \{0\}$. But then G is isomorphic to \mathbb{Z}^z and $z \geq 1$ so by Lemma 3.5(2) G^u and G^v are not isomorphic. \square

Lemma 3.10. *Let $(G, +)$ be an infinite abelian group, let $u \in \mathbb{N}$, and let p be a nonminimal idempotent in $\beta(G^u)$. The center of $p + \beta(G^u) + p$ is $G^u + p$ which is isomorphic to G^u .*

Proof. By [14, Theorem 6.63] the center of $p + \beta(G^u) + p$ is $(G^u + p) \cap (p + \beta(G^u) + p)$. By [14, Theorem 6.54] the center of $\beta(G^u)$ is G^u so $G^u + p = G^u + p + p = p + G^u + p \subseteq p + \beta(G^u) + p$ and thus the center of $p + \beta(G^u) + p$ is $G^u + p$. Define $\varphi : G^u \rightarrow G^u + p$ by $\varphi(\vec{x}) = \vec{x} + p$.

Then φ is a surjective homomorphism which is injective by [14, Lemma 6.28(1)]. \square

Theorem 3.11. *Let G be an infinite finitely generated abelian group and let $u, v \in \mathbb{N}$ with $u \neq v$.*

- (1) $\beta(G^u)$ and $\beta(G^v)$ are not isomorphic.
- (2) $(G^u)^*$ and $(G^v)^*$ are not isomorphic.

Proof. (1) By [14, Theorem 6.54], the centers of $\beta(G^u)$ and $\beta(G^v)$ are G^u and G^v respectively and these are not isomorphic by Lemma 3.9.

(2) This is nearly identical with the proof of Theorem 3.8(2) using Lemma 3.10. \square

Note that if G is an infinite finitely generated abelian group, then βG has a nonminimal idempotent so Theorem 3.2 applies to G .

No result similar to Theorem 3.2(3) can apply to $\beta(G^u)$ and $\beta(H^v)$ for finitely generated abelian groups G and H for the simple reason that G^u is another finitely generated abelian group.

It is obvious why we need nonminimal idempotents for the proof of Theorem 3.2.

It is less obvious why nonminimal idempotents are needed for the proofs of Theorems 3.8(2) and 3.11(2). The reason is that we don't know what the centers of the maximal groups in βS are. It is not even known whether the center of $p + \beta\mathbb{N} + p$ is isomorphic to \mathbb{Z} when p is a minimal idempotent in $\beta\mathbb{N}$.

We now investigate the situation where βS does not have nonminimal idempotents.

Definition 3.12. Let S be a semigroup and let $e \in E(S)$. Then $G_e = \{s \in S : se = s\}$.

Recall that a semigroup S is a left zero semigroup provided $xy = x$ for all $x, y \in S$ and S is a right zero semigroup provided $xy = y$ for all $x, y \in S$.

Lemma 3.13. *Let S be an infinite left cancellative semigroup and let $e \in E(S)$.*

- (1) e is a left identity of βS . In particular, $E(S)$ is a right zero semigroup.
- (2) G_e is a left ideal of S .

Proof. (1) Given $s \in S$, $es = ees$ so $s = es$. Since λ_e and the identity function on βS agree on S , they agree on βS .

(2) Since $e \in G_e$, $G_e \neq \emptyset$. Given $s \in G_e$ and $t \in S$, $tse = ts$, so $ts \in G_e$. \square

We will frequently use without mention the fact that every idempotent in S is a left identity for βS .

Given a semigroup S and $a \in S$, we say that the *order* of a is $|\{a^n : n \in \mathbb{N}\}|$. Recall that a *right group* S is a left cancellative semigroup which is right simple, meaning that it has no proper right ideals.

Lemma 3.14. *Let S be a left cancellative semigroup such that every element of S has finite order. Then S is a right group.*

Proof. Let R be a right ideal of S and pick $a \in R$. Pick $m < n$ such that $a^m = a^n$. Then $a^m \cdot a = a^m \cdot a \cdot a^{n-m}$ so $a = a \cdot a^{n-m}$. Let $e = a^{n-m}$. Then $a \cdot e = a \cdot a^{n-m} \cdot e$ so $e = e \cdot e$. Then $e \in R$ and e is a left identity for S so $R = S$. \square

Theorem 3.15. *Let S be an infinite left cancellative semigroup such that every element of S has finite order. There is a group $G \subseteq S$ such that S is isomorphic to $G \times E(S)$, $S = G \cdot E(S)$, $E(S)$ is a right zero semigroup, and for every $e \in E(S)$, G_e is isomorphic to G .*

Proof. Since S is a right group, we have by [5, Theorem 1.27] that there is a group G such that $S = G \cdot E(S)$ and S is isomorphic to $G \times E(S)$. (The statement of the cited theorem says that there is a group G and a right zero semigroup E such that S is isomorphic to $G \times E$. But the proof shows that $E = E(S)$ and $S = G \cdot E(S)$.)

To complete the proof, let $e \in E(S)$. We show first that $G_e = G \cdot e$. Given $x \in G$, $x \cdot e \cdot e = x \cdot e$ so $x \cdot e \in G_e$. Now let $x \in G_e$. Pick $y \in G$ and $f \in E(S)$ such that $x = y \cdot f$. Then $x = x \cdot e = y \cdot f \cdot e = y \cdot e \in G \cdot e$.

Now we claim that the restriction of ρ_e to G is an isomorphism from G onto G_e . We have just seen that $\rho_e[G] = G_e$. To see that the restriction is a homomorphism, let $a, b \in G$. Then $\rho_e(a) \cdot \rho_e(b) = a \cdot e \cdot b \cdot e = a \cdot b \cdot e = \rho_e(a \cdot b)$. To see that the restriction is one-to-one, let $a, b \in G$ and assume that $\rho_e(a) = \rho_e(b)$. Let f be the identity of G . Then $a = a \cdot f = a \cdot e \cdot f = b \cdot e \cdot f = b \cdot f = b$. \square

Lemma 3.16. *Let S be an infinite semigroup. If S^* has no nonminimal idempotents, then every element of S has finite order and S does not contain an infinite group.*

Proof. If either S has an element of infinite order or S contains an infinite group, then S contains a countably infinite cancellative semigroup T . By Lemma 3.4 S^* contains a nonminimal idempotent. \square

The proofs of both parts of the next lemma are easy exercises.

Lemma 3.17. *Let S be an infinite semigroup.*

- (1) *If L is a left ideal of S , then \bar{L} is a left ideal of βS .*

(2) If U is a clopen subset of βS , then $U = \overline{U \cap S}$.

Theorem 3.18. *Let S be an infinite left cancellative semigroup. The following statements are equivalent.*

- (a) βS has no nonminimal idempotents.
- (b) S^* has no nonminimal idempotents.
- (c) There exists a finite group G such that $S = G \cdot E(S)$.
- (d) There exists a finite group G such that $S = G \cdot E(S)$, S is isomorphic to $G \times E(S)$, $E(S)$ is a right zero semigroup, $K(\beta S) = \beta S$, every idempotent in βS is a left identity for βS , $E(\beta S) = \overline{E(S)}$, $\beta S = G \cdot E(\beta S)$, and βS is isomorphic to $G \times E(\beta S)$.
- (e) $K(\beta S) = \beta S$.

Proof. (a) \Rightarrow (b). Suppose there is a nonminimal idempotent p in S^* and pick by [14, Theorem 1.60] a minimal idempotent $q \in S^*$ such that $q < p$. Then q and p are in βS , so p is nonminimal in βS .

(b) \Rightarrow (c). Assume that $E(S^*) \subseteq K(S^*)$. By Lemma 3.16, all elements of S have finite order and S contains no infinite groups. By Theorem 3.15, pick a group G such that $S = G \cdot E(S)$. Then G is finite.

(c) \Rightarrow (d). Pick a finite group G such that $S = G \cdot E(S)$. By Lemma 3.13(1), $E(S)$ is a right zero semigroup. Let e be the identity of G . We claim that $G = G_e$. Trivially $G \subseteq G_e$. Let $a \in G_e$ and pick $b \in G$ and $f \in E(S)$ such that $a = b \cdot f$. Then $a = a \cdot e = b \cdot f \cdot e = b \cdot e = b$.

Define $\varphi : G \times E(S) \rightarrow S$ by $\varphi(a, f) = a \cdot f$. Since $S = G \cdot E(S)$, φ is onto S . To see that φ is a homomorphism, let $(a, f), (b, g) \in G \times E(S)$. Then $\varphi(a, f) \cdot \varphi(b, g) = a \cdot f \cdot b \cdot g = a \cdot b \cdot g = a \cdot b \cdot f \cdot g = \varphi((a, f) \cdot (b, g))$. To see that φ is one-to-one, assume that $(a, f), (b, g) \in G \times E(S)$ and $a \cdot f = b \cdot g$. Then $a = a \cdot e = a \cdot f \cdot e = b \cdot g \cdot e = b \cdot e = b$. Then $a \cdot f = a \cdot g$ so $f = g$.

By Lemma 3.13(2), G is a left ideal of S so by Lemma 3.17(1), \overline{G} is a left ideal of βS . Since G is finite, $G = \overline{G}$ so G is a left ideal of βS . We claim that G is a minimal left ideal of βS . So assume we have a left ideal L of βS with $L \subseteq G$. Pick $a \in L$. To see that $G \subseteq L$, let $b \in G$. Pick $c \in G$ such that $c \cdot a = b$. Then $b \in L$. Since G is a minimal left ideal of βS , $e \in K(\beta S)$ so $\beta S = e \cdot \beta S \subseteq K(\beta S)$. That is, $K(\beta S) = \beta S$.

Now, if $f \in E(\beta S)$, then f is in the minimal right ideal $e \cdot \beta S$, so $f \cdot \beta S$ is a right ideal contained in $e \cdot \beta S$ so $f \cdot \beta S = e \cdot \beta S = \beta S$, so by [14, Lemma 1.30(b)], f is a left identity for βS .

Next we note that if $f \in \beta S$, then $f \in E(\beta S)$ if and only if $f \cdot e = e$. We have just established the necessity. So assume that $f \cdot e = e$. Then $f = e \cdot f = f \cdot e \cdot f = f \cdot f$. Thus we have that $E(\beta S) = \rho_e^{-1}[\{e\}]$. Since ρ_e is continuous and $\{e\}$ is clopen in βS , we have that $E(\beta S)$ is clopen. Thus by Lemma 3.17(2) $E(\beta S) = \overline{E(S)}$.

Now $\beta S = \overline{S} = \overline{G \cdot E(S)} = \bigcup_{x \in G} \lambda_x[\overline{E(S)}] = \bigcup_{x \in G} \lambda_x[E(\beta S)] = G \cdot E(\beta S)$.

Finally, define $\varphi : G \cdot E(\beta S) \rightarrow \beta S$ by $\varphi(a, f) = a \cdot f$. The proof that φ is an isomorphism is the same as before, using the fact that every $f \in E(\beta S)$ is a left identity for βS .

It is trivial that (d) implies (e) and (e) implies (a). \square

Theorem 3.19. *Let S be an infinite left cancellative semigroup. The following statements are equivalent.*

- (a) S^* has a nonminimal idempotent.
- (b) βS has a nonminimal idempotent.
- (c) S contains either a copy of $(\mathbb{N}, +)$ or an infinite group.
- (d) S contains a countably infinite cancellative subsemigroup.
- (e) S contains a countably infinite right cancellative and weakly left cancellative subsemigroup.

Proof. Statements (a) and (b) are equivalent by Theorem 3.18.

(b) \Rightarrow (c). Assume that βS has a nonminimal idempotent and that S does not contain a copy of $(\mathbb{N}, +)$. Then every element of S has finite order, so by Theorem 3.15, pick a group G such that $S = G \cdot E(S)$. By Theorem 3.18, G is not finite.

It is trivial that (c) implies (d) and that (d) implies (e). The fact that (e) implies (a) is Lemma 3.4. \square

We saw in Theorem 3.2 that for countable left cancellative semigroups S and T , one could not have $(\beta S)^u$ and $(\beta T)^v$ isomorphic if $u \neq v$ and one of βS and βT had a nonminimal idempotent. We see that we cannot get so strong a result if neither βS nor βT has a nonminimal idempotent.

If R is an infinite right zero semigroup, it is an easy exercise (which is [14, Exercise 4.2.2]) to show that βR is a right zero semigroup.

Theorem 3.20. *Let R be an infinite right zero semigroup. Then for each $u, v \in \mathbb{N}$, $(\beta R)^u$, $(\beta R)^v$, $(R^*)^u$, $(R^*)^v$, $\beta(R^u)$, and $\beta(R^v)$ are all isomorphic.*

Proof. If $|R| = \kappa$ and $u \in \mathbb{N}$, then $(\beta R)^u$, $(R^*)^u$, and $\beta(R^u)$ are right zero semigroups of cardinality 2^{2^κ} . \square

Theorem 3.21. *Let R be an infinite right zero semigroup, let $S = \mathbb{Z}_2 \times R$ and let $T = \mathbb{Z}_2 \times \mathbb{Z}_2 \times R$. Then $(\beta S)^2$ and βT are isomorphic.*

Proof. By Theorem 3.18, $(\beta S)^2$ is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \beta R \times \beta R$ and βT is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \beta R$. By Theorem 3.20, $\beta R \times \beta R$ is isomorphic to βR . \square

We do not need to assume countability in the following result.

Theorem 3.22. *Let S be an infinite semigroup such that βS has no nonminimal idempotents and let u and v be distinct positive integers.*

- (1) $(\beta S)^u$ is isomorphic to $(\beta S)^v$ if and only if S is a right zero semigroup.
- (2) $(S^*)^u$ is isomorphic to $(S^*)^v$ if and only if S is a right zero semigroup.
- (3) $\beta(S^u)$ is isomorphic to $\beta(S^v)$ if and only if S is a right zero semigroup.

Proof. (1) The sufficiency is Theorem 3.20. Assume that S is not a right zero semigroup. By Theorem 3.18, pick a finite group G such that $S = G \cdot E(S)$. Since S is not a right zero semigroup, we have that $|G| > 1$. Since βS is isomorphic to $G \times E(\beta S)$ and $(E(\beta S))^u$ is isomorphic to $E(\beta S)$, we have $(\beta S)^u$ is isomorphic to $G^u \times E(\beta S)$. Thus the maximal groups in $(\beta S)^u$ have cardinality $|G|^u$ while the maximal groups of $(\beta S)^v$ have cardinality $|G|^v$.

After noting that S^* is isomorphic to $G \times E(S^*)$, the proof of (2) may be taken verbatim from the proof of (1) by replacing all occurrences of βS by S^* .

(3) The sufficiency is Theorem 3.20. Assume that S is not a right zero semigroup. By Theorem 3.18, pick a finite group G such that S is isomorphic to $G \times E(S)$ and note that $|G| > 1$. It suffices to show that the maximal groups of $\beta(S^u)$ are isomorphic to G^u . We have

$$\begin{aligned} S^u &\approx (G \times E(S))^u \\ &\approx G^u \times (E(S))^u \\ &\approx G^u \times E(S^u), \end{aligned}$$

where the last equivalence holds because $(E(S))^u$ and $E(S^u)$ are right zero semigroups of the same cardinality. Applying (c) \Rightarrow (d) of Theorem 3.18 to the semigroup S^u , we have $\beta(S^u)$ is isomorphic to $G^u \times E(\beta(S^u))$. \square

The next result is similar to Theorem 3.2 (and the proofs are also similar). We do not need the assumption of countability, nor do we need to assume that T is left cancellative, but we do need to add the assumption that the supposed isomorphism is continuous (and thus a homeomorphism).

Theorem 3.23. *Let S be an infinite left cancellative semigroup such that βS contains a nonminimal idempotent, let T be an arbitrary infinite semigroup, and let u and v be distinct positive integers. Then $(\beta S)^u$ and $(\beta T)^v$ are not topologically isomorphic.*

Proof. Suppose that $(\beta S)^u$ and $(\beta T)^v$ are topologically isomorphic and let $\varphi : (\beta S)^u \rightarrow (\beta T)^v$ be a continuous isomorphism.

We show first that βT has a nonminimal idempotent and T is left cancellative. Exactly as in of the proof of Theorem 3.2(1), we see that βT has a nonminimal idempotent.

Since S^u is the set of isolated points of $(\beta S)^u$, T^v is the set of isolated points of $(\beta T)^v$, and φ is a homeomorphism (since it is a continuous bijection) we have that $\varphi[S^u] = T^v$ so that T^v is left cancellative and therefore T is left cancellative.

Since we now have that S and T satisfy the same hypotheses, we may assume without loss of generality that $u > v$.

We have by Theorem 3.19 that S contains a countably infinite cancellative subsemigroup S_0 . Pick a nonminimal idempotent p in βS_0 and a minimal idempotent q in βS_0 such that $q < p$. As in the proof of 3.2 define for each $i \in \{1, 2, \dots, u\}$ a nonminimal idempotent \vec{r}_i in $(\beta S_0)^u$ and a minimal idempotent \vec{q} in $(\beta S_0)^u$ such that $\vec{r}_i \cdot \vec{r}_j = \vec{q}$ whenever i and j are distinct members of $\{1, 2, \dots, u\}$.

For $k \in \{1, 2, \dots, v\}$, let $T_k = \pi_k[\varphi[(S_0)^u]]$. Then T_k is a countable subsemigroup of T , and for $i \in \{1, 2, \dots, u\}$, $\varphi(\vec{r}_i)(k) \in \overline{T_k} = \beta T_k$.

We claim that, given $k \in \{1, 2, \dots, v\}$, there is at most one $i \in \{1, 2, \dots, u\}$ such that $\varphi(\vec{r}_i)(k) \neq \varphi(\vec{q})(k)$. Suppose instead we have $i \neq j$ in $\{1, 2, \dots, u\}$ such that $\varphi(\vec{r}_i)(k) \neq \varphi(\vec{q})(k)$ and $\varphi(\vec{r}_j)(k) \neq \varphi(\vec{q})(k)$. We have that

$$\varphi(\vec{r}_i)(k) \cdot \varphi(\vec{r}_j)(k) = \varphi(\vec{r}_j)(k) \cdot \varphi(\vec{r}_i)(k) = \varphi(\vec{q})(k)$$

so $\beta T_k \cdot \varphi(\vec{r}_i)(k) \cap \beta T_k \cdot \varphi(\vec{r}_j)(k) \neq \emptyset$. Therefore by Lemma 3.1 without loss of generality, $\varphi(\vec{r}_i)(k) = \varphi(\vec{r}_i)(k) \cdot \varphi(\vec{r}_j)(k) = \varphi(\vec{q})(k)$, a contradiction.

For $k \in \{1, 2, \dots, v\}$, pick $f(k) \in \{1, 2, \dots, u\}$ so that $\varphi(\vec{r}_i)(k) = \varphi(\vec{q})(k)$ for all $i \in \{1, 2, \dots, u\} \setminus \{f(k)\}$. Pick $i \in \{1, 2, \dots, u\} \setminus \{f(k) : k \in \{1, 2, \dots, v\}\}$. Then $\varphi(\vec{r}_i) = \varphi(\vec{q})$ and therefore $\vec{r}_i = \vec{q}$, a contradiction. \square

We note that it is at least consistent that there are left cancellative semigroups S and T such that βS and βT are isomorphic but not homeomorphic. Indeed by [17, Chapter 2, Theorem 2.18], if $\kappa \geq \omega$ and $MA(\kappa)$ holds, then $2^\kappa = 2^\omega$. So if we assume that $MA(\omega_1)$ holds and let S and T be right zero semigroups of cardinality ω and ω_1 , then βS and βT are right zero semigroups of cardinality 2^{2^ω} , so are isomorphic, while their sets of isolated points have different cardinality, so they are not homeomorphic.

Most of the results of this section require that S be left cancellative. For some of them, “left cancellative” cannot be weakened to “weakly left and weakly right cancellative”. If $S = (\mathbb{N}, \vee)$ where $n \vee m = \max\{n, m\}$, then

S^* is right zero, $K(\beta S) = S^*$, and every element of S is an idempotent. So statement (b) of Theorem 3.18 holds while none of the other statements hold and statement (b) of Theorem 3.19 holds while none of the other statements hold.

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