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# Central Sets and Their Combinatorial Characterization

by

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**ABSTRACT**. Central sets in semigroups are known to have very rich combinatorial structure, described by the "Central Sets Theorem". It has been unknown whether the Central Sets Theorem in fact characterizes central sets, and if not whether some other combinatorial characterization could be found. We derive here a combinatorial characterization of central sets and of the weaker notion of quasi-central sets. We show further that in $(\mathbb{N}, +)$ these notions are different and strictly stronger than the characterization provided by the Central Sets Theorem. In addition, we derive an algebraic characterization of sets satisfying the conclusion of the Central Sets Theorem and use this characterization to show that the conclusion of the Central Sets Theorem is a partition regular property in any commutative semigroup.

## 1. Introduction

1. **Introduction**. The notion of *central* subsets of the set $\mathbb{N}$ of positive integers was introduced by Furstenberg in [7] where he proved the "Central Sets Theorem" [7, Proposition 8.21]. This theorem is mildly complicated but has several easily derivable consequences. For example, any central set has solutions to any partition regular system of homogeneous linear equations with rational coefficients. Also, given any sequence $\langle x_n \rangle_{n=1}^{\infty}$ and any central set $A$, there exist arbitrarily long arithmetic progressions in $A$ whose increment comes from $FS(\langle x_n \rangle_{n=1}^{\infty}) = \{\Sigma_{n \in F} x_n : F$ is a finite nonempty subset of $\mathbb{N}\}$ . (See [7, pp. 169-174] for both of these consequences.)

The definition of "central" in [7] was in terms of dynamical systems, and the definition makes sense in any semigroup. In [3] (with the assistance of B. Weiss) that definition was shown to be equivalent to a much simpler algebraic characterization if the semigroup is countable. It is this algebraic characterization which we take as the definition for all semigroups.

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The algebraic characterization of a central set in $(S, \cdot)$ is in the setting of $(\beta S, \cdot)$ where $\beta S$ is the Stone-Čech compactification of the discrete space $S$ and $\cdot$ is the extension of the operation on $S$ to $\beta S$ making $\beta S$ a right topological semigroup with $S$ contained in its topological center. (By "right topological" we mean that for each $p \in \beta S$, the function $\rho_p : \beta S \to \beta S$ is continuous where $\rho_p(q) = q \cdot p$. By the "topological center" we mean the set of points $p$ such that $\lambda_p$ is continuous, where $\lambda_p(q) = p \cdot q$.)

As a compact right topological semigroup, $\beta S$ has a smallest two sided ideal denoted $K(\beta S)$. Further $K(\beta S)$ is the union of all minimal right ideals of $\beta S$ and is also the union of all minimal left ideals. (See [5, Chapter 1] for these and any other unfamiliar facts about compact right topological semigroups.) Any compact right topological semigroup has an idempotent and one can define a partial ordering of the idempotents by $p \leq q$ if and only if $p = p \cdot q = q \cdot p$. An idempotent $p$ is "minimal" if and only if $p$ is minimal with respect to the order $\leq$. Equivalently an idempotent $p$ is minimal if and only if $p \in K(\beta S)$.

1.1 **Definition**. Let $(S, \cdot)$ be a semigroup and let $A \subseteq S$. Then $A$ is *central* if and only if there is some minimal idempotent $p \in \beta S$ with $p \in c\ell A$.

We take the points of $\beta S$ to be the ultrafilters on $S$, identifying the principal ultrafilters with the points of $S$. Then given $A \subseteq S$ one has $c\ell A = \{p \in \beta S : A \in p\}$ and the topology on $\beta S$ has as a basis $\{c\ell A : A \subseteq S\}$. Accordingly a subset $A$ of $S$ is central if and only if $A$ is a member of some minimal idempotent.

Now it is well known that a subset $A$ of $S$ is a member of some idempotent in $\beta S$ if and only if there is some sequence $\langle x_n \rangle_{n=1}^{\infty}$ in S with $FP(\langle x_n \rangle_{n=1}^{\infty}) \subseteq A$ where $FP(\langle x_n \rangle_{n=1}^{\infty}) = \{\Pi_{n \in F}\ x_n : F$ is a finite nonempty subset of $\mathbb{N}\}$, the products being taken in increasing order of indices. Likewise it is known that $A$ is a member of some $p \in K(\beta S)$ if and only if $A$ is "piecewise syndetic". (See Definition 3.1.)

Since members of idempotents and members of minimal ultrafilters (i.e. those ultrafilters in K($\beta S$)) both have simple combinatorial characterizations, it is natural to hope for a combinatorial characterization of their combination, members of minimal idempotents. In particular, one can ask whether the powerful "Central Sets Theorem" characterizes central sets.

In Section 2 we present a proof of the strongest version of the Central Sets Theorem for commutative semigroups of which we are aware. We define a *rich set* (Definition 2.4) as one satisfying the conclusion of the Central Sets Theorem. We derive an algebraic characterization of rich sets (Corollary 2.11) and use this characterization to show that rich sets are partition regular in the sense that whenever a rich set is partitioned into finitely many parts, one of these parts must be a rich set.

We also conclude from the algebraic characterization of rich sets that any member of any idempotent in $c\ell K(\beta S)$ is a rich set. This suggests a definition.

1.2 **Definition**. Let $(S,\cdot)$ be a semigroup and let $A \subseteq S$. Then $A$ is *quasi-central* if and only if there is some idempotent $p \in c\ell K(\beta S)$ with $p \in c\ell A$.

Thus an additional question presents itself. Namely are all quasi-central sets in fact central? (Equivalently are all idempotents in $c\ell K(\beta S)$ in fact in $c\ell\{p : p$ is a minimal idempotent of $\beta S\}$ ?)

In Section 3 we provide some combinatorial characterizations of central sets as well as similar characterizations of quasi-central sets. In Sections 4 and 5 we use these characterizations to show that in the semigroup $(\mathbb{N},+)$, there are quasi-central sets that are not central and there are rich sets that are not quasi-central.

We have already remarked that we take the points of $\beta S$ to be the ultrafilters on $S$. We mention now a characterization of the operations $\cdot$ on $\beta S$ that we will utilize. Given $p$ and $q$ in $\beta S$ and $A \subseteq S$, one has $A \in p\cdot q$ if and only if $\{x \in S : x^{-1}A \in q\} \in p$, where $x^{-1}A = \{y \in S : x\cdot y \in S\}$ . (We are *not* assuming S is embeddable in a group.) See [9] for a detailed description of the semigroup $(\beta S,\cdot)$, with the caution that there $\beta S$ is taken to be left topological rather than right topological. This problem exists throughout the literature. There are in fact four different choices that can be made, and all four do in fact appear in the literature. (One may choose either of the two kinds of continuity and one may choose what one *calls* it. That is, what we call right topological is called by some authors left topological.) The connection between the two operations is as follows: Let $\cdot_\ell$ denote the operation making $\beta S$ left topological. Then given $p$ and $q$ in $\beta S$ and $A \subseteq S$ one has $A \in p\cdot_\ell q$ if and only if $\{x \in S : Ax^{-1} \in p\} \in q$ where $Ax^{-1} = \{y \in S : y\cdot x \in A\}$. Thus if one defines an operation $*$ on $S$ by $x*y = y\cdot x$, one has for all $p,q \in \beta S$ that $p\cdot q = q *_\ell p$. (And in particular, if $S$ is commutative, then $p\cdot q = q\cdot_\ell p$.) If $S$ is not commutative it is known ([1] and [6]) that the left topological and right topological structures can be quite different. We point out in Section 2 that being a member of a minimal idempotent in $(\beta S,\cdot_\ell)$ (being "left central") differs from the notion of central (or "right central"). We don't present a separate treatment of "left central" because the characterizations are identical with all operations reversed.

2. **The Central Sets Theorem**. We establish here that any quasi-central set in a commutative semigroup $(S,\cdot)$ satisfies a strong version of the Central Sets Theorem

involving infinitely many prespecified sequences. We further provide and utilize an algebraic characterization of sets which satisfy this strong conclusion.

The restriction to commutative semigroups is not essential, but the Central Sets Theorem for non-commutative semigroups is much more complicated to state. See [4, Theorem 2.8] for a statement of this theorem with finitely many prespecified sequences in an arbitrary semigroup.

The algebraic proof of the Central Sets Theorem is based on ideas developed by Furstenberg and Katznelson in the context of enveloping semigroups. We begin by quoting a well known result.

2.1 **Lemma**. *Let $n \in \mathbb{N}$ and let $T_1, T_2, \ldots, T_n$ be compact right topological semigroups and let $Y = \times_{i=1}^n T_i$ with the product topology and coordinatewise operations. Then $K(Y) = \times_{i=1}^n K(T_i)$.*

**Proof**. By [5, Proposition 1.3.6] $Y$ is a compact right topological semigroup (so $K(Y)$ exists). Thus [5, Corollary 1.2.6] applies. $\square$

The following lemma encodes an important part of the ideas of Furstenberg and Katznelson which we will use repeatedly in this paper.

2.2 **Lemma**. *Let $(D, \leq)$ be a directed set and let $(S, \cdot)$ be a semigroup and let $\ell \in \mathbb{N}$. For each $i \in D$ let $E_i$ and $I_i$ be subsets of $\times_{t=1}^{\ell} S$ such that*

*(1) for each $i \in D$, $\emptyset \neq I_i \subseteq E_i$;*

*(2) for each $i, j \in D$ if $i \leq j$, then $I_j \subseteq I_i$ and $E_j \subseteq E_i$;*

*(3) for each $i \in D$ and each $\vec{x} \in I_i$ there exists $j \in D$ such that $\vec{x} \cdot E_j \subseteq I_i$; and*

*(4) for each $i \in D$ and each $\vec{x} \in E_i \backslash I_i$ there exists $j \in D$ such that $\vec{x} \cdot E_j \subseteq E_i$ and $\vec{x} \cdot I_j \subseteq I_i$.*

*Let $E = \bigcap_{i \in D} c\ell E_i$ and $I = \bigcap_{i \in D} c\ell I_i$ where the closures are taken in $\times_{t=1}^{\ell} \beta S$. Then $E$ is a compact right topological semigroup and $I$ is an ideal of $E$.*

**Proof**. It is a routine exercise to show that $\times_{t=1}^{\ell} \beta S$ is a compact right topological semigroup and that for $\vec{x} \in \times_{t=1}^{\ell} S$, $\lambda_{\vec{x}}$ is continuous. By conditions (1) and (2) we have $\emptyset \neq I \subseteq E$.

To complete the proof we let $\vec{p}, \vec{q} \in E$ and show that $\vec{p} \cdot \vec{q} \in E$ and if either $\vec{p} \in I$ or $\vec{q} \in I$, then $\vec{p} \cdot \vec{q} \in I$. To this end, let $U$ be an open neighborhood of $\vec{p} \cdot \vec{q}$ and let $i \in D$ be given. We show that $U \cap E_i \neq \emptyset$ and if $\vec{p} \in I$ or $\vec{q} \in I$, then $U \cap I_i \neq \emptyset$. Pick a neighborhood $V$ of $\vec{p}$ such that $V \cdot \vec{q} \subseteq U$ and pick $\vec{x} \in E_i \cap V$ with $\vec{x} \in I_i$ if $\vec{p} \in I$. If $\vec{x} \in I_i$ pick $j \in D$ such that $\vec{x} \cdot E_j \subseteq I_i$. If $\vec{x} \in E_i \backslash I_i$, pick $j \in D$ such that $\vec{x} \cdot E_j \subseteq E_i$

and $\vec{x}\cdot I_j \subseteq I_i$. Now $\vec{x}\cdot\vec{q}\in U$ so pick a neighborhood $W$ of $\vec{q}$ such that $\vec{x}\cdot W\subseteq U$ and pick $\vec{y}\in W\cap E_j$ with $\vec{y}\in I_j$ if $\vec{q}\in I$. Then $\vec{x}\cdot\vec{y}\in U\cap E_i$ and if either $\vec{p}\in I$ or $\vec{q}\in I$, then $\vec{x}\cdot\vec{y}\in U\cap I_i$. $\square$

2.3 **Definition**. (a) Let $A$ be any set. Then $\mathcal{P}_f(A)=\{B : B$ is a finite nonempty subset of $A\}$.

(b) $\Phi=\{f:\ f:\mathbb{N}\to\mathbb{N}$ and for all $n\in\mathbb{N},\ f(n)\leq n\}$.

We now introduce an ideal $J$ of $\beta S$ which is of interest in its own right.

2.4 **Definition**. Let $(S,\cdot)$ be a commutative semigroup.

(a) $\mathcal{Y}=\{\langle\langle y_{i,t}\rangle_{t=1}^{\infty}\rangle_{i=1}^{\infty}$ : for each $i$ and $t$, $y_{i,t}\in S\}$.

(b) Given $Y=\langle\langle y_{i,t}\rangle_{t=1}^{\infty}\rangle_{i=1}^{\infty}$ in $\mathcal{Y}$ and $A\subseteq S$, $A$ is a $J_Y$-set if and only if for each $n\in\mathbb{N}$ there exist $a\in S$ and $H\in\mathcal{P}_f(\mathbb{N})$ with $\min H\geq n$ such that for all $i\in\{1,2,\ldots,n\}$, $a\cdot\Pi_{t\in H}\ y_{i,t}\in A$.

(c) Given $Y\in\mathcal{Y}$, $J_Y=\{p\in\beta S$ : for all $A\in p$, $A$ is a $J_Y$-set$\}$.

(d) $J=\bigcap_{Y\in\mathcal{Y}} J_Y$

(e) A set $A$ is a *rich set* if and only if $A\subseteq S$ and for each $Y=\langle\langle y_{i,t}\rangle_{t=1}^{\infty}\rangle_{i=1}^{\infty}$ in $\mathcal{Y}$, there exist sequences $\langle a_n\rangle_{n=1}^{\infty}$ in $S$ and $\langle H_n\rangle_{n=1}^{\infty}$ in $\mathcal{P}_f(\mathbb{N})$, with $\max H_n<\min H_{n+1}$ for all $n$, such that for all $f\in\Phi$, $FP(\langle a_n\cdot\Pi_{t\in H_n}y_{f(n),t}\rangle_{n=1}^{\infty})\subseteq A$.

Note that rich sets are precisely those sets satisfying the conclusion of the Central Sets Theorem.

If the semigroup $(S,\cdot)$ is $(\mathbb{N},+)$ and $A\subseteq\mathbb{N}$ with $J\cap c\ell A\neq\emptyset$, then $A$ has the following interesting property: Given any sequence $\langle x_t\rangle_{t=1}^{\infty}$ in $\mathbb{N}$, there exist arbitrarily long arithmetic progressions in $A$ with increment $d\in FS(\langle x_t\rangle_{t=1}^{\infty})$. (To see this, let $y_{i,t}=i\cdot x_t$ for each $i\in\{1,2,\ldots,\ell\}$.) Consequently in $(\beta\mathbb{N},+)$ one has $J\subseteq\mathcal{AP}=\{p\in\beta\mathbb{N}$ : for all $A\in p, A$ contains arbitrarily long arithmetic progressions$\}$.

We will see in Theorem 2.6 that $J$ is an ideal of $\beta S$, from which it follows that $K(\beta S)\subseteq J$. However we need the following lemma in order to conclude that $J\neq\emptyset$.

2.5 **Lemma**. *Let $(S,\cdot)$ be a commutative semigroup and let $Y\in\mathcal{Y}$. Then $K(\beta S)\subseteq J_Y$.*

**Proof**. Let $p\in K(\beta S)$ and let $A\in p$. Let $Y=\langle\langle y_{i,t}\rangle_{t=1}^{\infty}\rangle_{i=1}^{\infty}$ in $\mathcal{Y}$ and let $n\in\mathbb{N}$ be given. Let $W=\times_{i=1}^{n}\beta S$ and let $\vec{p}=(p,p,\ldots.p)$. By Lemma 2.1, $\vec{p}\in K(W)$.

For $k\in\mathbb{N}$, let $I_k=\{(a\cdot\Pi_{t\in H}\ y_{1,t},a\cdot\Pi_{t\in H}\ y_{2,t},\ldots,a\cdot\Pi_{t\in H}\ y_{n,t}) : a\in S$ and $H\in\mathcal{P}_f(\mathbb{N})$ and $\min H\geq k\}$ and let $E_k=I_k\cup\{(a,a,\ldots,a) : a\in S\}$. Let $E=\bigcap_{k=1}^{\infty}c\ell E_k$ and $I=\bigcap_{k=1}^{\infty}c\ell I_k$. We claim that $E$ is a subsemigroup of $W$ and $I$ is an ideal of $E$. Given $k \in \mathbb{N}$ and $\vec{x} = (a \cdot \Pi_{t \in H}\ y_{1,t}, a \cdot \Pi_{t \in H}\ y_{2,t}, \ldots, a \cdot \Pi_{t \in H}\ y_{n,t})$ with $\min H \geq k$, let $m = \max H + 1$. Then, using the fact that $S$ is commutative, we have that $\vec{x} \cdot E_m \subseteq I_k$. If $\vec{x} = (a, a, \ldots, a)$, then $\vec{x} \cdot E_k \subseteq E_k$ and $\vec{x} \cdot I_k \subseteq I_k$. Thus Lemma 2.2 applies.

Now $\vec{p} \in E$. (Given $k \in \mathbb{N}$ and $B_1, B_2, \ldots, B_n \in p$ pick $a \in \bigcap_{i=1}^{n} B_i$. Then $(a, a, \ldots, a) \in E_k \cap \times_{i=1}^{n} c\ell B_i$.) Thus $K(W) \cap E \neq \emptyset$ so by [5, Corollary 1.2.15], $K(E) = K(W) \cap E$ and hence $\vec{p} \in K(E) \subseteq I$. Then $\times_{i=1}^{n} c\ell A \cap I_n \neq \emptyset$ so pick $a \in S$ and $H \in \mathcal{P}_f(\mathbb{N})$ with $\min H \geq n$ and $(a \cdot \Pi_{t \in H}\ y_{1,t}, a \cdot \Pi_{t \in H}\ y_{2,t}, \ldots, a \cdot \Pi_{t \in H}\ y_{n,t}) \in \times_{i=1}^{n} A$. $\square$

2.6 **Theorem**. *Let $(S, \cdot)$ be a commutative semigroup and let $Y \in \mathcal{Y}$. Then $J_Y$ is a closed two sided ideal of $\beta S$. Consequently $J$ is a closed two sided ideal of $\beta S$.*

**Proof**. By Lemma 2.5, $J_Y \neq \emptyset$. Since it is defined as the set of all ultrafilters all of whose members satisfy a given property, $J_Y$ is closed. Let $p \in J_Y$ and let $q \in \beta S$. To see that $q \cdot p \in J_Y$, let $A \in q \cdot p$ and let $n \in \mathbb{N}$. Since $A \in q \cdot p$ pick $x \in S$ such that $x^{-1}A \in p$. Pick $a \in S$ and $H \in \mathcal{P}_f(\mathbb{N})$ with $\min H \geq n$ such that $a \cdot \Pi_{t \in H}\ y_{i,t} \in x^{-1}A$ for each $i \in \{1, 2, \ldots, n\}$. Then $x \cdot a \cdot \Pi_{t \in H}\ y_{i,t} \in A$ for each $i \in \{1, 2, \ldots, n\}$.

To see that $p \cdot q \in J_Y$, let $A \in p \cdot q$ and let $n \in \mathbb{N}$. Since $\{x \in S : x^{-1}A \in q\} \in p$, pick $a \in S$ and $H \in \mathcal{P}_f(\mathbb{N})$ with $\min H \geq n$ such that $a \cdot \Pi_{t \in H}\ y_{i,t} \in \{x \in S : x^{-1}A \in q\}$ for each $i \in \{1, 2, \ldots, n\}$. Pick $x \in \bigcap_{i=1}^{n} (a \cdot \Pi_{t \in H}\ y_{i,t})^{-1} A$. Then $x \cdot a \cdot \Pi_{t \in H}\ y_{i,t} \in A$ for each $i \in \mathbb{N}$. $\square$

2.7 **Lemma**. *Let $(S, \cdot)$ be a commutative semigroup, and let $Y = \langle \langle y_{i,t} \rangle_{t=1}^{\infty} \rangle_{i=1}^{\infty}$ be in $\mathcal{Y}$. Let $p$ be an idempotent in $J_Y$. Then for all $A \in p$ there exist a sequence $\langle a_n \rangle_{n=1}^{\infty} \in S$ and a sequence $\langle H_n \rangle_{n=1}^{\infty} \in \mathcal{P}_f(\mathbb{N})$, with $\max H_n < \min H_{n+1}$ for all $n$, such that for all $f \in \Phi$, $FP(\langle a_n \cdot \Pi_{t \in H_n}\ y_{f(n),t} \rangle_{n=1}^{\infty}) \subseteq A$.*

**Proof**. Let $A_1 = A$ and let $B_1 = A_1 \cap \{x \in S : x^{-1}A_1 \in p\}$. Since $p = p \cdot p$ we have $B_1 \in p$ so, since $p \in J_Y$, pick $a_1$ and $H_1$ such that $a_1 \cdot \Pi_{t \in H_1}\ y_{1,t} \in B_1$. Let $A_2 = A_1 \cap (a_1 \cdot \Pi_{t \in H_1}\ y_{1,t})^{-1} A_1$. Inductively, given $A_n$, let $B_n = A_n \cap \{x \in S : x^{-1}A_n \in p\}$. Let $m = \max(H_{n-1} \cup \{n\}) + 1$. Since $B_n \in p$, pick $a_n \in S$ and $H_n \in \mathcal{P}_f(\mathbb{N})$ such that $\min H_n \geq m$ and for all $i \in \{1, 2, \ldots, m\}$, $a_n \cdot \Pi_{t \in H_n}\ y_{i,t} \in B_n$. Let $A_{n+1} = A_n \cap \bigcap_{k=1}^{n} (a_n \cdot \Pi_{t \in H_n}\ y_{k,t})^{-1} A_n$.

Let $f \in \Phi$. We show by induction on $|F|$ that if $F \in \mathcal{P}_f(\mathbb{N})$ and $m = \min F$ then $\Pi_{n \in F}(a_n \cdot \Pi_{t \in H_n}\ y_{f(n),t}) \in A_m$. If $F = \{m\}$ we have $a_m \cdot \Pi_{t \in H_m}\ y_{f(m),t} \in B_m \subseteq A_m$. So assume $|F| > 1$, let $G = F \backslash \{m\}$ and let $r = \min G$. Then $\Pi_{n \in G}(a_n \cdot \Pi_{t \in H_n}\ y_{f(n),t}) \in A_r \subseteq A_{m+1} \subseteq (a_m \cdot \Pi_{t \in H_m}\ y_{f(m),t})^{-1} A_m$ so $\Pi_{n \in F}(a_n \cdot \Pi_{t \in H_n}\ y_{f(n),t}) \in A_m$. $\square$

The following theorem is the "Central Sets Theorem".

2.8 **Theorem**. *Let $(S,\cdot)$ be a commutative semigroup and let $A \subseteq S$. If for each $Y \in \mathcal{Y}$ there is some idemptotent $p \in J_Y \cap c\ell A$, then $A$ is a rich set.*

**Proof**. Apply Lemma 2.7. □

The original Central Sets Theorem ([7, Proposition 8.21]) had $S = \mathbb{N}$ and allowed the (finitely many) sequences $\langle y_{i,t}\rangle_{t=1}^{\infty}$ to take values in $\mathbb{Z}$. Since any idempotent minimal in $(\beta\mathbb{N},+)$ is also minimal in $(\beta\mathbb{Z},+)$, and hence any central set in $(\mathbb{N},+)$ is central in $(\mathbb{Z},+)$, this version follows from Theorem 2.8.

2.9 **Corollary** *Let $(S,\cdot)$ be a commutative semigroup and let $A \subseteq S$. If there is an idempotent $p \in J \cap c\ell A$, then $A$ is a rich set. In particular each quasi-central set (and hence each central set) is a rich set.* □

We now prove the converse of Theorem 2.8.

2.10 **Theorem** *Let $(S,\cdot)$ be a commutative semigroup and let $A \subseteq S$ be a rich set. Then for each $Y \in \mathcal{Y}$ there is an idempotent $p \in J_Y \cap c\ell A$.*

**Proof**. Pick sequences $\langle a_n\rangle_{n=1}^{\infty}$ in S and $\langle H_n\rangle_{n=1}^{\infty}$ in $\mathcal{P}_f(\mathbb{N})$, with $\max H_n < \min H_{n+1}$ for all $n$, such that for each $f \in \Phi$, $FP(\langle a_n \cdot \Pi_{t\in H_n}\, y_{f(n),t}\rangle_{n=1}^{\infty}) \subseteq A$. For each $r \in \mathbb{N}$, let $M_r = \bigcup\{FP(\langle a_n \cdot \Pi_{t\in H_n}\, y_{f(n),t}\rangle_{n=r+1}^{\infty}) : f \in \Phi\}$, and let $M = \bigcap_{r=1}^{\infty} c\ell M_r$. Note that $M \subseteq c\ell A$ since $M_1 \subseteq A$.

We claim that $M$ is a subsemigroup of $\beta S$. To this end, let $p, q \in M$ and let $B \in p \cdot q$ and let $r \in \mathbb{N}$. We show that $B \cap M_r \neq \emptyset$. Let $C = \{x \in S : x^{-1}B \in q\}$. Then $C \in p$ and $p \in c\ell M_r$ so $C \cap M_r \neq \emptyset$ so pick $f \in \Phi$ and $F \in \mathcal{P}_f(\mathbb{N})$ with $\min F > r$ such that $x = \Pi_{n\in F}(a_n \cdot \Pi_{t\in H_n}\, y_{f(n),t}) \in C$. Let $s = \max F$. Now $x^{-1}B \in q$ and $q \in c\ell M_s$ so $x^{-1}B \cap M_s \neq \emptyset$. Pick $g \in \Phi$ and $G \in \mathcal{P}_f(\mathbb{N})$ with $\min G > s$ such that $y = \Pi_{n\in G}(a_n \cdot \Pi_{t\in H_n}\, y_{g(n),t}) \in x^{-1}B$. Let $h(n) = g(n)$ if $n \in G$ and let $h(n) = f(n)$ otherwise. Then $x \cdot y = \Pi_{n\in F\cup G}(a_n \cdot \Pi_{t\in H_n}\, y_{h(n),t}) \in B \cap M_r$.

Pick an idempotent $p \in K(M)$. Then $p \in c\ell A$ so to complete the proof we show that $p \in J_Y$. To this end let $B \in p$. We need to show that $B$ is a $J_Y$-set, so let $n \in \mathbb{N}$. Let $W = \times_{i=1}^{n} M$. For each $r \in \mathbb{N}$, let $I_r = \{(a \cdot x_1, a \cdot x_2, \ldots, a \cdot x_n)$ : there exist $D$ and $D'$ in $\mathcal{P}_f(\mathbb{N})$ and $f \in \Phi$ with $D \cap D' = \emptyset$ and $\min(D \cup D') > r$ and $a = \Pi_{m\in D}(a_m \cdot \Pi_{t\in H_m}\, y_{f(m),t})$ and for each $i \in \{1, 2, \ldots, n\}$, $x_i = \Pi_{m\in D'}(a_m \cdot \Pi_{t\in H_m}\, y_{i,t})\}$. For $r \in \mathbb{N}$ let $E_r = I_r \cup \{(a, a, \ldots, a) : a \in M_r\}$.

Let $E = \bigcap_{r=1}^{\infty} c\ell E_r$ and $I = \bigcap_{r=1}^{\infty} c\ell I_r$, where the closures are taken in $W$. Note that each $E_r \subseteq \times_{i=1}^{n} M_r$ and consequently $E \subseteq W$. We now show that $E$ is a semigroup and that $I$ is an ideal of $E$, using Lemma 2.2. Let $r \in \mathbb{N}$ be given and let $\vec{x} \in I_r$ and pick $D$, $D'$, and $f$ as in the definition of $I_r$ and let $s = \max(D \cup D')$. Then $\vec{x} \cdot E_s \subseteq I_r$. Now let $\vec{x} \in E_r \backslash I_r$, i.e., $\vec{x} = (a, a, \ldots, a)$ for some $a \in M_r$. Pick $D \in \mathcal{P}_f(\mathbb{N})$ and $f \in \Phi$ such that $a = \Pi_{n \in D}(a_n \cdot \Pi_{t \in H_n} \; y_{f(n),t})$. Let $s = \max D$. Then $\vec{x} \cdot E_s \subseteq E_r$ and $\vec{x} \cdot I_s \subseteq I_r$.

Let $\vec{p} = (p, p, \ldots, p)$. We claim $\vec{p} \in E$. To see this, let $C \in p$ and let $r \in \mathbb{N}$. Then $C \cap M_r \neq \emptyset$ so pick $a \in C \cap M_r$. Then $(a, a, \ldots, a) \in E_r \cap \times_{i=1}^{n} C$. Thus we have $\vec{p} \in E \cap \times_{i=1}^{n} K(M)$ so by Lemma 2.1, $\vec{p} \in E \cap K(W)$. Then by [5, Corollary 1.2.15] $K(E) = E \cap K(W)$ so $\vec{p} \in K(E) \subseteq I$ and hence $I_n \cap \times_{i=1}^{n} B \neq \emptyset$. (Recall that we are showing that $B$ is a $J_Y$ set.) Pick $\vec{z} = (a \cdot x_1, a \cdot x_2, \ldots, a \cdot x_n) \in I_n \cap \times_{i=1}^{n} B$. Pick $D$ and $D'$ in $\mathcal{P}_f(\mathbb{N})$ with $D \cap D' = \emptyset$ and $\min(D \cup D') > n$ and pick $f \in \Phi$ such that $a = \Pi_{m \in D}(a_m \cdot \Pi_{t \in H_m} \; y_{f(m),t})$ and for $i \in \{1, 2, \ldots, n\}$, $x_i = \Pi_{m \in D'}(a_m \cdot \Pi_{t \in H_m} \; y_{i,t})$. Let $b = a \cdot \Pi_{m \in D'} \; a_m$ and let $G = \bigcup_{m \in D'} \; H_m$ and note that $\min G \geq \min D' > n$. Then for each $i \in \{1, 2, \ldots, n\}$, $b \cdot \Pi_{t \in G} \; y_{i,t} \in B$. □

2.11 **Corollary**. *Let $(S, \cdot)$ be a commutative semigroup and let $A \subseteq S$. Then $A$ is a rich set if and only if for every $Y \in \mathcal{Y}$ there is an idempotent $p \in J_Y \cap c\ell A$.*

**Proof**. Theorems 2.8 and 2.10. □

As an additional, fortuitous, corollary we see that the property of being a rich set is partition regular.

2.12 **Corollary**. *Let $(S, \cdot)$ be a commutative semigroup and let $A \subseteq S$ be a rich set. If $r \in \mathbb{N}$ and $A = \bigcup_{i=1}^{r} B_i$, then for some $i \in \{1, 2, \ldots, r\}$, $B_i$ is a rich set.*

**Proof**. Direct $\mathcal{Y}$ by agreeing for $Y = \langle\langle y_{i,t}\rangle_{t=1}^{\infty}\rangle_{i=1}^{\infty}$ and $Z = \langle\langle z_{i,t}\rangle_{t=1}^{\infty}\rangle_{i=1}^{\infty}$ in $\mathcal{Y}$, that $Y \leq Z$ if and only if for each $i \in \mathbb{N}$ there exists $j \in \mathbb{N}$ with $\langle y_{i,t}\rangle_{t=1}^{\infty} = \langle z_{j,t}\rangle_{t=1}^{\infty}$. (It is routine to verify that this relation does direct $\mathcal{Y}$, and that $J_Z \subseteq J_Y$ whenever $Y \leq Z$.) For each $Y \in \mathcal{Y}$ pick an idempotent $p_Y \in J_Y \cap c\ell A$, which one can do by Theorem 2.10. Let $p \in \beta S$ be a cluster point of the net $\langle p_Y\rangle_{Y \in \mathcal{Y}}$ and pick $i \in \{1, 2, \ldots, r\}$ with $B_i \in p$. Then given any $Y \in \mathcal{Y}$ pick some $Z \geq Y$ in $\mathcal{Y}$ with $p_Z \in c\ell B_i$. Then $p_Z \in J_Y \cap c\ell B_i$. Thus by Theorem 2.8, $B_i$ is a rich set. □

We see now that if our commutative semigroup is countable, then rich sets satisfy an even stronger combinatorial statement.

2.13 **Theorem** *Let $(S, \cdot)$ be a countable commutative semigroup and let $A$ be a rich set. There is a sequence $\langle x_n\rangle_{n=1}^{\infty}$ such that $FP(\langle x_n\rangle_{n=1}^{\infty}) \subseteq A$ and for each $m \in \mathbb{N}$, $A \cap \bigcap\{y^{-1}A : y \in FP(\langle x_n\rangle_{n=1}^{m})\}$ is a rich set.*

**Proof**. Given $B \subseteq S$ and $Y \in \mathcal{Y}$, let $X(B,Y) = \{x \in B$ : there exists an idempotent $p \in J_Y \cap c\ell B \cap c\ell x^{-1}B\}$. Note first that if $B$ is a rich set, then for each $Y \in \mathcal{Y}$, $X(B,Y) \neq \emptyset$. Indeed by Theorem 2.10, we have some idempotent $p \in J_Y \cap c\ell B$. Then $\{x \in S : x^{-1}B \in p\} \in p$ and $B \in p$ so pick $x \in B$ such that $x^{-1}B \in p$. Then $x \in X(B,Y)$.

Next we show that if $B$ is a rich set, then $\bigcap_{Y\in\mathcal{Y}} X(B,Y) \neq \emptyset$. Indeed suppose $\bigcap_{Y\in\mathcal{Y}} X(B,Y) = \emptyset$ and for each $x \in B$, pick $Y_x \in \mathcal{Y}$ such that $x \notin X(B,Y_x)$. Direct $\mathcal{Y}$ as in the proof of Corollary 2.12 above. Since $B$ is countable, pick $Z \in \mathcal{Y}$ such that $Y_x \leq Z$ for each $x \in B$. (It is an easy exercise to show that one can do this.) Then $J_Z \subseteq J_{Y_x}$ for each $x \in B$ so $X(B,Z) \subseteq X(B,Y_x)$ for each $x \in B$. But $X(B,Z) \neq \emptyset$ so there is some $x \in X(B,Y_x)$, a contradiction.

As a final preliminary observation we note that whenever $B$ is a rich set and $x \in \bigcap_{Y\in\mathcal{Y}} X(B,Y)$ one has $B \cap x^{-1}B$ is a rich set. Indeed, one has directly from the definition that for each $Y \in \mathcal{Y}$ there is an idempotent $p \in J_Y \cap c\ell(B \cap x^{-1}B)$ so Theorem 2.8 applies.

Now we construct $\langle x_n\rangle_{n=1}^{\infty}$ inductively. Choose $x_1 \in \bigcap_{Y\in\mathcal{Y}} X(A,Y)$. Then $A \cap x_1^{-1}A$ is a rich set. Inductively let $m \in \mathbb{N}$ and assume we have chosen $\langle x_n\rangle_{n=1}^{m}$ such that $FP(\langle x_n\rangle_{n=1}^{m}) \subseteq A$ and $A \cap \bigcap\{y^{-1}A : y \in FP(\langle x_n\rangle_{n=1}^{m})\}$ is a rich set. Let $B = A \cap \bigcap\{y^{-1}A : y \in FP(\langle x_n\rangle_{n=1}^{m})\}$ and pick $x_{m+1} \in \bigcap_{Y\in\mathcal{Y}} X(B,Y)$. Since $x_{m+1} \in B$ we have $FP(\langle x_n\rangle_{n=1}^{m+1}) \subseteq A$. Also $B \cap x_{m+1}^{-1}B$ is a rich set and $B \cap x_{m+1}^{-1}B \subseteq A \cap \bigcap\{y^{-1}A : y \in FP(\langle x_n\rangle_{n=1}^{m+1})\}$. □

As we have already remarked, we restrict our attention to commutative $S$ in Theorem 2.8 because the conclusion becomes *much* more complicated when $S$ is not commutative. (In the proof in Lemma 2.5 that $E$ is a semigroup, one uses the fact that if $H \cap G = \emptyset$, then $a \cdot \Pi_{t\in H}\, y_{i,t} \cdot b \cdot \Pi_{t\in G}\, y_{i,t} = a \cdot b \cdot \Pi_{t\in H\cup G}\, y_{i,t}$.) There *is* however a Central Sets Theorem for noncommutative semigroups. Or rather there are two such theorems: one for members of idempotents minimal in $(\beta S, \cdot)$, the other for members of idempotents minimal in $(\beta S, \cdot_\ell)$. (As we have remarked, the reader can see [4, Theorem 2.8] for the latter, at least for finitely many given sequences.) To convert between such theorems one merely interchanges the order of all operations.

We conclude this section with a demonstration that the left and right notions can be quite different. The notion we are using for central should properly be called "right central".

2.14 **Theorem**. *Let $S$ be the free semigroup on two generators $a$ and $b$. There is a central subset $B$ of $S$ such that whenever $p \in S^* = \beta S\backslash S$ and $q \in \beta S\backslash c\ell\{b^n : n \in \mathbb{N}\}$,* *one has $B \notin p \cdot_\ell q$.*

**Proof**. Let $B = \{sab^n : n \in \mathbb{N}$ and $s \in S$ and $\text{length}(s) < n\}$. To see that $B$ is central we show that $c\ell B$ contains a left ideal of $\beta S$, which suffices since each left ideal contains a minimal idempotent. To see this, let $p \in S^* \cap c\ell\{ab^n : n \in \mathbb{N}\}$. Then given any $s \in S$, $s \cdot p \in c\ell B$ so $\beta S \cdot p \subseteq c\ell B$.

Now let $p \in S^*$ and let $q \in \beta S \backslash c\ell\{b^n : n \in \mathbb{N}\}$ and suppose that $B \in p \cdot_\ell q$. Pick $w \in S \backslash \{b^n : n \in \mathbb{N}\}$ such that $Bw^{-1} \in p$. Since $w \notin \{b^n : n \in \mathbb{N}\}$ there exist some $s \in S \cup \{\emptyset\}$ and some $n \in \mathbb{N} \cup \{0\}$ such that $w = sab^n$, where $b^0 = \emptyset$. Then $\{u \in S : \text{length}(u) > n\} \in p$, and $\{u \in S : \text{length}(u) > n\} \cap Bw^{-1} = \emptyset$, a contradiction. □

2.15 **Corollary**. *Let $S$ be the free semigroup on two generators. There is a central subset $B$ of $S$ which is not a member of any idempotent which is minimal in $(\beta S, \cdot_\ell)$, in fact is not a member of any idempotent in $(\beta S, \cdot_\ell)$.*

**Proof**. Let $B$ be the set produced in the proof of Theorem 2.14 and let $p$ be an idempotent in $(\beta S, \cdot_\ell)$. Suppose that $B \in p$. Since $B \cap \{b^n : n \in \mathbb{N}\} = \emptyset$, $\{b^n : n \in \mathbb{N}\} \notin p$ so by Theorem 2.14, $B \notin p \cdot_\ell p = p$, a contradiction. □

3. **Combinatorial characterizations of central and quasi-central**. Our characterizations of central and quasi-central utilize a notion from topological dynamics, namely that of being *piecewise syndetic*. In $\mathbb{N}$, a set is *syndetic* if it has bounded gaps. It is *piecewise syndetic* if there is a fixed bound and arbitrarily long intervals in which the set has gaps bounded by this fixed bound. The generalization to arbitrary semigroups is less intuitive, but standard. We are not restricting our attention to commutative semigroups so there will be two notions, one from each side. Thus what we are calling "piecewise syndetic" could be called "right piecewise syndetic".

We also include a generalization to arbitrary families of sets. Given subsets $A$ and $B$ of a semigroup $S$, we write $B^{-1}A = \bigcup_{t \in B} t^{-1}A$. (Thus $x \in B^{-1}A$ if and only if there is some $t \in B$ with $t \cdot x \in A$.)

3.1 **Definition**. Let $(S, \cdot)$ be a semigroup.

(a) A set $A \subseteq S$ is *syndetic* if and only if there exists $G \in \mathcal{P}_f(S)$ with $S \subseteq G^{-1}A$.

(b) A set $A \subseteq S$ is *piecewise syndetic* if and only if there exists $G \in \mathcal{P}_f(S)$ such that $\{y^{-1}G^{-1}A : y \in S\}$ has the finite intersection property.

(c) A family $\mathcal{A} \subseteq \mathcal{P}(S)$ is *collectionwise piecewise syndetic* if and only if there exists a function $G : \mathcal{P}_f(\mathcal{A}) \to \mathcal{P}_f(S)$ such that $\{y^{-1}(G(\mathcal{F}))^{-1}(\bigcap \mathcal{F}) : y \in S$ and $\mathcal{F} \in \mathcal{P}_f(\mathcal{A})\}$ has the finite intersection property.

Observe that $A$ is piecewise syndetic if and only if there is some $G \in \mathcal{P}_f(S)$ such that for every $F \in \mathcal{P}_f(S)$ there is some $x \in S$ with $F \cdot x \subseteq \bigcup_{t \in G} t^{-1}A$. Observe also that $\mathcal{A}$ is collectionwise piecewise syndetic if and only if there exist functions $G : \mathcal{P}_f(\mathcal{A}) \to \mathcal{P}_f(S)$ and $x : \mathcal{P}_f(\mathcal{A}) \times \mathcal{P}_f(S) \to S$ such that for all $F \in \mathcal{P}_f(S)$ and all $\mathcal{F}$ and $\mathcal{H}$ in $\mathcal{P}_f(\mathcal{A})$ with $\mathcal{F} \subseteq \mathcal{H}$ one has $F \cdot x(\mathcal{H}, F) \subseteq \bigcup_{t \in G(\mathcal{F})} t^{-1}(\bigcap \mathcal{F})$.

Note that a subset $A$ is piecewise syndetic if and only if $\{A\}$ is collectionwise piecewise syndetic. The importance of these notions is exhibited by the following theorem.

3.2 **Theorem**. *Let $(S, \cdot)$ be an infinite semigroup and let $\mathcal{A} \subseteq \mathcal{P}(S)$. There exists $p \in K(\beta S)$ with $\mathcal{A} \subseteq p$ if and only if $\mathcal{A}$ is collectionwise piecewise syndetic. In particular, given $A \subseteq S$, $K(\beta S) \cap c\ell A \neq \emptyset$ if and only if $A$ is piecewise syndetic.*

**Proof**. After the appropriate left-right switches, this is [10, Theorem 2.1]. □

The following is not directly relevant to our characterization of central sets, but does show some of the connections among the notions we are studying.

3.3 **Theorem**. *Let $(S, \cdot)$ be an infinite semigroup and let $A \subseteq S$. The following statements are equivalent.*

*(a) $A$ is piecewise syndetic.*

*(b) $\{x \in S : x^{-1}A$ is central$\}$ is syndetic.*

*(c) There is some $x \in S$ such that $x^{-1}A$ is central.*

**Proof** (a) ⇒ (b). Pick by Theorem 3.2 some $p \in K(\beta S)$ with $A \in p$. Now $K(\beta S)$ is the union of all minimal left ideals of $\beta S$. (Recall that the reference for basic facts about compact right topological semigroups is [5].) So pick a minimal left ideal $L$ of $\beta S$ with $p \in L$ and pick an idempotent $e \in L$. Then $p = p \cdot e$ so pick $y \in S$ such that $y^{-1}A \in e$.

Now by [8, Corollary 3.6] we have $B = \{z \in S : z^{-1}(y^{-1}A) \in e\}$ is syndetic, so pick finite $G \subseteq S$ such that $S = G^{-1}B$. Let $D = \{x \in S : x^{-1}A$ is central$\}$. We claim that $S = (y \cdot G)^{-1}D$. Indeed, let $x \in S$ be given and pick $t \in G$ such that $t \cdot x \in B$. Then $(t \cdot x)^{-1}(y^{-1}A) \in e$ so $(t \cdot x)^{-1}(y^{-1}A)$ is central. But $(t \cdot x)^{-1}(y^{-1}A) = (y \cdot t \cdot x)^{-1}A$. Thus $y \cdot t \cdot x \in D$ so $x \in (y \cdot t)^{-1}D$ as required.

(b) ⇒ (c). Trivial.

(c) ⇒ (a). Pick $x \in S$ such that $x^{-1}A$ is central and pick an idempotent $p \in K(\beta S)$ such that $x^{-1}A \in p$. Then $A \in x \cdot p$ and $x \cdot p \in K(\beta S)$ so by Theorem 3.2, $A$ is piecewise syndetic. □

Our combinatorial characterizations of *central* and *quasi-central* are based on an analysis of the usual proof (due to F. Galvin) that any member of any idempotent contains $FP(\langle x_n\rangle_{n=1}^{\infty})$ for some sequence $\langle x_n\rangle_{n=1}^{\infty}$ in $S$. Let's review that proof now.

Let $p = p{\cdot}p$ in $\beta S$ and let $A \in p$. Let $A_1 = A$ and let $B_1 = A_1 \cap \{x \in S : x^{-1}A_1 \in p\}$. Since $p \cdot p = p$ one has $B_1 \in p$. Pick $x_1 \in B_1$ and let $A_2 = A_1 \cap {x_1}^{-1}A_1$. Let $B_2 = A_2 \cap \{x \in S : x^{-1}A_2 \in p\}$ and pick $x_2 \in B_2$ and continue in this way. One has then, for example, that $x_1{\cdot}x_4{\cdot}x_5{\cdot}x_{10} \in A$ as follows. First $x_{10} \in B_{10} \subseteq A_{10} \subseteq A_9 \subseteq \ldots \subseteq A_6 \subseteq {x_5}^{-1}A_5$. Then $x_5 \cdot x_{10} \in A_5 \subseteq {x_4}^{-1}A_4$ so $x_4 \cdot x_5 \cdot x_{10} \in A_4 \subseteq A_3 \subseteq A_2 \subseteq {x_1}^{-1}A_1$. Now the important thing to notice about this proof is that when one chooses $x_n$ one in fact has a large number of choices. That is, one can draw a tree, branching infinitely often at each node, so that any path through that tree yields a sequence $\langle x_n\rangle_{n=1}^{\infty}$ with $FP(\langle x_n\rangle_{n=1}^{\infty}) \subseteq A$. (Recall that in $FP(\langle x_n\rangle_{n=1}^{\infty})$, the products are taken in increasing order of indices.)

We formalize the notion of "tree" below. We write $\omega = \{0, 1, 2, 3, \ldots\}$, the first infinite ordinal and recall that each ordinal is the set of its predecessors. (So $3 = \{0, 1, 2\}$ and $0 = \emptyset$ and, if $f$ is the function $\{(0, 3), (1, 5), (2, 9), (3, 7), (4, 5)\}$, then $f_{|3} = \{(0, 3), (1, 5), (2, 9)\}$.)

3.4 **Definition**. $T$ is a *tree in* $A$ if and only if $T$ is a set of functions and for each $f \in T$, domain$(f) \in \omega$ and range$(f) \subseteq A$ and if domain$(f) = n > 0$, then $f_{|n-1} \in T$. $T$ is a *tree* if and only if for some $A$, $T$ is a tree in $A$.

The last requirement in the definition is not essential; we utilize it nowhere in our proofs. Further, any set of functions with domains in $\omega$ can be converted to a tree by adding in all restrictions to initial segments. We include the requirement in the definition for aesthetic reasons – it is not nice for branches at some late level to appear from nowhere.

3.5 **Definition**. (a) Let $f$ be a function with domain$(f) = n \in \omega$ and let $x$ be given. Then $f^\frown x = f \cup \{(n, x)\}$.

(b) Given a tree $T$ and $f \in T$, $B_f = B_f(T) = \{x : f^\frown x \in T\}$.

(c) Let $(S, \cdot)$ be a semigroup and let $A \subseteq S$. Then $T$ is a $*$-*tree* in $A$ if and only if $T$ is a tree in $A$ and for all $f \in T$ and all $x \in B_f$, $B_{f^\frown x} \subseteq x^{-1}B_f$.

(d) Let $(S, \cdot)$ be a semigroup and let $A \subseteq S$. Then $T$ is a *FP-tree in* $A$ if and only if $T$ is a tree in $A$ and for all $f \in T$, $B_f = \{\Pi_{t\in F}\ g(t) : g \in T$ and $f \subsetneq g$ and $\emptyset \neq F \subseteq \text{domain}(g)\backslash\text{domain}(f)\}$.

The idea of the terminology is that a FP-tree is a tree of finite products. It is this notion which provides the most fundamental combinatorial characterization of both "central" and "quasi-central". A $*$-tree arises more directly from the proof outlined above.

3.6 **Lemma**. *Let $(S,\cdot)$ be an infinite semigroup and let $A \subseteq S$. Let $p$ be an idempotent in $\beta S$ with $A \in p$. There is a FP-tree $T$ in $A$ such that for each $f \in T$, $B_f \in p$.*

**Proof**. We will define the initial segments $T_n = \{f \in T : \text{domain}(f) = n\}$ inductively. Let $T_0 = \{\emptyset\}$ (of course) and let $C_\emptyset = A \cap \{x \in S : x^{-1}A \in p\}$ and note that $C_\emptyset \in p$. Let $T_1 = \{(0,x) : x \in C_\emptyset\}$.

Inductively assume we have $n \in \mathbb{N}$ and have defined $T_n$ so that for each $f \in T_n$ and all $x \in FP(\langle f(t)\rangle_{t=0}^{n-1})$ one has $x \in A$ and $x^{-1}A \in p$. Given $f \in T_n$, write $P_f = FP(\langle f(t)\rangle_{t=0}^{n-1})$ and note that, given $x \in P_f$, since $x^{-1}A \in p$ one also has $\{y \in S : y^{-1}(x^{-1}A) \in p\} \in p$. Let $C_f = A \cap \{y \in S : y^{-1}A \in p\} \cap \bigcap_{x\in P_f}(x^{-1}A) \cap \bigcap_{x \in P_f}\{y \in S : y^{-1}(x^{-1}A) \in p\}$, and note that $C_f \in p$. Let $T_{n+1} = \{f^\frown y : f \in T_n \text{ and } y \in C_f\}$. To see that our induction hypothesis is satisfied, let $g \in T_{n+1}$ and let $z \in FP(\langle g(t)\rangle_{t=0}^{n})$. Pick nonempty $F \subseteq \{0,1,\ldots,n\}$ such that $z = \Pi_{t\in F}\ g(t)$. Let $y = g(n)$ and let $f = g_{|n}$ (so $g = f^\frown y$ and $f \in T_n$). Now if $n \notin F$ one has $z \in FP(\langle f(t)\rangle_{t=0}^{n-1})$ so $z \in A$ and $z^{-1}A \in p$ by the induction hypothesis. If $F = \{n\}$, then $z = y$ and $y \in C_f$ so $y \in A$ and $y^{-1}A \in p$. Thus assume $\{n\}$ is properly contained in $F$ and let $G = F\backslash\{n\}$. Let $x = \Pi_{t\in G}\ f(t)$ so that $x \in FP(\langle f(t)\rangle_{t=0}^{n-1})$. Now $y \in x^{-1}A$ so $z = x\cdot y \in A$. Also $z^{-1}A = y^{-1}(x^{-1}A) \in p$.

The induction being complete, let $T = \bigcup_{n=0}^{\infty} T_n$. Then $T$ is a tree in $A$. One sees immediately from the construction that for each $f \in T$, $B_f = C_f$. We need to show that for each $f \in T$ one has $B_f = \{\Pi_{t\in F}\ g(t) : g \in T \text{ and } f \subsetneqq g \text{ and } \emptyset \neq F \subseteq \text{domain}(g)\backslash\text{domain}(f)\}$. Given $f \in T$ and $x \in B_f$, let $g = f^\frown x$ and let $F = \text{domain}(g)\backslash\text{domain}(f)$ (which is a singleton). For the other inclusion we first observe that if $f,h \in T$ with $f \subseteq h$ then $P_f \subseteq P_h$ so $B_h \subseteq B_f$. Let $f \in T_n$ and let $x \in \{\Pi_{t\in F}\ g(t) : g \in T \text{ and } f \subsetneqq g \text{ and } \emptyset \neq F \subseteq \text{domain}(g)\backslash\text{domain}(f)\}$. Pick $g \in T$ with $f \subsetneqq g$ and pick $F$ with $\emptyset \neq F \subseteq \text{domain}(g)\backslash\text{domain}(f)$ such that $x = \Pi_{t\in F}\ g(t)$. First assume $F = \{m\}$. Then $m \geq n$. Let $h = g_{|m}$. Then $f \subseteq h$ and $h^\frown x = g_{|m+1} \in T$. Hence $x \in B_h \subseteq B_f$ as required. Now assume $|F| > 1$, let $m = \max F$, and let $G = F\backslash\{m\}$. Let $h = g_{|m}$, let $x = \Pi_{t\in G}\ g(t)$, and let $y = g(m)$. Then $y \in B_h$. Let $P_f = FP(\langle f(t)\rangle_{t=0}^{n-1})$ and $P_h = FP(\langle h(t)\rangle_{t=0}^{m-1})$. We need to show that $x\cdot y \in B_f$. That is, we need $x \cdot y \in A$, $(x\cdot y)^{-1}A \in p$, and for all $z \in P_f$, $x\cdot y \in z^{-1}A$ and $(x\cdot y)^{-1}(z^{-1}A) \in p$. Now $x \in P_h$ and $y \in B_h$ so $y \in x^{-1}A$ and $y^{-1}(x^{-1}A) \in p$ so

$x \cdot y \in A$ and $(x \cdot y)^{-1}A \in p$. Let $z \in P_f$. Then $z \cdot x \in P_h$ and $y \in B_h$ so $y \in (z \cdot x)^{-1}A$ and $y^{-1}((z \cdot x)^{-1}A) \in p$ so $x \cdot y \in z^{-1}A$ and $(x \cdot y)^{-1}(z^{-1}A) \in p$. $\square$

3.7 **Theorem**. *Let $(S, \cdot)$ be an infinite semigroup and let $A \subseteq S$. Statements (1), (2), (3), and (4) are equivalent and are implied by statment (5). If $S$ is countable, then all five statements are equivalent.*

*(1) $A$ is quasi-central.*

*(2) There is a FP-tree $T$ in $A$ such that for each $F \in \mathcal{P}_f(T)$, $\bigcap_{f \in F} B_f$ is piecewise syndetic.*

*(3) There is a $*$-tree $T$ in $A$ such that for each $F \in \mathcal{P}_f(T)$, $\bigcap_{f \in F} B_f$ is piecewise syndetic.*

*(4) There is a downward directed family $\langle C_F \rangle_{F \in I}$ of subsets of $A$ such that*

*(a) for each $F \in I$ and each $x \in C_F$ there exists $G \in I$ with $C_G \subseteq x^{-1}C_F$ and*

*(b) for each $F \in I$, $C_F$ is piecewise syndetic.*

*(5) There is a decreasing sequence $\langle C_n \rangle_{n=1}^{\infty}$ of subsets of $A$ such that*

*(a) for each $n \in \mathbb{N}$ and each $x \in C_n$, there exists $m \in \mathbb{N}$ with $C_m \subseteq x^{-1}C_n$ and*

*(b) for each $n \in \mathbb{N}$, $C_n$ is piecewise syndetic.*

**Proof**. (1) $\Rightarrow$ (2). Pick an idempotent $p \in c\ell K(\beta S)$ with $A \in p$. By Lemma 3.6 pick a FP-tree $T$ in $A$ with $B_f \in p$ for each $f \in T$. Then given $F \in \mathcal{P}_f(T)$, one has $\bigcap_{f \in F} B_f \in p$ so $K(\beta S) \cap c\ell \bigcap_{f \in F} B_f \neq \emptyset$ so by Theorem 3.2, $\bigcap_{f \in F} B_f$ is piecewise syndetic.

(2) $\Rightarrow$ (3). Let $T$ be a FP-tree. Then given $f \in T$ and $x \in B_f$, we claim that $B_{f^\frown x} \subseteq x^{-1}B_f$. To this end let $y \in B_{f^\frown x}$ and pick $g \in T$ with $f^\frown x \subsetneq g$ and pick $F \subseteq \text{domain}(g) \setminus \text{domain}(f^\frown x)$ such that $y = \Pi_{t \in F}\, g(t)$. Let $n = \text{domain}(f)$ and let $G = F \cup \{n\}$. Then $x \cdot y = \Pi_{t \in G}\, g(t)$ and $G \subseteq \text{domain}(g) \setminus \text{domain}(f)$, so $x \cdot y \in B_f$ as required.

(3) $\Rightarrow$ (4). Let $T$ be the given $*$-tree. Let $I = \mathcal{P}_f(T)$ and for each $F \in I$, let $C_F = \bigcap_{f \in F} B_f$. Then immediately each $C_F$ is piecewise syndetic. Let $F \in I$ and let $x \in C_F$. Let $G = \{f^\frown x : f \in F\}$. Now for each $f \in F$ we have $B_{f^\frown x} \subseteq x^{-1}B_f$ so $C_G \subseteq x^{-1}C_F$.

(4) $\Rightarrow$ (1). Let $M = \bigcap_{F \in I} c\ell C_F$. It suffices to show that $M \cap c\ell K(\beta S) \neq \emptyset$ and $M$ is a subsemigroup of $\beta S$. For by [8, Corollary 4.6] $c\ell K(\beta S)$ is a semigroup. So one concludes that $M \cap c\ell K(\beta S)$ is a compact right topological semigroup and one then has that there is an idempotent $p \in M \cap c\ell K(\beta S)$. Since $M \subseteq c\ell A$ one has $A \in p$.

To see that $M \cap c\ell K(\beta S) \neq \emptyset$ it suffices, since $\langle C_F \rangle_{F \in I}$ is a downward directed family, to show that for each $F \in I$, $c\ell C_F \cap c\ell K(\beta S) \neq \emptyset$. But this is an immediate consequence of Theorem 3.2. To see that $M$ is a subsemigroup of $\beta S$, let $p, q \in M$ and let $F \in I$. We claim that $C_F \subseteq \{x \in S : x^{-1}C_F \in q\}$ (and hence $C_F \in p \cdot q$). So let $x \in C_F$ and pick $G \in I$ such that $C_G \subseteq x^{-1}C_F$. Then $C_G \in q$ so $x^{-1}C_F \in q$.

That (5) $\Rightarrow$ (4) is trivial.

Finally assume that $S$ is countable. We show that (3) $\Rightarrow$ (5). So let $T$ be the given $*$-tree in $A$. Then $T$ is countable so enumerate $T$ as $\langle f_n \rangle_{n=1}^{\infty}$. For each $n \in \mathbb{N}$, let $C_n = \bigcap_{k=1}^{n} B_{f_k}$. Then immediately each $C_n$ is piecewise syndetic. Let $n \in \mathbb{N}$ and let $x \in C_n$. Pick $m \in \mathbb{N}$ such that $\{f_k{}^\frown x : k \in \{1, 2, \ldots, n\}\} \subseteq \{f_t : t \in \{1, 2, \ldots, m\}\}$. Then $C_m \subseteq x^{-1}C_n$.$\square$

We have a nearly identical characterization of central sets.

3.8 **Theorem**. *Let $(S, \cdot)$ be an infinite semigroup and let $A \subseteq S$. Statements (1), (2), (3), and (4) are equivalent and are implied by statment (5). If $S$ is countable, then all five statements are equivalent.*

*(1) $A$ is central.*

*(2) There is a FP-tree $T$ in $A$ such that $\{B_f : f \in T\}$ is collectionwise piecewise syndetic.*

*(3) There is a $*$-tree $T$ in $A$ such that $\{B_f : f \in T\}$ is collectionwise piecewise syndetic.*

*(4) There is a downward directed family $\langle C_F \rangle_{F \in I}$ of subsets of $A$ such that*

*(a) for each $F \in I$ and each $x \in C_F$ there exists $G \in I$ with $C_G \subseteq x^{-1}C_F$ and*

*(b) $\{C_F : F \in I\}$ is collectionwise piecewise syndetic.*

*(5) There is a decreasing sequence $\langle C_n \rangle_{n=1}^{\infty}$ of subsets of $A$ such that*

*(a) for each $n \in \mathbb{N}$ and each $x \in C_n$, there exists $m \in \mathbb{N}$ with $C_m \subseteq x^{-1}C_n$ and*

*(b) $\{C_n : n \in \mathbb{N}\}$ is collectionwise piecewise syndetic.*

**Proof**. (1) $\Rightarrow$ (2). Pick an idempotent $p \in K(\beta S)$ with $A \in p$. By Lemma 3.6 pick a FP-tree with $\{B_f : f \in T\} \subseteq p$. By Theorem 3.2 $\{B_f : f \in T\}$ is collectionwise piecewise syndetic.

(2) $\Rightarrow$ (3). This is identical to the corresponding proof in Theorem 3.7.

(3) $\Rightarrow$ (4). This is identical to the corresponding proof in Theorem 3.7 except that one notes that since $\{B_f : f \in T\}$ is collectionwise piecewise syndetic, so is $\{\bigcap_{f \in F} B_f : F \in \mathcal{P}_f(T)\}$.

(4) $\Rightarrow$ (1). Let $M = \bigcap_{F\in I} c\ell C_F$. Exactly as in the proof of Theorem 3.7 we have that $M$ is a subsemigroup of $\beta S$. Since $\{C_F : F \in I\}$ is collectionwise piecewise syndetic, we have by Theorem 3.2 that $M \cap K(\beta S) \neq \emptyset$ so we may pick a minimal left ideal $L$ of $\beta S$ with $L \cap M \neq \emptyset$. Then $L \cap M$ is a compact subsemigroup of $\beta S$ which thus contains an idempotent, which is necessarily minimal.

That (5) $\Rightarrow$ (4) is trivial.

The proof that (3) $\Rightarrow$ (5) when $S$ is countable is nearly identical to the corresponding part of Theorem 3.7. □

We close this section by pointing out two Ramsey-Theoretic consequences of our characterizations.

3.9 **Corollary**. *Let $(S,\cdot)$ be an infinite semigroup, let $r \in \mathbb{N}$, and let $S = \bigcup_{i=1}^{r} A_i$. There exist $i \in \{1,2,\ldots,r\}$ and a FP-tree $T$ in $A_i$ such that $\{B_f : f \in T\}$ is collectionwise piecewise syndetic.*

**Proof**. Pick an idempotent $p \in K(\beta S)$ and pick $i \in \{1,2,\ldots,r\}$ such that $A_i \in p$. Apply Theorem 3.8. □

3.10 **Corollary** *Let $(S,\cdot)$ be an infinite commutative semigroup. Let $A \subseteq S$ and assume there is a $*$-tree $T$ in $A$ such that for each $F \in \mathcal{P}_f(T)$ one has $\bigcap_{f\in F} B_f$ is piecewise syndetic. Then whenever $r \in \mathbb{N}$ and $A = \bigcup_{i=1}^{r} D_i$ one has some $D_i$ is a rich set.*

**Proof**. By Theorem 3.7 pick an idempotent $p \in c\ell K(\beta S)$ with $A \in p$. Then if $A = \bigcup_{i=1}^{r} D_i$ one has some $D_i \in p$ and hence $D_i$ is quasi-central. Apply Corollary 2.9. □

4. **Quasi-central need not imply central**. It is easy to see that in some semigroups the notions of central and quasi-central are identical. For example, if $(S,\cdot)$ is a left-zero semigroup (so that $x \cdot y = x$ for all $x$ and $y$ in $S$) then so is $\beta S$, and hence $K(\beta S) = \beta S$ so every subset of $S$ is central.

For a slightly less trivial example, consider $(\mathbb{N}, \vee)$ where $x \vee y = \max\{x, y\}$. Then given $p \in \mathbb{N}^* = \beta\mathbb{N}\backslash\mathbb{N}$ and $x \in \mathbb{N}$, $x \vee p = p \vee x = p$ while given $p, q \in \mathbb{N}^*$ one has $q \vee p = p$. Then $K(\beta\mathbb{N}, \vee) = \mathbb{N}^*$ so the notions "central", "quasi-central", and "infinite" are synonymous.

We show in this section that in the semigroup $(\mathbb{N}, +)$ there is a subset which is quasi-central but not central. (The semigroup $(\mathbb{N},+)$ is the granddaddy of all semigroups and is the one wherein many of the most interesting combinatorial applications of the algebraic structure of $\beta S$ lie.)

Since the operation here is "+" we will write $FS(\langle x_n\rangle_{n=1}^{\infty}) = \{\Sigma_{n\in F}\ x_n : F \in \mathcal{P}_f(\mathbb{N})\}$ in lieu of $FP(\langle x_n\rangle_{n=1}^{\infty})$ and $-x+A$ in lieu of $x^{-1}A$. Also, since $(\mathbb{N},+)$ is commutative we have $-x+A = A-x = \{y\in\mathbb{N} : y+x\in A\}$.

We define now the set which is quasi-central, but not central.

4.1 **Definition**. (a) Let $n\in\mathbb{N}$. Then $X_n = \{2^{2^n\cdot(2b+1)} + a\cdot 2^{2n} + 2^{2n-2} : a,b\in\mathbb{N}$ and $a\le b\}$.

(b) $X = \bigcup_{n=1}^{\infty} X_n$.

(c) Given $x\in\mathbb{N}$, $\mathrm{supp}(x)$ is the finite subset of $\mathbb{N}\cup\{0\}$ such that $x = \Sigma_{t\in\mathrm{supp}(x)}\ 2^t$.

(d) $D = \{\Sigma_{i=1}^{\ell}\ x_i : \ell\in\mathbb{N}$ and for each $i\in\{1,2,\ldots,\ell\}$, $x_i\in X$ and if $\ell>1$ then for each $i\in\{1,2,\ldots,\ell-1\}$, $\max\ \mathrm{supp}(x_i) < \min\ \mathrm{supp}(x_{i+1})\}$.

(e) Let $r\in\mathbb{N}$. Then $D_r = D\cap\mathbb{N}2^r$.

4.2 **Lemma**. *Let $u\in\mathbb{N}$ and assume there exists $c\in D$ such that $u+c\in D$ and $\max\ \mathrm{supp}(u) < \min\ \mathrm{supp}(c)$. Then $u\in D$.*

**Proof**. Pick $c$ and $d$ in $D$ with $\max\ \mathrm{supp}(u) < \min\ \mathrm{supp}(c)$ and $u+c=d$ and $\max\ \mathrm{supp}(c)$ as small as possible among all such pairs. Pick $\ell$ and $m$ in $\mathbb{N}$ such that $c = \Sigma_{i=1}^{\ell}\ x_i$ and $d = \Sigma_{i=1}^{m}\ y_i$ where $x_1,x_2,\ldots,x_\ell$ and $y_1,y_2,\ldots,y_m$ are as in the definition of $D$. It suffices to show that $x_\ell = y_m$. For then, by the minimality of $\max\ \mathrm{supp}(c)$ we must have $\ell=1$ and hence $u = \Sigma_{i=1}^{m-1}\ y_i$.

Now $x_\ell = 2^{2^n\cdot(2b+1)} + a\cdot 2^{2n} + 2^{2n-2}$ for some $a,b,n\in\mathbb{N}$ with $a\le b$. Also $y_m = 2^{2^k\cdot(2f+1)} + g\cdot 2^{2k} + 2^{2k-2}$ for some $k,f,g\in\mathbb{N}$ with $g\le f$. Now there is no carrying when the $x_i$'s are added in binary and similarly there is no carrying when the $y_i$'s are added in binary. Consequently $\max\ \mathrm{supp}(u+c) = 2^n\cdot(2b+1)$ and $\max\ \mathrm{supp}(d) = 2^k\cdot(2f+1)$ so $n=k$ and $b=f$. Now also $\max\ \mathrm{supp}(u+c-x_\ell) < 2n-2$ and $\max\ \mathrm{supp}(d-y_m) < 2n-2$ so $a\cdot 2^{2n} = g\cdot 2^{2n}$ so $a=g$. That is $y_m = x_\ell$ as required. $\square$

4.3 **Lemma**. *If $p\in\beta\mathbb{N}$ and $q\in c\ell D\cap\bigcap_{n=1}^{\infty} c\ell\mathbb{N}2^n$ and $p+q\in c\ell D$, then $p\in c\ell D$.*

**Proof**. We have that $D\in p+q$. We claim that $\{u\in\mathbb{N} : -u+D\in q\}\subseteq D$ so let $u\in\mathbb{N}$ such that $-u+D\in q$. Let $m = \max\ \mathrm{supp}(u)$ and pick $c\in(-u+D)\cap D\cap\mathbb{N}2^{m+1}$. Then $u+c\in D$ and $\max\ \mathrm{supp}(u) < \min\ \mathrm{supp}(c)$ so by Lemma 4.2 $u\in D$. $\square$

4.4 **Theorem**. *$D$ is quasi-central but not central.*

**Proof**. To see that $D$ is quasi-central we show that $\langle D_n\rangle_{n=1}^{\infty}$ satisfies statement (5) of Theorem 3.7. Given $n\in\mathbb{N}$ and $a\in D_n$, let $m=\max\operatorname{supp}(a)$. Then $D_{m+1}\subseteq -a+D_n$. Given $n\in\mathbb{N}\backslash\{1\}$ we have that $X_n\subseteq D_n$ so to see that $D_n$ is piecewise syndetic it suffices to show that $X_n$ is piecewise syndetic. Let $G=\{1,2,\ldots,2^{2n}\}$. Then given any finite $F\subseteq\mathbb{N}$, let $b=\max F$ and let $x=2^{2^n\cdot(2b+1)}+2^{2n-2}$. Then $F+x\subseteq\bigcup_{t\in G}-t+X_n$.

Now suppose that $D$ is central and pick an idempotent $q\in K(\beta\mathbb{N})\cap c\ell D$. Since $q=q+q$ we have $q\in\bigcap_{n=1}^{\infty}c\ell\mathbb{N}2^n$. (There is some $i\in\{0,1,\ldots,2^n-1\}$ with $\mathbb{N}2^n+i\in q$ and $\mathbb{N}2^n+2i\in q+q$ so $i=0$.) Pick any $p\in\mathbb{N}^*\cap c\ell\{2^{2m+1}:m\in\mathbb{N}\}$. Since $q\in K(\beta\mathbb{N})$ we have $L=\beta\mathbb{N}+q$ is a minimal left ideal of $\beta\mathbb{N}$ and $p+q\in L$ so $L=\beta\mathbb{N}+p+q$ so $q\in\beta\mathbb{N}+p+q$, so pick $r\in\beta\mathbb{N}$ such that $q=r+p+q$. By Lemma 4.3, $r+p\in c\ell D$. So pick $x\in\mathbb{N}$ such that $-x+D\in p$. Also $\{2^{2m+1}:m>\max\ \operatorname{supp}(x)\}\in p$ so pick $m>\max\ \operatorname{supp}(x)$ with $2^{2m+1}\in -x+D$. But then $x+2^{2m+1}\in D$ so $\max\ \operatorname{supp}(x+2^{2m+1})$ is even, a contradiction. □

One should note that the " $2^{2n-2}$ " terms in the definition of $X_n$ play a crucial role. If one instead defines ${X_n}'=\{2^{2^n\cdot(2b+1)}+a\cdot 2^{2n}:a,b\in\mathbb{N}\text{ and }a\leq b\}$ one has that ${X_1}'$ contains arbitrarily long blocks of $\mathbb{N}4$ and it is not hard to see (using, say, Theorem 3.8) that any set containing arbitrarily long blocks of $\mathbb{N}n$ for any fixed $n$ is central.

The reader may want to amuse himself by figuring out where our proof breaks down if ${X_n}'$ replaces $X_n$ in the definition of $D$.

5. **Sets satisfying the Central Sets Theorem need not be quasi-central**. We show in fact that there is a subset $A$ of $\mathbb{N}$ which is a rich set but is not even piecewise syndetic (so is certainly not quasi-central since $K(\beta\mathbb{N})\cap c\ell A=\emptyset$). This same set $A$ is in fact a member of an idempotent in $J$ so we obtain as a corollary that $J\neq c\ell K(\beta\mathbb{N})$.

5.1 **Lemma**. *Let $n,m,k\in\mathbb{N}$ and for each $i\in\{1,2,\ldots,n\}$, let $\langle y_{i,t}\rangle_{t=1}^{\infty}$ be a sequence in $\mathbb{N}$. Then there exists $H\in\mathcal{P}_f(\mathbb{N})$ with* $\min\ H>m$ *such that for each $i\in\{1,2,\ldots,n\}$, $\Sigma_{t\in H}\ y_{i,t}\in\mathbb{N}2^k$.*

**Proof**. Choose an infinite set $G_1\subseteq\mathbb{N}$ such that for all $t,s\in G_1$, $y_{1,t}\equiv y_{1,s}\pmod{2^k}$. Inductively, given $i\in\{1,2,\ldots,n-1\}$ and $G_i$, choose an infinite subset $G_{i+1}$ of $G_i$ such that for all $t,s\in G_{i+1}$, $y_{i+1,t}\equiv y_{i+1,s}\pmod{2^k}$. Then for all $i\in\{1,2,\ldots,n\}$ and all $t,s\in G_n$ one has $y_{i,t}\equiv y_{i,s}$ $(mod\ 2^k)$. Now pick $H\subseteq G_n$ with $\min\ H>m$ and $|H|=2^k$.□

5.2 **Lemma**. *There is a set $A\subseteq\mathbb{N}$ such that*

*(1) $A$ is not piecewise syndetic.*

*(2) For all $x \in A$ there exists $n \in \mathbb{N}$ such that $\emptyset \neq A \cap \mathbb{N}2^n \subseteq -x + A$.*

*(3) For each $n \in \mathbb{N}$ and any $n$ sequences $\langle y_{1,t}\rangle_{t=1}^{\infty}, \langle y_{2,t}\rangle_{t=1}^{\infty}, \ldots, \langle y_{n,t}\rangle_{t=1}^{\infty}$ in $\mathbb{N}$ there exist $a \in \mathbb{N}$ and $H \in \mathcal{P}_f(\mathbb{N})$ such that for all $i \in \{1, 2, \ldots, n\}$, $a + \Sigma_{t\in H}\, y_{i,t} \in A \cap \mathbb{N}2^n$.*

**Proof**. For each $k \in \mathbb{N}$ let $B_k = \{2^k, 2^k{+}1, 2^k{+}2, \ldots, 2^{k+1}{-}1\}$ and let $A = \{n \in \mathbb{N} :$ for each $k \in \mathbb{N}$, $B_k \backslash \mathrm{supp}(n) \neq \emptyset\}$. Then one recognizes that $n \in A$ by looking at the binary expansion of $n$ and noting that there is at least one 0 between positions $2^k$ and $2^{k+1}$ for each $k \in \mathbb{N}$.

To show that $A$ is not piecewise syndetic we need to show that for each $g \in \mathbb{N}$ there is some $b \in \mathbb{N}$ such that for any $x \in \mathbb{N}$ there is some $y \in \{x+1, x+2, \ldots, x+b\}$ with $\{y+1, y+2, \ldots, y+g\} \cap A = \emptyset$. To this end let $g \in \mathbb{N}$ be given and pick $k \in \mathbb{N}$ such that $2^{2^k} > g$. Let $b = 2^{2^{k+1}}$. Let $x \in \mathbb{N}$ be given and pick the least $a \in \mathbb{N}$ such that $a \cdot 2^{2^{k+1}} - 2^{2^k} > x$ and let $y = a \cdot 2^{2^{k+1}} - 2^{2^k}$. Then $x < y \leq x + b$ and for each $t \in \{1, 2, \ldots, 2^{2^k} - 1\}$ one has $\{2^k, 2^k{+}1, 2^k{+}2, \ldots, 2^{k+1}{-}1\} \subseteq \mathrm{supp}(y{+}t)$ so $y{+}t \notin A$.

To verify conclusion (2), let $x \in \mathbb{N}$ and pick $k \in \mathbb{N}$ such that $2^{2^k - 1} > x$. Let $n = 2^k$. Then $\emptyset \neq A \cap \mathbb{N}2^n \subseteq -x + A$.

Finally, to verify (3) let $n \in \mathbb{N}$ and let sequences $\langle y_{1,t}\rangle_{t=1}^{\infty}, \langle y_{2,t}\rangle_{t=1}^{\infty}, \ldots, \langle y_{n,t}\rangle_{t=1}^{\infty}$ be given. We first observe that by Lemma 5.1 we can choose $H \in \mathcal{P}_f(\mathbb{N})$ such that for each $i \in \{1, 2, \ldots, n\}$, $\Sigma_{t\in H}\, y_{i,t} \in \mathbb{N}2^{n+1}$.

Next we observe that given any $z_1, z_2, \ldots, z_n$ in $\mathbb{N}$ and any $k$ with $2^k > n$, there exists $r \in B_k$ such that $B_k \backslash \mathrm{supp}(2^r + z_i) \neq \emptyset$ for each $i \in \{1, 2, \ldots, n\}$. Indeed, if $r \in B_k$ and $B_k \subseteq \mathrm{supp}(2^r + z)$ then $\mathrm{supp}(z) \cap B_k = B_k \backslash \{r\}$. Consequently $|\{r \in B_k :$ there is some $i \in \{1, 2, \ldots, n\}$ with $B_k \subseteq \mathrm{supp}(2^r + z_i)\}| \leq n$.

For $i \in \{1, 2, \ldots, n\}$, let $z_{0,i} = \Sigma_{t\in H}\, y_{i,t}$. Pick the least $\ell$ such that $2^\ell > n$. Now given $i$ we have $2^{n+1} | z_{0,i}$ and $2^{\ell-1} < n + 1$ so $2^{\ell-1} \in B_{\ell-1} \backslash \mathrm{supp}(z_{0,i})$, where $B_{\ell-1}$ is defined as in the paragraph above. Pick $r_0 \in B_\ell$ such that $B_\ell \backslash \mathrm{supp}(2^{r_0} + z_{0,i}) \neq \emptyset$ for each $i \in \{1, 2, \ldots, n\}$ and let $z_{1,i} = z_{0,i} + 2^{r_0}$. Inductively choose $r_j \in B_{\ell+j}$ such that $B_{\ell+j} \backslash \mathrm{supp}(2^{r_j} + z_{j,i}) \neq \emptyset$ for each $i \in \{1, 2, \ldots, n\}$ and let $z_{j+1,i} = z_{j,i} + 2^{r_j}$. Continue the induction until $\ell + j = k$ where $2^{2^k} > \Sigma_{t\in H}\, y_{i,t}$ for each $i \in \{1, 2, \ldots, n\}$ and let $a = 2^{r_0} + 2^{r_1} + \ldots + 2^{r_{k-\ell}}$. $\square$

5.3 **Theorem**. *There is a set $A \subseteq \mathbb{N}$ such that $A$ is a rich set but $K(\beta\mathbb{N}) \cap c\ell A = \emptyset$.*

**Proof**. Let $A$ be as in Lemma 5.2. Since $A$ is not piecewise syndetic we have by Theorem 3.2 that $K(\beta\mathbb{N}) \cap c\ell A = \emptyset$. We could appeal to Lemma 5.4 below and Theorem 2.7 to conclude that $A$ is a rich set but this fact has a very easy elementary proof which we present now.

Let $Y = \langle\langle y_{i,t}\rangle_{t=1}^{\infty}\rangle_{i=1}^{\infty}$ be given. Choose by condition (3) of Lemma 5.2 some $a_1$ and $H_1$ with $a_1 + \Sigma_{t\in H_1}\ y_{1,t} \in A$. Inductively, given $a_n$ and $H_n$, let $\ell = \max\ H_n$ and pick $m > n$ in $\mathbb{N}$ such that for all $i \in \{1,2,\ldots,n\}$, $a_n + \Sigma_{t\in H_n}\ y_{i,t} < 2^{2^m}$. Pick by condition (3) of Lemma 5.2 (applied to the sequences $\langle y_{1,\ell+t}\rangle_{t=1}^{\infty}, \langle y_{2,\ell+t}\rangle_{t=1}^{\infty}, \ldots, \langle y_{m,\ell+t}\rangle_{t=1}^{\infty}$, so that $\min\ H_{n+1} > \ell = \max\ H_n$.) some $a_{n+1}$ and $H_{n+1}$ such that for all $i \in \{1,2,\ldots,m\}$, $a_{n+1} + \Sigma_{t\in H_{n+1}}\ y_{i,t} \in A \cap \mathbb{N}2^{2^m}$. Observe that if $x, y \in A$ and for some $k$, $x < 2^{2^k}$ and $2^{2^k}|y$, then $x+y \in A$. Consequently for $f \in \Phi$ one has $FS(\langle a_n + \Sigma_{t\in H_n}\ y_{f(n),t}\rangle_{n=1}^{\infty}) \subseteq A$. □

Now we turn our attention to showing that there is an idempotent in $J\backslash c\ell K(\beta\mathbb{N})$.

5.4 **Lemma**. *Assume $A \subseteq \mathbb{N}$ and*

*(1) for all $x \in A$ there is some $n \in \mathbb{N}$ such that $\emptyset \neq A \cap \mathbb{N}2^n \subseteq -x + A$ and*

*(2) for all $n \in \mathbb{N}$ and all sequences $\langle y_{1,t}\rangle_{t=1}^{\infty}, \langle y_{2,t}\rangle_{t=1}^{\infty}, \ldots, \langle y_{n,t}\rangle_{t=1}^{\infty}$ in $\mathbb{N}$ there exist $a \in \mathbb{N}$ and $H \in \mathcal{P}_f(\mathbb{N})$ such that for all $i \in \{1,2,\ldots,n\}$, $a + \Sigma_{t\in H}\ y_{i,t} \in A \cap \mathbb{N}2^n$.*

*Then there is some idempotent $p$ in $J \cap c\ell A$.*

**Proof**. Let $T = c\ell A \cap \bigcap_{n=1}^{\infty} c\ell\mathbb{N}2^n$. Then condition (1) guarantees that $T$ is a semigroup. (Since each $A \cap \mathbb{N}2^n \neq \emptyset$, $T \neq \emptyset$. Given $p$ and $q$ in $T$ we have $p + q \in \bigcap_{n=1}^{\infty} c\ell\mathbb{N}2^n$ and $A \subseteq \{x \in \mathbb{N} : -x + A \in q\}$ so $A \in p + q$.)

Pick a minimal idempotent $p$ in $T$. We claim $p \in J$. To this end let $y \in \mathcal{Y}$ be given and let $B \in p$. To see that $B$ is a $J_Y$-set, let $n \in \mathbb{N}$. For each $r \in \mathbb{N}$ let $I_r = \{(a + \Sigma_{t\in H}\ y_{1,t}, a + \Sigma_{t\in H}\ y_{2,t}, \ldots, a + \Sigma_{t\in H}\ y_{n,t}) : a \in \mathbb{N}$ and $H \in \mathcal{P}_f(\mathbb{N})$ and $\min H > r$ and for each $i \in \{1,2,\ldots,n\}$, $a + \Sigma_{t\in H}\ y_{i,t} \in A \cap \mathbb{N}2^r\}$ and let $E_r = I_r \cup \{(a,a,\ldots,a) : a \in A \cap \mathbb{N}2^r\}$. Let $E = \bigcap_{r=1}^{\infty} c\ell E_r$ and $I = \bigcap_{r=1}^{\infty} c\ell I_r$ where the closures are taken in $W = \times_{i=1}^{n}(\beta\mathbb{N})$.

Observe first that $E \subseteq \times_{i=1}^{n} T$, since each $E_r \subseteq \times_{i=1}^{n}(A \cap \mathbb{N}2^r)$. To see that $I \neq \emptyset$ (and hence $E \neq \emptyset$) we notice that each $I_r \neq \emptyset$ by condition (2) (using the sequences $\langle y'_{1,t}\rangle_{t=1}^{\infty}, \langle y'_{2,t}\rangle_{t=1}^{\infty}, \ldots, \langle y'_{n,t}\rangle_{t=1}^{\infty}$ where $y'_{i,t} = y_{i,r+t}$).

Now we show that $E$ is a semigroup and $I$ is an ideal of $E$ using Lemma 2.2. Let $r \in \mathbb{N}$ and let $\vec{x} \in I_r$. Pick $a \in \mathbb{N}$ and $H \in \mathcal{P}_f(\mathbb{N})$ with $\min H > r$ and $a + \Sigma_{t\in H}\ y_{i,t} \in A \cap \mathbb{N}2^r$ for each $i \in \{1,2,\ldots,n\}$ such that $\vec{x} = (a + \Sigma_{t\in H}\ y_{1,t}, a + \Sigma_{t\in H}\ y_{2,t}, \ldots, a + \Sigma_{t\in H}\ y_{n,t})$. Choose by condition (1) for each $i \in \{1,2,\ldots,n\}$ some $k_i \in \mathbb{N}$ such that $A \cap \mathbb{N}2^{k_i} \subseteq -(a + \Sigma_{t\in H}\ y_{i,t}) + A$. Let $s = \max\ (\{k_1, k_2, \ldots, k_n\} \cup H) + 1$. It is routine to show that $\vec{x} + E_s \subseteq I_r$.

Now let $\vec{x} = (a, a, \ldots, a)$ where $a \in A \cap \mathbb{N}2^r$. Choose by condition (1) $k \in \mathbb{N}$ such that $A \cap \mathbb{N}2^k \subseteq -a + A$. Let $s = \max\{k, r\}$. Then one can see that $\vec{x} + E_s \subseteq E_r$ and

$\vec{x} + I_s \subseteq I_r$ so Lemma 2.2 applies.

Now let $\vec{p} = (p, p, \ldots, p)$. Then given $C \in p$ and $r \in \mathbb{N}$ we have $C \cap A \cap \mathbb{N}2^r \neq \emptyset$ so picking $a \in C \cap A \cap \mathbb{N}2^r$ we have $(a, a, \ldots, a) \in (\times_{i=1}^n C) \cap E_r$. Thus $\vec{p} \in E$ so, as before, $\vec{p}$ is minimal in $E$ so $\vec{p} \in I$.

Thus $(\times_{i=1}^n B) \cap I_n \neq \emptyset$. That is there exist $a \in \mathbb{N}$ and $H \in \mathcal{P}_f(\mathbb{N})$ with $\min H \geq n$ and for each $i \in \{1, 2, \ldots, n\}$, $a + \Sigma_{t \in H}\ y_{i,t} \in B$. □

5.5 **Theorem** *There is an idempotent* $p \in J \backslash c\ell K(\beta\mathbb{N})$. *In particular* $J \neq c\ell K(\beta\mathbb{N})$.

**Proof**. Lemma 5.2, Theorem 5.3, and Lemma 5.4. □

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