A history of central sets

Neil Hindman [*][†]

This paper is dedicated to Vitaly Bergelson
on the occasion of his 65th birthday.

Abstract

We survey results about, and results using, central sets since their
introduction in 1981.

1   Introduction

Let me begin with some personal remarks. As indicated above, this paper is
dedicated to Vitaly Bergelson. But more than that, it is an attempt on my part
to thank him for introducing me to central sets.

In the summer of 1984 I got a letter from Bruce Rothschild, telling me about
this brilliant student of Hillel Furstenberg that he met in Israel who could prove
some remarkable things in *Ramsey Theory*, a subject in which I was also very
interested. That student was, of course, Vitaly Bergelson, and I immediately
began corresponding with him. Our lengthy collaboration began in earnest in
the fall of 1986 when Vitaly was visiting at the University of Maryland which
is just a few miles down the road from my house.

A typical statement in Ramsey Theory begins by finitely partitioning some
structure and showing that one can guarantee that one cell will contain struc-
tures of a particular kind. For example, consider a result close to my heart,
namely the Finite Sums Theorem.

Given a semigroup $(S, +)$ and a sequence $\langle x_n \rangle_{n=1}^{\infty}$ in S, let $FS(\langle x_n \rangle_{n=1}^{\infty}) = \{\sum_{n \in F} x_n : F \in \mathcal{P}_f(\mathbb{N})\}$, where for any set X, $\mathcal{P}_f(X)$ is the set of finite
nonempty subsets of X. We let $\mathbb{N}$ be the set of positive integers and $\omega = \mathbb{N} \cup \{0\}$.

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[*]Department of Mathematics, Howard University, Washington, DC 20059, USA.
nhindman@aol.com

The reader who is not familiar with semigroups and is only interested in applications in $\mathbb{N}$ can assume that the semigroup is $(\mathbb{N}, +)$ or occasionally $(\mathbb{N}, \cdot)$.

Theorem 1.1 (Finite Sums Theorem). *Let $(S, +)$ be a semigroup, let $r \in \mathbb{N}$, and let $S = \bigcup_{i=1}^{r} A_i$. There exist $i \in \{1, 2, \ldots, r\}$ and a sequence $\langle x_n \rangle_{n=1}^{\infty}$ in S such that $FS(\langle x_n \rangle_{n=1}^{\infty}) \subseteq A_i$.*

In the alternative coloring terminology used in Ramsey Theory, one says that if S is finitely colored, there is a sequence $\langle x_n \rangle_{n=1}^{\infty}$ with $FS(\langle x_n \rangle_{n=1}^{\infty})$ *monochromatic*. (Formally a finite coloring of S is a function with domain S and finite range, and a set is monochromatic provided the function is constant on that set.)

Similarly, one defines $FP(\langle x_n \rangle_{n=1}^{\infty}) = \{\prod_{n \in F} x_n : F \in \mathcal{P}_f(\mathbb{N})\}$ specifing that products are in increasing order of indices in the event the operation is not commutative. One defines $FS(\langle x_n \rangle_{n=1}^{k})$ and $FP(\langle x_n \rangle_{n=1}^{k})$ analogously.

At the time I met Vitaly, I was an enthusiastic missionary on behalf of the utility of the algebraic structure of the Stone-Čech compactification of a discrete semigroup in proving results in Ramsey Theory. (The original combinatorial proof of the Finite Sums Theorem was very complicated. The proof is essentially trivial using the algebraic structure of βS.)

Vitaly was, if not the first, certainly one of the most enthusiastic of my converts. Not long after I had begun proving theorems with him, he told me about Furstenberg's Central Sets Theorem (Theorem 3.1 below) and suggested that we ought to be able to prove the conclusion for members of minimal idempotents in $\beta \mathbb{N}$. He was right. And thus began what has been for me, over a quarter century, a love affair with central sets. They are the subject of this paper.

In order to talk more about central sets, I need to give a brief introduction to the algebraic structure of βS. Let $(S, +)$ be a discrete semigroup. There is an extension of the operation to the Stone-Čech compactification βS of S making $(\beta S, +)$ a *right topological* semigroup (meaning that for each $p \in \beta S$, the function $\rho_p : \beta S \to \beta S$ defined by $\rho_p(q) = q + p$ is continuous) with S contained in its topological center (meaning that for each $x \in S$, the function $\lambda_x : \beta S \to \beta S$ defined by $\lambda_x = x + p$ is continuous). (Note that, even when S is commutative, βS is very unlikely to be commutative.)

It should be noted that, when I was in the process of converting Vitaly, I took βS to be left topological. I changed to agree with most of the references when the first edition of [53] was written, but when Vitaly writes about βS he continues to use the left topological structure. (I have been told that I should buy mirrors for my early students and Vitaly.)

We take the points of βS to be the ultrafilters on S, identifying the point x of S with the principal ultrafilter $\{A \subseteq S : x \in A\}$. Given $p, q \in \beta S$ and $A \subseteq S$, one has that $A \in p + q$ if and only if $\{x \in S : -x + A \in q\} \in p$, where $-x + A = \{y \in S : x + y \in A\}$. If multiplicative notation is being used, one has $A \in p \cdot q$ if and only if $\{x \in S : x^{-1} A \in q\} \in p$, where $x^{-1} A = \{y \in S : x \cdot y \in A\}$.

As with any compact Hausdorff right topological semigroup, $(\beta S, +)$ has a smallest two sided ideal $K(\beta S)$ which is the union of all of the minimal left ideals and is also the union of all of the minimal right ideals. The intersection of any minimal left ideal and any minimal right ideal is a group. In particular, there are idempotents in $K(\beta S)$. There is an ordering of the idempotents, whereby $p \leq q$ if and only $p = p + q = q + p$. An idempotent is *minimal* with respect to this ordering if and only if it is a member of $K(\beta S)$. Equivalently p is minimal if and only if it is a member of a minimal left ideal (if and only if it is a member of a minimal right ideal).

In the remaining sections of the paper, we cover first the definitions of central sets and the versions of the Central Sets Theorem. However, it should be emphasized that the main reason central sets are interesting is because of the remarkable amount of combinatorial structure that they must contain. Most of the results about semigroups that will be presented here are valid for arbitrary semigroups but I have not carefully checked which ones might not be. Accordingly, throughout this paper all hypothesized semigroups are assumed to be infinite.

I have been in a quandry regarding naming the originators of the theorems presented. Partly because so many of the papers cited have Hindman as an author, I have decided, with few exceptions, not to name the originators in the body of the text, expecting the interested reader to check the list of authors of the cited papers. The exceptions are Theorem 4.3 which qualifies because the result is known as "Rado's Theorem", Theorem 2.8 which qualifies because Weiss is not a listed author of the cited paper, and Theorem 3.1 because it somehow seemed appropriate that the author of the original Central Sets Theorem ought to be named.

Terms are defined as they are needed. Accordingly, there is an index of definitions at the end of the paper.

2   The definitions of central

Central subsets of the set $\mathbb{N}$ of positive integers were introduced by Furstenberg [31] in 1981. The definition depended on some notions from topological dynamics.

Definition 2.1. A *dynamical system* is a pair $(X, \langle T_s \rangle_{s \in S})$ such that

(1) X is a compact Hausdorff space;

(2) $(S, \cdot)$ is a semigroup;

(3) for each $s \in S$, T_s is a continuous function from X to X; and

(4) for s and t in S, $T_s \circ T_t = T_{s \cdot t}$.

Of course, if the operation is denoted by $+$, item (4) becomes "for s and t in S, $T_s \circ T_t = T_{s+t}$." I will not make a habit of remarking on such things.

Definition 2.2. Let $(S, \cdot)$ be a semigroup and let $A \subseteq S$. Then A is *syndetic* if and only if there exists $F \in \mathcal{P}_f(S)$ such that $S = \bigcup_{t \in F} t^{-1}A$.

In the semigroup $(\mathbb{N}, +)$, A is syndetic if and only if the gaps between successive members of A are bounded.

Definition 2.3. Let $(X, \langle T_s \rangle_{s \in S})$ be a dynamical system. A point $y \in X$ is *uniformly recurrent* if and only if for each neighborhood U of y, $\{s \in S : T_s(y) \in U\}$ is syndetic.

In [31], Furstenberg assumed that the *phase space* X of a dynamical system was a metric space.

Definition 2.4. Let $(X, \langle T_s \rangle_{s \in S})$ be a dynamical system, where (X, d) is a metric space. The points x and y of X are *proximal* if and only if there exists a sequence $\langle s_n \rangle_{n=1}^{\infty}$ in S such that $\lim_{n \to \infty} d\big(T_{s_n}(x), T_{s_n}(y)\big) = 0$.

Definition 2.5. A set $A \subseteq \mathbb{N}$ is *central* if and only if there exist a compact metric space (X, d) and a continuous function $T : X \to X$ such that, letting $T_n = T^n$ for each $n \in \mathbb{N}$, one has points x and y in X and a neighborhood U of y such that

(1) the point y is uniformly recurrent;

(2) the points x and y are proximal; and

(3) $A = \{n \in \mathbb{N} : T^n(x) \in U\}$.

That is, A is the set of returns of the point x to a neighborhood of y. It always seemed weird to me that one would come up with such a definition and expect anything nice to come of it. I said so, not for the first time, when I spoke at the conference celebrating Vitaly's accomplishments. Afterwords, Hillel explained to me that he had in mind some of the properties that he wanted central sets to satisfy, before he had defined them. Then he (and B. Weiss) figured out what properties they would have to have for them to be able to complete the proofs. (The definition of central set was anticipated in their result [32, Theorem 4.4], which was about a set satisfying Definition 2.5 for the space $\Omega = {}^{\mathbb{Z}}\{1, 2, \ldots, a\}$ for some $a \in \mathbb{N}$ and the function $T : \Omega \to \Omega$ defined by $T(x)(n) = x(n+1)$.) Here ${}^{\mathbb{Z}}\{1, 2, \ldots, a\}$ denotes the set of functions from $\mathbb{Z}$ to $\{1, 2, \ldots, a\}$. More generally, given sets A and B, ${}^{A}B$ is the set of functions from A to B.

Furstenberg proved [31, Theorem 8.8] that if $\mathbb{N}$ is partitioned into finitely many cells, then one of them contains a central set and [31, Proposition 8.9] that any central set contains arbitrarily long arithmetic progressions. He proved the original version of the Central Sets Theorem [31, Proposition 8.21]. (For the

detailed statement of this and other versions see Section 3.) Using this theorem, he established [31, Theorem 8.22] that any finite system of equations satisfying Rado's *columns condition* has solutions in any central set.

As described in the introduction, Vitaly suggested that the conclusion of the Central Sets Theorem could be derived for members of minimal idempotents.

Definition 2.6. Let S be a discrete semigroup. A set $A \subseteq S$ is *central* if and only if there is a minimal idempotent $p \in \beta S$ such that $A \in p$. That is, A is central if and only if there is an idempotent in $\overline{A} \cap K(\beta S)$.

Naturally, one would like to not have conflicting definitions of the same notion.

Theorem 2.7. *Let S be a countable semigroup (not necessarily commutative), let $(X, \langle T_s \rangle_{s \in S})$ be a dynamical system with X a metric space, let y be a uniformly recurrent point of X, and let x be proximal to y. There is a minimal idempotent $p \in \beta S$ such that, whenever U is a neighborhood of y in X, $\{s \in S : T_s(x) \in U\} \in p$.*

Proof. [12, Theorem 6.8]. □

Thus, if A is a subset of S which is central by the obvious adjustment to Defintion 2.5 wherein $\langle T_n \rangle_{n \in \mathbb{N}}$ is replaced by $\langle T_s \rangle_{s \in S}$, then it is central by Definition 2.6. We also presented the proof of the converse which was due to B. Weiss.

Theorem 2.8 (Weiss)**.** *Let S be a countable semigroup (not necessarily commutative), let p be a minimal idempotent in βS, and let $A \in p$. There exist a metric dynamical system $(X, \langle T_s \rangle_{s \in S})$, a uniformly recurrent point y of X, a point x of X which is proximal to y, and an open neighborhood U of y such that $A = \{s \in S : T_s(x) \in U\}$.*

Proof. [12, Theorem 6.11]. (Earlier Glasner [35, Proposition 4.6] had proved this result under the assumption that S was a countable abelian group.) □

In the proof of Theorem 2.8, the space X is the product space $^{S \cup \{0\}}\{0,1\}$, where $\{0,1\}$ has the discrete topology. If S is uncountable, this space is not metrizable – in fact not even first countable. In hope of extending the equivalence to uncountable semigroups, the dynamical definition of central was extended to allow a non-metric phase space. In so doing, the definition of *proximal* needed to be modified. Note that if X is a metric space, the two definitions agree.

Definition 2.9. Let $(X, \langle T_s \rangle_{s \in S})$ be a dynamical system. The points x and y of X are *proximal* if and only if there exists a net $\langle s_i \rangle_{i \in I}$ in S such that the nets $\langle T_{s_i}(x) \rangle_{i \in I}$ and $\langle T_{s_i}(y) \rangle_{i \in I}$ converge to the same point in X.

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Then the modified dynamical definition of central becomes:

Definition 2.10. Let S be a semigroup. A set $A \subseteq S$ is *central* if and only if there exist a dynamical system $(X, \langle T_s \rangle_{s \in S})$, a uniformly recurrent point y of X, a point x of X which is proximal to y, and a neighborhood U of y such that $A = \{s \in S : T_s(x) \in U\}$.

The equivalence of the notions of central for an arbitrary semigroup was established in [64].

Theorem 2.11. *Let S be a semigroup and let $A \subseteq S$. Then A satisfies Definition 2.6 if and only if A satisfies Definition 2.10.*

Proof. [64]. $\qquad \square$

Notice that as a consequence of Theorem 2.11, one has the non-obvious fact that the dynamical version of central is closed under passage to supersets.

Given a property P of sets, we say that a set has property P* if and only if it has nonempty intersection with any set with property P. Thus, for example, a set is central* if and only if it is a member of every minimal idempotent (since if it is not a member of p, its complement is).

3 Versions of the Central Sets Theorem

We begin with the original version of the Central Sets Theorem.

Theorem 3.1 (Furstenberg). *Let A be a central subset of $\mathbb{N}$, let $k \in \mathbb{N}$, and for each $i \in \{1, 2, \ldots, k\}$, let $\langle y_{i,n} \rangle_{n=1}^{\infty}$ be a sequence in $\mathbb{Z}$. There exist sequences $\langle a_n \rangle_{n=1}^{\infty}$ in $\mathbb{N}$ and $\langle H_n \rangle_{n=1}^{\infty}$ in $\mathcal{P}_f(\mathbb{N})$ such that*

(1) for each n, $\max H_n < \min H_{n+1}$ and

(2) for each $i \in \{1, 2, \ldots, k\}$ and each $F \in \mathcal{P}_f(\mathbb{N})$, $\sum_{n \in F}(a_n + \sum_{t \in H_n} y_{i,t}) \in A$.

Proof. [31, Proposition 8.21]. $\qquad \square$

In [12, Theorem 4.11], a version of Theorem 3.1 was proved which was valid for a class of commutative semigroups which generalizes the semigroups $(\mathbb{N}, +)$ and $(\mathbb{N} \setminus \{1\}, \cdot)$. I will not go into the definition of that class, because other later versions apply to all semigroups.

It should be emphasized that the basic idea behind all of the newer versions of the Central Sets Theorem is based on an idea developed by Furstenberg and Katznelson [33] in the context of enveloping semigroups.

In [13, Theorem 2.8], a version was proved which is valid for arbitrary, not necessarily commutative, semigroups. A slightly strengthened version of this will be presented as Theorem 3.4 below. (The strengthening is in the fact that countably many sequences are considered at once.) To ease into that rather complicated statement, we present the commutative version first.

Definition 3.2. $\Phi = \{f \in {}^{\mathbb{N}}\mathbb{N} : f(n) \leq n \text{ for all } n \in \mathbb{N}\}$

Theorem 3.3. *Let $(S, +)$ be a commutative semigroup, let A be a central set in S, and for each $\ell \in \mathbb{N}$, let $\langle y_{\ell,n} \rangle_{n=1}^{\infty}$ be a sequence in S. There exist a sequence $\langle a_n \rangle_{n=1}^{\infty}$ in S and a sequence $\langle H_n \rangle_{n=1}^{\infty}$ in $\mathcal{P}_f(\mathbb{N})$ such that $\max H_n < \min H_{n+1}$ for each $n \in \mathbb{N}$ and such that for each $f \in \Phi$,*

$$FS(\langle a_n + \sum_{t \in H_n} y_{f(n),t} \rangle_{n=1}^{\infty}) \subseteq A.$$

Proof. [53, Theorem 14.11]. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad\square$

As applied to $\mathbb{N}$, Theorem 3.3 is stronger than Theorem 3.1 in that infinitely many sequences are considered and in that the choice of the sequence for which the sum over H_n is taken is allowed to change when n changes. It is superficially weaker in that the sequences have entries restricted to $\mathbb{N}$ rather than being allowed to range over $\mathbb{Z}$. To see that Theorem 3.3 implies Theorem 3.1, note that by [53, Exercise 4.3.8], any set central in $\mathbb{N}$ is also central in $\mathbb{Z}$, so apply Theorem 3.3 with $S = \mathbb{Z}$.

For the noncommutative case, we switch to multiplicative notation. Recall that, if $F \in \mathcal{P}_f(\mathbb{N})$, then $\prod_{t \in F} x_t$ is the product in increasing order of indices. The situation now is more complicated because the a_n's need to be split into several parts. Thus a term like $a_3 \cdot \prod_{t \in H_3} y_{2,t}$ in Theorem 3.3 becomes something like

$$(a_{3,1} \cdot \prod_{t \in H_{3,1}} y_{2,t}) \cdot (a_{3,2} \cdot \prod_{t \in H_{3,2}} y_{2,t}) \cdot (a_{3,3} \cdot \prod_{t \in H_{3,3}} y_{2,t}) \cdot (a_{3,4} \cdot \prod_{t \in H_{3,4}} y_{2,t}) \cdot a_{3,5}.$$

Theorem 3.4. *Let $(S, \cdot)$ be a semigroup, let A be a central set in S, and for each $\ell \in \mathbb{N}$, let $\langle y_{\ell,n} \rangle_{n=1}^{\infty}$ be a sequence in S. There exist sequences $\langle m(n) \rangle_{n=1}^{\infty}$, $\langle \langle a_n(i) \rangle_{i=1}^{m(n)+1} \rangle_{n=1}^{\infty}$, and $\langle \langle H_n(i) \rangle_{i=1}^{m(n)} \rangle_{n=1}^{\infty}$ such that*

(1) for each $n \in \mathbb{N}$, $m(n) \in \mathbb{N}$;

(2) for each $n \in \mathbb{N}$ and each $i \in \{1, 2, \ldots, m(n)+1\}$, $a_n(i) \in S$;

(3) for each $n \in \mathbb{N}$ and each $i \in \{1, 2, \ldots, m(n)\}$, $H_n(i) \in \mathcal{P}_f(\mathbb{N})$;

(4) for each $n \in \mathbb{N}$ and each $i \in \{1, 2, \ldots, m(n)-1\}$,
    $\max H_n(i) < \min H_n(i+1)$;

(5) for each $n \in \mathbb{N}$, $\max H_n\big(m(n)\big) < \min H_n(1)$; and

(6) for each $F \in \mathcal{P}_f(\mathbb{N})$ and each $f \in \Phi$,

$$\prod_{n \in F} \left(\prod_{i=1}^{m(n)} (a_n(i) \cdot \prod_{t \in H_n(i)} y_{f(n),t}) \cdot a_n(m(n)+1) \right) \in A.$$

A significant strengthening of the Central Sets Theorem was obtained in [26] when a version was produced which deals with all sequences in S at once, rather than just countably many. Again, the commutative version is simpler to state, so we state it first.

Theorem 3.5. *Let $(S,+)$ be a commutative semigroup and let A be a central subset of S. There exist functions $\alpha : \mathcal{P}_f(^{\mathbb{N}}S) \to S$ and $H : \mathcal{P}_f(^{\mathbb{N}}S) \to \mathcal{P}_f(\mathbb{N})$ such that*

(1) *if $F, G \in \mathcal{P}_f(^{\mathbb{N}}S)$ and $F \subsetneqq G$, then $\max H(F) < \min H(G)$ and*

(2) *whenever $m \in \mathbb{N}$, $G_1, G_2, \ldots, G_m \in \mathcal{P}_f(^{\mathbb{N}}S)$, $G_1 \subsetneqq G_2 \subsetneqq \ldots \subsetneqq G_m$, and for each $i \in \{1, 2, \ldots, m\}$, $f_i \in G_i$, one has*
$\sum_{i=1}^{m} \left(\alpha(G_i) + \sum_{t \in H(G_i)} f_i(t) \right) \in A.$

Proof. [26, Theorem 2.2]. $\qquad\square$

Notice that Theorem 3.3 is an easy consequence of Theorem 3.5. To see this, observe that one can assume that the sequences in the statement of Theorem 3.3 are distinct. Then given such sequences, for each $n \in \mathbb{N}$, let $F_n = \{\langle y_{1,t} \rangle_{t=1}^{\infty}, \langle y_{2,t} \rangle_{t=1}^{\infty}, \ldots, \langle y_{n,t} \rangle_{t=1}^{\infty}\}$ and let $a_n = \alpha(F_n)$ and $H_n = H(F_n)$.

The version of the noncommutative Central Sets Theorem proved in [26] separated $\alpha(G_i)$ into several parts so that a term like $\alpha(G_2) \cdot \prod_{t \in H(G_2)} f_2(t)$ became something like

$$\alpha(G_2)(1) \cdot \prod_{t \in H(G_2)(1)} f_2(t) \cdot \alpha(G_2)(2) \prod_{t \in H(G_2)(2)} f_2(t) \cdot \alpha(G_2)(3),$$

where in this case $\alpha(G_2)$ was split into three parts (and $H(G_2)$ was split into two parts). We will not present this version because it was greatly simplified in [58] where it was shown that one could presume that each $H(G_i)(j)$ could in fact be taken to be a singleton.

Definition 3.6. Let $m \in \mathbb{N}$. Then $\mathcal{J}_m = \left\{ \big(t(1), t(2), \ldots, t(m)\big) \in \mathbb{N}^m : t(1) < t(2) < \ldots < t(m) \right\}$.

Theorem 3.7. *Let $(S, \cdot)$ be a semigroup and let C be a central subset of S. There exist*

$$m : \mathcal{P}_f(^{\mathbb{N}}S) \to \mathbb{N}, \alpha \in \bigtimes_{F \in \mathcal{P}_f(^{\mathbb{N}}S)} S^{m(F)+1}, \text{ and } \tau \in \bigtimes_{F \in \mathcal{P}_f(^{\mathbb{N}}S)} \mathcal{J}_{m(F)}$$

such that

(1) *if $F, G \in \mathcal{P}_f(^{\mathbb{N}}S)$ and $F \subsetneqq G$, then $\max H(F) < \min H(G)$ and*

(2) *whenever $n \in \mathbb{N}$, $G_1, G_2, \ldots, G_n \in \mathcal{P}_f(^{\mathbb{N}}S)$, $G_1 \subsetneq G_2 \subsetneq \ldots \subsetneq G_n$, and for each $i \in \{1, 2, \ldots, n\}$, $f_i \in G_i$, one has*

$$\textstyle\prod_{i=1}^{n} \left(\left(\prod_{j=1}^{m(G_i)} \alpha(G_i)(j) \cdot f_i\big(\tau(G_i)(j)\big) \right) \cdot \alpha(G_i)(m(G_i) + 1) \right) \in A \,.$$

Proof. [58, Corollary 3.3]. □

4 Finite combinatorial applications

Central sets were first invented for the semigroup $(\mathbb{N}, +)$ and we begin our list of applications with properties that must be satisfied by any central set in $\mathbb{N}$. (When we say a set is central in $\mathbb{N}$ without modification, we mean it is central in $(\mathbb{N}, +)$.) The first of these results was the following.

Theorem 4.1. *Let C be a central subset of $\mathbb{N}$. Then C contains arbitrarily long arithmetic progressions.*

Proof. [31, Proposition 8.9]. □

The first result which used the Central Sets Theorem dealt with systems of homogeneous linear equations. In [63], R. Rado characterized those systems of homogenous linear equations which are partition regular over $\mathbb{N}$. This characterization was based on the *columns condition*.

Definition 4.2. Let $u, v \in \mathbb{N}$, let A be a $u \times v$ matrix with entries from $\mathbb{Q}$, and let $\vec{c_1}, \vec{c_2}, \ldots, \vec{c_v}$ be the columns of A. Let $R = \mathbb{Z}$ or $R = \mathbb{Q}$. The matrix A satisfies the *columns condition* over R if and only if there exist $m \in \mathbb{N}$ and $I_1, I_2, \ldots, I_m$ such that

(1) $\{I_1, I_2, \ldots, I_m\}$ is a partition of $\{1, 2, \ldots, v\}$;

(2) $\sum_{i \in I_1} \vec{c_i} = \vec{0}$; and

(3) if $m > 1$ and $t \in \{2, 3, \ldots, m\}$, then $\sum_{i \in I_t} \vec{c_i}$ is a linear combination of $\{\vec{c_i} : i \in \bigcup_{j=1}^{t-1} I_j\}$ with coefficients from R.

Theorem 4.3 (Rado). *Let $u, v \in \mathbb{N}$ and let A be a $u \times v$ matrix with entries from $\mathbb{Q}$. The following statements are equivalent.*

(a) *Whenever $\mathbb{N}$ is finitely colored, there exists $\vec{x}$ such that $A\vec{x} = \vec{0}$ and the entries of $\vec{x}$ are monochromatic.*

(b) *The matrix A satisfies the columns condition over $\mathbb{Q}$.*

Proof. [63, Satz IV]. □

A matrix satisfying conclusion (a) of Theorem 4.3 will be called *kernel partition regular*.

Theorem 4.4. *Let $u, v \in \mathbb{N}$, let A be a $u \times v$ matrix with entries from $\mathbb{Q}$ which satisfies the columns condition over $\mathbb{Q}$, and let C be a central subset of $\mathbb{N}$. There exists $\vec{x} \in C^v$ such that $A\vec{x} = \vec{0}$.*

Proof. [31, Theorem 8.22]. (Using the relationship between kernel partition regularity and certain other matrices established in [28], this can also be deduced from [32, Theorem 4.4] which was published before the invention of central sets.) □

In other words, any system of partition regular homogeneous linear equations not only has monochromatic solutions given any finite coloring of $\mathbb{N}$, but such solutions must be found in any of the color classes that are central. (And, of course, there must be at least one such.)

Theorem 4.4 was extended in [56]. In dealing with an arbitrary commutative semigroup, things like $-2x$ may not mean anything, so, rather than asking whether $3x - 2y + z = 0$, we ask whether $3x + z = 2y$.

Theorem 4.5. *Let $(S, +)$ be a commutative semigroup with identity 0, let $u, v \in \mathbb{N}$, let A and B be $u \times v$ matrices with entries from ω, let $C = A - B$, and let D be central in S.*

(a) *If C satisfies the columns condition over $\mathbb{Z}$, then there exists $\vec{x} \in D^v$ such that $A\vec{x} = B\vec{x}$.*

(b) *If for each $d \in \mathbb{N}$, dS is central* in S and C satisfies the columns condition over $\mathbb{Q}$, then there exists $\vec{x} \in D^v$ such that $A\vec{x} = B\vec{x}$.*

Proof. [56, Theorem 2.5]. □

The "other matrices" mentioned in the proof of Theorem 4.4 are examples of *image partition regular* matrices.

Definition 4.6. Let $u, v \in \mathbb{N}$, let A be a $u \times v$ matrix with rational entries and finitely many nonzero entries per row, and let S be a nontrivial subsemigroup of $(\mathbb{R}, +)$. The matrix A is *image partition regular over S* if and only if whenever $S \setminus \{0\}$ is finitely colored, there exists $\vec{x} \in (S \setminus \{0\})^v$ such that the entries of $A\vec{x}$ are monochromatic.

Many natural results in Ramsey Theory are easily stated using image partition regular matrices. For example, Theorem 4.1 is the assertion that for each

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$k \in \mathbb{N}$, any central subset of $\mathbb{N}$ contains an image of the matrix

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ \vdots & \vdots \\ 1 & k \end{pmatrix}.$$

Definition 4.7. Let $u, v \in \mathbb{N}$ and let A be a $u \times v$ matrix with entries from $\mathbb{Q}$. Then A is a *first entries matrix* if and only if no row of A is $\vec{0}$ and whenever $i, j \in \{1, 2, \ldots, u\}$ and $k = \min\{t \in \{1, 2, \ldots, v\} : a_{i,t} \neq 0\} = \min\{t \in \{1, 2, \ldots, v\} : a_{j,t} \neq 0\}$, then $a_{i,k} = a_{j,k} > 0$. An element b of $\mathbb{Q}$ is a *first entry* of A if and only if there is some row i of A such that $b = a_{i,k}$ where $k = \min\{t \in \{1, 2, \ldots, v\} : a_{i,t} \neq 0\}$.

Characterizations of those matrices that are image partition regular over $\mathbb{N}$ were first obtained in [43, Theorem 3.1]. Since then many other characterizations have been obtained, and quite a number of them involve central sets. We present some of these now.

Theorem 4.8. *Let $u, v \in \mathbb{N}$ and let A be a $u \times v$ matrix with entries from $\mathbb{Q}$. The following statements are equivalent.*

(a) *A is image partition regular over $\mathbb{N}$.*

(b) *There exist $m \in \mathbb{N}$ and a $u \times m$ first entries matrix such that given any $\vec{y} \in \mathbb{N}^m$, there is some $\vec{x} \in \mathbb{N}^v$ with $A\vec{x} = B\vec{y}$.*

(c) *For every central set C in $\mathbb{N}$, there exists $\vec{x} \in \mathbb{N}^v$ such that $A\vec{x} \in C^u$.*

(d) *For every central set C in $\mathbb{N}$, $\{\vec{x} \in \mathbb{N}^v : A\vec{x} \in C^u\}$ is central in $\mathbb{N}^v$.*

(e) *Whenever $m \in \mathbb{N}$, and $\phi_1, \phi_2, \ldots, \phi_m$ are nonzero linear mappings from $\mathbb{Q}^v$ to $\mathbb{Q}$, there exists $\vec{b} \in \mathbb{Q}^m$ such that, whenever C is central in $\mathbb{N}$, there exists $\vec{x} \in \mathbb{N}^v$ for which $A\vec{x} \in C^u$ and, for each $i \in \{1, 2, \ldots, m\}$, $b_i\phi_i(\vec{x}) \in C$, and in particular $\phi_i(\vec{x}) \neq 0$.*

(f) *For every central set C in $\mathbb{N}$, there exists $\vec{x} \in \mathbb{N}^v$ such that $\vec{y} = A\vec{x} \in C^u$, all entries of $\vec{x}$ are distinct, and for all $i, j \in \{1, 2, \ldots, u\}$, if rows i and j of A are unequal, then $y_i \neq y_j$.*

Proof. These are parts of [46, Theorem 2.10]. Unfortunately, as published there were gaps in the proof. For a complete proof see the version of [46] posted on my web page, `nhindman.us`, or [53, Theorem 15.24]. □

Statement (d) of Theorem 4.8 is reminiscent of many Ramsey Theoretic results obtained using ergodic theory – not only are there monochromatic good solutions, there are lots of them.

In [39, Theorem 4.1] it was shown that Theorem 4.8 remains valid if the entries of A are allowed to come from $\mathbb{R}$ and all occurrences of $\mathbb{N}$ in statements (a) through (f) are replaced by $\mathbb{R}^+ = \{x \in \mathbb{R} : x > 0\}$. (If $\mathbb{N}$ is replaced by $\mathbb{R}$, it is shown in [39, Theorem 3.1] that statements (a) and (b) are equivalent.)

We look now at a consequence of the noncommutative Central Sets Theorem, namely an extension of the Hales-Jewett Theorem [38, Theorem 1].

Definition 4.9. Let A be a set. A *variable word* over A is a word over the alphabet $A \cup \{v\}$ in which v occurs, where v is a *variable* not in A. If w is a variable word over A and $a \in A$, then $w(a)$ is the word obtained by replacing each occurrence of v in w by a.

For example, if $A = \{1, 2, 3\}$ and $w = 12v2v1vv$, then $w(1) = 1212111$ and $w(3) = 12323133$. The *free semigroup* over A is the set of all words over the alphabet A. (For a formal definition of "word", see [53, Definition 1.3].) The Hales-Jewett Theorem says that if the free semigroup over a finite nonempty alphabet A is finitely colored, then there is a variable word w such that $\{w(a) : a \in A\}$ is monochromatic.

The following is [13, Corollary 3.6]. I will present the proof to show how easy a consequence it is of the noncommutative Central Sets Theorem.

Theorem 4.10. *Let A be a nonempty finite alphabet, let S be the free semigroup over A, and let C be central in S. There is a variable word w over A such that $\{w(a) : a \in A\} \subseteq C\}$.*

Proof. Let m, α, and τ be as guaranteed for S and C by Theorem 3.7. (We will only use the $n = 1$ case of the conclusion of Theorem 3.7.) For $a \in A$, let $\bar{a}$ be the sequence constantly equal to a and let $G_1 = \{\bar{a} : a \in A\}$. Let $k = m(G_1)$, let $\delta = \alpha(G_1)$, and let $t = \tau(G_1)$. Let $w = \left(\prod_{j=1}^{k} \delta(j)v \right) \delta(k+1)$. Then given $a \in A$, $w(a) = \left(\prod_{j=1}^{k} \delta(j)\bar{a}\big(t(j)\big) \right) \delta(k+1) \in A$. $\qquad\square$

The Graham-Rothschild Parameter Sets Theorem [36] is a complicated theorem, having many previously known results as consequences, as well as several new applications. The Hales-Jewett Theorem is its simplest instance. And we have just seen that sets satisfying the conclusion of the Hales-Jewett Theorem can be found in any central set. The next simplest instance of the Graham-Rothschild Parameter Sets Theorem is one of the few finitistic Ramsey Theoretic results I can think of that are not guaranteed to have sets satisfying their conclusions in any central set. I will introduce now just enough terminology to state that next simplest instance.

Definition 4.11. Let A be a set. A *two variable word* over A is a word w over the alphabet $A \cup \{v_1, v_2\}$, where v_1 and v_2 are distinct variables not in A such that

(1) both v_1 and v_2 occur in w and

(2) the first occurrence of v_1 precedes the first occurrence of v_2.

If w is a two variable word over A and $u = l_1l_2$ is a length two one variable word over A, then $w(u)$ is the result of replacing each occurrence of v_i by l_i for $i \in \{1, 2\}$.

The second simplest instance referred to above is the assertion that if the set of one variable words over A is finitely colored, then there is a two variable word w such that $\{w(u) : u$ is a length two one variable word$\}$ is monochromatic.

Theorem 4.12. *Let A be a nonempty finite alphabet and let S be the semigroup of one variable words over A. Pick $a \in A$ and let $M = \{u \in S : |\{i : u_i = v\}| > |\{i : u_i = a\}|\}$. Then M is central in S and there does not exist a two variable word w such that $\{w(u) : u$ is a length two one variable word$\} \subseteq M$.*

Proof. [24, Theorem 3.6]. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad$ $\square$

The following theorem was described in [19] as a "multi-dimensional van der Waerden theorem". (If one converts to additive notation, one sees that the points g form a kind of an increment.) The theorem says that the set of "good" points g is a member of any minimal idempotent. That is, it is a central* set.

Theorem 4.13. *Let G be a countable amenable group with the discrete topology and let p be a minimal idempotent in βG. For any finite coloring of $G \times G \times G$, $\{g \in G : (\exists(a, b, c) \in G \times G \times G)$*
$(\{(a, b, c), (ag, b, c), (ag, bg, c), (ag, bg, cg)\}$ is monochromatic$)\} \in p$.

Proof. [19, Theorem 4.4]. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad$ $\square$

We conclude this section with some results about $(\mathbb{N}, \cdot)$ and its relation with $(\mathbb{N}, +)$. Given a $u \times v$ matrix A, we write $\vec{x}^A = \vec{1}$ to represent the equations $\prod_{j=1}^{v} x_{a_{i,j}} = 1$ for $i \in \{1, 2, \ldots, u\}$.

By Theorem 4.5(b), if it were true that for each $d \in \mathbb{N}$, $\{x^d : x \in \mathbb{N}\}$ is central* in $(\mathbb{N}, \cdot)$, then the columns condition over $\mathbb{Q}$ would be sufficient for the existence of solutions in any central set. But by [12, Theorem 5.3], $\{x^2 : x \in \mathbb{N}\}$ is not central in $(\mathbb{N}, \cdot)$, so its complement is a central set containing no solutions to $x_1^2 x_2^{-2} x_3 = 1$, while $(\ 2\quad -2\quad 1\ )$ satisfies the columns condition over $\mathbb{Q}$.

Theorem 4.14. *Let $u, v \in \mathbb{N}$ and let C be a $u \times v$ matrix with entries from $\mathbb{Z}$. The following statements are equivalent.*

(a) *Whenever D is a central subset of $(\mathbb{N}, \cdot)$, there exists $\vec{x} \in D^v$ such that $\vec{x}^D = \vec{1}$.*

(b) *The matrix C satisfies the columns condition over $\mathbb{Z}$.*

Proof. [56, Theorem 3.10]. □

It is also shown in [56, Theorem 4.3] that the columns condition over $\mathbb{Q}$ is necessary and sufficient for kernel partition regularity over $(\mathbb{N}, \cdot)$. In the same vein is the following result.

Theorem 4.15. *Let S be a dense subsemigroup of $\big((0,1), \cdot\big)$, let $u, v \in \mathbb{N}$, let C be a $u \times v$ matrix with entries from $\mathbb{Z}$, and let D be a central subset of S.*

(a) *If C satisfies the columns condition over $\mathbb{Z}$, then there exists $\vec{x} \in D^v$ such that $\vec{x}^D = \vec{1}$.*

(b) *If $S = (0,1)$ and C satisfies the columns condition over $\mathbb{Q}$, then there exists $\vec{x} \in D^v$ such that $\vec{x}^D = \vec{1}$.*

Proof. [17, Theorem 3.5], □

One of the nice things about working in $\beta\mathbb{N}$ is that the same space has both additive and multiplicative structure.

Theorem 4.16. *Given any finite coloring of $\mathbb{N}$, there is one color class which is both additively and multiplicatively central.*

Proof. [12, Corollary 5.5]. □

In particular, one color class must satisfy both the additive and multiplicative versions of the Central Sets Theorem.

The question arrises as to how much multiplicative structure an additively central set must have and how much additive structure a multiplicatively central set must have. The answers are "none" and "quite a bit" respectively.

Theorem 4.17. *There is an additively central subset A of $\mathbb{N}$ for which there exist no x and y in $\mathbb{N}$ with $\{x, y, xy\} \subseteq A$.*

Proof. [14, Theorem 3.4]. □

Theorem 4.18. *Let A be a multiplicatively central subset of $\mathbb{N}$. For each $m \in \mathbb{N}$ there exists a sequence $\langle y_i \rangle_{i=1}^m$ such that $FS(\langle y_i \rangle_{i=1}^m) \subseteq A$.*

Proof. [14, Theorem 3.5]. □

The last two results of this section deal with arithmetic and geometric progressions.

Theorem 4.19. *Let C be a central subset of $(\mathbb{N}, \cdot)$, let $k \in \mathbb{N}$, and let G be a finite subset of $\mathbb{N}$. There exist $a, b, d \in C$ such that*

$$\big\{b(a + di)^j : i \in G \text{ and } j \in \{0, 1, \ldots, k\}\big\} \cup \big\{bd^j : j \in \{0, 1, \ldots, k\}\big\}$$
$$\cup \{a + di : i \in G\} \subseteq C.$$

Proof. [8, Corollary 4.4]. □

Theorem 4.20. *Let $m, k \in \mathbb{N}$. Let C_1 be central in $(\mathbb{N}, +)$ and let C_2 be central in $(\mathbb{N}, \cdot)$. For each $i \in \{0, 1, \ldots, k\}$ let $\langle x_{i,t} \rangle_{t=1}^{\infty}$ and $\langle y_{i,t} \rangle_{t=1}^{\infty}$ be sequences in $\mathbb{N}$. Let $\mathbb{N} = \bigcup_{s=1}^{m} A_s$. Then there exist $s \in \{1, 2, \ldots, m\}$, $F \in \mathcal{P}_f(\mathbb{N})$, and $a, b \in \mathbb{N}$ such that*

$$\{ba\} \cup \big\{b(a + \textstyle\sum_{t \in F} x_{i,t}) : i \in \{0, 1, \ldots, k\}\big\} \cup$$
$$\big\{ba \cdot \textstyle\prod_{t \in F} y_{j,t} : j \in \{0, 1, \ldots, k\}\big\} \cup$$
$$\big\{b(a + \textstyle\sum_{t \in F} x_{i,t}) \cdot (\prod_{t \in F} y_{j,t}) : i, j \in \{0, 1, \ldots, k\}\big\} \subseteq A_s,$$
$$\{a\} \cup \big\{a + \textstyle\sum_{t \in F} x_{i,t} : i \in \{0, 1, \ldots, k\}\big\} \subseteq C_1, \text{ and}$$
$$\{b\} \cup \big\{b \cdot \textstyle\prod_{t \in F} y_{j,t} : j \in \{0, 1, \ldots, k\}\big\} \subseteq C_2.$$

Proof. [8, Corollary 5.9]. □

5 Infinite combinatorial applications

The first infinite application of central sets was established by Furstenberg in the section where central sets were invented. The proof did not use the Central Sets Theorem. An *IP set* is a set which contains $FS(\langle x_n \rangle_{n=1}^{\infty})$ for some sequence $\langle x_n \rangle_{n=1}^{\infty}$.

Theorem 5.1. *Any central subset of $\mathbb{N}$ is an IP set.*

Proof. [31, Proposition 8.10]. □

In fact, using the algebraic characterization of central, this fact holds in much greater generality.

Theorem 5.2. *Any central subset of any semigroup is an IP set.*

Proof. The Galvin-Glazer proof of the Finite Sum Theorem establishes that any member of any idempotent is an IP set. (See the notes to [53, Chapter 5] for the history of this proof.) □

The next result was obtained using *partial semigroups*, a notion that we will explore in Section 12, and is a corollary to a much more general result which would require the introduction of too much notation to present.

Given $q \in \mathbb{N}$ written in base ten and $t \in \{1, 2, \ldots, 9\}$, let $q(t)$ be the base ten number obtained by replacing each occurrence of 9 by t. For example, if $q = 309219950$, then $q(4) = 304214450$, $q(5) = 305215550$, and $q(9) = 309219950 = q$. If q does not have any occurrence of 9 in its base ten expansion, let $\varphi(q)$ be the number represented by the decimal digits of q viewed as a base nine expansion. (Thus if, written in base ten, $q = 3208$, then $\varphi(q) = 3 \cdot 9^3 + 2 \cdot 9^2 + 8$.)

Theorem 5.3. *Let $r \in \mathbb{N}$ and let $\mathbb{N} = \bigcup_{i=1}^{r} A_i$. There exist $i, j \in \{1, 2, \ldots, r\}$ and a sequence $\langle q_n \rangle_{n=1}^{\infty}$ in $\mathbb{N}$ such that*

(1) *for each $n \in \mathbb{N}$, q_n has at least one 9 in its base ten expansion;*

(2) *for each n and m in $\mathbb{N}$, if $10^m \leq q_n$, then 10^{m+1} divides q_{n+1};*

(3) *for each $F \in \mathcal{P}_f(\mathbb{N})$ and each $f : F \to \{1, 2, \ldots, 9\}$,*

    (a) *if $9 \in f[F]$, then $\sum_{n \in F} q_n(f(n)) \in A_i$;*

    (b) *if $9 \notin f[F]$, then $\sum_{n \in F} q_n(f(n)) \in A_j$;*

(4) *A_i is central in $\mathbb{N}$; and*

(5) *if $X = \{q \in \mathbb{N} : q$ has no occurrence of 9 in its base ten expansion$\}$, then $\varphi[A_j \cap X]$ is central in $\mathbb{N}$.*

Proof. [10, Corollary 4.7]. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ □

Of course, there is nothing special about base ten in Theorem 5.3; a corresponding result holds for any base bigger than two.

There are no known characterizations of either kernel or image partition regularity for infinite matrices. A *finite sums matrix* is a matrix whose rows are all possible rows with all 0's and 1's and only finitely many 1's. Theorem 5.1 says that a finite sums matrix has an image contained in any central set, and is in particular image partition regular. A large class of image partition regular matrices are the *Milliken-Taylor* matrices. They are so named because the fact that they are image partition regular over $\mathbb{N}$ follows easily from the Milliken-Taylor Theorem [61, Theorem 2.2], [65, Lemma 2.2].

Definition 5.4. Let $k \in \omega$ and let $\vec{a} = \langle a_0, a_1, \ldots, a_k \rangle$ be a sequence in $\mathbb{R}$ such that $\vec{a} \neq \vec{0}$. The sequence $\vec{a}$ is *compressed* if and only if no $a_i = 0$ and for each $i \in \{0, 1, \ldots, k-1\}$, $a_i \neq a_{i+1}$. The sequence $c(\vec{a}) = \langle c_0, c_1, \ldots, c_m \rangle$ is the compressed sequence obtained from $\vec{a}$ by first deleting all occurrences of 0 and then deleting any entry which is equal to its successor. Then $c(\vec{a})$ is called the *compressed form of* $\vec{a}$. And $\vec{a}$ is said to be a *compressed sequence* if $\vec{a} = c(\vec{a})$.

Definition 5.5. Let $k \in \omega$, let $\vec{a} = \langle a_0, a_1, \ldots, a_k \rangle$ be a compressed sequence in $\mathbb{R} \setminus \{0\}$, and let A be an $\omega \times \omega$ matrix. Then A is an $MT(\vec{a})$-matrix if and only if the rows of A are all rows $\vec{r} \in \mathbb{R}^\omega$ with finitely many nonzero entries such that $c(\vec{r}) = \vec{a}$. The matrix A is a *Milliken-Taylor matrix* if and only if it is an $MT(\vec{a})$-matrix for some $\vec{a}$.

Note that a finite sums matrix ia an $MT(\langle 1 \rangle)$-matrix. Milliken-Taylor matrices provide a major contrast with the finite image partition regular matrices. The only Milliken-Taylor matrices that are guaranteed to have images in any central subset of $\mathbb{N}$ are the $MT(\langle a \rangle)$-matrices for $a \in \mathbb{Q}^+ = \{x \in \mathbb{Q} : x > 0\}$ – that is the ones which are positive rational multiples of the finite sums matrix.

Theorem 5.6. *Let $\vec{a}$ and $\vec{b}$ be compressed sequences in $\mathbb{Q} \setminus \{0\}$ such that $\vec{b}$ is not a multiple of $\vec{a}$, let A be an $MT(\vec{a})$-matrix and let B be an $MT(\vec{b})$-matrix. There exists a partition $\{C_1, C_2\}$ of $\mathbb{Q} \setminus \{0\}$ such that there do not exist $i \in \{1, 2\}$ and $\vec{x}$ and $\vec{y}$ in $\mathbb{Q}^\omega$ with all entries of $A\vec{x}$ and all entries of $B\vec{y}$ in C_i.*

Proof. [54, Corollary 4.5]. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

As a consequence, for any compressed sequence $\vec{b}$ with more than one term, there is a central set which does not contain an image of an $MT(\vec{b})$-matrix. We were therefore quite surprised with the following theorem which shows that one can come close to getting images in any central set. The only special requirement is that the last term of the compressed sequences be a 1.

Definition 5.7. A matrix B with entries from $\mathbb{Q}$ is *centrally image partition regular* provided that any central subset of $\mathbb{N}$ contains an image of B.

Theorem 5.8. *Let $m \in \omega$ and for each $i \in \{0, 1, \ldots, m\}$, let $k(i) \in \mathbb{N}$, let $\vec{a}_i = \langle a_{i,0}, a_{i,1}, \ldots, a_{i,k(i)} \rangle$ be a compressed sequence in $\mathbb{Z} \setminus \{0\}$ with $a_{i,k(i)} = 1$, and let M_i be an $MT(\vec{a}_i)$-matrix. Let $\bar{0}$ and $\bar{1}$ be the length ω constant column vectors. Then*

$$
B = \begin{pmatrix}
\bar{1} & \bar{0} & \ldots & \bar{0} & M_0 \\
\bar{0} & \bar{1} & \ldots & \bar{0} & M_1 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
\bar{0} & \bar{0} & \ldots & \bar{1} & M_m \\
\bar{0} & \bar{0} & \ldots & \bar{0} & \mathbf{F}
\end{pmatrix}
$$

is centrally image partition regular.

Proof. [47, Corollary 6.4]. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

An image of the matrix which we define now includes an image of every finite image partition regular matrix, as well as all finite sums choosing at most one entry from each such image.

Definition 5.9. Fix an enumeration $\langle B_n \rangle_{n=0}^\infty$ of the finite matrices with rational entries that are image partition regular over $\mathbb{N}$. For each n, assume that B_n is a $u(n) \times v(n)$ matrix. For each $i \in \mathbb{N}$, let $\vec{0}_i$ be the 0 vector with i entries. Let $\mathbf{D}$ be an $\omega \times \omega$ matrix with all rows of the form $\vec{r}_0 \frown \vec{r}_1 \frown \vec{r}_2 \frown \ldots$ where each $\vec{r}_i$ is either $\vec{0}_{v(i)}$ or is a row of B_i, and all but finitely many are $\vec{0}_{v(i)}$.

It was shown in [29] that **D** is image partition regular. The proof used an idempotent ultrafilter. Neither of the authors of [29] were aware of the fact that central sets are characterized by idempotent ultrafilters.

Definition 5.10. A matrix B is *strongly centrally image partition regular* if and only if whenever C is a central set in $\mathbb{N}$, there exists $\vec{x}$ of the appropriate dimension such that the entries of $B\vec{x}$ are in C, the entries of $\vec{x}$ are distinct, and entries of $B\vec{x}$ corresponding to distinct rows of B are distinct.

Theorem 5.11. *The matrix* **D** *is strongly centrally image partition regular.*

Proof. Except for the requirement that the entries ov $\vec{x}$ be distinct, this follows from [50, Theorem 3.2]. The full statement is [47, Theorem 2.6]. □

Now we turn our attention to kernel partition regularity of infinite matrices.

Definition 5.12. . Let B be a matrix with entries from $\mathbb{Q}$ and let R be a subsemigroup of $(\mathbb{Q}, +)$.

(a) The matrix B is *centrally kernel partition regular* over R if and only if for every central subset C of R, there is a vector $\vec{x}$ with entries from C such that $B\vec{x} = \vec{0}$.

(b) The matrix B is *strongly centrally kernel partition regular* over R if and only if for every central subset C of R, there is an injective vector $\vec{x}$ with entries from C such that $B\vec{x} = \vec{0}$.

Recall that kernel partition regularity of a finite matrix is determined by the columns condition, the first requirement of which is that some nonempty set of columns sums to zero. It is easy to see that, for infinite matrices, the columns condition is not sufficient for kernel partition regularity. Consider for example the matrix

$$
B = \begin{pmatrix}
1 & -1 & -1 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 1 & -1 & -1 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 1 & -1 & -1 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix} .
$$

This matrix is not partition regular over $\mathbb{N}$ for the trivial reason that there is no $\vec{x} \in \mathbb{N}^\omega$ with $B\vec{x} = \vec{0}$. (That would require that $x_{2n} > x_{2n+2}$ for each n.) To see that B satisfies the columns condition, let $I_0 = \{0, 2, 4, 6, \ldots\}$ and $I_1 = \{1, 3, 5, \ldots\}$. Then $\sum_{i \in I_0} \vec{c}_i = \vec{0}$ and $\sum_{i \in I_1} \vec{c}_i = \sum_{t=1}^{\infty} t\vec{c}_{2t}$.

The question remained whether the columns condition was necessary for kernel partition regularity.

Theorem 5.13. *Let*

$$A = \begin{pmatrix} 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 2 & 0 & 1 & 1 & 1 & 1 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

and let I be the $\omega \times \omega$ identity matrix. The $\omega \times (\omega + \omega)$ matrix $\begin{pmatrix} A & -I \end{pmatrix}$ is centrally kernel partition regular and has no nonempty set of columns summing to 0.

Proof. [3, Theorem 2.1]. □

We conclude this section with results establishing that sets central in subrings of $\mathbb{Q}$ are good enough to distinguish between any two subrings of $\mathbb{Q}$ with identities.

Definition 5.14. Let $\mathbb{P}$ be the set of primes and let $F \subseteq \mathbb{P}$. Then $\mathbb{G}_F = \{a/b : a \in \mathbb{Z}, b \in \mathbb{N} \text{ and all prime factors of } b \text{ are in } F\}$.

Thus $\mathbb{G}_\emptyset = \mathbb{Z}$, $\mathbb{G}_{\{2\}} = \mathbb{D}$, the set of dyadic rationals, and $\mathbb{G}_\mathbb{P} = \mathbb{Q}$. It is easy to check that the $\mathbb{G}_F$ are precisely the subrings of $\mathbb{Q}$ with identity. (Given a subring R of $\mathbb{Q}$ with identity, let $F = \{p \in \mathbb{P} : \frac{1}{p} \in R\}$. Given $\frac{a}{b} \in R$ with $(a, b) = 1$, pick k and l in $\mathbb{Z}$ such that $1 = ka + lb$. Then $\frac{1}{b} = k \cdot \frac{a}{b} + l \in R$.)

Theorem 5.15. *Let F and H be subsets of $\mathbb{P}$ with $H \setminus F \neq \emptyset$ and pick $q \in H \setminus F$. Let*

$$B = \begin{pmatrix} 1/q & 1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 1/q^2 & 0 & 0 & 0 & 1 & 1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 1/q^3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & -1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} .$$

Then B is strongly centrally kernel partition regular over $\mathbb{G}_H$ but is not kernel partition regular over $\mathbb{G}_F$.

Proof. [2, Theorem 4.3]. □

It is shown in [2, Theorem 4.4] that there is a similar system (with different first column) which is strongly centrally kernel partition regular over $\mathbb{Q}$ but is not partition regular over any subring of $\mathbb{Q}$ with identity.

6   Products and disjoint central sets

We first note that central sets are preserved under cartesian products of arbitrary semigroups.

Theorem 6.1. *Let S and T be semigroups, let A be a central subset of S, and let B be a central subset of T. Then $A \times B$ is central in $S \times T$.*

Proof. [52, Corollary 2.2]. □

To characterize when arbitrary products are central, we need the notion of *weakly thick*.

Definition 6.2. Let $(S, \cdot)$ be a semigroup and let $A \subseteq S$.

(a) The set A is *thick* if and only if for each $F \in \mathcal{P}_f(S)$ there exists $x \in S$ such that $Fx \subseteq A$.

(b) The set A is *weakly thick* if and only if there exists $s \in S$ such that $s^{-1}A$ is thick.

It is easy to see that any thick set is weakly thick and that the notions are equivalent for commutative semigroups.

Theorem 6.3. *Let I be a set and for each $\alpha \in I$, let S_α be a semigroup and let $A_\alpha \subseteq S_\alpha$. Let $S = \bigtimes_{\alpha \in I} S_\alpha$, let $A = \bigtimes_{\alpha \in I} A_\alpha$, and let $J = \{\alpha \in I : A_\alpha$ is not weakly thick in $S_\alpha\}$. Then A is central in S if and only if J is finite and for each $\alpha \in I$, A_α is central in S_α.*

Proof. [52, Theorem 3.8]. □

In [45, Corollary 2.13], it was shown that any central subset of $\mathbb{N}$ can be divided into two disjoint central sets. This was significantly extended in the next result, for which we need the notion of a tree. Here we are taking $n \in \omega$ to be an ordinal, that is the set of its predecessors.

Definition 6.4. (a) T is a *tree* if and only if T is a nonempty set of functions, for each $g \in T$, domain$(g) \in \omega$, and if domain$(g) = n > 0$, then $g_{|n-1} \in T$.

(b) Let g be a function with domain$(g) = n \in \omega$ and let x be given. Then $g \frown x = g \cup \{(n, x)\}$.

(c) Given a tree T, g is a *path through* T if and only if g is a function, domain$(g) = \omega$, and for each $n \in \omega$, $g_{|n} \in T$.

Theorem 6.5. *Let D be a central subset of $\mathbb{N}$. Then there exist a choice of $D_{i,j}$ for each $i, j \in \omega$ and a tree T such that*

(1) for each $i, j \in \omega$, $D_{i,j}$ is a central subset of D;

(2) if $i, j, l, m \in \omega$ and $(i, j) \neq (l, m)$, then $D_{i,j} \cap D_{l,m} = \emptyset$;

(3) for each $g \in T$, if domain$(g) = n$, then there exist $U_0, U_1, \ldots, U_n$ such that $\{\vec{x} : g \frown \vec{x} \in T\} = U_0 \times U_1 \times \ldots \times U_n$, each U_i is central, and each $U_i \subseteq D_{i,i}$; and

(4) if g is a path through T and for each $n \in \omega$, $g(n) = (x_{0,n}, x_{1,n}, \dots, x_{n,n})$, then whenver $F \in \mathcal{P}_f(\omega)$, $f : F \to \{0, 1, \dots, \min F\}$, $i = f(\min F)$, and $j = f(\max F)$, one has $\sum_{n \in F} x_{f(n),n} \in D_{i,j}$.

Proof. [23, Theorem 2.8]. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

Of course, one cannot get uncountably many pairwise disjoint central subsets of $\mathbb{N}$. However, as a consequence of Theorem 6.8, one can get $\mathfrak{c}$ *almost disjoint* central subsets of $\mathbb{N}$.

Definition 6.6. Let X be an infinite set. A set $\mathcal{A}$ is a *set of almost disjoint subsets of X* if and only if $\mathcal{A} \subseteq \mathcal{P}(X)$, for each $A \in \mathcal{A}$, $|A| = |X|$, and for $A \neq B$ in $\mathcal{A}$, $|A \cap B| < |X|$.

Definition 6.7. Let S be a semigroup with cardinality κ. A subset A of S is a *left solution set of S* if and only if there exist $w, z \in S$ such that $A = \{x \in S : w = zx\}$. We shall say that S is *very weakly left cancellative* if the union of fewer than κ left solution sets of S must have cardinality less than κ.

If $|S| = \kappa$, an ultrafilter $p \in \beta S$ is *uniform* if and only if every member of p has cardinality κ.

Theorem 6.8. *Let κ be an infinite cardinal and let S be a very weakly left cancellative semigroup with cardinality κ, let p be a minimal idempotent of βS which is uniform, and let $C \in p$. If there is a family of μ almost disjoint subsets of κ, then C contains μ almost disjoint sets each of which is a member of a uniform minimal idempotent in βS.*

Proof. [22, Theorem 3.3]. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

7 C-, D-, and J-sets

In this section we address some notions closely related to central sets.

Definition 7.1. Let $(S, \cdot)$ be a semigroup.

(a) A subset A of S is a *J-set* if and only if for each $F \in \mathcal{P}_f(^{\mathbb{N}}S)$, there exist $m \in \mathbb{N}$, $\alpha \in S^{m+1}$, and $t \in \mathcal{J}_m$ such that for each $f \in F$,
$$\left(\prod_{j=1}^{m} \alpha(j) f\big(t(j)\big) \right) \alpha(m+1) \in A.$$

(b) $J(S) = \{p \in \beta S : (\forall A \in p)(A \text{ is a } J\text{-set})\}$.

(c) A subset C of S is a *C-set* if and only if it satisfies the conclusion of Theorem 3.7.

If $(S, +)$ is commutative, the definition of J-set reduces to the assertion that for all $F \in \mathcal{P}_f({}^{\mathbb{N}}S)$ there exist some $a \in S$ and some $H \in \mathcal{P}_f(\mathbb{N})$ such that for all $f \in F$, $a + \sum_{t \in F} f(t) \in A$. And in that case, A is a C-set if and only if it satisfies the conclusion of Theorem 3.5.

Theorem 7.2. *Let S be a semigroup.*

  (a) *$J(S)$ is a compact two sided ideal of βS.*

  (b) *A set $A \subseteq S$ is a C-set if and only if there is an idempotent $p \in J(S) \cap \overline{A}$.*

Proof. (a) [58, Theorem 2.8].

  (b) [58, Theorems 3.2 and 3.5]. $\qquad\square$

We see that, like central sets, J-sets and C-sets are preserved under cartesian products.

Theorem 7.3. *Let S and T be semigroups, let $A \subseteq S$, and let $B \subseteq T$. If A and B are J-sets, so is $A \times B$. If A and B are C-sets, so is $A \times B$.*

Proof. [52, Theorems 2.11 and 2.16]. $\qquad\square$

For the remainder of this section we restrict our attention to $\mathbb{N}$.

Definition 7.4. Let $A \subseteq \mathbb{N}$. Then $d^*(A) = \sup\{\alpha \in [0, 1] : (\forall k \in \mathbb{N})(\exists n \geq k)(\exists a \in \mathbb{N})(|A \cap \{a, a + 1, \ldots, a + n - 1\}| \geq \alpha \cdot n)\}$ and $\Delta^* = \{p \in \beta\mathbb{N} : (\forall A \subseteq \mathbb{N})(\overline{A} \in p \Rightarrow d^*(A) > 0)\}$.

While $d^*(A)$ was introduced by Polya in [62], it is commonly called "Banach density". It is easy to see that Δ^* is a compact two sided ideal of $\beta\mathbb{N}$ and consequently $K(\beta\mathbb{N}) \subseteq \Delta^*$. In particular, if A is a central subset of $\mathbb{N}$, then $d^*(A) > 0$.

Definition 7.5. A set $A \subseteq \mathbb{N}$ is a D-set if and only if there is an idempotent $p \in \Delta^* \cap \overline{A}$.

Theorem 7.6. *If $A \subseteq \mathbb{N}$ and $d^*(A) > 0$, then A is a J-set. In particular $\Delta^* \subseteq J(\mathbb{N})$.*

Proof. This is [6, Theorem 7] where it is referred to as the *IP Szemerédi Theorem*. It is derived as a consequence of [34, Theorem A]. $\qquad\square$

Theorem 7.7. *Let $A \subseteq \mathbb{N}$. If A is a D-set, then A is a C-set.*

Proof. In [6, Theorem 11] it is shown that any D-set satisfies the conclusion of Theorem 3.1. To verify the full conclusion of Theorem 3.7, assume that A is a C-set and pick an idempotent $p \in \Delta^* \cap \overline{A}$. Then by Theorem 7.6, $p \in J(\mathbb{N})$ so by Theorem 7.2(b), A is a C-set. $\qquad\square$

It was shown in [40, Theorem 2.1] that there is a C-set $A \subseteq \mathbb{N}$ such that $d^*(A) = 0$ so C-sets need not be D-sets.

8   Piecewise syndetic sets

Very closely related to the notion of central, is that of *piecewise syndetic*.

Definition 8.1. Let $(S, \cdot)$ be a semigroup and let $A \subseteq S$. Then A is *piecewise syndetic* if and only if there is some $G \in \mathcal{P}_f(S)$ such that for every $F \in \mathcal{P}_f(S)$, there exists $x \in S$ such $Fx \subseteq \bigcup_{t \in G} t^{-1} A$.

In $(\mathbb{N}, +)$ a set A is piecewise syndetic if and only if there exist a bound b and arbitrarily long intervals in which A has no gaps longer than b. The importance of the notion of piecewise syndetic is that $\overline{A} \cap K(\beta S) \neq \emptyset$ if and only if A is piecewise syndetic [53, Theorem 4.40].

Theorem 8.2. *Let $(S, \cdot)$ be a semigroup and let $A \subseteq S$. The following statements are equivalent.*

(a) *A is piecewise syndetic.*

(b) *The set $\{x \in S : x^{-1} A \text{ is central}\}$ is syndetic.*

(c) *There is some $x \in S$ such that $x^{-1} A$ is central.*

Proof. [53, Theorem 4.43]. □

We again see a relationship between the additive and multiplicative structures on $\mathbb{N}$.

Theorem 8.3. *Let $A \subseteq \mathbb{N}$ and assume that $\mathbb{N} \setminus A$ is not piecewise syndetic in $(\mathbb{N}, +)$. Then for all $t \in \mathbb{Z}$, $c\ell K(\beta \mathbb{N}, +) \subseteq \overline{(t + A) \cap \mathbb{N}}$. In particular, for all $t \in \mathbb{Z}$, $(t + A) \cap \mathbb{N}$ is central in $(\mathbb{N}, +)$ and is central in $(\mathbb{N}, \cdot)$.*

Proof. [7, Theorem 3.11]. □

The notions that we have been discussing in this section and Sections 6 and 7 are all one sided notions. Since the notions given go naturally with βS having the right topological structure let me refer to them as "right thick" and so on. We now introduce the left versions of three of these notions.

Definition 8.4. Let $(S, \cdot)$ be a semigroup and let $A \subseteq S$.

(a) The set A is *left thick* if and only if for each $F \in \mathcal{P}_f(S)$ there exists $x \in S$ such that $xF \subseteq A$.

(b) The set A is *left piecewise syndetic* if and only if there is some $G \in \mathcal{P}_f(S)$ such that for every $F \in \mathcal{P}_f(S)$, there exists $x \in S$ such $xF \subseteq \bigcup_{t \in G} At^{-1}$, where $At^{-1} = \{x \in S : xt \in A\}$.

(c) The set A is *left central* if and only if A is a member of a minimal idempotent in βS when βS is given the operation making it a left topological semigroup with ρ_s continuous for each $s \in S$.

Theorem 8.5. *Let κ be a finite or infinite cardinal with $\kappa > 1$ and let S be either the free group or free semigroup on κ generators. There is a set $A \subseteq S$ such that A is left thick (and therefore left central and left piecewise syndetic) but A is not right piecewise syndetic (and therefore not right central and not right thick).*

Proof. [42, Theorems 2.3, 2.4, 2.5, and 2.6]. $\qquad\qquad\square$

9 Characterizations of central sets and related notions

We will begin by presenting combinatorial characterizations of some of the notions related to central, namely C-sets, D-sets, and *quasi-central* sets, the last of which we have not yet defined.

Definition 9.1. Let S be a semigroup and let $A \subseteq S$. Then A is a *quasi-central* set if and only if there is an idempotent in $\overline{A} \cap c\ell\big(K(\beta S)\big)$.

Since $\overline{A} \cap K(\beta S) \neq \emptyset$ if and only if A is piecewise syndetic, we have that A is quasi-central if and only if A is a member of an idempotent, all of whose members are piecewise syndetic.

We thus have three notions which are characterized as follows. There is a set $\mathcal{R} \subseteq \mathcal{P}(S)$ such that A has the specified property if and only if A is a member of an idempotent p such that $p \subseteq \mathcal{R}$. (For C-sets, $\mathcal{R} = \{A \subseteq S : A$ is a J-set$\}$; for D-sets, $\mathcal{R} = \{A \subseteq \mathbb{N} : d^*(A) > 0\}$; for quasi-central sets, $\mathcal{R} = \{A \subseteq S : A$ is piecewise syndetic$\}$.)

The case of Z-sets as quasi-central sets in the following theorem is [49, Theorem 3.7] and the case of Z-sets as C-sets is [51, Theorem 2.7]. I believe that the case of Z-sets as D-sets is new.

Theorem 9.2. *Let $(S, \cdot)$ be a semigroup, let $\mathcal{R} \subsetneq \mathcal{P}(S)$, let $\mathcal{T} = \{p \in \beta S : p \subseteq \mathcal{R}\}$, and assume that*

(1) if $B \cup C \in \mathcal{R}$, then $B \in \mathcal{R}$ or $C \in \mathcal{R}$; and

(2) $\mathcal{T}$ is a subsemigroup of βS.

Say that a subset A of S is a Z-set if and only if A is a member of an idempotent $p \in \mathcal{T}$. Let $A \subseteq S$. Statements (a) and (b) are equivalent and are implied by statement (c). If S is countable, all three statements are equivalent.

(a) *A is a Z-set.*

(b) *There is a downward directed family $\langle C_F \rangle_{F \in I}$ of subsets of A such that*

    (i) *for all $F \in I$ and all $x \in C_F$, there exists $G \in I$ such that $C_G \subseteq x^{-1}C_F$ and*

    (ii) *for each $F \in I$, $C_F \in \mathcal{R}$.*

(c) *There is a decreasing sequence $\langle C_n \rangle_{n=1}^{\infty}$ of subsets of A such that*

    (i) *for all $n \in \mathbb{N}$ and all $x \in C_n$, there exists $m \in \mathbb{N}$ such that $C_m \subseteq x^{-1}C_n$ and*

    (ii) *for all $n \in \mathbb{N}$, $C_n \in \mathcal{R}$.*

Proof. To see that (a) implies (b), pick an idempotent $p \in \mathcal{T} \cap \overline{A}$ and let $A^{\star} = \{t \in S : t^{-1}A \in p\}$. By [53, Lemma 4.14], if $x \in A^{\star}$, then $x^{-1}A^{\star} \in p$. Let $I = \mathcal{P}_f(A^{\star})$ and for $F \in I$, let $C_F = \bigcap_{t \in F} t^{-1}A^{\star}$. Then conclusion (ii) holds directly. To verify (i), let $F \in I$ and let $x \in C_F$. Let $G = Fx$. Then $G \subseteq A^{\star}$ so $G \in I$. To see that $C_G \subseteq x^{-1}C_F$, let $y \in C_G$. To see that $xy \in C_F$, let $t \in F$. Then $tx \in G$ so $txy \in A^{\star}$ so $xy \in C_F$ as required.

To see that (b) implies (a), let $\langle C_F \rangle_{F \in I}$ be as guaranteed by (b). Let $\mathcal{M} = \bigcap_{F \in I} \overline{C_F}$. By condition (i) and [53, Theorem 4.20], $\mathcal{M}$ is a subsemigroup of βS. By assumption (1) and [53, Theorem 3.11], $\mathcal{M} \cap \mathcal{T} \neq \emptyset$ and thus $\mathcal{M} \cap \mathcal{T}$ is a compact semigroup which therefore has an idempotent p. Since $\mathcal{M} \subseteq \overline{A}$, we have that A is a Z-set.

Trivially (c) implies (b). Now assume that S is countable. To see that (a) implies (c), pick an idempotent $p \in \mathcal{T} \cap \overline{A}$ and let $A^{\star} = \{t \in S : t^{-1}A \in p\}$. Enumerate $A^{\star}$ as $\langle t_n \rangle_{n=1}^{\infty}$. For $n \in \mathbb{N}$, let $C_n = \bigcap_{i=1}^{n} t_i^{-1}A^{\star}$. Then conclusion (ii) holds directly. To verify (i), let $n \in \mathbb{N}$ and let $x \in C_n$. Pick $m \in \mathbb{N}$ such that $\{t_1 x, t_2 x, \ldots, t_n x\} \subseteq \{t_1, t_2, \ldots, t_m\}$. As in the proof that (a) implies (b), one easily sees that $C_m \subseteq x^{-1}C_n$. $\qquad\square$

To see that Theorem 9.2 provides combinatorial characterizations of quasi-central sets, D-sets, and C-sets, we need to verify that the relevant sets $\mathcal{R}$ and $\mathcal{T}$ satisfy assumptions (1) and (2). It is elementary to verify that assumption (1) holds for piecewise syndetic sets and sets with positive Banach density. The (nontrivial) verification of (1) for J-sets is [58, Theorem 2.5]. For (2), each of the relevant versions of $\mathcal{T}$ is in fact a two sided ideal. For C-sets this is part of Theorem 7.2. As remarked before, the verification for D-sets is routine. And for quasi-central it is [53, Theorem 4.44].

There is a very similar characterization of central sets. However, it is more complicated and harder to work with. The reason is that the notions we have been discussing are determined by membership in an idempotent in a compact subsemigroup, whereas "central" is determined by membership in an idempotent in $K(\beta S)$, which is almost never compact.

Theorem 9.3. *Let $(S, \cdot)$ be a semigroup and let $A \subseteq S$. Statements (a) and (b) are equivalent and are implied by statement (c). If S is countable, all three statements are equivalent.*

(a) *A is a central set.*

(b) *There is a downward directed family $\langle C_F \rangle_{F \in I}$ of subsets of A such that*

    (i) *for all $F \in I$ and all $x \in C_F$, there exists $G \in I$ such that $C_G \subseteq x^{-1}C_F$ and*

    (ii) *$\{C_F : F \in I\}$ is collectionwise piecewise syndetic.*

(c) *There is a decreasing sequence $\langle C_n \rangle_{n=1}^{\infty}$ of subsets of A such that*

    (i) *for all $n \in \mathbb{N}$ and all $x \in C_n$, there exists $m \in \mathbb{N}$ such that $C_m \subseteq x^{-1}C_n$ and*

    (ii) *$\{C_n : n \in \mathbb{N}\}$ is collectionwise piecewise syndetic.*

Proof. [49, Theorem 3.8]. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

The astute reader may have noticed that I have not defined *collectionwise piecewise syndetic*, nor do I intend to. It is very complicated. The important thing for the proof (which is very similar to the proof of Theorem 9.2) is that a set $\mathcal{A} \subseteq \mathcal{P}(S)$ is collectionwise piecewise syndetic if and only if there is some $p \in K(\beta S)$ with $\mathcal{A} \subseteq p$ [48, Theorem 2.1]. For the actual combinatorial definition, the masochistic reader is referred to [53, Definition 14.19].

Recall that the original definition of central was in terms of a dynamical system, and then central sets were characterized as members of an idempotent in $K(\beta S)$. The notions of D-sets and quasi-central sets were defined as members of idempotents in certain specified subsemigroups. And C-sets, while defined in purely combinatorial terms, are characterized as members of idempotents in $J(S)$. The question of whether they have characterizations in terms of dynamical systems is then natural. A quite general such characterization has been obtained.

Theorem 9.4. *Let $(S, \cdot)$ be a semigroup, let $\mathcal{R} \subsetneq \mathcal{P}(S)$, let $\mathcal{T} = \{p \in \beta S : p \subseteq \mathcal{R}\}$, and assume that*

(1) *if $B \cup C \in \mathcal{R}$, then $B \in \mathcal{R}$ or $C \in \mathcal{R}$; and*

(2) *$\mathcal{T}$ is a subsemigroup of βS.*

Say that a subset A of S is a Z-set if and only if A is a member of an idempotent $p \in \mathcal{T}$. Let $A \subseteq S$. The following statements are equivalent.

(a) A is a Z-set.

(b) There exist a dynamical system $(X, \langle T_s \rangle_{s \in S})$ and points $x, y \in X$ such that

    (i) for each neighborhood V of y, $\{s \in S : T_s(x) \in V \text{ and } T_s(y) \in V\} \in \mathcal{R}$ and

    (ii) there exists a neighborhood U of y such that $A = \{s \in S : T_s(x) \in U\}$.

Proof. [57, Theorem 3.3]. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad$ $\square$

Exactly as with Theorem 9.2, Theorem 9.4 provides dynamical characterizations of quasi-central sets, C-sets, and D-sets. This same characterization of quasi-central sets was obtained earlier in [21, Theorem 3.4].

Note that both of Theorems 9.2 and 9.4 apply to other classes of sets that we have not discussed, like elements of idempotents in $\beta\mathbb{N}$, every member of which has positive upper asymptotic density.

Differently stated dynamical characterizations of C-sets and D-sets in $\mathbb{N}$ were obtained in [59] and [11]. The equivalence of these characterizations with those provided by Theorem 9.4 is not obvious to me.

Theorem 9.5. *Let $A \subseteq \mathbb{N}$. Then A is a C-set if and only if there exist a dynamical system (X, T) and points $x, y \in X$ such that*

(i) for every neighborhood V of y, $\{n \in \mathbb{N} : T^n(y) \in V\}$ is a J-set;

(ii) $(y, y) \in c\ell\{(T^n(x), T^n(y)) : n \in \mathbb{N}\}$; and

(iii) there is an open neighborhood U of y such that $A = \{n \in \mathbb{N} : T^n(x) \in U\}$.

Proof. [59]. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad$ $\square$

Theorem 9.6. *Let $A \subseteq \mathbb{N}$. Then A is a D-set if and only if there exist a dynamical system (X, T) and points $x, y \in X$ such that*

(i) for every neighborhood V of y, $d^*(\{n \in \mathbb{N} : T^n(y) \in V\}) > 0$;

(ii) $(y, y) \in c\ell\{(T^n(x), T^n(y)) : n \in \mathbb{N}\}$; and

(iii) there is an open neighborhood U of (y, y) in $X \times X$ such that $A = \{n \in \mathbb{N} : (T^n(x), T^n(y)) \in U\}$.

Proof. [11, Theorem 2.8]. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad$ $\square$

The contrast between Theorem 9.4 and Theorems 9.5 and 9.6 is not in the fact that the latter theorems assert that U is open. In the proof of the sufficiency of Theorem 9.4, the neighborhood U which is produced is open.

Finally for this section, in [11] is the following dynamical characterization of central sets which is not obviously equivalent to the original.

Theorem 9.7. *Let $A \subseteq \mathbb{N}$. Then A is central if and only if there exist a metrizable dynamical system (X, T) and points $x, y \in X$ such that*

(i) *y is uniformly recurrent;*

(ii) *$(y, y) \in c\ell\{(T^n(x), T^n(y)) : n \in \mathbb{N}\}$; and*

(iii) *there is an open neighborhood U of (y, y) in $X \times X$ such that $A = \{n \in \mathbb{N} : (T^n(x), T^n(y)) \in U\}$.*

Proof. [11, Theorem 2.8]. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

10   When central is equivalent to other notions

We present here some results regarding when a set being central is the same as satisfying some weaker or stronger properties. One of these stronger properties we define now.

Definition 10.1. Let $(S, \cdot)$ be a semigroup. A set $A \subseteq S$ is a *strongly central* set if and only if for every minimal left ideal L of βS, there is an idempotent $p \in L \cap c\ell A$.

Since $K(\beta S)$ is the union of all minimal left ideals, strongly central sets are central. A sequence $\langle x_n \rangle_{n=1}^{\infty}$ in $\mathbb{N}$ satisfies *uniqueness of finite sums* provided that whenever $F, H \in \mathcal{P}_f(\mathbb{N})$ and $\sum_{n \in F} x_n = \sum_{n \in H} x_n$, one must have $F = H$. The easiest way to arrange for this to happen is to require that for each $n \in \mathbb{N}$, $x_{n+1} > \sum_{i=1}^{n} x_i$.

Theorem 10.2. *Let $\langle x_n \rangle_{n=1}^{\infty}$ be a sequence in $\mathbb{N}$ such that for each $n \in \mathbb{N}$, $x_{n+1} > \sum_{i=1}^{n} x_i$. The following statements are equivalent.*

(a) *$FS(\langle x_n \rangle_{n=1}^{\infty})$ is piecewise syndetic.*

(b) *For all $m \in \mathbb{N}$, $FS(\langle x_n \rangle_{n=m}^{\infty})$ is piecewise syndetic.*

(c) *$FS(\langle x_n \rangle_{n=1}^{\infty})$ is strongly central.*

(d) *For all $m \in \mathbb{N}$, $FS(\langle x_n \rangle_{n=m}^{\infty})$ is strongly central.*

(e) *$FS(\langle x_n \rangle_{n=1}^{\infty})$ is syndetic.*

(f) *For all $m \in \mathbb{N}$, $FS(\langle x_n \rangle_{n=m}^{\infty})$ is syndetic.*

(g) *$FS(\langle x_n \rangle_{n=1}^{\infty})$ is central.*

(h) *For all $m \in \mathbb{N}$, $FS(\langle x_n \rangle_{n=m}^{\infty})$ is central.*

(i) *The sequence $\langle x_{n+1} - \sum_{i=1}^{n} x_i \rangle_{n=1}^{\infty}$ is bounded from above.*

Proof. [1, Corollary 4.2]. □

The other results in this brief section deal with words over a finite alphabet.

Definition 10.3. Let A be a finite nonempty set and let $X = {}^{\mathbb{N}}A$. Define $T : X \to X$ by, for $n \in \mathbb{N}$ and $x \in X$, $T(x)(n) = x(n+1)$ and let X have the product topology where A is discrete. A word $w \in X$ is a *uniformly recurrent word* if and only if it is a uniformly recurrent point in the dynamical system (X, T).

Definition 10.4. Let A be a finite nonempty set, let $X = {}^{\mathbb{N}}A$, let $m \in \mathbb{N}$, let $w \in X$ and let $u \in A^m$. Then $w_{|u} = \{n \in \mathbb{N} : w(n) = u(1), w(n+1) = u(2), \ldots, w(n+m-1) = u(m)\}$.

Thus $w_{|u}$ is the set of locations of occurrences of u in w.

Theorem 10.5. *Let A be a finite nonempty alphabet, let w be a uniformly recurrent word in ${}^{\mathbb{N}}A$, and let u be a finite nonempty word over A. Then $w_{|u}$ is an IP set if and only if $w_{|u}$ is a central set.*

Proof. [20, Theorem 3.12]. □

The next result involves a substantial amount of terminology. The reader is referred to [4, Section 2] for the relevant definitions.

Theorem 10.6. *Let τ be an irreducible primitive substitution of Pisot type. Then for any pair of fixed points x and y of τ, the following are equivalent.*

(a) *The points x and y are strongly coincident.*

(b) *There exists a minimal idempotent $p \in \beta\mathbb{N}$ such that $y = p\text{-}\lim_{n \in \mathbb{N}} T^n(x)$, where T is the shift map.*

(c) *For any prefix u of y, $x_{|u}$ is a central set.*

(d) *For any prefix u of y, $x_{|u}$ is an IP set.*

Proof. [4, Theorem 1 and Corollary 1]. □

11 Preservation of largeness

In this section we consider how the notion of central and well as several other related notions are preserved by certain transformations. The first notion we consider is, in terminology introduced in [37], the γ-nonhomogeneous spectrum of α.

Definition 11.1. Let $\alpha > 0$ and $0 < \gamma < 1$. Define $g_{\alpha,\gamma} : \mathbb{N} \to \mathbb{N}$ by, for $n \in \mathbb{N}$, $g_{\alpha,\gamma}(n) = \lfloor \alpha n + \gamma \rfloor$.

The γ-nonhomogeneous spectrum of α is $g_{\alpha,\gamma}[\mathbb{N}]$. These functions provide ways of producing nontrivial examples of central subsets of $\mathbb{N}$. For example, let A be any central subset of $\mathbb{N}$. Then as a consequence of the following theorem we have that $\{\lfloor n\sqrt{10} + \frac{1}{\pi} \rfloor : n \in A\}$ is central and therefore $\{\lfloor 7\lfloor \sqrt{10}n + \frac{1}{\pi} \rfloor + \sqrt{.5} \rfloor : n \in A\}$ is central.

Theorem 11.2. *Let $\alpha > 0$, let $0 < \gamma < 1$ and let $A \subseteq \mathbb{N}$. Let "large" be any of "central", "central*", "IP", "IP*", "strongly central", "J-set", or "C-set". If A is large, then so is $g_{\alpha,\gamma}[A]$.*

Proof. [16, Theorem 6.1] and [41, Theorems 3.4, 4.6, and 4.8]. $\square$

The next result combines spectra with images of polynomials.

Theorem 11.3. *Let $v, m \in \mathbb{N}$ and for $u \in \{1, 2, \ldots, v\}$, let P_u be a polynomial with real coefficients and zero constant term. For $t \in \{1, 2, \ldots, m\}$, let α_t be a positive real and let $0 < \gamma_t < 1$. For $u \in \{1, 2, \ldots, v\}$ let*

$$Q_u = P_u \circ g_{\alpha_1,\gamma_1} \circ \cdots \circ g_{\alpha_m,\gamma_m} .$$

For $x \in \mathbb{R}$, let $w(x) = x - \lfloor x + \frac{1}{2} \rfloor$. The following statements are equivalent.

(a) *Whenever U is an open subset of $(-\frac{1}{2}, \frac{1}{2})^v$ with $\overline{0} \in c\ell U$,*
   $\{x \in \mathbb{N} : \big(w\big(P_1(x)\big), w\big(P_2(x)\big), \ldots, w\big(P_v(x)\big) \big) \in U\}$ *is strongly central.*

(b) *Whenever U is an open subset of $(-\frac{1}{2}, \frac{1}{2})^v$ with $\overline{0} \in c\ell U$,*
   $\{x \in \mathbb{N} : \big(w\big(P_1(x)\big), w\big(P_2(x)\big), \ldots, w\big(P_v(x)\big) \big) \in U\}$ *is central.*

(c) *Whenever U is an open subset of $(-\frac{1}{2}, \frac{1}{2})^v$ with $\overline{0} \in c\ell U$,*
   $\{x \in \mathbb{N} : \big(w\big(P_1(x)\big), w\big(P_2(x)\big), \ldots, w\big(P_v(x)\big) \big) \in U\} \neq \emptyset$.

(d) *Any nontrivial linear combination of $\{P_u : u \in \{1, 2, \ldots, v\}\}$ over $\mathbb{Q}$ has at least one irrational coefficient.*

(e) *Whenever U is an open subset of $(-\frac{1}{2}, \frac{1}{2})^v$ with $\overline{0} \in c\ell U$,*
   $\{x \in \mathbb{N} : \big(w\big(Q_1(x)\big), w\big(Q_2(x)\big), \ldots, w\big(Q_v(x)\big) \big) \in U\}$ *is strongly central.*

(f) *Whenever U is an open subset of $(-\frac{1}{2}, \frac{1}{2})^v$ with $\overline{0} \in c\ell U$,*
   $\{x \in \mathbb{N} : \big(w\big(Q_1(x)\big), w\big(Q_2(x)\big), \ldots, w\big(Q_v(x)\big) \big) \in U\}$ *is central.*

(g) Whenever U is an open subset of $(-\frac{1}{2}, \frac{1}{2})^v$ with $\overline{0} \in c\ell U$,
$$\{x \in \mathbb{N} : \big(w(Q_1(x)), w(Q_2(x)), \ldots, w(Q_v(x))\big) \in U\} \neq \emptyset.$$

Proof. [18, Theorem 2.5]. $\qquad\square$

The next result shows that largeness is preserved by variable words. Recall that the Hales-Jewett Theorem says that if the free semigroup on the finite nonempty alphabet A is divided into finitely many cells, then there is some variable word $w(v)$ such that $\{w(a) : a \in A\}$ is contained in one of these cells.

Theorem 11.4. *Let $l \in \mathbb{N}$, let $A = \{a_1, a_2, \ldots, a_l\}$ be an alphabet on l letters, let S be the free semigroup on the alphabet A, let*

$$HJ_l = \{\big(w(a_1), w(a_2), \ldots, w(a_l)\big) : w(v) \text{ is a variable word}\},$$

and let $B \subseteq S$. Let "large" be any of "piecewise syndetic", "central", "central*", "thick", or "IP*". If B is large in S, then $B^l \cap HJ_l$ is large in HJ_l.

Proof. [15, Corollary 4.3]. $\qquad\square$

12   Central sets in partial semigroups

A *partial multiplication* on a set S is a function taking some subset D of $S \times S$ to S. If $(x, y) \in D$, we write $x * y$ for the value of the function at (x, y) and say that $x * y$ is *defined*. A *partial semigroup* is a pair $(S, *)$ where $*$ is a partial multiplication on S and for all $x, y, z \in S$, $x * (y * z) = (x * y) * z$ in the sense that if either side is defined, then so is the other and they are equal.

Definition 12.1. Let $(S, *)$ be a partial semigroup.

(a) For $F \in \mathcal{P}_f(S)$, let $\sigma(F) = \{y \in S : (\forall x \in F)(x * y \text{ is defined})\}$.

(b) $(S, *)$ is an *adequate* partial semigroup if and only if, for each $F \in \mathcal{P}_f(S)$, $\sigma(F) \neq \emptyset$.

(c) $\delta S = \bigcap_{F \in \mathcal{P}_f(S)} \overline{\sigma(F)}$.

Natural examples of partial semigroups are abundant. For example, let $S = \{f : \big(\exists F \in \mathcal{P}_f(\mathbb{N})\big)(f : F \to \mathbb{N})\}$ and define $f * g = f \cup g$, defining $*$ only when $f \cap g = \emptyset$. Most examples one easily thinks of are adequate. One natural example of a nonadequate partial semigroup is the set of finite matrices over $\mathbb{R}$ with the usual matrix multiplication.

Note that $\delta S \neq \emptyset$ if and only if S is adequate. There is a natural operation, also denoted by $*$, on δS making δS a compact right topological semigroup, and thus δS has all of the structure guaranteed to compact Hausdorff right topological semigroups. (If S is a semigroup, in which case $\delta S = \beta S$, the operations coincide.)

Definition 12.2. Let $(S, *)$ be an adequate partial semigroup and let $A \subseteq S$. Then A is central in S if and only if A is a member of an idempotent $p \in K(\delta S)$.

Theorem 12.4 is a Central Sets Theorem for partial semigroups. (If S is $(\mathbb{N}, +)$, then Theorem 3.1 is the special case in which the functions f are constant.)

Definition 12.3. Let S be an adequate partial semigroup and let $\langle y_n \rangle_{n=1}^{\infty}$ be a sequence in S. Then $\langle y_n \rangle_{n=1}^{\infty}$ is *adequate* if and only if $\prod_{n \in F} y_n$ is defined for each $F \in \mathcal{P}_f(\mathbb{N})$ and for every $K \in \mathcal{P}_f(S)$, there exists $m \in \mathbb{N}$ such that $FP(\langle y_n \rangle_{n=m}^{\infty}) \subseteq \sigma(K)$.

Theorem 12.4. *Let $(S, +)$ be a commutative adequate partial semigroup, let A be a central subset of S, and for each $l \in \{1, 2, \ldots, k\}$, let $\langle y_{l,n} \rangle_{n=1}^{\infty}$ be an adequate sequence in S. There exist a sequence $\langle a_n \rangle_{n=1}^{\infty}$ in S and a sequence $\langle H_n \rangle_{n=1}^{\infty}$ in $\mathcal{P}_f(\mathbb{N})$ such that*

(1) $\max H_n < \min H_{n+1}$ for each $n \in \mathbb{N}$ and

(2) for each $f : \mathbb{N} \to \{1, 2, \ldots, k\}$, $FS(\langle a_n + \sum_{t \in H_n} y_{f(n),t} \rangle_{n=1}^{\infty}) \subseteq A$.

Proof. [60, Theorem 3.3]. $\qquad\square$

We close this section with an application of one of the main results of [30], which was obtained using minimal idempotents in partial semigroups.

For a finite alphabet A let $W(A)$ denote the free semigroup (with identity e) on the alphabet A.

Note that every homomorphism $f : W(A) \to W(A)$ is uniquely determined by its restriction to A. If $A = \{a, b, c\}$ and $x, y, z \in \{a, b, c, e\}$, then let f_{xyz} be the homomorphism from $W(\{a, b, c\})$ to $W(\{a, b, c\})$ that is uniquely determined by $f(a) = x$, $f(b) = y$, and $f(c) = z$.

Theorem 12.5. *For every $r \in \mathbb{N}$ and every partition $W(\{a, b, c\}) = \bigcup_{j=1}^{r} C_j$ there exist an infinite sequence $\langle x_n \rangle_{n=1}^{\infty}$ in $W(\{a, b, c\}) \setminus W(\{a, b\})$ and a function $\gamma : \{a, b, c\} \to \{1, 2, \ldots, r\}$ such that if $\sigma \in \{f_{eab}, f_{aeb}, f_{aab}\}$ and*

$$\mathcal{F} = \{f_{abc}, f_{abb}, f_{aba}, f_{abe}, \sigma\} \cup \{f_{xyz} : x, y, z \in \{a, e\}\},$$

then we have

(1) $\{\prod_{n \in F} g_n(x_n) : F \in \mathcal{P}_f(\mathbb{N}), \text{ and for each } n \in F, g_n \in \mathcal{F}\}$
$\cap \big(W(\{a, b, c\}) \setminus W(\{a, b\})\big) \subseteq C_{\gamma(a)}$;

(2) $\{\prod_{n \in F} g_n(x_n) : F \in \mathcal{P}_f(\mathbb{N}), \text{ and for each } n \in F, g_n \in \mathcal{F}\}$
$\cap \big(W(\{a, b\}) \setminus W(\{a\})\big) \subseteq C_{\gamma(b)}$; and

(3) $\{\prod_{n \in F} g_n(x_n) : F \in \mathcal{P}_f(\mathbb{N}), \text{ and for each } n \in F, g_n \in \mathcal{F}\}$
$\cap \big(W(\{a\}) \setminus \{e\}\big) \subseteq C_{\gamma(c)}$.

Proof. [30, Corollary 3.14]. $\qquad\square$

13 Sets central near zero

We deal in this section with the set of ultrafilters on $(0, 1)$ that converge to 0 in the usual topology. Of course, when dealing with the Stone-Čech compactification, we deal with $(0, 1)_d$, that is $(0, 1)$ with the discrete topology.

Definition 13.1. $0^+ = \bigcap_{\epsilon > 0} c\ell_{\beta(0,1)_d}(0, \epsilon)$.

With the restriction of the operations from $(\beta\mathbb{R}_d, +)$ and $(\beta\mathbb{R}_d, \cdot)$, 0^+ is a semigroup under both addition and multiplication. It is in fact an ideal of $(\beta(0, 1)_d, \cdot)$ and, consequently, $K(0^+, \cdot) = K(\beta(0, 1)_d, \cdot)$. In particular, a set is central in $\big((0, 1), \cdot\big)$ if and only if it is a member of a minimal idempotent in $(0^+, \cdot)$. On the other hand, $0^+ \cap K(\beta(0, \infty)_d, +) = \emptyset$.

Definition 13.2. Let S be a dense subsemigroup of $\big((0, \infty), +\big)$ and let $A \subseteq S$. Then A is *central near zero* if and only if there is an idempotent in $K(0^+, +) \cap \overline{A}$.

In [44, Theorem 4.7], a combinatorial characterization of sets central near zero in $(0, \infty)$ was obtained which is very similar to that of Theorem 9.3. It unfortunately involves the very complicated notion of *collectionwise piecewise syndetic near zero*, and will not be stated here.

The following is a Central Sets Theorem, very similar to Theorem 3.3, valid for sets central near zero.

Theorem 13.3. *Let S be a dense subsemigroup of $\big((0, \infty), +\big)$ and let A be a subset of S which is central near zero. For each $l \in \mathbb{N}$, let $\langle y_{l,n} \rangle_{n=1}^{\infty}$ be a sequence in $S \cup -S \cup \{0\}$ such that $\sum_{n=1}^{\infty} |y_{i,n}|$ converges. There exist sequences $\langle a_n \rangle_{n=1}^{\infty}$ in S and $\langle H_n \rangle_{n=1}^{\infty}$ in $\mathcal{P}_f(\mathbb{N})$ such that*

(1) for each $n \in \mathbb{N}$, $a_n < \frac{1}{n}$ and $\max H_n < \min H_{n+1}$ and

(2) for each $f \in \Phi$, $FS(\langle a_n + \sum_{t \in H_n} y_{f(n),t} \rangle_{n=1}^{\infty}) \subseteq A$.

Proof. [44, Theorem 4.11]. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

There are some results about *image partition regularity near zero*.

Definition 13.4. Let S be a dense subsemigroup of $\big((0, \infty), +\big)$, let $u, v \in \mathbb{N}$, and let M be a $u \times v$ matrix with entries from $\mathbb{R}$. Then M is *image partition regular over S near zero* if and only if, whenever S is finitely colored and $\delta > 0$, there exists $\vec{x} \in S^v$ such that the entries of $A\vec{x}$ are monochromatic and lie in $(-\delta, \delta)$.

We saw in Theorem 4.8 that first entries matrices are image partition regular over $\mathbb{N}$. A subset D of S is *central* near zero* if and only if for every subset C of S which is central near zero, $C \cap D$ is central near zero.

Theorem 13.5. *Let $u, v \in \mathbb{N}$ and let A be a $u \times v$ first entries matrix. Let S be a dense subsemigroup of $\big((0, \infty), +\big)$. If there is a subgroup T of $(\mathbb{R}, +)$ such that $S = T \cap (0, \infty)$, assume that the entries of A come from $\mathbb{Z}$. Otherwise, assume that the entries of A come from ω. Assume that for every first entry c of A, cS is central* near zero. Then A is image partition regular over S near zero if and only if for every set C which is central near zero there exists $\vec{x} \in S^v$ such that $A\vec{x} \in C^u$.*

Proof. [25, Theorem 4.10]. □

The following theorem is more general in that it applies to all image partition regular matrices and allows the entries of the matrix to come from $\mathbb{R}$ and less general in that it is restricted to $S = (0, \infty)$.

Theorem 13.6. *Let $u, v \in \mathbb{N}$ and let A be a $u \times v$ matrix with entries from $\mathbb{R}$. The following statements are equivalent.*

(a) *A is image partition regular over $(0, \infty)$.*

(b) *A is image partition regular over $(0, \infty)$ near zero.*

(c) *For every set C which is central in $(0, \infty)$, there exists $\vec{x} \in (0, \infty)^v$ such that $A\vec{x} \in C^u$.*

(d) *For every set C which is central near zero in $(0, \infty)$, there exists $\vec{x} \in (0, \infty)^v$ such that $A\vec{x} \in C^u$.*

Proof. [27, Theorem 2.3]. □

14 Variations on the Central Sets Theorem

The following variation on Theorem 3.5 applies to sets central near zero.

Theorem 14.1. *Let S be a dense subsemigroup of $\big((0, \infty), +\big)$. Let $\mathcal{T}$ be the set of sequences $\langle y_n \rangle_{n=1}^{\infty}$ in $S \cup \{0\}$ such that $\lim_{n \to \infty} y_n = 0$. Let C be a subset of S which is central near zero. Then there exist $\alpha : \mathcal{P}_f(\mathcal{T}) \to S$ and $H : \mathcal{P}_f(\mathcal{T}) \to \mathcal{P}_f(\mathbb{N})$ such that*

(1) *for each $F \in \mathcal{P}_f(\mathcal{T})$, $\alpha(F) \in (0, \frac{1}{|F|})$;*

(2) *if $F, G \in \mathcal{P}_f(\mathcal{T})$ and $F \subsetneq G$, then $\max H(F) < \min H(G)$; and*

(3) *if $m \in \mathbb{N}$, $G_1, G_2, \ldots, G_m \in \mathcal{P}_f(\mathcal{T})$, $G_1 \subsetneq G_2 \subsetneq \ldots \subsetneq G_m$, and for each $i \in \{1, 2, \ldots, m\}$, $\langle y_{i,t} \rangle_{t=1}^{\infty} \in G_i$, then $\sum_{i=1}^{m} (\alpha(G_i) + \sum_{t \in H(G_i)} y_{i,t}) \in C$.*

Proof. [25, Theorem 4.6]. □

The next variation extends Theorem 3.3 in much the same way as the Milliken-Taylor Theorem extends the Finite Sums Theorem. Given a set X and a cardinal k, let $[X]^k$ be the set of k-element subsets of X.

Theorem 14.2. *Let $(S, +)$ be a commutative cancellative semigroup. For each $l \in \mathbb{N}$, let $\langle y_{l,n} \rangle_{n=1}^{\infty}$ be a sequence in S. Let $k, r \in \mathbb{N}$ and let $[S]^k = \bigcup_{i=1}^{r} A_i$. There exist $i \in \{1, 2, \ldots, r\}$, a sequence $\langle a_n \rangle_{n=1}^{\infty}$ in S, and a sequence $\langle H_n \rangle_{n=1}^{\infty}$ in $\mathcal{P}_f(\mathbb{N})$ such that*

(1) for each $n \in \mathbb{N}$, $\max H_n < \min H_{n+1}$ and

(2) whenever $g \in \Phi$ and $\langle F_s \rangle_{s=1}^{k}$ is a sequence in $\mathcal{P}_f(\mathbb{N})$ with $\max F_s < \min F_{s+1}$ for $s < k$, one has

$$\{\textstyle\sum_{n \in F_1}(a_n + \sum_{t \in H_n} y_{g(n),t}), \ldots, \sum_{n \in F_k}(a_n + \sum_{t \in H_n} y_{g(n),t})\} \in A_i .$$

Proof. [5, Theorem 1.1]. $\qquad\square$

The statement of Theorem 14.2 does not mention central sets, but the proof uses a minimal idempotent.

15 Minimal idempotents

Since central sets are defined as members of minimal idempotents, it seems appropriate to conclude this paper with some algebraic results about minimal idempotents. For the relationship between minimal idempotents and minimal dynamical systems, see [9, Section 2].

Recall that each minimal idempotent p is the identity of the group $L \cap R$, where L is the minimal left ideal to which p belongs and R is the minimal right ideal to which p belongs.

Theorem 15.1. *Let p be a minimal idempotent in $(\beta\mathbb{N}, +)$ and let L and R be respectively the minimal left and minimal right ideals of $(\beta\mathbb{N}, +)$ with $p \in L \cap R$. Then for each $C \in p$, there are $2^{\mathfrak{c}}$ minimal idempotents in $L \cap \overline{C}$ and $2^{\mathfrak{c}}$ minimal idempotents in $R \cap \overline{C}$.*

Proof. [45, Theorem 2.12]. $\qquad\square$

The next result combines algebraic and topological features.

Theorem 15.2. *Let D be a central subset of $\mathbb{N}$. There exists a sequence $\langle p_i \rangle_{i=1}^{\infty}$ of idempotents in $K(\beta\mathbb{N}) \cap \overline{D}$ such that $p_i + p_j \neq p_l + p_m$ whenever $(i, j) \neq (l, m)$, $p_i + p_j + p_l = p_i + p_l$ for all $i, j, l \in \mathbb{N}$, and $\{p_i + p_j : i, j \in \mathbb{N}\}$ is discrete.*

Proof. [23, Theorem 2.6]. $\qquad\square$

If p and q are idempotents in $(\beta S, +)$, then $p \leq_L q$ if and only if $p = p + q$ and $p \leq_R q$ if and only if $p = q + p$. Thus $p \leq q$ if and only if $p \leq_L q$ and $p \leq_R q$. In any semigroup, an idempotent is minimal with respect to any one of these orders if and only if it is minimal with respect to all of them. (Here, for example, to say that p is minimal with respect to $\leq_L$ means that if $q \leq_l p$ one must have $p \leq_L q$.) In the case of βS, if p and q are minimal idempotents with $p \leq_L q$, one has that p and q are in the same minimal left ideal.

Definition 15.3. Let $(S, \cdot)$ be a semigroup.

(a) S is a *left zero semigroup* if and only if for all $x, y \in S$, $xy = x$.

(b) S is a *right zero semigroup* if and only if for all $x, y \in S$, $yx = x$.

(c) S is a *rectangular semigroup* if and only if S is isomorphic to the cartesian product of a left zero semigroup and a right zero semigroup.

Note that any rectangular semigroup consists entirely of idempotents.

Theorem 15.4. *Let D be a semigroup consisting entirely of idempotents. There is a copy of D contained in $K(\beta\mathbb{N}, +)$ if and only if D is rectangular and $|D| \leq 2^{\mathfrak{c}}$.*

Proof. [55, Corollary 3.13]. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Theorem 15.5. *Let L be a left zero semigroup with $2^{\mathfrak{c}}$ elements, let R be a right zero semigroup with $2^{\mathfrak{c}}$ elements, and let F be a free group on $2^{\mathfrak{c}}$ generators. Then $K(\beta\mathbb{N}, +)$ contains an algebraic copy of $L \times R \times F$.*

Proof. [55, Corollary 3.15]. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

It is an easy fact that in any compact Hausdorff right topological semigroup there exist right maximal idempotents. In fact by [53, Theorem 2.12], if p is any idempotent, there is a right maximal idempotent q with $p \leq_R q$. And of course, if q is right maximal, i.e., maximal with respect to $\leq_R$, then q is maximal. It was known that Martin's Axiom implies that there exist left maximal idempotents in $\beta\mathbb{N}$, in fact idempotents that are both left maximal and right maximal. But it has been a longstanding open question whether one could show in ZFC that there are left maximal idempotents in $\beta\mathbb{N}$.

It has also been known for some time that given any nonminimal idempotent in $\beta\mathbb{N}$ there is a strictly decreasing sequence of idempotents below it. (See [53, Theorem 9.23].) But it was not known whether some idempotent might be both minimal and maximal. (I would certainly have bet the house that this was impossible.)

Theorem 15.6. *Let G be a countably infinite discrete group. There is an idempotent $p \in K(\beta G)$ such that p is left maximal. In particular p is both minimal and maximal.*

Proof. This was shown in [66, Corollary 5] assuming Martin's Axiom. It was established in ZFC in [67, Corollary 1.2]. □

References

[1] C. Adams, N. Hindman, and D. Strauss, *Largeness of the set of finite products in a semigroup*, Semigroup Forum **76** (2008), 276-296.

[2] B. Barber, N. Hindman, I. Leader, and D. Strauss, *Distinguishing subgroups of the rationals by their Ramsey properties*, J. Comb. Theory (Series A) **129** (2014), 93-104.

[3] B. Barber, N. Hindman, I. Leader, and D. Strauss, *Partition regularity without the columns property*, Proc. Amer. Math. Soc. **143** (2015), 3387-3399.

[4] M. Barge and L. Zamboni, *Central sets and substitutive dynamical systems*, Adv. Math. **248** (2013), 308-323.

[5] M. Beiglböck, *A multidimensional central sets theorem*, Combin. Probab. Comput. **15** (2006), 807-814.

[6] M. Beiglböck, V. Bergelson, T. Downarowicz, and A. Fish, *Solvability of Rado systems in D-sets*, Topology Appl. **156** (2009), 2565-2571.

[7] M. Beiglböck, V. Bergelson, N. Hindman, and D. Strauss, *Multiplicative structures in additively large sets*, J. Comb. Theory (Series A) **113** (2006), 1219-1242.

[8] M. Beiglböck, V. Bergelson, N. Hindman, and D. Strauss, *Some new results in multiplicative and additive Ramsey Theory*, Trans. Amer. Math. Soc. **360** (2008), 819-847.

[9] V. Bergelson, *Minimal idempotents and ergodic Ramsey theory*, Topics in dynamics and ergodic theory, 8-39, London Math. Soc. Lecture Note Ser. **310**, Cambridge Univ. Press, Cambridge, 2003.

[10] V. Bergelson, A. Blass, and N. Hindman, *Partition theorems for spaces of variable words*, Proc. London Math. Soc. **68** (1994), 449-476.

[11] V. Bergelson and T. Downarowicz, *Large sets of integers and hierarchy of mixing properties of measure preserving systems*, Colloq. Math. **110** (2008), 117-150.

[12] V. Bergelson and N. Hindman, *Nonmetrizable topological dynamics and Ramsey Theory*, Trans. Amer. Math. Soc. **320** (1990), 293-320.

[13] V. Bergelson and N. Hindman, *Ramsey Theory in non-commutative semigroups*, Trans. Amer. Math. Soc. **330** (1992), 433-446.

[14] V. Bergelson and N. Hindman, *On IP*-sets and central sets*, Combinatorica **14** (1994), 269-277.

[15] V. Bergelson and N. Hindman, *Partition regular structures contained in large sets are abundant*, J. Comb. Theory (Series A) **93** (2001), 18-36.

[16] V. Bergelson, N. Hindman, and B. Kra, *Iterated spectra of numbers – elementary, dynamical and algebraic approaches*, Trans. Amer. Math. Soc. **348** (1996), 893-912.

[17] V. Bergelson, N. Hindman, and I. Leader, *Additive and multiplicative Ramsey Theory in the reals and the rationals*, J. Comb. Theory (Series A) **85** (1999), 41-68.

[18] V. Bergelson, N. Hindman, and D. Strauss, *Polynomials at iterated spectra near zero*, Topology and its Applications **158** (2011), 1815-1830.

[19] V. Bergelson and R. McCutcheon, *Central sets and a non-commutative Roth theorem*, Amer. J. Math. **129** (2007), 1251-1275.

[20] M. Bucci, S. Puzynina, and L. Zamboni, *Central sets generated by uniformly recurrent words*, Ergodic Theory Dynam. Systems **35** (2015), 714-736.

[21] S. Burns and N. Hindman, *Quasi-central sets and their dynamical characterization*, Topology Proceedings **31** (2007), 445-455.

[22] T. Carlson, N. Hindman, J. McLeod, and D. Strauss, *Almost disjoint large subsets of semigroups*, Topology and its Applications **155** (2008), 433-444.

[23] T. Carlson, N. Hindman, and D. Strauss, *Ramsey theoretic consequences of some new results about algebra in the Stone-Čech compactification*, Integers **5** (2005), A04.

[24] T. Carlson, N. Hindman, and D. Strauss, *An infinitary extension of the Graham-Rothschild Parameter Sets Theorem*, Trans. Amer. Math. Soc. **358** (2006), 3239-3262.

[25] D. De and N. Hindman, *Image partition regularity near zero*, Discrete Math. **309** (2009), 3219-3232.

[26] D. De, N. Hindman, and D. Strauss, *A new and stronger Central Sets Theorem*, Fundamenta Mathematicae **199** (2008), 155-175.

[27] D. De and R. Paul, *Image partition regularity of matrices near 0 with real entries*, New York J. Math. **17** (2011), 149-161.

[28] W. Deuber, *Partitionen und lineare Gleichungssysteme*, Math. Zeit. **133** (1973), 109-123.

[29] W. Deuber and N. Hindman, *Partitions and sums of (m,p,c)-sets*, J. Comb. Theory (Series A) **45** (1987), 300-302.

[30] I. Farah, N. Hindman, and J. McLeod, *Partition theorems for layered partial semigroups*, J. Comb. Theory (Series A) **98** (2002), 268-311.

[31] H. Furstenberg, *Recurrence in ergodic theory and combinatorical number theory*, Princeton University Press, Princeton, 1981.

[32] H. Furstenberg and B. Weiss, *Topological dynamics and combinatorial number theory*, J. Analyse Math. **34** (1978), 61-85.

[33] H. Furstenberg and Y. Katznelson, *Idempotents in compact semigroups and Ramsey Theory*, Israel J. Math. **68** (1989), 257-270.

[34] H. Furstenberg and Y. Katznelson, *An ergodic Szemerédi theorem for IP-systems and combinatorial theory*, J. Analyse Math. **45** (1985), 117-168.

[35] S. Glasner, *Divisibility properties and the Stone-Čech compactification*, Canad. J. Math. **32** (1980), 993-1007.

[36] R. Graham and B. Rothschild, *Ramsey's Theorem for n-parameter sets*, Trans. Amer. Math. Soc. **159** (1971), 257-292.

[37] R. Graham, S. Lin, and C. Lin, *Spectra of numbers*, Math. Mag. **51** (1978), 174-176.

[38] A. Hales and R. Jewett, *Regularity and positional games*, Trans. Amer. Math. Soc. **106** (1963), 222-229.

[39] N. Hindman, *Image partition regularity over the reals*, New York J. Math., **9** (2003), 79-91.

[40] N. Hindman, *Small sets satisfying the Central Sets Theorem*, Combinatorial number theory, 57-63, Walter de Gruyter, Berlin, 2009.

[41] N. Hindman and J. Johnson, *Images of C sets and related large sets under nonhomogeneous spectra*, Integers **12A** (2012), A2.

[42] N. Hindman, L. Jones, and M. Peters, *Left large subsets of free semigroups and groups that are not right large*, Semigroup Forum **90** (2015), 374-385.

[43] N. Hindman and I. Leader, *Image partition regularity of matrices*, Comb. Prob. and Comp. **2** (1993), 437-463.

[44] N. Hindman and I. Leader, *The semigroup of ultrafilters near* 0, Semigroup Forum **59** (1999), 33-55.

[45] N. Hindman, I. Leader, and D. Strauss, *Infinite partition regular matrices – solutions in central sets*, Trans. Amer. Math. Soc. **355** (2003), 1213-1235.

[46] N. Hindman, I. Leader, and D. Strauss, *Image partition regular matrices – bounded solutions and preservation of largeness*, Discrete Math. **242** (2002), 115-144.

[47] N. Hindman, I. Leader, and D. Strauss, *Extensions of infinite partition regular systems*, Electronic J. Comb. **22(2)** (2015), #P2.29.

[48] N. Hindman and A. Lisan, *Points very close to the smallest ideal of βS*, Semigroup Forum **49** (1994), 137-141.

[49] N. Hindman, A. Maleki, and D. Strauss, *Central sets and their combinatorial characterization*, J. Comb. Theory (Series A) **74** (1996), 188-208.

[50] N. Hindman and D. Strauss, *Infinite partition regular matrices, II – extending the finite results*, Topology Proceedings **25** (2000), 217-255.

[51] N. Hindman and D. Strauss, *A simple characterization of sets satisfying the Central Sets Theorem*, New York J. Math. **15** (2009), 405-413.

[52] N. Hindman and D. Strauss, *Cartesian products of sets satisfying the Central Sets Theorem*, Topology Proceedings **35** (2010), 203-223.

[53] N. Hindman and D. Strauss, *Algebra in the Stone-Čech compactification: theory and applications, second edition*, de Gruyter, Berlin, 2012.

[54] N. Hindman and D. Strauss, *Separating Milliken-Taylor systems in $\mathbb{Q}$*, J. Combinatorics **5** (2014), 305-333.

[55] N. Hindman, D. Strauss, and Y. Zelenyuk, *Large rectangular semigroups in Stone-Čech compactifications*, Trans. Amer. Math. Soc. **355** (2003), 2795-2812.

[56] N. Hindman and W. Woan, *Central sets in semigroups and partition regularity of systems of linear equations*, Mathematika **40** (1993), 169-186.

[57] J. Johnson, *A dynamical characterization of C-sets*, arXiv:1112.0715.

[58] J. Johnson, *A new and simpler noncommutative central sets theorem*, Topology Appl. **189** (2015), 10-24.

[59] J. Li, *Dynamical characterization of C-sets and its application*, Fund. Math. **216** (2012), 259-286.

[60] J. McLeod, *Central sets in commutative adequate partial semigroups*, Topology Proc. **29** (2005), 567-576.

[61] K. Milliken, *Ramsey's Theorem with sums or unions*, J. Comb. Theory (Series A) **18** (1975), 276-290.

[62] G. Polya, *Untersuchungen über Lücken und Singularitaten von Potenzreihen*, Math. Zeit. **29** (1929), 549-640.

[63] R. Rado, *Studien zur Kombinatorik*, Math. Zeit. **36** (1933), 242-280.

[64] H. Shi and H. Yang, *Nonmetrizable topological dynamical characterization of central sets*, Fund. Math. **150** (1996), 1-9.

[65] A. Taylor, *A canonical partition relation for finite subsets of ω*, J. Comb. Theory (Series A) **21** (1976), 137-146.

[66] Y. Zelenyuk, *Principal left ideals of βG may be both minimal and maximal*, Bull. London Math. Soc. **45** (2013), 613-617.

[67] Y. Zelenyuk, *Left maximal idempotents in G^**, Adv. Math. **262** (2014), 593-603.