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CARTESIAN PRODUCTS OF SETS SATISFYING THE CENTRAL SETS THEOREM

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ABSTRACT. *Central* subsets of a discrete semigroup S have very strong combinatorial properties which are a consequence of the *Central Sets Theorem*. We show here that, not only is the Cartesian product of two central sets central, but in fact the Cartesian product of any two sets satisfying the conclusion of the Central Sets Theorem satisfies the conclusion of the Central Sets Theorem. Intimately related to the notion of a central set is something we call a J-set. These sets have many of the combinatorial properties of central sets and we show that this notion is also preserved under finite Cartesian products. Finally, we characterize when the Cartesian product of infinitely many sets is central.

1. INTRODUCTION

Central subsets of the set $\mathbb{N}$ of positive integers were introduced by H. Furstenberg [5], where they were defined in terms of notions from topological dynamics. Furstenberg showed that if $\mathbb{N}$ is partitioned into finitely many cells, one of these cells must be central, and he proved the original Central Sets Theorem.

Given a set X we write $\mathcal{P}_f(X)$ for the set of finite nonempty subsets of X. Given sets X and Y we write ${}^{X}Y$ for the set of functions from X to Y.

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Theorem 1.1 (Original Central Sets Theorem)**.** *Let A be a central subset of $\mathbb{N}$ and let $F \in \mathcal{P}_f\big({}^{\mathbb{N}}\mathbb{Z}\big)$. There exist a sequence $\langle a_n\rangle_{n=1}^\infty$ in $\mathbb{N}$ and a sequence $\langle H_n\rangle_{n=1}^\infty$ in $\mathcal{P}_f(\mathbb{N})$ such that*

(1) *for each $n \in \mathbb{N}$, $\max H_n < \min H_{n+1}$ and*
(2) *for each $K \in \mathcal{P}_f(\mathbb{N})$ and each $f \in F$,*
$\sum_{n\in K}\big(a_n + \sum_{t\in H_n} f(t)\big) \in A$.

Proof. [5, Proposition 8.21]. □

Central subsets of $\mathbb{N}$ have remarkably strong combinatorial properties, many of which were derived in [5]. For example, any central set contains solutions to any partition regular system of homogeneous linear equations as well as

$$FS(\langle x_n\rangle_{n=1}^\infty) = \{\textstyle\sum_{n\in F} x_n : F \in \mathcal{P}_f(\mathbb{N})\}$$

for some sequence $\langle x_n\rangle_{n=1}^\infty$.

V. Bergelson suggested that we could prove Theorem 1.1 for sets A which were members of minimal idempotents in the semigroup $(\beta\mathbb{N}, +)$ and he was right. (The algebraic structure of the Stone-Čech compactification of a discrete semigroup will be briefly described later in this introduction.) The dynamical definition of central can be extended to apply to an arbitrary semigroup. With the assistance of B. Weiss, it was shown [2] that for countable semigroups S a subset C of S satisfies the dynamical definition of central if and only if C is a member of a minimal idempotent in βS. (A portion of this argument was anticipated by S. Glasner in [6].) Later H. Shi and H. Yang [13] established the same equivalence for arbitrary semigroups. Furthermore, many of the strong combinatorial properties apply to central subsets of arbitrary semigroups. See [9, Part III] for a summary of some of these results. We will formally define central sets as members of minimal idempotents once we have described what this means.

Recently a new and stronger version of the Central Sets Theorem was obtained. The following is this theorem as applied to commutative semigroups. The full version for arbitrary semigroups will be stated in Section 2.

Theorem 1.2. *Let $(S, +)$ be a commutative semigroup and let $\mathcal{T} = {}^{\mathbb{N}}S$, the set of sequences in S. Let A be a central subset of S. There exist functions $\alpha : \mathcal{P}_f(\mathcal{T}) \to S$ and $H : \mathcal{P}_f(\mathcal{T}) \to \mathcal{P}_f(\mathbb{N})$ such that*

(1) *if* $F, G \in \mathcal{P}_f(\mathcal{T})$ *and* $F \subsetneq G$, *then* $\max H(F) < \min H(G)$ *and*

(2) *whenever* $n \in \mathbb{N}$, $G_1, G_2, \ldots, G_n \in \mathcal{P}_f(\mathcal{T})$, $G_1 \subsetneq G_2 \subsetneq \ldots \subsetneq G_m$, *and for each* $i \in \{1, 2, \ldots, n\}$, $f_i \in G_i$, *one has* $\sum_{i=1}^{n} \big(\alpha(G_i) + \sum_{t \in H(G_i)} f_i(t)\big) \in A$.

Proof. [4, Theorem 2.2]. □

Definition 1.3. Let $(S, +)$ be a commutative semigroup and let $A \subseteq S$. Then A is a C-set if and only if A satisfies the conclusion of Theorem 1.2.

Closely related to the concept of a C-set is the notion of J-set.

Definition 1.4. Let $(S, +)$ be a commutative semigroup and let $A \subseteq S$. Then A is a J-*set* if and only if whenever $F \in \mathcal{P}_f\big({}^{\mathbb{N}}S\big)$, there exist $a \in S$ and $H \in \mathcal{P}_f(\mathbb{N})$ such that for all $f \in F$, $a + \sum_{t \in H} f(t) \in A$.

The set $\mathcal{J}$ of ultrafilters, every member of which is a J-set, is a two sided ideal of βS, and a subset of S is a C-set if and only if it is a member of an idempotent in $\mathcal{J}$ [4, Theorems 3.5 and 3.8].

One has immediately that every central set is a C-set and every C-set is a J-set. Trivially the set of odd positive integers is a J-set which is not a C-set. And, as a consequence of [4, Theorem 3.8] and [8, Theorem 5.5], there is a subset of $\mathbb{N}$ which is a C-set but not a central set.

Notice that J-sets are already guaranteed to have substantial combinatorial content. For example, if A is a J-set in $(\mathbb{N}, +)$ and $\langle x_n \rangle_{n=1}^{\infty}$ is a sequence in $\mathbb{N}$, then A contains arbitrarily long arithmetic progressions with increment in $FS(\langle x_n \rangle_{n=1}^{\infty})$. (For $k, t \in \mathbb{N}$, define $f_k(t) = k \cdot x_t$ and for $l \in \mathbb{N}$ apply the definition of J-set to $F_l = \{f_1, f_2, \ldots, f_l\}$.)

If S is any commutative semigroup and A is a subset of S such that there is a left invariant mean μ on S with $\mu(\overline{A}) > 0$, then A is a J-set [11, Theorem 6.10].

We are concerned in this paper with Cartesian products of central sets, J-sets, and C-sets. In Section 2 we shall state the version of the (new) Central Sets Version which applies to possibly noncommutative semigroups. We will extend the definitions of J-sets and C-sets to arbitrary semigroups, and prove that for arbitrary semigroups, the product of two central sets is a central set, the product of two J-sets is a J-set, and the product of two C-sets is a C-set.

In Section 3 we will characterize precisely when the arbitrary Cartesian product of sets is a central set, as well as when such a product is *piecewise syndetic*. We shall also show that this characterization is not valid for J-sets or C-sets.

We now present our promised introduction to the algebra of βS. Let $(S,\cdot)$ be an infinite discrete semigroup. The Stone-Čech compactification βS of S is the set of ultrafilters on S, the principal ultrafilters being identified with the points of S. Given $A \subseteq S$ and $p \in \beta S$, $p \in c\ell A = \overline{A}$ if and only if $A \in p$. The operation extends to βS making $(\beta S,\cdot)$ a right topological semigroup (meaning that for each $p \in \beta S$ the function $\rho_p : \beta S \to \beta S$ defined by $\rho_p(q) = q \cdot p$ is continuous) with S contained in its topological center (meaning that for each $x \in S$ the function $\lambda_x : \beta S \to \beta S$ defined by $\lambda_x(q) = x \cdot q$ is continuous). Given $p,q \in \beta S$ and $A \subseteq S$, $A \in p \cdot q$ if and only if $\{x \in S : x^{-1}A \in q\} \in p$, where $x^{-1}A = \{y \in S : x \cdot y \in A\}$.

Like any compact topological semigroup, βS has a smallest two sided ideal $K(\beta S)$ which is the union of all of the minimal left ideals of βS and is also the union of all of the minimal right ideals of βS. The intersection of any minimal left ideal with any minimal right ideal is a group, and in particular there are idempotents in $K(\beta S)$. Such idempotents are said to be *minimal*. See [9] for an elementary introduction to the algebraic structure of βS as well as unfamiliar algebraic facts used here.

When we write "let S be a semigroup", we shall assume the operation is denoted by $\cdot$ (or simply by juxtaposition) unless we specify otherwise.

Definition 1.5. Let S be a semigroup and let $A \subseteq S$. Then A is *central* if and only if there is an idempotent $p \in K(\beta S)$ such that $A \in p$.

2. Finite Cartesian products of central sets, J-sets, and C-sets

We begin with the simple proof that the product of two central sets is central. Given the simplicity of its proof, it is surprising that no one (including us) seems to have noticed it before.

Theorem 2.1. *Let S and T be infinite discrete semigroups. Let $p \cdot p = p \in K(\beta S)$ and let $q \cdot q = q \in K(\beta T)$. Let*

$$\widetilde{\iota} : \beta(S \times T) \to \beta S \times \beta T$$

be the continuous extension of the identity function on $S \times T$ and let $M = \widetilde{\iota}^{-1}[\{(p,q)\}]$. Then M is a compact subsemigroup of $\beta(S \times T)$ and $K(M) \subseteq K\big(\beta(S \times T)\big)$.

Proof. We have that $\widetilde{\iota}$ is surjective and by [9, Corollary 4.22], (due to P. Milnes in [12]) $\widetilde{\iota}$ is a homomorphism. Consequently M is a compact subsemigroup of $\beta(S \times T)$. By [9, Exercise 1.7.3], $\widetilde{\iota}\big[K\big(\beta(S \times T)\big)\big] = K(\beta S \times \beta T)$. Also, by [9, Theorem 2.23],

$$K(\beta S \times \beta T) = K(\beta S) \times K(\beta T)$$

and so $(p,q) \in K(\beta S \times \beta T)$. Consequently $K\big(\beta(S \times T)\big) \cap M \neq \emptyset$ and therefore $K\big(\beta(S \times T)\big) \cap M$ is an ideal of M and so $K(M) \subseteq K\big(\beta(S \times T)\big) \cap M$. (In fact equality holds by [9, Theorem 1.65].) $\square$

Corollary 2.2. *Let S and T be infinite discrete semigroups, let A be a central subset of S and let B be a central subset of T. Then $A \times B$ is a central subset of $S \times T$.*

Proof. Pick $p = p \cdot p \in K(\beta S)$ and $q = q \cdot q \in K(\beta T)$ such that $A \in p$ and $B \in q$. Let M be as in Theorem 2.1 and pick $r = r \cdot r \in K(M)$. Then $\widetilde{\iota}(r) = (p,q)$ and so $A \times B \in r$. Since $r \in K\big(\beta(S \times T)\big)$ we have that $A \times B$ is central. $\square$

We extend the definitions of J-sets and C-sets to arbitrary semigroups and show that the Cartesian product of two J-sets is a J-set and the Cartesian product of two C-sets is a C-set. We introduce some special notation. The notation does not reflect all of the variables upon which it depends. In a noncommutative semigroup, when we write $\prod_{t \in F} x_t$ we mean the product taken in increasing order of indices.

We use $\prod$ for algebraic products and use $\times$ to denote Cartesian products.

Definition 2.3. Let S be a semigroup.

(a) $\mathcal{T} = {}^{\mathbb{N}}S$.
(b) For $m \in \mathbb{N}$, $\mathcal{I}_m = \{\big(H(1), H(2), \ldots, H(m)\big) :$ each $H(j) \in \mathcal{P}_f(\mathbb{N})$ and for any $j \in \{1, 2, \ldots, m-1\}$, $\max H(j) < \min H(j+1)\}$.
(c) Given $m \in \mathbb{N}$, $a \in S^{m+1}$, $H \in \mathcal{I}_m$, and $f \in \mathcal{T}$,

$$x(m, a, H, f) = \Big(\prod_{j=1}^{m} \big(a(j) \cdot \prod_{t \in H(j)} f(t)\big)\Big) \cdot a(m+1)\,.$$

(d) $A \subseteq S$ is a *J-set* if and only if for each $F \in \mathcal{P}_f(\mathcal{T})$ there exist $m \in \mathbb{N}$, $a \in S^{m+1}$, and $H \in \mathcal{I}_m$ such that for each $f \in F$, $x(m, a, H, f) \in A$.
(e) $J(S) = \{p \in \beta S : \text{for all } A \in p,\ A \text{ is a } J\text{-set}\}$.
(f) $A \subseteq S$ is a *C-set* if and only if there exist $m : \mathcal{P}_f(\mathcal{T}) \to \mathbb{N}$, $\alpha \in \times_{F \in \mathcal{P}_f(\mathcal{T})} S^{m(F)+1}$, and $H \in \times_{F \in \mathcal{P}_f(\mathcal{T})} \mathcal{I}_{m(F)}$ such that
  (1) if $F, G \in \mathcal{P}_f(\mathcal{T})$ and $F \subsetneq G$, then $\max H(F)\big(m(F)\big) < \min H(G)(1)$ and
  (2) whenever $n \in \mathbb{N}$, $G_1, G_2, \ldots, G_n \in \mathcal{P}_f(\mathcal{T})$, $G_1 \subsetneq G_2 \subsetneq \ldots \subsetneq G_n$, and for each $i \in \{1, 2, \ldots, n\}$, $f_i \in G_i$, one has $\prod_{i=1}^{n} x(m(G_i), \alpha(G_i), H(G_i), f_i) \in A$.

In [4] the definition of a J-set was superficially stronger because it required that for each $n \in \mathbb{N}$, there exist m, a and H as in Definition 2.3(d) with the additional requirement that $\min H(1) > n$. That version can be seen to be equivalent to the one given here by replacing each $f \in F$ by the function g_f defined by $g_f(k) = f(n+k)$.

We need to note at this point that if S is commutative, the definitions of J-set and C-set given here agree with those given earlier. If $A \subseteq S$ is a J-set as defined by Definition 2.3 it is clearly also a J-set as defined by Definition 1.4. The converse is not quite so trivial because, given that $a \cdot \prod_{t \in H} f(t) \in A$, one does not know that $a \in S \cdot S$.

Lemma 2.4. *Let S be a commutative semigroup and let $A \subseteq S$. Then A is a J-set as defined by Definition* 1.4 *if and only if A is a J-set as defined by Definition* 2.3. *Also A is a C-set as defined by Definition* 1.3 *if and only if A is a C-set as defined by Definition* 2.3.

Proof. We have already remarked that the sufficiency of the statement for J-sets is trivial. Assume that A is a J-set as defined by Definition 1.4. Let $F \in \mathcal{P}_f(\mathcal{T})$. Pick $c \in S$ and for $f \in F$, define $g_f \in \mathcal{T}$ by $g_f(n) = c \cdot f(n)$. Pick $b \in S$ and $H \in \mathcal{P}_f(\mathbb{N})$ such that for each $f \in F$, $b \cdot \prod_{t \in H} g_f(t) \in A$. Let $k = |H|$, let $m = 1$, let $H(1) = H$, let $a(1) = b$, and let $a(2) = c^k$.

The proof of the assertion about C-sets is essentially the same, though notationally more cumbersome. □

As promised earlier, we state what is currently the most general version of the Central Sets Theorem.

Theorem 2.5. *Let S be an infinite semigroup and let A be a central subset of S. Then A is a C-set.*

Proof. [4, Corollary 3.10]. □

We now turn our attention to proving that the product of two J-sets is a J-set. To accomplish this, we use the notion of an *adequate partial semigroup*.

Definition 2.6. (a) A *partial semigroup* is a set S together with an operation $*$ that maps a subset D of $S \times S$ into S and satisfies the associative law $(x * y) * z = x * (y * z)$ in the sense that if either side is defined then so is the other and they are equal.

(b) Given a partial semigroup S, the set $D \subseteq S \times S$ on which $*$ is defined, and $x \in S$, $\varphi(x) = \{y \in S : (x, y) \in D\}$.

(c) The partial semigroup S is *adequate* if and only if for each $F \in \mathcal{P}_f(S)$, $\bigcap_{x \in F} \varphi(x) \neq \emptyset$.

(d) If S is an adequate partial semigroup, then $\delta S = \bigcap_{x \in S} c\ell_{\beta S} \varphi(x)$.

(e) Let S be a partial semigroup, let $A \subseteq S$, and let $x \in S$. Then $x^{-1}A = \{y \in \varphi(x) : x * y \in A\}$.

(f) Let S be an adequate partial semigroup and let $p, q \in \delta S$. Then $p * q = \{A \subseteq S : \{x \in S : x^{-1}A \in q\} \in p\}$.

Lemma 2.7. *Let S be an adequate partial semigroup. Then $(\delta S, *)$ is a compact right topological semigroup.*

Proof. [1, Proposition 2.6]. □

We shall need the following extension of [9, Lemma 4.14].

Lemma 2.8. *Let S be an adequate partial semigroup, let $p * p = p \in \delta S$, let $A \in p$, and let $A^\star = \{x \in A : x^{-1}A \in p\}$. If $x \in A^\star$, then $x^{-1}A^\star \in p$.*

Proof. Let $x \in A^\star$ and let $B = x^{-1}A$. Then $B \in p$ so $B^\star \in p$. We claim that $B^\star \subseteq x^{-1}A^\star$. Let $y \in B^\star$. Then $y \in B$ so $y \in \varphi(x)$ and $x * y \in A$. We need to show that $(x * y)^{-1}A \in p$. Since $y \in B^\star$, $y^{-1}B \in p$ so it suffices to show that $y^{-1}B \subseteq (x * y)^{-1}A$. So let $z \in y^{-1}B$. Then $z \in \varphi(y)$ and $y * z \in B = x^{-1}A$ so $y * z \in \varphi(x)$ and $x * (y * z) \in A$. Therefore, $z \in \varphi(x * y)$ and $(x * y) * z = x * (y * z) \in A$ as required. □

Definition 2.9. Let $\mathcal{I} = \bigcup_{m=1}^{\infty} \mathcal{I}_m$ and define a partial operation on $\mathcal{I}$ by

$$\begin{aligned}&\big(H(1), H(2), \ldots, H(m)\big) * \big(K(1), K(2), \ldots, K(n)\big) = \\ &\big(H(1), H(2), \ldots, H(m), K(1), K(2), \ldots, K(n)\big)\end{aligned}$$

provided $\max H(m) < \min K(1)$ with the operation undefined otherwise.

It is trivial to verify that $(\mathcal{I}, *)$ is an adequate partial semigroup and that $\delta\mathcal{I} = \bigcap_{n=1}^{\infty} c\ell_{\beta\mathcal{I}}\{H \in \mathcal{I} : \min H(1) > n\}$.

Lemma 2.10. *Let S be a semigroup, let A be a J-set in S, and let $F \in \mathcal{P}_f\big({}^{\mathbb{N}}S\big)$. Let*

$$\begin{aligned}\mathcal{B} = \ &\big\{\big(H(1), H(2), \ldots, H(m)\big) \in \mathcal{I} : \\ &(\exists a \in S^{m+1})(\forall f \in F)(x(m, a, H, f) \in A)\big\}.\end{aligned}$$

Let $p = p * p \in \delta\mathcal{I}$. Then $\mathcal{B} \in p$.

Proof. Suppose $\mathcal{B} \notin p$. Let $\mathcal{C} = \mathcal{I} \setminus \mathcal{B}$. We claim that we can choose a sequence $\langle m(n)\rangle_{n=1}^{\infty}$ in $\mathbb{N}$ and a sequence $\langle H(n)\rangle_{n=1}^{\infty}$ such that for each $n \in \mathbb{N}$, $H(n) \in \mathcal{I}_{m(n)}$ and $\max H(n)\big(m(n)\big) < \min H(n+1)(1)$, and $FP(\langle H(n)\rangle_{n=1}^{\infty}) \subseteq \mathcal{C}$, where $FP(\langle H(n)\rangle_{n=1}^{\infty}) = \{\prod_{n\in K} H(n) : K \in \mathcal{P}_f(\mathbb{N})\}$.

By Lemma 2.8 pick $H(1) \in \mathcal{C}^\star$ and choose $m(1) \in \mathbb{N}$ such that $H(1) \in \mathcal{I}_{m(1)}$. Inductively let $n \in \mathbb{N}$ and assume that we have chosen $\langle m(t)\rangle_{t=1}^{n}$ in $\mathbb{N}$ and $\langle H(t)\rangle_{t=1}^{n}$ with each $H(t) \in \mathcal{I}_{m(t)}$ such that whenever $\emptyset \neq K \subseteq \{1, 2, \ldots, n\}$, $\prod_{t\in K} H(t) \in \mathcal{C}^\star$. Let $\mathcal{E} = \big\{\prod_{t\in K} H(t) : \emptyset \neq K \subseteq \{1, 2, \ldots, n\}\big\}$. Let $\mathcal{D} = \mathcal{C}^\star \cap \bigcap_{B\in\mathcal{E}} B^{-1}\mathcal{C}^\star$. Then $\mathcal{D} \in p$ so pick $H(n+1) \in \mathcal{D}$ and pick $m(n+1)$ such that

$H(n+1) \in \mathcal{I}_{m(n+1)}$. The induction being complete, the claim is established.

Pick $d \in S$. For $f \in F$, define $g_f \in {}^{\mathbb{N}}S$ by

$$g_f(n) = \prod_{t \in H(n)(1)} f(t)$$

if $m(n) = 1$ and

$$g_f(n) = \prod_{t \in H(n)(1)} f(t) \cdot \prod_{l=2}^{m(n)} \big(d \cdot \prod_{t \in H(n)(l)} f(t)\big)$$

if $m(n) > 1$. Pick $k \in \mathbb{N}$, $G \in \mathcal{I}_k$, and $a \in S^{k+1}$ such that for each $f \in F$, $x(k, a, G, g_f) \in A$. Let $s_1 = 1$ and for $i \in \{1, 2, \ldots, k\}$, let $s_{i+1} = s_i + \sum_{n \in G(i)} m(n)$. For $i \in \{1, 2, \ldots, k+1\}$, let $b(s_i) = a(i)$ and for $i \in \{1, 2, \ldots, k\}$ and $s_i < j < s_{i+1}$, let $b(j) = d$. Let

$$\begin{aligned}&\big(K(1), K(2), \ldots, K(s_{k+1} - 1)\big) = \\ &\prod_{i=1}^{k} \prod_{n \in G(i)} \Big(H(n)(1), H(n)(2), \ldots, H(n)\big(m(n)\big)\Big),\end{aligned}$$

the last product being computed in $\mathcal{I}$. Then

$$\big(K(1), K(2), \ldots, K(s_{k+1} - 1)\big) \in FP(\langle H(n) \rangle_{n=1}^{\infty})$$

and for each $f \in F$, $x(s_{k+1} - 1, b, K, f) = x(k, a, G, g_f) \in A$ so $FP(\langle H(n) \rangle_{n=1}^{\infty}) \cap \mathcal{B} \neq \emptyset$, a contradiction. $\square$

Theorem 2.11. *Let S and T be semigroups, let A be a J-set in S, and let B be a J-set in T. Then $A \times B$ is a J-set in $S \times T$.*

Proof. Let $F \in \mathcal{P}_f\big({}^{\mathbb{N}}(S \times T)\big)$. Pick $p = p * p \in \delta\mathcal{I}$. Let

$$\begin{aligned}\mathcal{B} = \ &\big\{\big((H(1), H(2), \ldots, H(m)\big) \in \mathcal{I} : \\ &(\exists a \in S^{m+1})(\forall f \in F)\big(x(m, a, H, \pi_1 \circ f) \in A\big)\big\}\end{aligned}$$

and let

$$\begin{aligned}\mathcal{C} = \ &\big\{\big((H(1), H(2), \ldots, H(m)\big) \in \mathcal{I} : \\ &(\exists b \in T^{m+1})(\forall f \in F)\big(x(m, a, H, \pi_2 \circ f) \in B\big)\big\}.\end{aligned}$$

Then $\mathcal{B} \cap \mathcal{C} \in p$ so pick $\big(H(1), H(2), \ldots, H(m)\big) \in \mathcal{B} \cap \mathcal{C}$. Let a and b be as guaranteed by the definitions of $\mathcal{B}$ and $\mathcal{C}$ respectively. Then given $f \in F$, $x(m, (a, b), H, f) \in A \times B$. $\square$

Finally, we shall show in Theorem 2.16 that the Cartesian product of two C-sets is a C-set. We need the following result from [4].

Theorem 2.12. *Let S be an infinite semigroup. Then $J(S)$ is an ideal of βS. A subset A of S is a C-set if and only if there is an idempotent $p \in J(S)$ such that $A \in p$.*

Proof. [4, Theorems 3.5 and 3.8]. $\square$

In the process of proving that the Cartesian product of two C-sets is a C-sets, we shall use the fact that the property of being a J-set is partition regular, a fact which we feel is interesting in its own right.

Lemma 2.13. *Let S be an infinite semigroup, let A be a J-set in S, let $F \in \mathcal{P}_f\big({}^{\mathbb{N}}S\big)$, and let $\langle H_n \rangle_{n=1}^{\infty}$ be a sequence in $\mathcal{P}_f(\mathbb{N})$ such that for each n, $\max H_n < \min H_{n+1}$. There exist $m \in \mathbb{N}$, $a \in S^{m+1}$, and $G \in \mathcal{I}_m$ such that for all $f \in F$,*

$$\prod_{j=1}^{m} \big(a(j) \cdot \prod_{k \in G(j)} \prod_{t \in H_k} f(t)\big) \cdot a(m+1) \in A\,.$$

Proof. For $f \in F$, define $g_f \in {}^{\mathbb{N}}S$ by $g_f(k) = \prod_{t \in H_k} f(t)$. Pick $m \in \mathbb{N}$, $a \in S^{m+1}$, and $G \in \mathcal{I}_m$ as guaranteed by Definition 2.3(d) for $\{g_f : f \in F\}$. $\square$

Theorem 2.14. *Let S be an infinite semigroup, let A be a J-set in S, and assume that $A = A_1 \cup A_2$. Either A_1 is a J-set in S or A_2 is a J-set in S.*

Proof. It suffices to let $F \in \mathcal{P}_f\big({}^{\mathbb{N}}S\big)$ and show that there exist $i \in \{1,2\}$, $u \in \mathbb{N}$, $c \in S^{m+1}$, and $K \in \mathcal{I}_u$ such that for all $f \in F$, $\prod_{j=1}^{u} \big(c(j) \cdot \prod_{t \in K(j)} f(t)\big) \cdot c(m+1) \in A_i$. (For if F_1 were a witness to the fact that A_1 is not a J-set and F_2 were a witness to the fact that A_2 is not a J-set, then $F = F_1 \cup F_2$ would fail to satisfy the above statement.) So let $F \in \mathcal{P}_f\big({}^{\mathbb{N}}S\big)$ be given.

Let $k = |F|$ and write $F = \{f_1, f_2, \ldots, f_k\}$. Pick by the Hales-Jewett Theorem ([7] or see [9, Section 14.2]) some $n \in \mathbb{N}$ such that whenever the length n words over the alphabet $\{1,2,\ldots,k\}$ are 2-colored, there is a variable word $w(v)$ such that $\big\{w(j) : j \in \{1,2,\ldots,k\}\big\}$ is monochromatic. By replacing $n+2$ by n, we can (and shall) assume that $w(v)$ begins and ends with a constant. (The only reason for doing this is to make it easier to compute the number u below.)

Let W be the set of length n words over $\{1,2,\ldots,k\}$. For $w = b_1 b_2 \cdots b_n \in W$ (where each $b_i \in \{1,2,\ldots,k\}$) define $g_w : \mathbb{N} \to S$

by, for $l \in \omega = \mathbb{N} \cup \{0\}$ and $i \in \{1, 2, \ldots, n\}$, $g_w(ln+i) = f_{b_i}(ln+i)$. For $l \in \omega$, let $H_l = \{ln+1, ln+2, \ldots, ln+n\}$. Pick by Lemma 2.13, $m \in \mathbb{N}$, $a \in S^{m+1}$, and $G \in \mathcal{I}_m$ such that for all $w \in W$,

$$\prod_{j=1}^m \big(a(j) \cdot \prod_{k \in G(j)} \prod_{t \in H_k} g_w(t)\big) \cdot a(m+1) \in A\,.$$

Define $\varphi : W \to \{1, 2\}$ by $\varphi(w) = 1$ if

$$\prod_{j=1}^m \big(a(j) \cdot \prod_{k \in G(j)} \prod_{t \in H_k} g_w(t)\big) \cdot a(m+1) \in A_1$$

and $\varphi(w) = 2$ otherwise. Pick a variable word $w(v)$ (beginning and ending with a constant) such that $\big\{w(j) : j \in \{1, 2, \ldots, k\}\big\}$ is monochromatic with respect to φ. Assume without loss of generality that $\varphi\big(w(j)\big) = 1$ for all $j \in \{1, 2, \ldots, k\}$. Let $w(v) = b_1 b_2 \cdots b_n$ where each $b_i \in \{1, 2, \ldots, k\} \cup \{v\}$, some $b_i = v$, and b_1 and b_n are in $\{1, 2, \ldots, k\}$.

Pick $r \in \mathbb{N}$, $L \in \mathcal{I}_{r+1}$, and $M \in \mathcal{I}_r$ such that for each $i \in \{1, 2, \ldots, r\}$, $\max L(i) < \min M(i)$ and $\max M(i) < \min L(i+1)$,

$$\begin{array}{rcl} \bigcup_{i=1}^{r+1} L(i) & = & \big\{j \in \{1, 2, \ldots, n\} : b_j \in \{1, 2, \ldots, k\}\big\}, \text{ and} \\ \bigcup_{i=1}^{r} M(i) & = & \{j \in \{1, 2, \ldots, n\} : b_j = v\}\,. \end{array}$$

(For example, if $k = 3$, $n = 8$, and $w(v) = 2vv31v12$, then $r = 2$, $L(1) = \{1\}$, $M(1) = \{2, 3\}$, $L(2) = \{4, 5\}$, $M(2) = \{6\}$, and $L(3) = \{7, 8\}$.)

Let $u = r \cdot \sum_{j=1}^m |G(j)|$. We claim that there exist $c \in S^{u+1}$ and $K \in \mathcal{I}_u$ such that for each $s \in \{1, 2, \ldots, k\}$,

$$\begin{array}{l} \prod_{j=1}^u \big(c(j) \cdot \prod_{t \in K(j)} f_s(t)\big) \cdot c(u+1) = \\ \prod_{j=1}^m \big(a(j) \cdot \prod_{k \in G(j)} \prod_{t \in H_k} g_{w(s)}(t)\big) \cdot a(m+1) \in A_1 \end{array}$$

which will complete the proof. Write

$$\bigcup_{j=1}^m G(j) = \{\gamma_0, \gamma_1, \ldots, \gamma_{d-1}\}$$

with $\gamma_0 < \gamma_1 < \ldots < \gamma_{d-1}$ (so that $u = rd$). For $y \in \{0, 1, \ldots, d-1\}$ and $i \in \{1, 2, \ldots, r\}$, let $K(yr+i) = \{\gamma_y n + s : s \in M(i)\}$.

(In the event that S is commutative, this portion of the proof is much simpler. One may let $u = 1$, let

$$K(1) = \bigcup_{y=0}^{d-1} \bigcup_{i=1}^r \{\gamma_y n + s : s \in M(i)\}\,,$$

let $c(1) = a(1) \cdot \prod_{y=0}^{d-1} \prod_{i=1}^{r+1} \prod_{t \in L(i)} f_{b_t}(\gamma_y n + t)$, and let $c(2) = a(m+1)$.)

If S is noncommutative, computing the exact values of $c(1), c(2), \ldots, c(u+1)$ is notationally very messy. (Actually $c(1)$ is easy. It is $a(1)\cdot\prod_{j\in L(0)} f_{b_j}(\gamma_0 n+j)$.) Instead of trying to spell it out, we shall give an illustration, from which we hope the general process will be clear. Say $n=6$, $m=2$, $G(1)=\{2,4\}$, $G(2)=\{5\}$, and $w(v)=b_1vb_3vvb_6$ where b_1, b_3, and b_6 are in $\{1,2,\ldots,k\}$. Thus $r=2$, $L(1)=\{1\}$, $L(2)=\{3\}$, $L(3)=\{6\}$, $M(1)=\{2\}$, $M(2)=\{4,5\}$, and $u=6$. Then $K(1)=\{14\}$, $K(2)=\{16,17\}$, $K(3)=\{26\}$, $K(4)=\{28,29\}$, $K(5)=\{32\}$, and $K(6)=\{34,35\}$. Given $s\in\{1,2,\ldots,k\}$ we have

$$\begin{aligned} & \textstyle\prod_{j=1}^{u}\big(c(j)\cdot\prod_{t\in K(j)} f_s(t)\big)\cdot c(u+1) \\ = & \textstyle\prod_{j=1}^{m}\big(a(j)\cdot\prod_{k\in G(j)}\prod_{t\in H_k} g_{w(s)}(t)\big)\cdot a(m+1) \\ = & \; a(1)\cdot f_{b_1}(13)\cdot f_s(14)\cdot f_{b_3}(15)\cdot f_s(16)\cdot f_s(17)\cdot f_{b_6}(18) \\ & \cdot f_{b_1}(25)\cdot f_s(26)\cdot f_{b_3}(27)\cdot f_s(28)\cdot f_s(29)\cdot f_{b_6}(30)\cdot a(2) \\ & \cdot f_{b_1}(31)\cdot f_s(32)\cdot f_{b_3}(33)\cdot f_s(34)\cdot f_s(35)\cdot f_{b_6}(36)\cdot a(3)\,. \end{aligned}$$

Then $c(1)=a(1)\cdot f_{b_1}(13)$, $c(2)=f_{b_3}(15)$, $c(3)=f_{b_6}(18)\cdot f_{b_1}(25)$, $c(4)=f_{b_3}(27)$, $c(5)=f_{b_6}(30)\cdot a(2)\cdot f_{b_1}(31)$, $c(6)=f_{b_3}(33)$, and $c(7)=f_{b_6}(36)\cdot a(3)$. □

Lemma 2.15. *Let S and T be semigroups, let p be an idempotent in $J(S)$, and let q be an idempotent in $J(T)$. Let $\widetilde{\iota}:\beta(S\times T)\to\beta S\times\beta T$ be the continuous extension of the identity function. Then $\widetilde{\iota}^{-1}[\{(p,q\}]\cap J(S\times T)\neq\emptyset$.*

Proof. We claim that $\{\overline{A\times B}\cap J(S\times T): A\in p$ and $B\in q\}$ has the finite intersection property. To see this, it suffices to let $A\in p$ and $B\in q$ and show that $\overline{A\times B}\cap J(S\times T)\neq\emptyset$. By Theorem 2.11 $A\times B$ is a J-set in $S\times T$. By Theorem 2.14, the property of being a J-set is partition regular. So by [9, Theorem 3.11] there is an ultrafilter r on $S\times T$ such that $A\times B\in r$ and every member of r is a J-set. That is, $r\in\overline{A\times B}\cap J(S\times T)$. □

Theorem 2.16. *Let S and T be semigroups, let A be a C-set in S, and let B be a C-set in T. Then $A\times B$ is a C-set in $S\times T$.*

Proof. Pick idempotents $p\in J(S)$ and $q\in J(T)$ such that $A\in p$ and $B\in q$. By Lemma 2.15, $\widetilde{\iota}^{-1}[\{(p,q\}]\cap J(S\times T)$ is a compact subsemigroup of $\beta(S\times T)$ so pick an idempotent

$$r\in\widetilde{\iota}^{-1}[\{(p,q\}]\cap J(S\times T)\,.$$

By Theorem 2.12, $A\times B$ is a C-set in $S\times T$. □

3. Infinite Cartesian products

In this section we shall determine when the Cartesian products of subsets of semigroups is central. We shall also see that the same conditions are sufficient but not necessary for the infinite Cartesian product of C-sets to be a C-set and likewise are not necessary for the infinite Cartesian product of J-sets to be a J-set.

We shall utilize two notions of size of a semigroup that originate in topological dynamics (and introduce a new one). Notice that all of these notions are one sided.

Definition 3.1. Let S be a semigroup and let $A \subseteq S$.

(a) The set A is *thick* if and only if for each $F \in \mathcal{P}_f(S)$ there exists $x \in S$ such that $Fx \subseteq A$.
(b) The set A is *piecewise syndetic* if and only if there exists $H \in \mathcal{P}_f(S)$ such that $\bigcup_{t \in H} t^{-1}A$ is thick.
(c) The set A is *weakly thick* if and only if there exists $s \in S$ such that $s^{-1}A$ is thick.

There are simple algebraic characterizations of the first two of these notions. (And the algebraic characterization of thick immediately translates into an algebraic characterization of weakly thick.)

Lemma 3.2. *Let S be a semigroup and let $A \subseteq S$.*

(a) *The set A is thick if and only if there is a left ideal L of βS such that $L \subseteq \overline{A}$.*
(b) *The set A is piecewise syndetic if and only if $\overline{A} \cap K(\beta S) \neq \emptyset$.*

Proof. (a) [3, Theorem 2.9(c)].
(b) [9, Theorem 4.40]. □

From these algebraic characterizations it is immediate that any thick set is central and any central set is piecewise syndetic. It is also trivially true that any thick set is weakly thick. And it is immediate from the definitions that any weakly thick set is piecewise syndetic.

If S is commutative, then it is immediate that the notions of thick and weakly thick are equivalent. It is a fact proved in [10] that the notions of thick and weakly thick are also equivalent for left amenable semigroups. (We will not need this latter fact.) On the other hand, if S is the free semigroup on the alphabet $\{a, b\}$ and $A = aS$, then for any $x \in S$, $\{b\}x \cap A = \emptyset$ while $a^{-1}A = S$, so A is weakly thick but not thick. The fact that any weakly thick set is central is not obvious.

Lemma 3.3. *Let S be a semigroup and let $A \subseteq S$. If A is weakly thick, then A is central.*

Proof. Pick $s \in S$ such that $s^{-1}A$ is thick and pick by Lemma 3.2(a) a left ideal L of βS such that $L \subseteq \overline{s^{-1}A}$. Since every left ideal contains a minimal left ideal, we may assume that L is a minimal left ideal. Pick a minimal right ideal $R \subseteq s\beta S$. Then $L \cap R$ is a group so pick an idempotent $p \in L \cap R$. Since $L \subseteq \overline{s^{-1}A}$ we have that $sL \subseteq \overline{A}$. Since $p \in s\beta S$, pick $q \in \beta S$ such that $p = sq$. Then $p = pp = sqp$ and $qp \in L$ so $p \in sL \subseteq \overline{A}$ and thus $p \in K(\beta S) \cap \overline{A}$. □

Lemma 3.4. *Let I be a set and for each $\alpha \in I$, let S_α be a semigroup and let $A_\alpha \subseteq S_\alpha$. Let $S = \times_{\alpha \in I} S_\alpha$ and let $A = \times_{\alpha \in I} A_\alpha$.*

(a) *If for each $\alpha \in I$, A_α is thick in S_α, then A is thick in S.*
(b) *If for each $\alpha \in I$, A_α is weakly thick in S_α, then A is weakly thick in S.*
(c) *If A is central in S, then for each $\alpha \in I$, A_α is central in S_α.*
(d) *If A is piecewise syndetic in S, then for each $\alpha \in I$, A_α is piecewise syndetic in S_α.*
(e) *If A is a J-set in S, then for each $\alpha \in I$, A_α is a J-set in S_α.*
(f) *If A is a C-set in S, then for each $\alpha \in I$, A_α is a C-set in S_α.*

Proof. The proofs of all of these statements except statement (c) are completely elementary and routine and we omit them. To verify statement (c), assume that A is central in S and let $\alpha \in I$. Since π_α is a homomorphism from S onto S_α, its continuous extension $\widetilde{\pi_\alpha} : \beta S \to \beta S_\alpha$ is a surjective homomorphism by [9, Corollary 4.22]. By [9, Exercise 1.7.3], $\pi_\alpha[K(\beta S)] = K(\beta S_\alpha)$. Pick an idempotent $p \in K(\beta S)$ such that $A \in p$. Then $\widetilde{\pi_\alpha}(p)$ is an idempotent in $K(\beta S_\alpha)$ and $A_\alpha \in \widetilde{\pi_\alpha}(p)$. □

We omit the routine proof of the following lemma.

Lemma 3.5. *Let S and T be semigroups, let $A \subseteq S$, and let $B \subseteq T$. If A is piecewise syndetic in S and B is piecewise syndetic in T, then $A \times B$ is piecewise syndetic in $S \times T$.*

The following lemma is the key to our characterization of when the arbitrary product of sets is a central set.

Lemma 3.6. *Let I be a set and for each $\alpha \in I$, let S_α be a semigroup and let $A_\alpha \subseteq S_\alpha$. Let $S = \times_{\alpha \in I} S_\alpha$, let $A = \times_{\alpha \in I} A_\alpha$, and let $J = \{\alpha \in I : A_\alpha$ is not weakly thick in $S_\alpha\}$. If J is infinite, then A is not piecewise syndetic in S.*

Proof. Suppose instead that A is piecewise syndetic in S and pick $H \in \mathcal{P}_f(S)$ such that $\bigcup_{m \in H} m^{-1}A$ is thick. Pick an injection $\delta : H \to J$. Given $m \in H$, we have that $A_{\delta(m)}$ is not weakly thick in $S_{\delta(m)}$ so pick $K_m \in \mathcal{P}_f(S_{\delta(m)})$ such that for all $x \in S_{\delta(m)}$, $K_m x \setminus (m_{\delta(m)})^{-1} A_{\delta(m)} \neq \emptyset$. For each $\alpha \in I \setminus \delta[H]$, pick $z_\alpha \in S_\alpha$. Let

$$K = \{x \in S : \begin{array}{l} (\forall \alpha \in I \setminus \delta[H])(x_\alpha = z_\alpha) \\ \text{and } (\forall m \in H)(x_{\delta(m)} \in K_m)\} \,. \end{array}$$

Then $K \in \mathcal{P}_f(S)$ so pick $y \in S$ such that $Ky \subseteq \bigcup_{m \in H} m^{-1}A$. For each $m \in H$, $K_m y_{\delta(m)} \setminus (m_{\delta(m)})^{-1} A_{\delta(m)} \neq \emptyset$ so pick $w_m \in K_m$ such that $m_{\delta(m)} w_m y_{\delta(m)} \notin A_{\delta(m)}$. Define $u \in S$ by, for $\alpha \in I$,

$$u_\alpha = \begin{cases} z_\alpha & \text{if } \alpha \in I \setminus \delta[H] \\ w_m & \text{if } m \in H \text{ and } \alpha = \delta(m) \,. \end{cases}$$

Then $u \in K$ so pick $m \in H$ such that $muy \in A$. Then

$$m_{\delta(m)} w_m y_{\delta(m)} = m_{\delta(m)} u_{\delta(m)} y_{\delta(m)} \in A_{\delta(m)} \,,$$

a contradiction. □

Theorem 3.7. *Let I be a set and for each $\alpha \in I$, let S_α be a semigroup and let $A_\alpha \subseteq S_\alpha$. Let $S = \times_{\alpha \in I} S_\alpha$, let $A = \times_{\alpha \in I} A_\alpha$, and let $J = \{\alpha \in I : A_\alpha$ is not weakly thick in $S_\alpha\}$. Then A is piecewise syndetic in S if and only if J is finite and for each $\alpha \in I$, A_α is piecewise syndetic in S_α.*

Proof. Necessity. By Lemma 3.4(d), each A_α is piecewise syndetic in S_α and by Lemma 3.6, J is finite.

Sufficiency. If $J = \emptyset$ we have by Lemma 3.4(b) that A is weakly thick in S and hence is piecewise syndetic in S. So assume that

$J \neq \emptyset$. If $I \setminus J = \emptyset$, then I is finite so A is piecewise syndetic in S by Lemma 3.5. So assume also that $I \setminus J \neq \emptyset$.

Now $\times_{\alpha \in I \setminus J} A_\alpha$ is weakly thick in $\times_{\alpha \in I \setminus J} S_\alpha$ by Lemma 3.4(b) so is piecewise syndetic. Also $\times_{\alpha \in J} A_\alpha$ is piecewise syndetic in $\times_{\alpha \in J} S_\alpha$ by Lemma 3.5. Therefore $\times_{\alpha \in I \setminus J} A_\alpha \times \times_{\alpha \in J} A_\alpha$ is piecewise syndetic in $\times_{\alpha \in I \setminus J} S_\alpha \times \times_{\alpha \in J} S_\alpha$ by Lemma 3.5. □

We then have immediately our desired characterization of when the Cartesian product of sets is central.

Theorem 3.8. *Let I be a set and for each $\alpha \in I$, let S_α be a semigroup and let $A_\alpha \subseteq S_\alpha$. Let $S = \times_{\alpha \in I} S_\alpha$, let $A = \times_{\alpha \in I} A_\alpha$, and let $J = \{\alpha \in I : A_\alpha$ is not weakly thick in $S_\alpha\}$. Then A is central in S if and only if J is finite and for each $\alpha \in I$, A_α is central in S_α.*

Proof. Necessity. By Lemma 3.4(c), each A_α is central in S_α and, since central sets are piecewise syndetic, by Lemma 3.6, J is finite.

Sufficiency. If $J = \emptyset$ we have by Lemma 3.4(b), A is weakly thick in S and hence by Lemma 3.3, A is central in S. So assume that $J \neq \emptyset$. If $I \setminus J = \emptyset$, then I is finite so A is central in S by Corollary 2.2. So assume also that $I \setminus J \neq \emptyset$.

Now $\times_{\alpha \in I \setminus J} A_\alpha$ is weakly thick in $\times_{\alpha \in I \setminus J} S_\alpha$ by Lemma 3.4(b) so is central by Lemma 3.3. Also $\times_{\alpha \in J} A_\alpha$ is central in $\times_{\alpha \in J} S_\alpha$ by Corollary 2.2. Therefore $\times_{\alpha \in I \setminus J} A_\alpha \times \times_{\alpha \in J} A_\alpha$ is central in $\times_{\alpha \in I \setminus J} S_\alpha \times \times_{\alpha \in J} S_\alpha$ by Corollary 2.2. □

In one direction we have the corresponding results for J-sets and C-sets. The proofs are nearly identical to the suffiency parts of Theorems 3.7 and 3.8, so we omit them.

Theorem 3.9. *Let I be a set and for each $\alpha \in I$, let S_α be a semigroup and let $A_\alpha \subseteq S_\alpha$. Let $S = \times_{\alpha \in I} S_\alpha$, let $A = \times_{\alpha \in I} A_\alpha$, and let $J = \{\alpha \in I : A_\alpha$ is not weakly thick in $S_\alpha\}$. If J is finite and for each $\alpha \in I$, A_α is a J-set in S_α, then A is a J-set in S. If J is finite and for each $\alpha \in I$, A_α is a C-set in S_α, then A is a C-set in S.*

We now see that the requirement that J be finite is not necessary for the product of sets to be a C-set in $\times_{n=1}^{\infty} \mathbb{N}$. Since $(\mathbb{N}, +)$ is commutative, the notions of thick and weakly thick are equivalent in $\mathbb{N}$.

Theorem 3.10. *For each $n \in \mathbb{N} \setminus \{1\}$, let*

$$A_n = \{x \in \mathbb{N} : x \not\equiv -1 \pmod{n}\}\,.$$

Let $\varphi : \mathbb{N} \to \mathbb{N} \setminus \{1\}$ and assume that $\lim_{n\to\infty} \varphi(n) = \infty$. Let $A = \times_{n=1}^{\infty} A_{\varphi(n)}$ and let $S = \times_{n=1}^{\infty} \mathbb{N}$. Then each A_n is not thick in $(\mathbb{N}, +)$, but A is a C-set (and hence a J-set) in S.

Proof. Trivially each A_n is not thick. Pick any $p = p + p \in K(\beta\mathbb{N})$. Given any n, since $\mathbb{N}n \subseteq A_n$, we have that $A_n \in p$.

For each $r \in \mathbb{N}$, let $B_r = \times_{n=1}^{r} A_{\varphi(n)}$ and let $S_r = \times_{n=1}^{r} \mathbb{N}$. Let $\widetilde{\iota_r} : \beta S_r \to \times_{n=1}^{r} \beta\mathbb{N}$ be the continuous extension of the identity function on S_r. Pick by Theorem 2.1, $q_r = q_r + q_r \in K(\beta S_r)$ such that $\widetilde{\iota_r}(q_r) = (p, p, \ldots, p)$ and notice that $B_r \in q_r$. We claim that for each $x \in B_r$, $-x + B_r \in q_r$. To see this, let $x \in B_r$ and note that for each $n \in \{1, 2, \ldots, r\}$, $\mathbb{N}\varphi(n) \in p$ so $\times_{n=1}^{r} \mathbb{N}\varphi(n) \in q_r$. We claim that $\times_{n=1}^{r} \mathbb{N}\varphi(n) \subseteq -x + B_r$, so let $y \in \times_{n=1}^{r} \mathbb{N}\varphi(n)$. Given $n \in \{1, 2, \ldots, r\}$, $(x + y)_n = x_n + y_n \equiv x_n \not\equiv -1 \pmod{\varphi(n)}$ so $x + y \in B_r$.

Next notice that if $F \in \mathcal{P}_f(\mathbb{N})$ and $n \in \mathbb{N}$ with $n > |F|$, then there is some $a \in \{1, 2, \ldots, n\}$ such that $F + a \subseteq A_n$.

To see that A is a C-set in S, we shall define $\alpha(F) \in S$ and $H(F) \in \mathcal{P}_f(\mathbb{N})$ for $F \in \mathcal{P}_f\big({}^{\mathbb{N}}S\big)$ by induction on $|F|$ satisfying the following induction hypotheses:

(1) If $\emptyset \neq G \subsetneq F$, then $\max H(G) < \min H(F)$.
(2) If $m \in \mathbb{N}$, $\emptyset \neq G_1 \subsetneq G_2 \subsetneq \ldots \subsetneq G_m = F$ and $f_i \in G_i$ for each $i \in \{1, 2, \ldots, m\}$, then $\sum_{i=1}^{m} \big(\alpha(G_i) + \sum_{l \in H(G_i)} f_i(l)\big) \in A$.

Assume first that $F = \{f\}$. Define $H(F) = \{1\}$. For $n \in \mathbb{N}$, since $\varphi(n) > |F|$, pick $c_n \in \{1, 2, \ldots, \varphi(n)\}$ such that $c_n + f(1)(n) \in A_{\varphi(n)}$ and define $\alpha(F)(n) = c_n$.

Now assume that $|F| > 1$ and that $\alpha(G)$ and $H(G)$ have been defined whenever $\emptyset \neq G \subsetneq F$. Let $K = \bigcup\{H(G) : \emptyset \neq G \subsetneq F\}$ and let $k = \max K$. Let

$$\begin{aligned} M = \ & \{\textstyle\sum_{i=1}^{m} \big(\alpha(G_i) + \sum_{l \in H(G_i)} f_i(l)\big) : \\ & m \in \mathbb{N}, \emptyset \neq G_1 \subsetneq G_2 \subsetneq \ldots \subsetneq G_m \subsetneq F \\ & \text{and } f_i \in G_i \text{ for each } i \in \{1, 2, \ldots, m\}\}\,. \end{aligned}$$

Then $M \subseteq A$ by assumption.

Pick $r \in \mathbb{N}$ such that for all $n > r$, $\varphi(n) > (|M| + 1) \cdot |F|$. Let $M' = \{x_{|\{1,2,\ldots,r\}} : x \in M\}$. For $f \in F$, define $f' : \mathbb{N} \to S_r$ by, for $n \in \mathbb{N}$, $f'(n) = f(n)_{|\{1,2,\ldots,r\}}$. Then $M' \subseteq B_r$ and for all $x \in M'$, $-x + B_r \in q_r$. Let $C = B_r \cap \bigcap_{x\in M'} (-x + B_r)$. Then $C \in q_r$ and so C is piecewise syndetic in S_r. Pick by [4, Theorem 2.1] some $c \in S_r$ and $L \in \mathcal{P}_f(\mathbb{N})$ such that $\min L > k$ and for each $f \in F$, $c + \sum_{l\in L} f'(l) \in C$. Define $H(F) = L$.

Now for $n > r$, let

$$\begin{array}{rl} V_n = & \{x_n + \sum_{l\in L} f(l)(n) : x \in M \text{ and } f \in F\} \cup \\ & \{\sum_{l\in L} f(l)(n) : f \in F\}\,. \end{array}$$

Then $|V_n| \leq |M| \cdot |F| + |F| < \varphi(n)$ so pick $d_n \in \{1, 2, \ldots, \varphi(n)\}$ such that $V_n + d_n \subseteq A_{\varphi(n)}$. Define $\alpha(F)$ by, for $n \in \mathbb{N}$,

$$\alpha(F)(n) = \begin{cases} d_n & \text{if } n > r \\ c_n & \text{if } n \leq r\,. \end{cases}$$

We claim that $\alpha(F)$ and $H(F)$ are as required. Requirement (1) is satisfied directly. To verify (2), let $m \in \mathbb{N}$ and assume that $\emptyset \neq G_1 \subsetneq G_2 \subsetneq \ldots \subsetneq G_m = F$ and $f_i \in G_i$ for each $i \in \{1, 2, \ldots, m\}$.

Assume first that $m = 1$. We want to show that

$$\alpha(F) + \sum_{l\in H(F)} f_1(l) \in A\,.$$

So let $n \in \mathbb{N}$. We shall show that

$$\alpha(F)(n) + \sum_{l\in H(F)} f_1(l)(n) \in A_{\varphi(n)}\,.$$

If $n \leq r$, then

$$\alpha(F)(n) + \sum_{l\in H(F)} f_1(l)(n) = c_n + \sum_{l\in H(F)} f_1'(l)(n) \in A_{\varphi(n)}$$

because $c + \sum_{l\in H(F)} f_1'(l) \in B_r$. If $n > r$, then

$$\begin{array}{rcl} \alpha(F)(n) + \sum_{l\in H(F)} f_1(l)(n) & = & d_n + \sum_{l\in H(F)} f_1(l)(n) \\ & \in & d_n + V_n \subseteq A_{\varphi(n)}\,. \end{array}$$

Now assume that $m > 1$ and let

$$x = \sum_{i=1}^{m-1} \big(\alpha(G_i) + \sum_{l\in H(G_i)} f_i(l)\big)\,.$$

We want to show that $\alpha(F) + \sum_{l\in H(F)} f_m(l) + x \in A$. So let $n \in \mathbb{N}$. We need to show that

$$\alpha(F)(n) + \sum_{l\in H(F)} f_m(l)(n) + x_n \in A_{\varphi(n)}\,.$$

If $n \leq r$, then

$$\alpha(F)(n) + \sum_{l \in H(F)} f_m(l)(n) = c_n + \sum_{l \in H(F)} f'_m(l)(n)$$

and

$$c + \sum_{l \in H(F)} f'_m(l) \in C \subseteq -x_{|\{1,2,\ldots,r\}} + B_r$$

so $c_n + \sum_{l \in H(F)} f_m(l)(n) + x_n \in A_{\varphi(n)}$. If $n > r$, then

$$x_n + \alpha(F)(n) + \sum_{l \in H(F)} f_m(l)(n) \in d_n + V_n \subseteq A_{\varphi(n)}\,. \qquad \square$$

Notice that the set A in Theorem 3.10 is not a central set in S, in fact it is not a piecewise syndetic set in S by Lemma 3.6.

Notice that in Theorem 3.10 the size of sets witnessing that $A_{\varphi(n)}$ is not thick went to infinity, although that size could grow as slowly as we please. We see now that if the size did not go to infinity, then the Cartesian product could not be a J-set.

Theorem 3.11. *Let I be a set and for each $\alpha \in I$, let $(S_\alpha, +)$ be a commutative semigroup and let $A_\alpha \subseteq S_\alpha$. Let $S = \times_{\alpha \in I} S_\alpha$, let $A = \times_{\alpha \in I} A_\alpha$, and let $J = \{\alpha \in I : A_\alpha$ is not thick in $S_\alpha\}$. For $\alpha \in J$, choose $F_\alpha \in \mathcal{P}_f(S_\alpha)$ such that for all $x \in S_\alpha$, $(F_\alpha + x) \setminus A_\alpha \neq \emptyset$. If there exists $k \in \mathbb{N}$ such that $\{\alpha \in I : |F_\alpha| \leq k\}$ is infinite, then A is not a J-set in S.*

Proof. Pick by the pigeon hole principle some $k \in \mathbb{N}$ such that $\{\alpha \in I : |F_\alpha| = k\}$ is infinite and let L be a countably infinite subset of $\{\alpha \in I : |F_\alpha| = k\}$. Enumerate L as $\langle \delta(n) \rangle_{n=1}^{\infty}$. For $n \in \mathbb{N}$, let $F_{\delta(n)} = \{x_{n,1}, x_{n,2}, \ldots, x_{n,k}\}$. For each $\alpha \in I$ pick $b_\alpha \in S_\alpha$. For $i \in \{1, 2, \ldots, k\}$, define $f_i : \mathbb{N} \to S$ by, for $n \in \mathbb{N}$ and $\alpha \in I$,

$$f_i(n)(\alpha) = \begin{cases} x_{n,i} & \text{if } \alpha = \delta(n) \\ b_\alpha & \text{if } \alpha \neq \delta(n)\,. \end{cases}$$

Suppose that we have $a \in S$ and $H \in \mathcal{P}_f(\mathbb{N})$ such that for all $i \in \{1, 2, \ldots, k\}$, $a + \sum_{t \in H} f_i(t) \in A$. Pick $n \in H$ and let $w = a_{\delta(n)} + \sum_{t \in H \setminus \{n\}} b_{\delta(n)}$ (so, if $H = \{n\}$, then $w = a_{\delta(n)}$). Pick $i \in \{1, 2, \ldots, k\}$ such that $w + x_{n,i} \notin A_{\delta(n)}$. Then

$$\big(a + \sum_{t \in H} f_i(t)\big)\big(\delta(n)\big) = w + x_{n,i} \notin A_{\delta(n)}$$

so $a + \sum_{t \in H} f_i(t) \notin A$, a contradiction. $\square$

As a consequence of Theorem 3.11, for any $n > 1$, if A_n is as in Theorem 3.10, then $\times_{k=1}^{\infty} A_n$ is not a J-set in $\times_{k=1}^{\infty} \mathbb{N}$.

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