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SETS CENTRAL WITH RESPECT TO CERTAIN SUBSEMIGROUPS OF βS_d

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ABSTRACT. *Central* sets in a semigroup S are most simply characterized as members of minimal idempotents in the Stone-Čech compactification βS_d of S with the discrete topology. They are known to have remarkably strong combinatorial properties. In this paper we concentrate on members of idempotents in compact subsemigroups T of βS_d. We show that under reasonable hypotheses any member of a minimal idempotent in T is in fact a member of many distinct minimal idempotents in T. And we show that under certain assumptions which occur quite widely, the subsets of S whose closures contain T must contain images of all first entries matrices. For example, all of our results apply to the case in which $(S, +)$ is a commutative cancellative topological semigroup with an identity and T is the set of ultrafilters on S which converge to the identity.

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1. Introduction

Central subsets of $\mathbb{N}$ were defined by H. Furstenberg [5] in terms of notions from topological dynamics, and he proved the original *Central Sets Theorem*.

Theorem 1.1 (Furstenberg). *Let $l \in \mathbb{N}$ and for each $i \in \{1, 2, \ldots, l\}$, let $\langle y_{i,n} \rangle_{n=1}^{\infty}$ be a sequence in $\mathbb{Z}$. Let C be a central subset of $\mathbb{N}$. Then there exist sequences $\langle a_n \rangle_{n=1}^{\infty}$ in $\mathbb{N}$ and $\langle H_n \rangle_{n=1}^{\infty}$ in $\mathcal{P}_f(\mathbb{N}) = \{F \subseteq \mathbb{N} : F$ is finite and nonempty$\}$ such that*

(1) *for all n, $\max H_n < \min H_{n+1}$ and*
(2) *for all $F \in \mathcal{P}_f(\mathbb{N})$ and all $i \in \{1, 2, \ldots, l\}$, $\sum_{n \in F} \left(a_n + \sum_{t \in H_n} y_{i,t}\right) \in C$.*

Proof. [5, Proposition 8.21]. $\square$

This theorem was shown in [5] to have several powerful consequences. V. Bergelson suggested that we ought to be able to prove Theorem 1.1 for subsets of $\mathbb{N}$ that are members of idempotents in the smallest ideal of $(\beta\mathbb{N}, +)$, and it turned out that Bergelson was right.

We take the Stone-Čech compactification of a discrete space X to be the set of ultrafilters on X, the principal ultrafilters being identified with the points of X. Given a semigroup $(S, \cdot)$ with the discrete topology, the operation extends to the Stone-Čech compactification βS of S in such a way that $(\beta S, \cdot)$ is a right topological semigroup (meaning that for each $p \in \beta S$, the function $\rho_p : \beta S \to \beta S$ defined by $\rho_p(q) = q \cdot p$ is continuous) with S contained in its topological center (meaning that for each $x \in S$, the function $\lambda_x : \beta S \to \beta S$ defined by $\lambda_x(q) = x \cdot q$ is continuous). Since it is a compact Hausdorff right topological semigroup, $(\beta S, \cdot)$ has a smallest two sided ideal $K(\beta S)$ which is the union of all of the minimal left ideals of βS and is also the union of all of the minimal right ideals of βS. The intersection of any minimal left ideal with any minimal right ideal is a group, and in particular there are idempotents in $K(\beta S)$. An idempotent in $K(\beta S)$ is called simply a *minimal* idempotent. (See [10] for an elementary introduction to the algebraic structure of $(\beta S, \cdot)$.)

With the assistance of B. Weiss, it was shown [1] that for countable semigroups S (with metric phase space) a subset C of S satisfies the dynamical definition of central if and only if C is a member of a minimal idempotent in βS. Later H. Shi and H. Yang [12] established the same equivalence for arbitrary semigroups.

In many cases there are interesting compact subsemigroups of βS, which therefore have smallest ideals with the structure indicated above, but for which known theorems about $K(\beta S)$ provide no information. For example, if for each $n \in \mathbb{N}$, $A_n = (0, 1/n) \subseteq \mathbb{R}$, then $T = \bigcap_{n=1}^{\infty} c\ell_{\beta\mathbb{R}_d} A_n$ is a subsemigroup of $(\beta\mathbb{R}_d, +)$ and so has a smallest ideal $K(T)$. (Here $\mathbb{R}_d$ is $\mathbb{R}$ with the discrete topology.) If one had that $T \cap K(\beta\mathbb{R}_d) \neq \emptyset$, then by [10, Theorem 1.65] one would have that $K(T) = T \cap K(\beta\mathbb{R}_d)$. But $T \cap K(\beta\mathbb{R}_d) = \emptyset$, so no useful information about $K(T)$ is obtained from what is known about the structure of $K(\beta\mathbb{R}_d)$. This structure, and that associated with other subsemigroups of $\big((0, \infty), +\big)$, was investigated in [7].

Of particular interest has been the fact, as was shown in [5], that any central subset of $\mathbb{N}$ contains an image of any *first entries matrix*. (The notion is based on Deuber's (m, p, c)-sets [4].) We follow the usual custom of denoting the entries of a matrix by the lower case letter corresponding to the upper case name of the matrix.

Definition 1.2. Let $u, v \in \mathbb{N}$ and let M be a $u \times v$ matrix with entries from $\mathbb{Q}$. Then M is a *first entries matrix* if and only if no row of M is $\vec{0}$ and for all $i, j \in \{1, 2, \ldots, u\}$, if $k = \min\{t : m_{i,t} \neq 0\} = \min\{t : m_{j,t} \neq 0\}$, then $m_{i,k} = m_{j,k} > 0$. If $k = \min\{t : m_{i,t} \neq 0\}$, then $m_{i,k}$ is a *first entry* of M.

Theorem 1.3. *If C is a central subset of $\mathbb{N}$, $u, v \in \mathbb{N}$, and M is a first entries matrix with entries from $\mathbb{Q}$, then there exists some $\vec{x} \in \mathbb{N}^v$ such that $M\vec{x} \in C^u$.*

Proof. [5, page 174]. Or see [10, Theorem 15.5]. □

Many natural results in Ramsey Theory assert that whenever $\mathbb{N}$ is partitioned into finitely many sets, one of these contains an image of some first entries matrix. (See [10, Section 15.1].) Further the image partition regularity of any finite matrix is characterized in terms of first entries matrices. (See [10, Theorem 15.24].) Accordingly, it was interesting to see (as was done in [7]) that if for each $n \in \mathbb{N}$, $A_n = (0, 1/n)$ and $T = \bigcap_{n=1}^{\infty} c\ell_{\beta\mathbb{N}} A_n$, then any member of a minimal idempotent in T also satisfied the conclusion of Theorem 1.3.

We shall be concerned in this paper with compact subsemigroups T of βS_d. In Section 2 we shall establish that under reasonable hypotheses any member of a minimal idempotent of T is a member of very many minimal idempotents of T – as many as there are points in βS_d. A trivial consequence of this, namely that a central set in βS_d can be divided into two disjoint central sets was not known, even for $\beta\mathbb{N}$, until the publication of [8]. In this section, the "reasonable hypotheses" are that S is infinite and very weakly cancellative and that T is the intersection of at most $|S|$ sets of the form $\overline{A}$, where $A \subseteq S$ has the same cardinality as S.

In Section 3 we shall establish a version of the Central Sets Theorem which is valid for members of minimal idempotents in certain compact subsemigroups T of βS_d. The hypotheses of this theorem are valid in the case in which S is a commutative left topological semigroup with an identity and T is the set of ultrafilters in βS_d which converge to the identity. As a fortuitous corollary we obtain a version of the Central Sets Theorem for commutative semigroups which is superficially stronger than the strongest previously known version. (We shall establish that it is not actually stronger.)

In Section 4 we will derive conditions guaranteeing that members of minimal idempotents in T contain images of all first entries matrices with entries from $\omega = \mathbb{N} \cup \{0\}$, or at least all of those all of whose first entries are equal to 1. In the case where S is a group, the entries of the matrices are allowed to come from $\mathbb{Z}$. The hypotheses used in this section hold in the case in which S is a commutative left topological semigroup with an identity, such that the semigroup operation is jointly continuous at the identity, and T is the set of ultrafilters in βS_d which converge to the identity.

All hypothesized topological spaces are Hausdorff.

2. Many Minimal Idempotents

As we have already noted, central sets are guaranteed to have substantial combinatorial properties. And, since whenever S is partitioned into finitely many cells, one of these must be central, one obtains as a corollary in each case the corresponding Ramsey Theoretic result. One naturally wanted to know then whether every central set can be divided into two disjoint central sets. That question was answered in the affirmative for central subsets of $\mathbb{N}$ as a consequence of the following theorem.

Theorem 2.1. *Let p be a minimal idempotent in $(\beta\mathbb{N}, +)$ and let L and R be respectively the minimal left and minimal right ideals of $(\beta\mathbb{N}, +)$ with $p \in L \cap R$. Then for each $C \in p$, there are $2^{\mathfrak{c}}$ minimal idempotents in $L \cap \overline{C}$ and $2^{\mathfrak{c}}$ minimal idempotents in $R \cap \overline{C}$.*

Proof. [8, Theorem 2.12]. □

Our extension of this result uses a very weak version of cancellativity.

Definition 2.2. Let $(S, \cdot)$ be an infinite semigroup with cardinality κ. A subset A of S is a *left solution set of S* (respectively a *right solution set of S*) if and only if there exist $w, z \in S$ such that $A = \{x \in S : w = zx\}$ (respectively $A = \{x \in S : w = xz\}$). The semigroup S is *weakly left cancellative* if every left solution set is finite and S is *weakly right cancellative* if every right solution set is finite. The semigroup S is *very weakly left cancellative* if the union of fewer than κ left solution sets of S must have cardinality less than κ and S is *very weakly right cancellative* if the union of fewer than κ right solution sets of S must have cardinality less than κ. The semigroup S is *weakly cancellative* if it is both weakly left cancellative and weakly right cancellative and S is *very weakly cancellative* if it is both very weakly left cancellative and very weakly right cancellative.

We remark that if κ is regular, S is very weakly left cancellative if and only if every left solution set of S has cardinality less than κ. If κ is singular, S is very weakly left cancellative if and only if there is a cardinal less than κ which is an upper bound for the cardinalities of all left solution sets of S. Of course, weak left cancellativity implies very weak left cancellativity. The two notions are equivalent if $\kappa = \omega$.

The following theorem extends Theorem 2.1 except that we do not make any assertions about the number of minimal idempotents in a given left ideal. Given a topological space S and a subset A of S, we shall use the notation $\overline{A}$ exclusively to denote $c\ell_{\beta S_d} A$, the closure of A with respect to Stone-Čech compactification of

S_d, where S_d is S with the discrete topology. Recall that if X is any discrete space, $q \in \beta X$, Y is a compact Hausdorff space, $f : X \to Y$, and $y \in Y$, then $q\text{-}\lim\limits_{x \in X} f(x) = y$ if and only if for every neighborhood U of y in Y, $\{x \in X : f(x) \in U\} \in q$.

Theorem 2.3. *Let $(S, \cdot)$ be an infinite very weakly cancellative semigroup with cardinality κ and assume that $\mathcal{A} \subseteq \mathcal{P}(S)$ such that $\mathcal{A}$ is closed under finite intersections, $|\mathcal{A}| \leq \kappa$, and for all $A \in \mathcal{A}$, $|A| = \kappa$. Let $T = \bigcap_{A \in \mathcal{A}} \overline{A}$, and assume that T is a subsemigroup of $(\beta S_d, \cdot)$. Let p be an idempotent in $K(T)$, let $C \in p$, and let R be the minimal right ideal of T to which p belongs. Then there are 2^{2^κ} idempotents in $K(T) \cap R \cap \overline{C}$.*

Proof. Let $U_\kappa(S)$ be the set of κ-uniform ultrafilters on S. By [2, Lemma 3.1] $U_\kappa(S)$ is an ideal of S. By [10, Theorem 3.62], $U_\kappa(S) \cap T \neq \emptyset$ so $U_\kappa(S) \cap T$ is an ideal of T so $p \in U_\kappa(S)$. Let $C^\star = \{x \in C : x^{-1}C \in p\}$, where $x^{-1}C = \{y \in S : xy \in C\}$. By [10, Lemma 4.14], if $x \in C^\star$, then $x^{-1}C^\star \in p$. Choose $\varphi : \kappa \to \mathcal{P}_f(C^\star) \times \mathcal{A}$ such that for each $F \in \mathcal{P}_f(C^\star)$ and each $B \in \mathcal{A}$, $|\varphi^{-1}[\{(F, B)\}]| = \kappa$. Enumerate S as $\{s_\alpha : \alpha < \kappa\}$. We inductively choose $\langle t_\alpha \rangle_{\alpha < \kappa}$ such that for each $\alpha < \kappa$

(i) $t_\alpha \in C^\star \cap \bigcap_{a \in \pi_1(\varphi(\alpha))} a^{-1}C^\star$,
(ii) $t_\alpha \in \pi_2\big(\varphi(\alpha)\big)$, and
(iii) if $\gamma, \delta, \beta < \alpha$, then $s_\gamma t_\alpha \neq s_\delta t_\beta$.

This is possible since $C^\star \cap \pi_2\big(\varphi(\alpha)\big) \cap \bigcap_{a \in \pi_1(\varphi(\alpha))} a^{-1}C^\star \in p$, and therefore has cardinality κ, and S is very weakly cancellative.

For $\alpha < \kappa$, let $X_\alpha = \big\{\beta < \kappa : \pi_1\big(\varphi(\alpha)\big) \subseteq \pi_1\big(\varphi(\beta)\big)$ and $\pi_2\big(\varphi(\beta)\big) \subseteq \pi_2\big(\varphi(\alpha)\big)\big\}$. We claim that $\{X_\alpha : \alpha < \kappa\}$ has the κ-uniform finite intersection property. To see this, let $F \in \mathcal{P}_f(\kappa)$, let $G = \bigcup_{\alpha \in F} \pi_1\big(\varphi(\alpha)\big)$ and let $B = \bigcap_{\alpha \in F} \pi_2\big(\varphi(\alpha)\big)$. Then

$$\varphi^{-1}[\{(G, B)\}] \subseteq \textstyle\bigcap_{\alpha \in F} X_\alpha \,.$$

Let $Q = \{q \in \beta\kappa_d : \{X_\alpha : \alpha < \kappa\} \subseteq q\}$. (Here κ_d is κ with the discrete topology.) By [10, Theorem 3.62], $|Q| = 2^{2^\kappa}$. For $q \in Q$, let $p_q = q\text{-}\lim\limits_{\alpha < \kappa} t_\alpha$.

Let $V = \overline{C^\star} \cap \bigcap_{a \in C^\star} \overline{a^{-1}C^\star}$. Then $p \in V$. We claim that V is a subsemigroup of βS. To see this, it suffices by [10, Theorem 4.20] to note that if $a \in C^\star$ and $b \in a^{-1}C^\star$, then $ab \in C^\star$ and

$b\big((ab)^{-1}C^\star\big) \subseteq a^{-1}C^\star$. Now we claim that for all $q \in Q$, $p_q \in V \cap T$. To see that $p_q \in V$, let $a \in C^\star$. Pick $\alpha < \kappa$ such that $\pi_1\big(\varphi(\alpha)\big) = \{a\}$. Then for all $\beta \in X_\alpha$, $t_\beta \in C^\star \cap a^{-1}C^\star$. To see that $p_q \in T$, let $B \in \mathcal{A}$ and pick $\alpha < \kappa$ such that $\pi_2\big(\varphi(\alpha)\big) = A \cap B$. Then for all $\beta \in X_\alpha$, $t_\beta \in A \cap B$.

Now we claim that if q_1 and q_2 are distinct members of Q, then $Tp_{q_1} \cap Tp_{q_2} = \emptyset$. To see this, suppose instead one has $r, v \in T$ such that $rp_{q_1} = vp_{q_2}$. Pick $Y_1 \in q_1$ and $Y_2 \in q_2$ such that $Y_1 \cap Y_2 = \emptyset$. Then $\{s_\gamma t_\alpha : \gamma < \alpha \text{ and } \alpha \in Y_1\} \in rp_{q_1}$ and $\{s_\delta t_\beta : \delta < \beta \text{ and } \beta \in Y_2\} \in vp_{q_2}$. But by (iii), these sets are disjoint.

Given $q \in Q$, $(T \cap V)p_q$ is a left ideal of $T \cap V$ and $R \cap T \cap V$ is a right ideal of $T \cap V$. (It is nonempty since $p \in R \cap T \cap V$.) So pick an idempotent $r \in K(T \cap V) \cap (T \cap V)p_q \cap (R \cap T \cap V)$. Since $p \in K(T) \cap V$, $K(T \cap V) = K(T) \cap V$ by [10, Theorem 1.65] so $r \in K(T)$. □

Corollary 2.4. *Let $(S, \cdot)$ be an infinite very weakly cancellative semigroup with cardinality κ, let p be an idempotent in $K(\beta S_d)$, let $C \in p$, and let R be the minimal right ideal of βS_d to which p belongs. Then there are 2^{2^κ} idempotents in $K(\beta S_d) \cap R \cap \overline{C}$.*

Proof. Let $\mathcal{A} = \{S\}$ and apply Theorem 2.3. □

We now see that there are many examples where Theorem 2.3 applies. By a "left topological" semigroup, we mean a semigroup $(S, \cdot)$ with topology such that for each $x \in S$ the function $\lambda_x : S \to S$, defined by $\lambda_x(y) = x \cdot y$, is continuous.

Corollary 2.5. *Let $(S, \cdot)$ be a nondiscrete weakly cancellative left topological semigroup with identity 1, let*

$$\kappa = \min\{|U| : U \text{ is a neighborhood of } 1\},$$

and assume that $\mathcal{A}$ is a basis of open neighborhoods at 1 with $|\mathcal{A}| \leq \kappa$. Let $T = \bigcap_{U \in \mathcal{A}} \overline{U}$. Then T is a compact subsemigroup of βS_d. Let p be an idempotent in $K(T)$, let $C \in p$, and let R be the minimal right ideal of T to which p belongs. Then there are 2^{2^κ} idempotents in $K(T) \cap R \cap \overline{A}$.

Proof. Pick $U \in \mathcal{A}$ such that $|U| = \kappa$. We may then presume that $|S| = \kappa$, since S could be replaced by the subsemigroup generated by U and $\mathcal{A}$ by $\{U \cap V : V \in \mathcal{A}\}$. It suffices to show that T is a subsemigroup of βS_d, since then Theorem 2.3 applies. By [10, Theorem 4.20], it suffices to note that for each $A \in \mathcal{A}$ and each $a \in A$, there exists $B \in \mathcal{A}$ such that $a \cdot B \subseteq A$. $\square$

We see now that we can get a good deal of the conclusion of Corollary 2.5 even if there is no basis of neighborhoods of 1 of cardinality at most κ.

Corollary 2.6. *Let $(S, \cdot)$ be a nondiscrete weakly cancellative left topological semigroup with identity* 1*, let*

$$\kappa = \min\{|U| : U \text{ is a neighborhood of } 1\}\,,$$

and assume that $\mathcal{B}$ is a family of neighborhoods of 1 *with $|\mathcal{B}| \leq \kappa$ and $|U| = \kappa$ for some $U \in \mathcal{B}$. Then $\bigcap_{U \in \mathcal{B}} \overline{U}$ contains a compact subsemigroup T of βS_d with the property that there are 2^{2^κ} idempotents in every minimal right ideal of T.*

Proof. We may assume that the members of $\mathcal{B}$ are open, since each member can be replaced by its interior. Pick $U \in \mathcal{B}$ such that $|U| = \kappa$. We may then assume that $|S| = \kappa$, since S could be replaced by the subsemigroup generated by U and $\mathcal{B}$ by $\{U \cap V : V \in \mathcal{B}\}$.

We shall now define a family $\mathcal{B}_n$ of open neighborhoods of 1 for every $n \in \omega$. We let $\mathcal{B}_0 = \{\bigcap \mathcal{F} : \mathcal{F} \in \mathcal{P}_f(\mathcal{B})\}$. Now let $n \in \mathbb{N}$ and assume that we have chosen $\mathcal{B}_m$ for each $m \in \{0, 1, \ldots, n-1\}$ such that

1. $\mathcal{B}_m$ is closed under finite intersections;
2. $|\mathcal{B}_m| \leq \kappa$;
3. if $m < n-1$, then $\mathcal{B}_m \subseteq \mathcal{B}_{m+1}$;
4. if $m < n-1$, $B \in \mathcal{B}_m$, and $s \in B$, then there exists $V \in \mathcal{B}_{m+1}$ such that $sV \subseteq B$.

For every $B \in \mathcal{B}_{n-1}$ and every $t \in B$ we choose an open neighborhood $V_{t,B}$ of 1 satisfying $tV_{t,B} \subseteq B$ and we put

$$\mathcal{C} = \mathcal{B}_{n-1} \cup \{V_{t,B} : B \in \mathcal{B}_{n-1} \text{ and } t \in B\}$$

and $\mathcal{B}_n = \{\bigcap \mathcal{F} : \mathcal{F} \in \mathcal{P}_f(\mathcal{C}\}$.

We now let $\mathcal{A} = \bigcup_{n=0}^{\infty} \mathcal{B}_n$. Then $\mathcal{A}$ satisfies the hypotheses of Theorem 2.3. Thus, if $T = \bigcap_{V \in \mathcal{A}} \overline{V}$, it suffices to show that T is a subsemigroup of βS_d, since then Theorem 2.3 applies. By [10, Theorem 4.20], it suffices to note that for each $A \in \mathcal{A}$ and each $a \in A$, there exists $B \in \mathcal{A}$ such that $a \cdot B \subseteq A$. $\square$

In the above two corollaries we assumed that S was weakly cancellative rather than just very weakly cancellative because in the proof we passed to the semigroup generated by U. It is not true in general that a subsemigroup of a very weakly cancellative semigroup is itself very weakly cancellative. (For example, define an operation $*$ on ω by $x * y = 0$, let M be any uncountable cancellative semigroup with identity 1, let $S = \omega \times M$, and let $T = \omega \times \{1\}$. Then any left or right solution set in S is countable, so S is very weakly cancellative, while T is not.)

Corollary 2.7. *Let S be an infinite discrete weakly cancellative semigroup of cardinality κ and let p be a κ-uniform ultrafilter on S which is an idempotent in βS. Suppose that $\mathcal{B}$ is a set of members of p of cardinality at most κ. Then $\bigcap_{B \in \mathcal{B}} \overline{B}$ contains a compact subsemigroup T of βS with the property that every minimal right ideal of T contains 2^{2^κ} idempotents. In particular, every member of p is a member of 2^{2^κ} idempotents.*

Proof. We may suppose that S has an identity 1 since an identity may be adjoined. By [9, Lemma 2.11, Lemma 2.12, and Theorem 2.16] we can define a nondiscrete left invariant topology $\mathcal{T}_p$ on S for which a subset V of S is a neighborhood of 1 if and only if $1 \in V$ and $V \in p$. Our claim now follows from Corollary 2.6. □

The condition of [10, Theorem 4.20] that was used in the proof of Corollaries 2.5 and 2.6 to establish that T is a subsemigroup of βS_d is quite simple. And we shall need to adopt this requirement on $\mathcal{A}$ in subsequent sections. We see now that this condition is not necessary to conclude that T is a semigroup. Given $x \in \mathbb{N}$ we define $\operatorname{supp}(x) \in \mathcal{P}_f(\omega)$ by $x = \sum_{t \in \operatorname{supp}(x)} 2^t$. Given $p, q \in \beta S_d$ and $A \subseteq S$, one has that $A \in p \cdot q$ if and only if $\{x \in S : x^{-1}A \in q\} \in p$. If the operation is denoted by $+$, we have $A \in p + q$ if and only if $\{x \in S : -x + A \in q\} \in p$, where $-x + A = \{y \in S : x + y \in A\}$.

Theorem 2.8. *For each $n \in \mathbb{N}$, let $A_n = \{x \in \mathbb{N} : |\operatorname{supp}(x)| > n\}$, let $\mathcal{A} = \{A_n : n \in \mathbb{N}\}$, and let $T = \bigcap_{A \in \mathcal{A}} \overline{A}$. Then T is a subsemigroup of $(\beta\mathbb{N}, +)$ but $\mathcal{A}$ does not satisfy the condition of* [10, Theorem 4.20]. *In fact, there is no $n \in \mathbb{N}$ such that for all $x \in A_n$ there exists $m \in \mathbb{N}$ with $x + A_m \subseteq A_1$.*

Proof. We verify the last assertion first. So let $n \in \mathbb{N}$ and suppose that for all $x \in A_n$ there exists $m \in \mathbb{N}$ with $x + A_m \subseteq A_1$. Let

$x = 2^{n+1} - 1$. Then $|\mathrm{supp}(x)| = n + 1 > n$ so $x \in A_n$. Pick $m \in \mathbb{N}$ such that $x + A_m \subseteq A_1$. Let $y = 1 + 2^{m+n+1} - 2^{n+1}$. Then $|\mathrm{supp}(y)| = m + 1$ so $y \in A_m$. But $x + y = 2^{m+n+1}$ so $x + y \notin A_1$.

Now to see that T is a semigroup, let $p, q \in T$ and suppose that $p + q \notin T$. Pick $n \in \mathbb{N}$ such that

$$B = \{x \in \mathbb{N} : |\mathrm{supp}(x)| \leq n\} \in p + q\,.$$

Let $C = \{x \in \mathbb{N} : -x + B \in q\}$. Then $C \in p$. Pick $x \in C$ and let $k = \max \mathrm{supp}(x)$. Pick $y \in C \cap A_{n+k+2}$ and let $G = \mathrm{supp}(y)$. Then $|G \setminus \{0, 1, \ldots, k\}| \geq n + 2$. Let $l = \max G$ and pick

$$z \in (-x + B) \cap (-y + B) \cap A_{n+l+1}\,.$$

Let $H = \mathrm{supp}(z)$. We claim that $\{k + 1, k + 2, \ldots, l\} \subseteq H$. For otherwise, if say $t \in \{k+1, k+2, \ldots, l\} \setminus H$, then there is no carrying past position t when x and z are added so

$$H \setminus \{0, 1, \ldots, l\} \subseteq \mathrm{supp}(x + z)$$

so $|\mathrm{supp}(x + z)| > n$, a contradiction. But now, $|\mathrm{supp}(y + z) \cap \{k+1, k+2, \ldots, l\}| \geq |G \setminus \{0, 1, \ldots, k\}| - 1 \geq n+1$, a contradiction. $\square$

3. A New Central Sets Theorem for Subsemigroups

The original Central Sets Theorem (Theorem 1.1) applied to finitely many sequences in $\mathbb{Z}$. And the version presented in [10] as Theorem 14.11 dealt with countably many sequences in an arbitrary commutative semigroup. (There are also versions for noncommutative semigroups, but they are much more complicated and we will not deal with them in this paper.)

In [3, Theorem 2.2] we proved a new version of the Central Sets Theorem which applies to all sequences in a commutative semigroup $(S, +)$ simultaneously. (We are switching to additive notation because from here on we will be restricting our attention to commutative semigroups and additive notation is much more convenient when dealing with partition regularity of matrices as we shall do in the next section.)

Theorem 3.1. *Let $(S, +)$ be a commutative semigroup and let $\mathcal{T} = {}^{\mathbb{N}}S$, the set of sequences in S. Let C be a central subset of S. There exist functions $\alpha : \mathcal{P}_f(\mathcal{T}) \to S$ and $H : \mathcal{P}_f(\mathcal{T}) \to \mathcal{P}_f(\mathbb{N})$ such that*

(1) *if* $F, G \in \mathcal{P}_f(\mathcal{T})$ *and* $F \subsetneq G$, *then* $\max H(F) < \min H(G)$ *and*

(2) *whenever* $m \in \mathbb{N}$, $G_1, G_2, \ldots, G_m \in \mathcal{P}_f(\mathcal{T})$, $G_1 \subsetneq G_2 \subsetneq \ldots \subsetneq G_m$, *and for each* $i \in \{1, 2, \ldots, m\}$, $f_i \in G_i$, *one has* $\sum_{i=1}^{m} \big(\alpha(G_i) + \sum_{t \in H(G_i)} f_i(t)\big) \in C$.

Proof. [3, Theorem 2.2]. □

In this section we shall primarily be interested in a version of the Central Sets Theorem applied to members of minimal idempotents in $T = \bigcap_{A \in \mathcal{A}} \overline{A}$ for a suitable family $\mathcal{A}$ of subsets of S. This will be Theorem 3.4. However, the case $\mathcal{A} = \{S\}$ yields a strengthening of Theorem 3.1, which we shall present as Corollary 3.5. We shall then show that the "strengthening" is only superficial.

The proof of the following lemma uses a technique which has become standard in proofs of versions of the Central Sets Theorem. This technique was originally due to H. Furstenberg and Y. Katznelson in [6].

Lemma 3.2. *Let* $(S, +)$ *be a discrete commutative semigroup, let* $c \in \mathbb{N}$, *let* $\mathcal{A} \subseteq \mathcal{P}(S)$, *let* $T = \bigcap_{A \in \mathcal{A}} \overline{A}$, *let* $(D, \leq)$ *be a directed set, and let* $\mathcal{T} \subseteq {}^{D}S$, *the set of functions from* D *to* S. *Assume that*

(1) $\mathcal{A} \neq \emptyset$ *and* $\emptyset \notin \mathcal{A}$;

(2) $(\forall A \in \mathcal{A})(\forall B \in \mathcal{A})(A \cap B \in \mathcal{A})$;

(3) $(\forall A \in \mathcal{A})(\forall a \in A)(\exists B \in \mathcal{A})(a + B \subseteq A)$; *and*

(4) $\big(\forall A \in \mathcal{A}\big)\big(\forall d \in D\big)\big(\forall F \in \mathcal{P}_f(\mathcal{T})\big)\big(\exists d' \in D\big)\big(d < d'$ *and* $(\forall f \in F)(f(d') \in A)\big)$.

Let $C \subseteq S$ *and assume that* $\overline{C} \cap K(T) \cap \bigcap_{A \in \mathcal{A}} \overline{cA} \neq \emptyset$. *Then for all* $F \in \mathcal{P}_f(\mathcal{T})$, *all* $d \in D$, *and all* $A \in \mathcal{A}$, *there exist* $a \in A$, $m \in \mathbb{N}$, *and* $d_1, d_2, \ldots, d_m$ *in* D *such that* $d < d_1 < d_2 < \ldots < d_m$, $ca \in C$, *and for all* $f \in F$, $ca + \sum_{j=1}^{m} f(d_j) \in C$.

Proof. Pick $p \in \overline{C} \cap K(T) \cap \bigcap_{A \in \mathcal{A}} \overline{cA}$. Let $F \in \mathcal{P}_f(\mathcal{T})$ be given. Let $l = |F|$ and enumerate F as $\{f_1, f_2, \ldots, f_l\}$. Let $Y = \times_{i=1}^{l+1} T$ and $Z = \times_{i=1}^{l+1} \beta S$. Then by [10, Theorem 2.22], Y and Z are compact right topological semigroups and if $\vec{x} \in \times_{i=1}^{l+1} S$, then $\lambda_{\vec{x}} : Z \to Z$ is continuous.

For $d \in D$ and $A \in \mathcal{A}$, let

$$\begin{array}{rcl} I_{A,d} & = & \big\{\big(ca + \sum_{j=1}^m f_1(d_j), \ldots, ca + \sum_{j=1}^m f_l(d_j), ca\big) : \\ & & a \in A\,,\, ca \in A\,,\, m \in \mathbb{N}\,, d_1, d_2, \ldots, d_m \in D\,, \\ & & d < d_1 < d_2 < \ldots < d_m, \text{ and} \\ & & (\forall i \in \{1,2,\ldots,l\})(ca + \sum_{j=1}^m f_i(d_j) \in A)\big\} \end{array}$$

and let $E_{A,d} = I_{A,d} \cup \{(ca, ca, \ldots, ca) : a \in A \text{ and } ca \in A\}$.

Let $I = \bigcap_{A \in \mathcal{A}} \bigcap_{d \in D} c\ell_Z I_{A,d}$ and let $E = \bigcap_{A \in \mathcal{A}} \bigcap_{d \in D} c\ell_Z E_{A,d}$. Trivially $I \subseteq E$. We claim that E is a subsemigroup of Y and I is an ideal of E.

We show first that each $I_{A,d} \neq \emptyset$. From this it follows that $I \neq \emptyset$ since if $d, d' \in D$, $A, A' \in \mathcal{A}$, and $e \in D$ such that $d \leq e$ and $d' \leq e$, then $I_{A \cap A',e} \subseteq I_{A,d} \cap I_{A',d'}$. So let $A \in A$ and $d \in D$. Now $A \cap cA \in p$ so pick $a \in A$ such that $ca \in A$. Pick $B \in \mathcal{A}$ such that $ca + B \subseteq A$. Pick by (4) some $d' \in D$ such that $d < d'$ and for all $i \in \{1, 2, \ldots, l\}$, $f_i(d') \in B$. Then $\big(ca + f_1(d'), ca + f_2(d'), \ldots, ca + f_l(d'), ca\big) \in I_{A,d}$.

To see that $E \subseteq Y$, let $\vec{q} = (q_1, q_2, \ldots, q_{l+1}) \in E$. Let $i \in \{1, 2, \ldots, l+1\}$. To see that $q_i \in T$ let $A \in \mathcal{A}$. Pick any $d \in D$. If $A \notin q_i$, then $\pi_i^{-1}[\overline{S \setminus A}]$ would be a neighborhood of $\vec{q}$ missing $E_{A,d}$.

Now let $\vec{q}, \vec{r} \in E$. We show that $\vec{q} + \vec{r} \in E$ and if either $\vec{q} \in I$ or $\vec{r} \in I$, then $\vec{q} + \vec{r} \in I$. Let $A \in \mathcal{A}$, let $d \in D$, and let U be an open neighborhood of $\vec{q} + \vec{r}$. Pick a neighborhood V of $\vec{q}$ such that $V + \vec{r} \subseteq U$. Pick $\vec{x} \in V \cap E_{A,d}$ with $\vec{x} \in I_{a,d}$ if $\vec{q} \in I$.

If $\vec{x} \in I_{A,d}$, pick $a \in A$, $m \in \mathbb{N}$, and $d_1, d_2, \ldots, d_m \in D$ such that $d < d_1 < d_2 < \ldots < d_m$, $ca \in A$, $x_{l+1} = ca$, and for all $i \in \{1, 2, \ldots, l\}$, $x_i = ca + \sum_{j=1}^m f_i(d_j) \in A$. Pick $B_0 \in \mathcal{A}$ such that $a + B_0 \subseteq A$ and $ca + B_0 \subseteq A$. For $i \in \{1, 2, \ldots, l\}$, pick $B_i \in \mathcal{A}$ such that $ca + \sum_{j=1}^m f_i(d_j) + B_i \subseteq A$. Let $B = \bigcap_{i=0}^l B_i$ and let $e = d_m$.

If $\vec{x} \in E_{A,d} \setminus I_{a,d}$, pick $a \in A$ such that $ca \in A$ and $\vec{x} = (ca, ca, \ldots, ca)$. Pick $B \in \mathcal{A}$ such that $a + B \subseteq A$ and $ca + B \subseteq A$ and let $e = d$.

Now $\vec{x} + \vec{r} \in U$ and $\vec{x} \in \times_{i=1}^{l+1} S$ so pick a neighborhood W of $\vec{r}$ such that $\vec{x} + W \subseteq U$. Pick $\vec{y} \in W \cap E_{B,e}$ with $\vec{y} \in I_{B,e}$ if $\vec{r} \in I$.

If $\vec{y} \in I_{B,e}$, pick $b \in B$, $n \in \mathbb{N}$, and $e_1, e_2, \ldots, e_n \in D$ such that $cb \in B$, $e < e_1 < e_2 < \ldots < e_n$, $y_{l+1} = cb$, and for $i \in \{1, 2, \ldots, l\}$, $y_i = cb + \sum_{j=1}^n f_i(e_j) \in B$.

If $\vec{y} \in E_{B,e} \setminus I_{B,e}$, pick $b \in B$ such that $cb \in B$ and $\vec{y} = (cb, cb, \ldots, cb)$.

Then $\vec{x}+\vec{y} \in U \cap E_{A,d}$ and if $\vec{x} \in I_{A,d}$ or $\vec{y} \in I_{B,e}$, then $\vec{x}+\vec{y} \in U \cap I_{A,d}$.

By [10, Theorem 2.23], $K(Y) = \times_{i=1}^{l+1} K(T)$. Let $\overline{p} = (p, p, \ldots, p)$. It suffices to show that $\overline{p} \in E$. For then $\overline{p} \in E \cap K(Y)$ and so by [10, Theorem 1.65] $K(E) = E \cap K(Y)$ and thus $\overline{p} \in K(E) \subseteq I$. Since $\times_{i=1}^{l+1} \overline{C}$ is a neighborhood of $\overline{p}$ and will therefore meet $I_{A,d}$ for any $A \in \mathcal{A}$ and any $d \in D$, this will complete the proof.

So let $A \in \mathcal{A}$, let $d \in D$, and let U be a neighborhood of $\overline{p}$. Pick $M \in p$ such that $\times_{i=1}^{l+1} \overline{M} \subseteq U$. Now $M \cap A \cap cA \in p$ so pick $a \in A$ such that $ca \in M \cap A$. Then $(ca, ca, \ldots, ca) \in U \cap E_{A,d}$. □

Definition 3.3. Let $(D, \leq)$ be a directed set. Then $\mathcal{P}_f^{\text{lin}}(D) = \{H : H \text{ is a finite nonempty linearly ordered subset of } D\}$.

Theorem 3.4. *Let $(S,+)$ be a discrete commutative semigroup, let $\mathcal{A} \subseteq \mathcal{P}(S)$, let $T = \bigcap_{A \in \mathcal{A}} \overline{A}$, let $(D, \leq)$ be a directed set, and let $\mathcal{T} \subseteq {}^{D}S$. Assume that*

(1) *$\mathcal{A} \neq \emptyset$ and $\emptyset \notin \mathcal{A}$;*
(2) *$(\forall A \in \mathcal{A})(\forall B \in \mathcal{A})(A \cap B \in \mathcal{A})$;*
(3) *$(\forall A \in \mathcal{A})(\forall a \in A)(\exists B \in \mathcal{A})(a + B \subseteq A)$; and*
(4) *$\big(\forall A \in \mathcal{A}\big)\big(\forall d \in D\big)\big(\forall F \in \mathcal{P}_f(\mathcal{T})\big)\big(\exists d' \in D\big)\big(d < d'$ and $(\forall f \in F)(f(d') \in A)\big)$.*

Let $C \subseteq S$ and assume that there is an idempotent p in $\overline{C} \cap K(T)$. Let $\varphi : \mathcal{P}_f(\mathcal{T}) \to \mathcal{A}$. Then there exist $\alpha : \mathcal{P}_f(\mathcal{T}) \to C$ and $H : \mathcal{P}_f(\mathcal{T}) \to \mathcal{P}_f^{\text{lin}}(D)$ such that

(a) *$\big(\forall F \in \mathcal{P}_f(\mathcal{T})\big)\big(\alpha(F) \in \varphi(F)\big)$;*
(b) *$\big(\forall F \in \mathcal{P}_f(\mathcal{T})\big)\big(\forall f \in F\big)\big(\alpha(F) + \sum_{t \in H(F)} f(t) \in C \cap \varphi(F)\big)$;*
(c) *if $F, G \in \mathcal{P}_f(\mathcal{T})$ and $G \subsetneq F$, then $\max H(G) < \min H(F)$; and*
(d) *if $m \in \mathbb{N}$, $G_1, G_2, \ldots, G_m \in \mathcal{P}_f(\mathcal{T})$, $G_1 \subsetneq G_2 \subsetneq \ldots \subsetneq G_m$, and for each $i \in \{1, 2, \ldots, m\}$, $f_i \in G_i$, then $\sum_{i=1}^{m} \big(\alpha(G_i) + \sum_{t \in H(G_i)} f_i(t)\big) \in C$.*

Proof. Let $C^\star = \{x \in C : -x + C \in p\}$. Then by [10, Lemma 4.14], for each $x \in C^\star$, $-x + C^\star \in p$. We define $\alpha(F)$ and $H(F)$ by induction on $|F|$ such that

(i) $\alpha(F) \in \varphi(F) \cap C^\star$;
(ii) for all $f \in F$, $\alpha(F) + \sum_{t \in H(F)} f(t) \in C^\star \cap \varphi(F)$;

(iii) if $\emptyset \neq G \subsetneq F$, then $\max H(G) < \min H(F)$; and
(iv) if $m \in \mathbb{N}$, $G_1, G_2, \ldots, G_m \in \mathcal{P}_f(\mathcal{T})$, $G_1 \subsetneq G_2 \subsetneq \ldots \subsetneq G_m = F$, and for each $i \in \{1, 2, \ldots, m\}$, $f_i \in G_i$, then $\sum_{i=1}^{m} \big(\alpha(G_i) + \sum_{t \in H(G_i)} f_i(t)\big) \in C^\star$.

Assume first that $F = \{f\}$. Pick by Lemma 3.2 (with $c = 1$) $\alpha(F) \in \varphi(F) \cap C^\star$, $m \in \mathbb{N}$, and $d_1, d_2, \ldots, d_m \in D$ such that $d_1 < d_2 < \ldots < d_m$ and $\alpha(F) + \sum_{j=1}^{m} f(d_j) \in C^\star \cap \varphi(F)$. Let $H(F) = \{d_1, d_2, \ldots, d_m\}$. All hypotheses hold, (iii) vacuously.

Now assume that $|F| > 1$ and $\alpha(G)$ and $H(G)$ have been defined for all G with $\emptyset \neq G \subsetneq F$. Let $L = \bigcup\{H(G) : \emptyset \neq G \subsetneq F\}$ and pick $d \in D$ such that if $\emptyset \neq G \subsetneq F$, then $\max H(G) \leq d$. Let

$$\begin{aligned} M \;=\; & \big\{ \textstyle\sum_{i=1}^{r} \big(\alpha(G_i) + \sum_{t \in H(G_i)} f_i(t)\big) : \\ & r \in \mathbb{N}\,,\, \emptyset \neq G_1 \subsetneq G_2 \subsetneq \ldots \subsetneq G_r \subsetneq F \\ & \text{and for each } i \in \{1, 2, \ldots, r\}\,,\, f_i \in G_i \big\}\,. \end{aligned}$$

By hypothesis (iv), M is a finite subset of $C^\star$. Let

$$B = C^\star \cap \varphi(F) \cap \textstyle\bigcap_{x \in M} (-x + C^\star)\,.$$

Then $B \in p$. Pick by Lemma 3.2 $\alpha(F) \in \varphi(F) \cap C^\star$, $n \in \mathbb{N}$, and $e_1, e_2, \ldots, e_n \in D$ such that $d < e_1 < e_2 < \ldots < e_n$ and for each $f \in F$, $\alpha(F) + \sum_{j=1}^{m} f(e_j) \in B$. Let $H(F) = \{e_1, e_2, \ldots, e_n\}$. Then hypotheses (i) and (ii) are satisfied directly. If $\emptyset \neq G \subsetneq F$, then $\max H(G) \leq d < e_1 = \min H(F)$, so hypothesis (iii) holds.

To verify hypothesis (iv), let $m \in \mathbb{N}$ and assume that $\emptyset \neq G_1 \subsetneq G_2 \subsetneq \ldots \subsetneq G_m = F$ and for each $i \in \{1, 2, \ldots, m\}$, $f_i \in G_i$. Assume first that $m = 1$. Then $\alpha(G_1) + \sum_{t \in H(G_1)} f_1(t) = \alpha(F) + \sum_{t \in H(F)} f_1(t) \in C^\star$. Now assume that $m > 1$ and let $r = m - 1$. Then $x = \sum_{i=1}^{r} \big(\alpha(G_i) + \sum_{t \in H(G_i)} f_i(t)\big) \in M$ and $\alpha(F) + \sum_{t \in H(F)} f_m(t) \in B \subseteq -x + C^\star$. □

Corollary 3.5. *Let $(S, +)$ be a discrete commutative semigroup, let $(D, \leq)$ be a directed set with no maximum element, let $\mathcal{T} = {}^{D}S$, and let C be a central subset of S. Then there exist $\alpha : \mathcal{P}_f(\mathcal{T}) \to C$ and $H : \mathcal{P}_f(\mathcal{T}) \to \mathcal{P}_f^{\mathrm{lin}}(D)$ such that*

(a) *if $F, G \in \mathcal{P}_f(\mathcal{T})$ and $G \subsetneq F$, then* $\max H(G) < \min H(F)$ *and*

(d) *if* $m \in \mathbb{N}$, $G_1, G_2, \ldots, G_m \in \mathcal{P}_f(\mathcal{T})$, $G_1 \subsetneq G_2 \subsetneq \ldots \subsetneq G_m$, *and for each* $i \in \{1, 2, \ldots, m\}$, $f_i \in G_i$, *then* $\sum_{i=1}^{m} \big(\alpha(G_i) + \sum_{t \in H(G_i)} f_i(t)\big) \in C$.

Proof. In Theorem 3.4 let $\mathcal{A} = \{S\}$ and of course let $\varphi(F) = S$ for each $F \in \mathcal{P}_f(\mathcal{T})$. □

Theorem 3.1 is the special case of Corollary 3.5 in which $D = \mathbb{N}$. We see now that in fact Corollary 3.5 is indeed derivable from Theorem 3.1.

Theorem 3.6. *Let* S *be a commutative semigroup, let* $(D, \leq)$ *be a directed set with no largest element, let* $\mathcal{T}_2 = {}^{D}S$, *and let* $C \subseteq S$ *satisfy the conclusion of Theorem* 3.1. *Then* C *also satisfies the conclusion of Corollary* 3.5.

Proof. Let $\mathcal{T}_1 = {}^{\mathbb{N}}S$ and pick $\alpha_1 : \mathcal{P}_f(\mathcal{T}_1) \to S$ and $H_1 : \mathcal{P}_f(\mathcal{T}_1) \to \mathcal{P}_f(\mathbb{N})$ as guaranteed by Theorem 3.1.

Pick an increasing sequence $\langle a_n \rangle_{n=1}^{\infty}$ in D. Define $\varphi : \mathcal{T}_2 \to \mathcal{T}_1$ by $\varphi(f)(n) = f(a_n)$.

We define $\gamma : \mathcal{P}_f(\mathcal{T}_2) \to \mathcal{P}_f(\mathcal{T}_1)$ so that

(1) for all $G \in \mathcal{P}_f(\mathcal{T}_2)$, $\varphi[G] \subseteq \gamma(G)$ and
(2) if $F, G \in \mathcal{P}_f(\mathcal{T}_2)$ and $F \subsetneq G$, then $\gamma(F) \subsetneq \gamma(G)$.

We do this for $G \in \mathcal{P}_f(\mathcal{T}_2)$ inductively on $|G|$. For $f \in \mathcal{T}_2$, let $\gamma(\{f\}) = \{\varphi(f)\}$. Now let $G \in \mathcal{P}_f(\mathcal{T}_2)$ with $|G| \geq 2$ and assume that $\gamma(F)$ has been defined for all F with $\emptyset \neq F \subsetneq G$. Let $K = \varphi[G] \cup \bigcup\{\gamma(F) : \emptyset \neq F \subsetneq G\}$, pick $f \in \mathcal{T}_1 \setminus K$, and let $\gamma(G) = K \cup \{f\}$.

Now define $\alpha_2 : \mathcal{P}_f(\mathcal{T}_2) \to S$ and $H_2 : \mathcal{P}_f(\mathcal{T}_2) \to \mathcal{P}_f^{\mathrm{lin}}(D)$ as follows. Given $G \in \mathcal{P}_f(\mathcal{T}_2)$, let $\alpha_2(G) = \alpha_1\big(\gamma(G)\big)$ and $H_2(G) = \big\{a_t : t \in H_1\big(\gamma(G)\big)\big\}$. If $F, G \in \mathcal{P}_f(\mathcal{T}_2)$ and $F \subsetneq G$, then $\gamma(F) \subsetneq \gamma(G)$, so $\max H_1\big(\gamma(F)\big) < \min H_1\big(\gamma(G)\big)$ and thus $\max H_2(F) < \min H_2(G)$.

Now assume that $m \in \mathbb{N}$, $G_1, G_2, \ldots, G_m \in \mathcal{P}_f(\mathcal{T}_2)$, $G_1 \subsetneq G_2 \subsetneq \ldots \subsetneq G_m$ and for each $i \in \{1, 2, \ldots, m\}$, $f_i \in G_i$. Then $\gamma(G_1) \subsetneq \gamma(G_2) \subsetneq \ldots \subsetneq \gamma(G_m)$ and for each $i \in \{1, 2, \ldots, m\}$, $\varphi(f_i) \in \gamma(G_i)$ so

$$\begin{array}{cl} & \sum_{i=1}^{m}\big(\alpha_2(G_i)+\sum_{s\in H_2(G_i)} f_i(s)\big) \\ = & \sum_{i=1}^{m}\big(\alpha_2(G_i)+\sum_{t\in H_1(\gamma(G_i))} f_i(a_t)\big) \\ = & \sum_{i=1}^{m}\big(\alpha_1\big(\gamma(G_i)\big)+\sum_{t\in H_1(\gamma(G_i))} \varphi(f_i)(t)\big) \\ \in & C\,. \end{array}$$

□

4. Ramsey Theoretic Consequences

We obtain conditions in this section guaranteeing that members of minimal idempotents in $T = \bigcap_{A\in\mathcal{A}} \overline{A}$ contain images of first entries matrices with entries from ω or $\mathbb{Z}$. We also show that these conditions are satisfied in commonly arising situations.

Lemma 4.1. *Let $(S,+)$ be a commutative semigroup, let $\mathcal{A} \subseteq \mathcal{P}(S)$, let $u,v \in \mathbb{N}$ and let M be a $u\times v$ matrix with entries from ω and no row equal to $\vec{0}$. Assume that*

(1) $(\forall A\in\mathcal{A})(\forall B\in\mathcal{A})(A\cap B\in\mathcal{A})$ *and*
(2) $(\forall A\in\mathcal{A})(\exists B\in\mathcal{A})(B+B\subseteq A)$.

Then for every $A\in\mathcal{A}$ there exists $B\in\mathcal{A}$ such that for all $\vec{x}\in B^v$, $M\vec{x}\in A^u$.

Proof. Define inductively $\varphi_l(A)$ by $\varphi_1(A) = A$ and $\varphi_{l+1}(A) = \varphi_l(A) + A$. One easily sees by induction that for all $l\in\mathbb{N}$ and all $A\in\mathcal{A}$, there exists $B\in\mathcal{A}$ such that $\varphi_l(B)\subseteq A$.

Let $A\in\mathcal{A}$ be given. For $i\in\{1,2,\ldots,u\}$, let $l_i = \sum_{j=1}^{v} m_{i,j}$ and pick $B_i\in\mathcal{A}$ such that $\varphi_{l_i}(B_i)\subseteq A$. Let $B = \bigcap_{i=1}^{l} B_i$. □

Definition 4.2. If M is a first entries matrix, then $P(M)$ is the set of first entries of M. If $P(M)=\{1\}$, then M is a *monic first entries matrix*.

Some of the classic results of Ramsey Theory assert that, given a fine partition $\mathbb{N}$, one cell contains the image of a monic first entries matrix. For example Schur's Theorem [11] and the length 4 version of van der Waerden's Theorem [13] are respectively the assertions that

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{pmatrix}$$

have that property.

Theorem 4.3. *Let $(S,+)$ be a commutative semigroup, let $\mathcal{A} \subseteq \mathcal{P}(S)$, let $u, v \in \mathbb{N}$, let M be a $u \times v$ first entries matrix with entries from ω, and assume that*

(1) $(\forall A \in \mathcal{A})(\forall B \in \mathcal{A})(A \cap B \in \mathcal{A})$;
(2) $\mathcal{A} \neq \emptyset$ *and* $\emptyset \notin \mathcal{A}$;
(3) $(\forall A \in \mathcal{A})(\forall a \in A)(\exists B \in \mathcal{A})(a + B \subseteq A)$; *and*
(4) $(\forall A \in \mathcal{A})(\exists B \in \mathcal{A})(B + B \subseteq A)$.

Let $T = \bigcap_{A \in \mathcal{A}} \overline{A}$. Then T is a subsemigroup of βS_d. If p is an idempotent in $K(T) \cap \bigcap_{A \in \mathcal{A}} \bigcap_{c \in P(M)} \overline{cA}$ and $C \in p$, then for all $A \in \mathcal{A}$ there exists $\vec{x} \in A^v$ such that $M\vec{x} \in C^u$.

Proof. Assume that p is an idempotent in $K(T) \cap \bigcap_{A \in \mathcal{A}} \bigcap_{c \in P(M)} \overline{cA}$ and $C \in p$. Let $C^\star = \{x \in C : -x + C \in p\}$. We may assume that M has no repeated rows. We proceed by induction on v. If $v = 1$, then $M = (c)$. Given $A \in \mathcal{A}$, $cA \cap C \in p$ so pick $x \in A$ such that $cx \in C$.

Now let $v \in \mathbb{N}$ and assume the result is valid for v. Let M be a $u \times (v+1)$ matrix. By rearranging rows and adding rows if need be, we have

$$M = \begin{pmatrix} c & \overline{0} \\ \overline{c} & M_1 \\ \overline{0} & M_2 \end{pmatrix}$$

where M_1 is a $u_1 \times v$ matrix with entries from ω and no row equal to $\vec{0}$ and M_2 is a $u_2 \times v$ first entries matrix. Also $P(M) = P(M_2) \cup \{c\}$. For $i \in \{1, 2, \ldots, u-1\}$ let $\vec{r}_i$ be the i^{th} row of $\begin{pmatrix} M_1 \\ M_2 \end{pmatrix}$.

Let $\mathcal{D}$ be the set of finite sequences in $\mathcal{A}$, ordering $\mathcal{D}$ by $\vec{G} < \vec{G}'$ if and only if $\vec{G}$ is a proper subsequence of $\vec{G}'$. Given $\vec{G} \in \mathcal{D}$, let $\ell(\vec{G})$ be the number of terms in $\vec{G}$. We claim that we can choose $\vec{x}(\vec{G}) \in S^v$ for each $\vec{G} \in \mathcal{D}$ so that

(a) $\vec{r}_i \cdot \vec{x}(\vec{G}) \in \bigcap_{j=1}^{\ell(\vec{G})} G_j$ for each $i \in \{1, 2, \ldots, u_1\}$ and
(b) if $m \in \mathbb{N}$, $\vec{G}_1, \vec{G}_2, \ldots, \vec{G}_m \in \mathcal{D}$, and $\vec{G}_1 < \vec{G}_2 < \ldots < \vec{G}_m$, then
 (i) $M_2\big(\vec{x}(\vec{G}_1) + \vec{x}(\vec{G}_2) + \ldots + \vec{x}(\vec{G}_m)\big) \in (C^\star)^{u_2}$ and
 (ii) $\vec{x}(\vec{G}_1) + \vec{x}(\vec{G}_2) + \ldots + \vec{x}(\vec{G}_m) \in \left(\bigcap_{i=1}^{\ell(\vec{G}_1)} G_{1,i}\right)^v$.

We proceed by induction on $\ell(\vec{G})$. Assume first that $\vec{G} = \langle A \rangle$. Pick by Lemma 4.1 some $B \in \mathcal{A}$ such that for all $\vec{y} \in B^v$, $M_1\vec{y} \in A^{u_1}$. Since $A \cap B \in \mathcal{A}$, pick $\vec{x}(\vec{G}) \in (A \cap B)^v$ such that $M_2\vec{x}(\vec{G}) \in (C^\star)^{u_2}$. The hypotheses are satisfied.

Now assume that $\ell(\vec{G}) > 1$ and we have chosen $\vec{x}(\vec{H})$ for every proper subsequence $\vec{H}$ of $\vec{G}$. Let $A = \bigcap_{i=1}^{\ell(\vec{G})} G_i$. Let

$$\begin{aligned} R = \{(\vec{H}_1, \vec{H}_2, \ldots, \vec{H}_k) : \ & k \in \mathbb{N}\,,\ \vec{H}_1, \vec{H}_2, \ldots, \vec{H}_k \in \mathcal{D}\,, \\ & \text{and } \vec{H}_1 < \vec{H}_2 < \ldots < \vec{H}_k < \vec{G}\}\,. \end{aligned}$$

Given $\mathcal{H} = (\vec{H}_1, \vec{H}_2, \ldots, \vec{H}_k) \in R$, we have

$$\vec{x}(\vec{H}_1) + \vec{x}(\vec{H}_2) + \ldots + \vec{x}(\vec{H}_k) \in \left(\bigcap_{i=1}^{\ell(\vec{H}_1)} H_{1,i}\right)^v$$

and $\bigcap_{i=1}^{\ell(\vec{H}_1)} H_{1,i} \in \mathcal{A}$. Applying assumptions (1) and (4) pick $D_\mathcal{H} \in \mathcal{A}$ such that $\vec{x}(\vec{H}_1)+\vec{x}(\vec{H}_2)+\ldots+\vec{x}(\vec{H}_k)+(D_\mathcal{H})^v \subseteq \left(\bigcap_{i=1}^{\ell(\vec{H}_1)} H_{1,i}\right)^v$.

Pick by Lemma 4.1 some $B \in \mathcal{A}$ such that for all $\vec{y} \in B^v$, $M_1\vec{y} \in A^{u_1}$. Let $E = A \cap B \cap \bigcap_{\mathcal{H}\in R} D_\mathcal{H}$. Let

$$\begin{aligned} Q \ = \ & \big\{\vec{r}_i \cdot \big(\vec{x}(\vec{H}_1) + \ldots + \vec{x}(\vec{H}_k)\big) : (\vec{H}_1, \vec{H}_2, \ldots, \vec{H}_k) \in R \\ & \text{and } i \in \{u_1+1, u_1+2, \ldots, u-1\}\big\}\,. \end{aligned}$$

Then $Q \subseteq C^\star$. Pick $\vec{x}(\vec{G}) \in E^v$ such that

$$M_2\vec{x}(\vec{G}) \in \big(C^\star \cap \textstyle\bigcap_{b\in Q}(-b + C^\star)\big)^{u_2}\,.$$

Since $\vec{x}(\vec{G}) \in B^v$, hypothesis (a) holds. To verify hypothesis (b), let $m \in \mathbb{N}$, let $\vec{H}_1, \vec{H}_2, \ldots, \vec{H}_m \in \mathcal{D}$, and assume that $\vec{H}_1 < \vec{H}_2 < \ldots < \vec{H}_m = \vec{G}$. If $m = 1$, we have $\vec{x}(\vec{G}) \in A^v$ and $M_2\vec{x}(\vec{G}) \in (C^\star)^{u_2}$ as required, so assume that $m > 1$ and let $\mathcal{H} = (\vec{H}_1, \vec{H}_2, \ldots, \vec{H}_{m-1}) \in R$. Then $\vec{x}(\vec{G}) \in (D_\mathcal{H})^v$ so

$$\vec{x}(\vec{H}_1) + \vec{x}(\vec{H}_2) + \ldots + \vec{x}(\vec{H}_{m-1}) + \vec{x}(\vec{G}) \in \left(\bigcap_{i=1}^{\ell(\vec{H}_1)} H_{1,i}\right)^v\,.$$

And for $i \in \{u_1+1, u_1+2, \ldots, u-1\}$,

$$b_i = \vec{r}_i \cdot \big(\vec{x}(\vec{H}_1) + \vec{x}(\vec{H}_2) + \ldots + \vec{x}(\vec{H}_k)\big) \in Q$$

so $\vec{r}_i \cdot \vec{x}(\vec{G}) \in -b + C^\star$ and thus

$$\vec{r}_i \cdot \big(\vec{x}(\vec{H}_1) + \vec{x}(\vec{H}_2) + \ldots + \vec{x}(\vec{H}_k) + \vec{x}(\vec{G})\big) \in C^\star\,.$$

Now for $i \in \{1, 2, \ldots, u_1\}$, define $f_i : \mathcal{D} \to S$ by $f_i(\vec{G}) = \vec{r}_i \cdot \vec{x}(\vec{G})$. Let $\mathcal{T} = \{f_1, f_2, \ldots, f_{u_1}\}$. We claim that the hypotheses of Lemma 3.2 are satisfied. Hypotheses (1), (2), and (3) are satisfied directly. To verify hypothesis (4), let $A \in \mathcal{A}$, $\vec{G} = \langle G_1, G_2, \ldots, G_k \rangle \in \mathcal{D}$, and $F \in \mathcal{P}_f(\mathcal{T})$ be given. Let $\vec{G}' = \langle G_1, G_2, \ldots, G_k, A \rangle$. Then $\vec{G} < \vec{G}'$ and if $f_i \in F$, then $f_i(\vec{G}') = \vec{r}_i \cdot \vec{x}(\vec{G}') \in A$ by condition (a).

To conclude the proof, let $A \in \mathcal{A}$ and let $\vec{G} = \langle A \rangle$. Pick by Lemma 3.2 $a \in A$, $m \in \mathbb{N}$, and $\vec{G}_1, \vec{G}_2, \ldots, \vec{G}_m$ in $\mathcal{D}$ such that $\vec{G} < \vec{G}_1 < \vec{G}_2 < \ldots < \vec{G}_m$, $ca \in C$, and for all $f \in F$, $ca + \sum_{j=1}^{m} f(\vec{G}_j) \in C$. Let $\vec{y} = \vec{x}(\vec{G}_1) + \vec{x}(\vec{G}_2) + \ldots + \vec{x}(\vec{G}_m)$. Then $\vec{y} \in A^v$ and $M_2\vec{y} \in C^{u_2}$. Also, if $i \in \{1, 2, \ldots, u_1\}$, then $ca + \vec{r}_i \cdot \vec{y} = ca + \sum_{j=1}^{m} f_i(\vec{G}_j) \in C$. So $\begin{pmatrix} a \\ \vec{y} \end{pmatrix} \in A^{v+1}$ and $M \begin{pmatrix} a \\ \vec{y} \end{pmatrix} \in C^u$. $\square$

It is possible that Theorem 4.3 is vacuous. That is, there might not be any idempotents in $K(T) \cap \bigcap_{A \in \mathcal{A}} \bigcap_{c \in P(M)} \overline{cA}$ – this set might even be empty. For example if $(S, +) = (\mathbb{N}, \cdot)$, $\mathcal{A} = \{\mathbb{N}\}$, and $M = (2)$, one has that the multiplicative analogue of $2\mathbb{N}$ is $\{x^2 : x \in \mathbb{N}\}$ and $K(\beta\mathbb{N}, \cdot) \cap \{x^2 : x \in \mathbb{N}\} = \emptyset$. (See [10, Exercise 15.1.2].) We are of course interested in knowing when the conclusion of Theorem 4.3 is both nonvacuous and nontrivial. We show first that for monic first entries matrices, this is easy to guarantee.

Since S is commutative, the hypothesis that S is left topological in the following theorem is equivalent to saying that S is semitopological, that is, the operation is separately continuous. Recall that $\overline{A}$ denotes the closure of A in βS_d, where S_d is the set S with the discrete topology.

Corollary 4.4. *Let $(S, +)$ be a commutative and weakly cancellative Hausdorff semitopological semigroup with identity 0 and assume that 0 is not an isolated point and $+$ is jointly continuous at 0. Let $\mathcal{A}$ denote the family of neighborhoods of 0, let $T = \bigcap_{A \in \mathcal{A}} \overline{A}$, let $u, v \in \mathbb{N}$, and let M be a $u \times v$ monic first entries matrix with entries from ω. Then T is a subsemigroup of βS_d and $K(T) \subseteq \beta S_d \setminus S$ so there are minimal idempotents of $K(T)$ in $\beta S_d \setminus S$. In particular, whenever $A \in \mathcal{A}$, $r \in \mathbb{N}$, and $A = \bigcup_{i=1}^{r} C_i$, there exist $i \in \{1, 2, \ldots, r\}$ and $\vec{x} \in (C_i)^v$ such that C_i is infinite and $M\vec{x} \in (C_i)^u$.*

Proof. The hypotheses of Theorem 4.3 are satisfied. By [10, Theorem 4.36] $\beta S_d \setminus S$ is an ideal of βS_d and, since each member of $\mathcal{A}$ is infinite, $K(T) \cap (\beta S_d \setminus S) \neq \emptyset$ and so $K(T) \subseteq \beta S_d \setminus S$. Pick an idempotent $p \in K(T)$, let $A \in \mathcal{A}$, let $r \in \mathbb{N}$, and let $A = \bigcup_{i=1}^r C_i$. Pick $i \in \{1, 2, \ldots, r\}$ such that $C_i \in p$ and note that C_i is infinite. Theorem 4.3 guarantees directly the existence of $\vec{x} \in A^v$ such that $M\vec{x} \in (C_i)^u$. To see that one may guarantee that $\vec{x} \in (C_i)^v$, note that we may assume that for each $j \in \{1, 2, \ldots, v\}$, there is a row of M whose j^{th} entry is 1 and all other entries are 0. $\square$

The following corollary provides a somewhat more complicated method of guaranteeing that one cell of a partition must contain the image of a first entries matrix which need not be monic.

Corollary 4.5. *Let $(S, +)$ be a commutative and weakly cancellative semigroup with identity 0, let $\mathcal{A} \subseteq \mathcal{P}(S)$, and assume that*

(1) $(\forall A \in \mathcal{A})(\forall B \in \mathcal{A})(A \cap B \in \mathcal{A})$;
(2) $\mathcal{A} \neq \emptyset$ *and* $\emptyset \notin \mathcal{A}$;
(3) $(\forall A \in \mathcal{A})(\forall a \in A)(\exists B \in \mathcal{A})(a + B \subseteq A)$; *and*
(4) $(\forall A \in \mathcal{A})(\exists B \in \mathcal{A})(B + B \subseteq A)$.

Let V be a subsemigroup of $(\mathbb{N}, \cdot)$ with $1 \in V$ and assume that $\{cA : A \in \mathcal{A}$ and $c \in V\}$ has the infinite finite intersection property. Let $u, v \in \mathbb{N}$ and let M be a $u \times v$ first entries matrix with entries from ω with $P(M) \subseteq V$. Let $T = \bigcap \{\overline{cA} : A \in \mathcal{A}$ and $c \in V\}$. Then T is a subsemigroup of βS_d for which $K(T) \subseteq \beta S_d \setminus S$. If p is any idempotent in $K(T)$ and if $C \in p$, there exists $\vec{x} \in S^v$ such that $M\vec{x} \in C^u$. In particular, if $r \in \mathbb{N}$ and $S = \bigcup_{i=1}^r C_i$, then there exists $i \in \{1, 2, \ldots, r\}$ such that $C_i \setminus \{0\}$ is infinite and contains an image of M.

Proof. Let $\mathcal{B} = \big\{ \bigcap \mathcal{F} : \mathcal{F}$ is a finite nonempty subset of $\{cA : c \in V$ and $A \in \mathcal{A}\}\big\}$. Then $\mathcal{B}$ satisfies (1), (2), (3), and (4). Also, using the fact that V is a subsemigroup of $(\mathbb{N}, \cdot)$,

$$\bigcap_{B \in \mathcal{B}} \overline{B} \cap \bigcap_{B \in \mathcal{B}} \bigcap_{c \in P(M)} \overline{cB} = \bigcap_{B \in \mathcal{B}} \overline{B}\,.$$

Since $T = \bigcap_{B \in \mathcal{B}} \overline{B}$, we can apply Theorem 4.3. As before, we observe that $\beta S_d \setminus S$ is an ideal of βS_d and, since each member of $\mathcal{B}$ is infinite, $T \cap (\beta S_d \setminus S) \neq \emptyset$ and so $K(T) \subseteq \beta S_d \setminus S$. $\square$

Corollary 4.6. *Let $(S, +)$ be an infinite weakly cancellative discrete commutative semigroup, and let V be a subsemigroup of* $(\mathbb{N}, \cdot)$

such that $1 \in V$ *and, for every* $n \in V$ *and* $s, t \in S$, $ns = nt$ *implies that* $s = t$. *Let* $T = \bigcap_{n \in V} \overline{nS}$. *Then* T *is a subsemigroup of* βS *and if* p *is any idempotent in* $K(T)$, $p \in \beta S_d \setminus S$ *and every member of* p *contains an image of every first entries matrix over* ω *whose first entries are in* V.

Proof. It is clear that, for every $n \in V$, nS is infinite. For any finite nonempty subset F of V, $\bigcap_{n \in F} nS$ is infinite because $(\prod_{n \in F} n)S \subseteq \bigcap_{n \in F} nS$. Our claim now follows from Corollary 4.5 with $\mathcal{A} = \{S\}$. □

We observe that the following corollary cannot be deduced from previously known theorems such as [10, Theorem 15.5], because it is possible that $T \cap K(\beta S) = \emptyset$. This is the case, as we have previously remarked, if $S = (\mathbb{N}, \cdot)$ and $\mathcal{A} = \big\{\{n^c : n \in S\} : c \in \mathbb{N}\big\}$. We remind the reader that $\{n^c : n \in S\}$ is the multiplicative analogue of cS. Also, the "image of a first entries matrix" in the case in which S is a subsemigroup of $(\mathbb{N}\cdot)$, must be interpreted multiplicatively. For example, if $M = \begin{pmatrix} 1 & 3 \\ 0 & 2 \end{pmatrix}$ and $\vec{x} \in S^2$, we interpret the image of M determined by $\vec{x}$ to be $\begin{pmatrix} x_1 \cdot (x_2)^3 \\ (x_2)^2 \end{pmatrix}$.

Corollary 4.7. *Let* S *be a subsemigroup of* $(\mathbb{N}, \cdot)$ *and let* $T = \bigcap_{c \in S} \overline{\{n^c : n \in S\}}$. *If* p *is an idempotent in* $K(T)$, *then* $p \in \beta S_d \setminus S$ *and every member of* p *contains an image of every first entries matrix over* ω *with entries in* S.

Proof. This follows immediately from Corollary 4.6 with $V = S$. □

We observe that hypothesis (3) in the following theorem is weaker than hypothesis (4) in the statement of Corollary 4.5. We note that the element p in the statement of Theorem 4.8 is not necessarily in $K(T)$ and is not neccessarily idempotent.

When we say that a semigroup $(S, +)$ is torsion free, we mean that whenever $n \in \mathbb{N}$ and s and t are distinct members of S, $ns \neq nt$. In this case, for every $n \in \mathbb{N}$ and every $s \in nS$ there is a unique element $t \in S$ for which $nt = s$. We put $\frac{m}{n}s = mt$ for every $m \in \mathbb{N}$. Thus multiplication by positive rational numbers is partially defined on S.

Theorem 4.8. *Let $(S,+)$ be an infinite commutative and weakly cancellative semigroup with identity 0. Suppose also that S is torsion free. Let $\mathcal{A}$ be a nonempty family of infinite subsets of S and assume that*

(1) $(\forall A \in \mathcal{A})(\forall B \in \mathcal{A})(A \cap B \in \mathcal{A})$;
(2) $(\forall A \in \mathcal{A})(\forall a \in A)(\exists B \in \mathcal{A})(a + B \subseteq A)$; *and*
(3) $(\forall A \in \mathcal{A})(\forall n \in \mathbb{N})(\exists B \in \mathcal{A})(nB \subseteq A)$.

Let $T = \bigcap\{\overline{nA} : n \in \mathbb{N}$ and $A \in \mathcal{A}\}$. Then T is a subsemigroup of βS_d, and there is an element $p \in T \setminus S$ such that every member of p contains an image of every first entries matrix M with nonnegative rational entries. In particular, if $A \in \mathcal{A}$ and if $A \setminus \{0\}$ is partitioned into a finite number of subsets, one member of the partition must contain an image of M.

Proof. To see that T is a subsemigroup of βS_d, we use [10, Theorem 4.20]. Let $n \in \mathbb{N}$, let $A \in \mathcal{A}$, and let $b \in nA$. Pick $s \in A$ such that $b = ns$ and pick $B \in \mathcal{A}$ such that $s + B \subseteq A$. Then $b + nB \subseteq nA$.

Next we claim that $T \setminus S \neq \emptyset$. By [10, Corollary 3.14] it suffices to show that $\{nA : n \in \mathbb{N}$ and $A \in \mathcal{A}\}$ has the infinite finite intersection property. To this end, let $A \in \mathcal{A}$ and let $F \in \mathcal{P}_f(\mathbb{N})$. For each $n \in F$, let $k_n = \prod_{t \in F \setminus \{n\}} t$ and pick $B_n \in \mathcal{A}$ such that $k_n B_n \subseteq A$. Let $C = \bigcap_{n \in F} B_n$ and let $r = \prod_{t \in F} t$. Given $n \in F$, $rC = nk_nC \subseteq nA$ so $rC \subseteq \bigcap_{n \in F} nA$.

Pick $r \in T \setminus S$ and let q be an idempotent in $K(\beta\mathbb{N},+)$. Let $p = q\text{-}\lim_{n \in \mathbb{N}}(r\text{-}\lim_{s \in S} ns)$. We claim that $p \in T \setminus S$. To see that $p \in T$ suppose instead that we have some $n \in \mathbb{N}$ and $A \in \mathcal{A}$ such that $nA \notin p$. Then $\{m \in \mathbb{N} : r\text{-}\lim_{s \in S} ms \in \overline{S \setminus nA}\} \in q$ so pick $m \in \mathbb{N}$ such that $r\text{-}\lim_{s \in S} ms \in \overline{S \setminus nA}$. Then $\{s \in S : ms \in S \setminus nA\} \in r$. Pick $B \in \mathcal{A}$ such that $mB \subseteq A$. Then $nB \in r$ so pick $s \in nB$ such that $ms \in S \setminus A$. Then $s = nt$ for some $t \in B$ and $ms = nmt \in nA$, a contradiction.

To see that $p \notin S$ suppose instead that $p \in S$. Then $\{n \in \mathbb{N} : r\text{-}\lim_{s \in S} ns = p\} \in q$ so pick $n \neq m$ such that $r\text{-}\lim_{s \in S} ns = p$ and $r\text{-}\lim_{s \in S} ms = p$. Then there is some $s \in S$ such that $ns = ms$, contradicting the assumption that S is torsion free.

Let M be a $u \times v$ first entries matrix with nonnegative rational entries. Let $P \in p$. Since $p \notin S$, we may assume that $0 \notin P$. If

$Q = \{n \in \mathbb{N} : n \odot r \in P\}$, then $Q \in q$. By Theorem 1.3 pick $\vec{x} \in \mathbb{N}^v$ such that, if $M\vec{x} = \vec{y} \in \mathbb{N}^u$, then $y_i \in Q$ for every $i \in \{1, 2, \cdots, u\}$. Now $\{s \in S : y_i s \in P\} \in r$ for every $i \in \{1, 2, \cdots, u\}$ and so we can choose $s \in S \setminus \{0\}$ such that $y_i s \in P$ for every $i \in \{1, 2, \cdots, u\}$. Let $\vec{t} = (y_1 s, y_2 s, \cdots, y_u s)^T \in (S \setminus \{0\})^u$. Then $M\vec{t}$ is defined and all its entries are in P. $\square$

Corollary 4.9. *Let $(S, +)$ be an infinite commutative and weakly cancellative semigroup with identity 0 which is torsion-free. Then there exists $p \in (\beta S_d \setminus S) \cap \bigcap_{n \in \mathbb{N}} \overline{nS}$ such that every member of p contains an image of every first entries matrix with nonnegative rational entries.*

Proof. This is immediate from Theorem 4.8, with $\mathcal{A} = \{S\}$. $\square$

In an arbitrary commutative semigroup $(S, +)$, it may make no sense to allow entries of a matrix to be negative since $-x$ may not mean anything in S. If S is a group, it does make sense, and we get a nearly verbatim version of Theorem 4.3 allowing entries of M to be negative. (Though recall that in a first entries matrix, all first entries are positive.)

Theorem 4.10. *Let $(S, +)$ be an abelian group, let $\mathcal{A} \subseteq \mathcal{P}(S)$, let $u, v \in \mathbb{N}$, let M be a $u \times v$ first entries matrix with entries from $\mathbb{Z}$, and assume that*

(1) $(\forall A \in \mathcal{A})(\forall B \in \mathcal{A})(A \cap B \in \mathcal{A})$;
(2) $\mathcal{A} \neq \emptyset$ *and* $\emptyset \notin \mathcal{A}$;
(3) $(\forall A \in \mathcal{A})(\forall a \in A)(\exists B \in \mathcal{A})(a + B \subseteq A)$; *and*
(4) $(\forall A \in \mathcal{A})(\exists B \in \mathcal{A})(B + B \subseteq A)$.

Let $T = \bigcap_{A \in \mathcal{A}} \overline{A}$. Then T is a subsemigroup of βS_d. If p is an idempotent in $K(T) \cap \bigcap_{A \in \mathcal{A}} \bigcap_{c \in P(M)} \overline{cA}$ and $C \in p$, then for all $A \in \mathcal{A}$ there exists $\vec{x} \in A^v$ such that $M\vec{x} \in C^u$. In particular, if M is monic, then every member of every idempotent in $K(T)$ contains an image of M.

If we assume, in addition, that S is torsion free, there is an ultrafilter $q \in T$ with the property that every member of q contains an image of every first entries matrix over $\mathbb{Q}$.

Proof. Let $k = 1 + \max\big\{|m_{i,j}| : i \in \{1, 2, \ldots, u\}$ and $j \in \{1, 2, \ldots, v\}\big\}$. Define a $v \times v$ matrix E by, for $t, j \in \{1, 2, \ldots, v\}$,

$$e_{t,j} = \begin{cases} k^{j-t} & \text{if } j \geq t \\ 0 & \text{if } j < t\,. \end{cases}$$

Then E is a monic first entries matrix with entries from ω, ME is a first entries matrix with entries from ω, and $P(ME) = P(M)$. (This is easy to verify, or see [10, Lemma 15.14].)

Let $A \in \mathcal{A}$. By Lemma 4.1 pick $B \in \mathcal{A}$ such that for all $\vec{y} \in B^v$, $E\vec{y} \in A^v$. By Theorem 4.3 pick $\vec{y} \in B^v$ such that $ME\vec{y} \in C^u$. Let $\vec{x} = E\vec{y}$.

The claim made in the case in which S is torsion free follows in the same way from Theorem 4.8. □

The following corollary follows from Theorem 4.10 in the same way that Corollary 4.4 followed from Theorem 4.3.

Corollary 4.11. *Let $(S, +)$ be a nondiscrete abelian Hausdorff topological group. Let $\mathcal{A} = \{A : A$ is a neighborhood of $0\}$, let $T = \bigcap_{A\in\mathcal{A}} \overline{A}$, let $u, v \in \mathbb{N}$, and let M be a $u \times v$ monic first entries matrix with entries from $\mathbb{Z}$. Then T is a subsemigroup of βS_d and $K(T) \subseteq \beta S_d \setminus S$ so there are minimal idempotents of $K(T)$ in $\beta S_d \setminus S$. In particular, whenever $A \in \mathcal{A}$, $r \in \mathbb{N}$, and $A = \bigcup_{i=1}^{r} C_i$, there exist $i \in \{1, 2, \ldots, r\}$ and $\vec{x} \in (C_i)^v$ such that C_i is infinite and $M\vec{x} \in (C_i)^u$.*

If, in addition, S is torsion free, there is an element $q \in T$ with the property that every member of q contains an image of every first entries matrix over $\mathbb{Q}$.

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