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Chains of Idempotents in βN

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Abstract. We show that any non-minimal idempotent in the semigroup $(\beta N, +)$ lies in a sequence of idempotents each smaller than its predecessor and each maximal among all idempotents smaller than its predecessor.

1. Introduction. Given any semigroup (S, \cdot) one can define two pre-orders \leq_L and \leq_R on the idempotents of S by $x \leq_L y$ if and only if $x = x \cdot y$ and $x \leq_R y$ if and only if $x = y \cdot x$. (Thus $x \leq_L y$ if and only if x is a member of every left ideal of S that includes y and $x \leq_R y$ if and only if x is a member of every right ideal of S that includes y .) Then defining $x \leq y$ if and only if $x \leq_L y$ and $x \leq_R y$ one obtains a (reflexive, transitive and anti-symmetric) partial order on the set of idempotents.

In the event that S is a compact right topological semigroup (that is for each $x \in S$ the function $\rho_x: S \rightarrow S$ defined by $\rho_x(y) = y \cdot x$ is continuous) idempotents minimal with respect to each of these orders exist. (To say for example that x is \leq_L -minimal means that for all $y \in S$, if $y \leq_L x$ then $x \leq_L y$.) In fact one has the following.

1.1 Theorem (Ruppert). *Let S be a compact right topological semigroup.*

(a) *Given any idempotent $x \in S$ there is a \leq -minimal idempotent y with $y \leq x$.*

(b) *Given any idempotent $x \in S$, the following statements are equivalent.*

(i) *x is minimal with respect to \leq ;*

(ii) *x is minimal with respect to \leq_L ;*

(iii) *x is minimal with respect to \leq_R ;*

(iv) *$x \cdot S$ is a minimal right ideal of S ;*

(v) *$S \cdot x$ is a minimal left ideal of S ;*

(vi) *x is a member of the smallest two sided ideal of S .*

Proof. All of the information is provided by [14, Theorem 3.5 and Corollary 3.9] except for the fact that (b)(i) implies (b)(ii). To see this let x be \leq -minimal and let y be an idempotent with $y \leq_L x$. Let $z = x \cdot y$. Then $z \cdot z = x \cdot y \cdot x \cdot y = x \cdot y \cdot y = x \cdot y = z$ so z is an idempotent. Also $z = x \cdot y \cdot x$ so $z \leq x$ so $z = x$. Then $x \leq_L y$ as required. \square

As a consequence of Theorem 1.1 the term “minimal idempotent” is unambiguous. We will frequently use without specific mention the fact [5, Corollary 2.10] that any compact right topological semigroup has idempotents.

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We shall be concerned here with a particular compact right topological semigroup, namely the semigroup $(\beta N, +)$ where βN is the Stone-Ćech compactification of the discrete set N of positive integers and $+$ is the right continuous extension of ordinary addition to βN which has the property that for all x in N , $\lambda_x : \beta N \rightarrow \beta N$ is continuous where $\lambda_x(y) = x + y$. (A word of caution is in order about the terminology. What we call “right continuity”, namely the continuity of ρ_x for each $x \in \beta N$, many authors call “left continuity”.)

The semigroup $(\beta N, +)$ has had numerous applications in combinatorial number theory, (see for example the survey [8]). In particular the existence of minimal idempotents is a powerful combinatorial tool [2]. This semigroup is also of significant interest in the study of dynamical systems. (See for example [1].)

While “going down” (as in Theorem 1.1(a)) is easy, it is notoriously difficult to go up in βN . For example it is easy to see that any left ideal of $(\beta N, +)$ contains a minimal left ideal and somewhat harder to see, but still true, that any right ideal of $(\beta N, +)$ contains a minimal right ideal (see [3, Corollary 1.3.12]). On the other hand it is a notoriously difficult question as to whether each left ideal of βN is contained in a maximal left ideal. In fact, even the simpler question as to whether any strictly increasing chain of left ideals exists in βN remains open. (See [12].)

In [15] it was shown that given any $p \in N^* = \beta N \setminus N$ if $N^* + p$ is not a minimal left ideal then there is a strictly decreasing chain of left ideals of the form $N^* + q$ of order type ω_1 contained in $N^* + p$ with the property that each member of the chain is maximal subject to being strictly included in all of its predecessors. In a similar vein we show here in Section 3 that given any non-minimal idempotent p there exist 2^c non-minimal idempotents immediately smaller than p and consequently there is a sequence $\langle q_n \rangle_{n=1}^\infty$ of idempotents with $q_1 = p$ and each $q_{n+1} < q_n$ and each q_{n+1} maximal among all idempotents which are less than q_n . We also obtain a chain of idempotents $\langle q_\sigma \rangle_{\sigma < \omega_1}$ with $q_\sigma <_L q_\tau$ (meaning $q_\sigma \leq_L q_\tau$ and it is not the case that $q_\tau \leq_L q_\sigma$) whenever $\tau < \sigma$. This construction unfortunately sheds no light on the corresponding problem of when one can find idempotents bigger than a given one.

The results of Section 3 depend on a result which we believe is of independent interest which we present in Section 2. Let K denote the smallest ideal of $(\beta N, +)$. As the smallest ideal of a compact right topological semigroup, K is the union of all minimal left ideals and is also the union of all minimal right ideals. Further, the intersection of any minimal right ideal with any minimal left ideal is a group. (See [3, Theorem 1.3.11].)

It is known that K contains 2^c idempotents and that $cl K$ is a two sided ideal of $(\beta N, +)$ [9, Lemma 3.5 and Theorem 3.8]. Ever since it was shown ([10, Theorem 7.6] and [9, Theorem 3.9]) that K misses the smallest ideal of $(\beta N, \cdot)$ while $cl K$ is a left ideal of $(\beta N, \cdot)$, it has been known that $(cl K) \setminus K$ contains significant algebraic structure. But it has been previously unknown whether all of the idempotents of $cl K$ are contained in K . We show in Section 2 that if p is any element of N^* such that $p \notin N^* + p$, then the smallest compact semigroup of βN including p misses K . Since such elements are known to exist in $cl K$ we then have that idempotents exist in $(cl K) \setminus K$. Combining this fact with the results of Section 3 we obtain long chains of idempotents in $(cl K) \setminus K$, showing that the algebraic structure of $(cl K) \setminus K$ is indeed very rich.

We take the points of βN to be the ultrafilters on N , the principal ultrafilters being identified with the points of N . Given p and q in βN , the sum $p + q$ is characterized as follows. For $A \subseteq N$, $A \in p + q$ if and only if $\{x \in N : A - x \in q\} \in p$, where $A - x = \{y \in N : y + x \in A\}$. Alternatively, if $\langle x_i \rangle_{i \in I}$ and $\langle y_j \rangle_{j \in J}$ are nets in N converging to p and q respectively, then $p + q = \lim_{i \in I} \lim_{j \in J} (x_i + y_j)$. For an elementary introduction to the semigroup $(\beta N, +)$, see [11] (with the caution that the operation is reversed there from the way we use it here).

We shall also have the occasion to use the semigroup $(\beta Z, +)$. We will gloss over the distinction between ultrafilters on N and those ultrafilters on Z with N as a member and pretend that $\beta N \subseteq \beta Z$.

2.Compact semigroups missing the smallest ideal. We show in this section that any right cancellable element of N^* is a member of a compact semigroup missing K , the smallest ideal of $(\beta N, +)$.

2.1 Definition. Let $\langle x_n \rangle_{n=1}^{\infty}$ be an increasing sequence in N and let $Y \subseteq N$.

(a) For each $m \in N$, $R_{m,Y} = R_{m,Y}(\langle x_n \rangle_{n=1}^{\infty}) = \{\sum_{j=1}^s x_{n(i)} : s \in N, x_{n(1)} > m, \{n(1), n(2), \dots, n(s)\} \subseteq Y, \text{ and for each } i \in \{1, 2, \dots, s\}, x_{n(i)+1} > m + \sum_{j=1}^i x_{n(j)}\}$

(b) $R_Y = R_Y(\langle x_n \rangle_{n=1}^{\infty}) = \bigcap_{m=1}^{\infty} cl R_{m,Y}$.

2.2 Lemma. Let $\langle x_n \rangle_{n=1}^{\infty}$ be an increasing sequence in N and let $Y \subseteq N$ and assume that $\{x_{n+1} - x_n : n \in Y\}$ is unbounded. Then R_Y is a compact semigroup of βN . Further, if $p, q \in \beta N$, $p + q \in R_Y$, and $q \in R_Y$, then $p \in R_Y$.

Proof. The assumption that $\{x_{n+1} - x_n : n \in Y\}$ is unbounded guarantees that $R_{m,Y} \neq \emptyset$ for each $m \in N$ and hence that $R_Y \neq \emptyset$. To see that R_Y is a semigroup, let $p, q \in R_Y$ and let $m \in N$ be given. We show that $p + q \in cl R_{m,Y}$, that is that $R_{m,Y} \in p + q$. To show this it suffices to show that $R_{m,Y} \subseteq \{y \in N : R_{m,Y} - y \in q\}$ so let $y \in R_{m,Y}$ and pick $s \in N$ and $n(1), n(2), \dots, n(s)$ as guaranteed by the definition of $R_{m,Y}$ with $y = \sum_{i=1}^s x_{n(i)}$. Let $k = m + \sum_{i=1}^s x_{n(i)}$. Then $R_{k,Y} \in q$ so it suffices to show that $R_{k,Y} \subseteq R_{m,Y} - y$. Let $z \in R_{k,Y}$ and pick $t \in N$ and $l(1), l(2), \dots, l(t)$ as guaranteed by the definition of $R_{k,Y}$ with $z = \sum_{i=1}^t x_{l(i)}$. Then letting $n(s+i) = l(i)$ for $i \in \{1, 2, \dots, t\}$ we have $z + y = \sum_{i=1}^{s+t} x_{n(i)}$ and $n(1), n(2), \dots, n(s+t)$ are as required to show that $z + y \in R_{m,Y}$.

Finally assume that $p + q \in R_Y$ and $q \in R_Y$. To see that $p \in R_Y$, let $m \in N$ be given. We show that $R_{m,Y} \in p$. Let $A = \{y \in N : R_{m,Y} - y \in q\}$. Since $R_{m,Y} \in p + q$ we have that $A \in p$ so it suffices to show that $A \subseteq R_{m,Y}$. So let $k \in A$ be given. Now $R_{k,Y} \in q$ so $R_{k,Y} \cap (R_{m,Y} - k) \neq \emptyset$. Let z be the smallest member of $R_{k,Y} \cap (R_{m,Y} - k)$. Pick $s \in N$ and $n(1), n(2), \dots, n(s)$ as guaranteed by the definition of $R_{m,Y}$ with $z + k = \sum_{i=1}^s x_{n(i)}$. Pick $t \in N$ and $l(1), l(2), \dots, l(t)$ as guaranteed by the definition of $R_{k,Y}$ with $z = \sum_{i=1}^t x_{l(i)}$.

We show now that $l(t) = n(s)$. Indeed if we had $n(s) > l(t)$ we would have $z + k \geq x_{n(s)} \geq x_{l(t)+1} > k + \sum_{j=1}^t x_{l(j)} = k + z$, a contradiction. Similarly if we had $n(s) < l(t)$ we would have $z \geq x_{l(t)} \geq x_{n(s)+1} > m + \sum_{j=1}^s x_{n(i)} = m + z + k$, a contradiction. Next we observe that $t = 1$, that is that $z = x_{l(t)} = x_{n(s)}$. For if $t > 1$ we would have $z - x_{l(t)} \in R_{k,Y} \cap (R_{m,Y} - k)$ contradicting the choice of z as its smallest member. Since $z = x_{n(s)}$, we have $k = \sum_{i=1}^{s-1} x_{n(i)}$ so that $k \in R_{m,Y}$ as required. \square

We recall from [4, Theorem 2.1] that an element $p \in N^*$ is right cancellable in N^* if and only if $p \notin N^* + p$ and from [16, Theorem 2] that p is right cancellable if and only if there is an increasing sequence $\langle x_n \rangle_{n=1}^{\infty}$ with the property that for each $k \in N$, $\{x_n : n \in N \text{ and } x_{n+1} > x_n + k\} \in p$.

2.3 Theorem. Let p be any right cancellable element of N^* . There is a compact subsemigroup R of βN such that $p \in R$ and $R \cap K = \emptyset$.

Proof. Choose an increasing sequence $\langle x_n \rangle_{n=1}^{\infty}$ such that for each $k \in N$, $A_k = \{x_n : n \in N \text{ and } x_{n+1} > x_n + k\} \in p$. Then $\bigcap_{k=1}^{\infty} cl A_k$ is a nonempty G_δ subset of N^* which therefore cannot be a singleton. (Alternatively, inductively choose $n(k)$ such that $x_{n(k)+1} > x_{n(k)} + k$ and note that $(cl\{x_{n(k)} : k \in N\}) \setminus N \subseteq \bigcap_{k=1}^{\infty} cl A_k$.) Thus we can choose $q \in \bigcap_{k=1}^{\infty} cl A_k$ with $q \neq p$. Choose

disjoint subsets Y and Z of N with $\{x_n : n \in Y\} \in p$ and $\{x_n : n \in Z\} \in q$.

Let $R = R_Y$. By Lemma 2.2 we have that R is a compact subsemigroup of βN . Suppose that $R \cap K \neq \emptyset$ and pick $r \in R \cap K$. Now K is the union of the minimal left ideals of βN ([14] or see [3, Theorem 1.3.11]) so pick a minimal left ideal L of βN with $r \in L$. Then $q + r \in L$ so $\beta N + q + r$ is a left ideal contained in L so $L = \beta N + q + r$. In particular $r \in \beta N + q + r$ so choose some $s \in \beta N$ with $r = s + q + r$. Since $r \in R$ and $s + q + r \in R$, by Lemma 2.2 we have $s + q \in R$.

Let $B = \{k + x_n : n \in Z, k \in N, \text{ and } x_{n+1} > x_n + k\}$. We claim that $B \in s + q$, for which it suffices to show that for all $k \in N$, $B - k \in q$. Indeed, given $k \in N$ one has $A_k \cap \{x_n : n \in Z\} \subseteq B - k$ so $B - k \in q$.

Now $B \in s + q$ and $R_{1,Y} \in s + q$ so choose some $w \in B \cap R_{1,Y}$. Since $w \in B$, pick $k \in N$ and $m \in Z$ such that $x_{m+1} > x_m + k$ and $w = k + x_m$. Also $w \in R_{1,Y}$ so pick $t \in N$ and $n(1), n(2), \dots, n(t)$ such that $w = \sum_{i=1}^t x_{n(i)}$ and $\{n(1), n(2), \dots, n(t)\} \subseteq Y$ and for each $i \in \{1, 2, \dots, t\}$, $x_{n(i)+1} > 1 + \sum_{j=1}^i x_{n(j)}$.

Finally $n(t) \in Y$ and $m \in Z$ so $n(t) \neq m$. But if we had $m > n(t)$ we would have $w = k + x_m > x_{n(t)+1} > \sum_{j=1}^t x_{n(j)} = w$, a contradiction. Thus $m < n(t)$. But then $w = \sum_{i=1}^t x_{n(i)} \geq x_{n(t)} \geq x_{m+1} > x_m + k = w$. This contradiction establishes that $R \cap K = \emptyset$. \square

We have already remarked that $(clK) \setminus K$ has rich algebraic structure. Part of this structure is the existence of right cancellable elements in $(clK) \setminus K$. (It is an easy exercise to show that no elements of K are right cancellable.) The first proof of their existence [9, Theorem 4.6] was an intricate combinatorial construction. This was some years before the discovery [4] of the characterization of right cancellability of p as $p \notin N^* + p$. The existence of right cancellable elements of $(clK) \setminus K$ is much easier to establish using that characterization and the following often discovered result: If $p \in \beta N$ and $\{n \in N : N \in p\}$ is infinite, then for all $q, r, s \in N^* \cdot p \cdot q \neq r + s$. (This is originally in [10, Theorem 5.3] and was discovered later but independently by the second author of this paper and by Balcar and Kalášek.) Then given $p \in clK \cap \bigcap_{n=1}^{\infty} cl(N)$, one has $p \cdot p \in clK$ since clK is a left ideal of $(\beta N$ [9, Theorem 3.9] and $p \cdot p \in \bigcap_{n=1}^{\infty} cl(BbbNn)$). Thus $p \cdot p \notin N^* + N^*$ and in particular $p \cdot p \notin N^* + p \cdot p$.

Recall that c is the cardinality of the continuum.

2.4 Corollary. *There exist compact subsemigroups with 2^c elements contained in $(clK) \setminus K$. In particular there are idempotents in $(clK) \setminus K$.*

Proof. By [9, Theorem 4.6] there are right cancellable elements of βN in clK . Given a right cancellable element p of βN , the elements $p, p + p, p + p + p, \dots$, are all distinct. (If say one had $p + p + p = p + p + p + p$, by cancelling p on the right twice one would have $p = p + p + p$ so $p \in \beta N + p$, contradicting one of the characterizations of right cancellability cited above.) Let R be the compact semigroup guaranteed by Theorem 2.3. Then $R \cap clK$ is a semigroup which is infinite and compact, hence has 2^c elements [7, Theorem 9.11]. \square

We remark that by [6, Theorem 3.2], the subsemigroups of Corollary 2.4 each contain 2^c idempotents.

3. Decreasing chains of idempotents. We show here that given any non-minimal idempotent in N^* there is a decreasing sequence of non-minimal idempotents below it with respect to \leq , each maximal among all those smaller than its predecessor. We also show that we can find decreasing chains with respect to \leq_L of order type ω_1^{-1} below any non-minimal idempotent.

3.1 Theorem. *Let p be any non-minimal idempotent in N^* . There exist 2^c non-minimal idempotents immediately below p with respect to \leq .*

Proof. By [15, Corollary 1] there is a sequence $\langle x_n \rangle_{n=1}^\infty$ in N such that

- (1) for each n , x_n divides x_{n+1} and $n!$ divides x_{n+1} and
- (2) if $q \in (cl\{x_n : n \in N\}) \setminus N$, then $p \notin \beta Z + q + p$.

As was shown in the proof of [15, Theorem 2] if q_1 and q_2 are distinct members of $(cl\{x_n : n \in N\}) \setminus N$, then $(N^* + q_1 + p) \cap (N^* + q_2 + p) = \emptyset$. Consequently since $cl\{x_n : n \in N\}$ has 2^c members [7, Corollary 9.12] it suffices to show that for each $q \in (cl\{x_n : n \in N\}) \setminus N$ there is some $v \in N^* + q + p$ with $v < p$ and v maximal among all s with $s < p$. To this end let $q \in (cl\{x_n : n \in N\}) \setminus N$ be given.

By [15, Theorem 2] $q + p$ is right cancellable. Pick by Theorem 2.3 a compact subsemigroup R of βN with $q + p \in R$ and $R \cap K = \emptyset$. Then $R \cap (N^* + q + p)$ is a compact semigroup so pick some idempotent $t \in R \cap (N^* + q + p)$. Observe that $t = t + p$ since $t \in N^* + p$. Let $r = p + t$. Then $r + r = p + t + p + t = p + t + t = p + t = r$ so r is an idempotent. Since $r = p + t + p$, we have $r \leq p$. We claim that r is non-minimal. Suppose instead that $r \in K$. Then $q + r \in K$ since K is an ideal. But $q + r = q + p + t$ and $q + p \in R$ and $t \in R$ so $q + r \in R \cap K$, a contradiction.

Let $\mathcal{A} = \{\Gamma : \Gamma \text{ is a } \leq_R\text{-chain of idempotents and } r \in \Gamma \text{ and } \Gamma \subseteq N^* + q + p \text{ and for all } s \in \Gamma, s \leq p\}$. Then $\{r\} \in \mathcal{A}$ and the union of any chain in \mathcal{A} is again in \mathcal{A} so by Zorn's Lemma we may pick a maximal member Γ of \mathcal{A} .

Let $L = \{u \in N^* + q + p : \text{for all } s \in \Gamma, s = u + s\}$. Now $L = (N^* + q + p) \cap \bigcap_{s \in \Gamma} \rho_s^{-1}[\{s\}]$ so L is compact. To see that $L \neq \emptyset$ it suffices to observe that $\{N^* + q + p\} \cup \{\rho_s^{-1}[\{s\}] : s \in \Gamma\}$ has the finite intersection property. (Given a finite subset F of Γ pick a \leq_R -maximal member u of F . Then $u \in (N^* + q + p) \cap \bigcap_{s \in F} \rho_s^{-1}[\{s\}]$.) Then (since each $\rho_s^{-1}[\{s\}]$ is a semigroup) we have L is a compact semigroup so pick an idempotent $u \in L$ and let $v = p + u$. Since $u \in N^* + p$ we have $u + p = u$ and hence $v + v = v$. Also $v = p + u \in N^* + q + p$ and given $s \in \Gamma$ we have $v + s = p + u + s = p + s = s$ (since $s \leq p$) so $v \in L$.

Next we observe that v is not minimal. Indeed $r \in \Gamma$ and $v \in L$ so $r = v + r$. If we had $v \in K$ we would have $r \in K$.

Now $v = p + u + p$ so $v \leq p$. Also by [15, Theorem 3] $N^* + q + p$ is strictly contained in $N^* + p$ so $p \notin N^* + q + p$ so $v < p$.

Finally we show that v is maximal among all idempotents less than p . We do this in two steps, showing first that v is maximal among those idempotents in $N^* + q + p$ which are less than p . Assume we have $w \in N^* + q + p$ with $v \leq w < p$. Then for all $s \in \Gamma$, $s \leq_R v \leq_R w$ so $\Gamma \cup \{w\}$ is a \leq_R -chain of idempotents so $\Gamma \cup \{w\}$ is in \mathcal{A} so $w \in \Gamma$ so $w \leq_R v$ (since $v \in L$) so $w = v$.

For the last step suppose w is an idempotent with $v < w < p$. Then $v \in (N^* + q + p) \cap (N^* + w)$. Then by [17, Theorem 2] either $N^* + w \subseteq N^* + q + p$ or $N^* + q + p \subseteq N^* + w$. If $N^* + w \subseteq N^* + q + p$, then $w \in N^* + q + p$ and we have already seen that v is maximal among the idempotents in $N^* + q + p$ which are less than p . Thus we must have $N^* + q + p$ properly contained in $N^* + w$. But then by [15, Theorem 3] we have $N^* + w = N^* + p$. But then $w = p + w = p$, a contradiction. \square

3.2 Corollary. *Let p be any non-minimal idempotent of N^* . There is a sequence $\langle p_n \rangle_{n=1}^\infty$ of idempotents such that $p_1 = p$ and for each n , $p_{n+1} < p_n$ and p_{n+1} is maximal among all idempotents less than p_n . \square*

3.3 Corollary. *There are 2^c idempotents in $(cl K) \setminus K$.*

Proof. By Corollary 2.4 there is some idempotent $p \in (cl K) \setminus K$. By Theorem 3.1 p has 2^c non-minimal predecessors. As we have remarked $cl K$ is an ideal of βN so all of these predecessors

are in $cl K$. \square

Observe that in the following corollary, we do not guarantee maximality, even for the pre-order \leq_L , at the limit stages. We remark also that one cannot produce decreasing sequences of length greater than c with respect to \leq_L . Indeed, suppose one had such a sequence $\langle p_\sigma \rangle_{\sigma < \kappa}$ for some $\kappa > c$. For each $\sigma < \kappa$, $N^* + p_\sigma$ properly contains the compact set $N^* + p_{\sigma+1}$ so one can choose a clopen set U_σ in βN with $N^* + p_{\sigma+1} \subseteq U_\sigma$ and $(N^* + p_\sigma) \setminus U_\sigma \neq \emptyset$. But the clopen subsets of βN correspond exactly to the subsets of N and so there are exactly c of them.

3.4 Corollary. *Let p be a non-minimal idempotent in N^* . There exists an ω_1 -sequence $\langle p_\sigma \rangle_{\sigma < \omega_1}$ of idempotents such that*

- (1) $p_0 = p$;
- (2) for each $\sigma < \omega_1$, $p_{\sigma+1} < p_\sigma$ and $p_{\sigma+1}$ is maximal among all idempotents of N^* which are less than p_σ ;
- (3) for any $\sigma < \tau < \omega_1$, $p_\tau < p_\sigma$.

Proof. Let $p_0 = p$. At successor stages apply Theorem 3.1. So let τ be an infinite limit ordinal with $\tau < \omega_1$. Choose $\langle \sigma(n) \rangle_{n=1}^\infty$ cofinal in τ (with $\sigma(n+1) > \sigma(n)$). Then for each n , $p_{\sigma(n+1)} \leq_L p_{\sigma(n)}$ and it is not true that $p_{\sigma(n)} \leq_L p_{\sigma(n+1)}$. Thus $\langle N^* + p_{\sigma(n)} \rangle_{n=1}^\infty$ is a strictly decreasing sequence of left ideals. By [15, Theorem 4] pick some right cancellable $q \in N^*$ with $N^* + q \subseteq \bigcap_{n=1}^\infty (N^* + p_{\sigma(n)})$. By Theorem 2.3 pick a compact semigroup R with $q \in R$ and $R \cap K = \emptyset$. Then pick an idempotent p_τ in $R \cap (N^* + q)$. Given any $\delta < \tau$ pick n such that $\delta < \sigma(n)$. Then $p_\tau \in N^* + p_{\sigma(n+1)} \subset N^* + p_{\sigma(n)} \subseteq N^* + p_\delta$. Since $N^* + p_{\sigma(n+1)} \neq N^* + p_{\sigma(n)}$, this completes the proof. \square

There are two improvements we would like to make in Corollary 3.4. First in conclusion (3) we would like to replace \leq_L by \leq . Second we would like to know whether at limit stages p_τ can be chosen maximal among those q with $q \leq_L p_\sigma$ for each $\sigma < \tau$.

One can define an equivalence relation \approx_L on the idempotents of N^* by $p \approx_L q$ if and only if $p \leq_L q$ and $q \leq_L p$. Now Theorem 2 of [17] says that if $(N^* + p) \cap (N^* + q) \neq \emptyset$, then $p = q$ or $p \in N^* + q$ or $q \in N^* + p$ (so in particular $N^* + p \subseteq N^* + q$ or $N^* + q \subseteq N^* + p$). One thus deduces that as one heads upward through the ordering \leq_L induces on the \approx_L equivalence classes, there is never any branching. On the other hand, Theorem 3.1 guarantees that there is a great deal of branching going down.

We close by mentioning an extremally annoying gap in our knowledge. We do not know whether or not it is possible for an idempotent to be both maximal and minimal with respect to \leq . We do know that no minimal idempotent is \leq_R -maximal. Indeed, by [14, Lemma I.2.6], any idempotent lies below a \leq_R -maximal idempotent. (Be cautioned that the continuity is reversed in [14] from that which we are using.) By [13, Theorem 3.3], for any \leq_R -maximal idempotent p , $\{x \in \beta N : p = x + p\}$ is finite. But if $p \in K$, then any of the 2^c idempotents in $p + \beta N$ is a left identity for p .

References

1. J. Auslander, *Minimal flows and their extensions*, North-Holland, Amsterdam, 1988.
2. V. Bergelson, W. Deuber, N. Hindman, and H. Lefmann, *Rado's Theorem for commutative rings*, J. Comb. Theory (Series A), to appear.
3. J. Berglund, H. Junghenn, and P. Milnes, *Analysis on semigroups*, Wiley, New York, 1989.
4. A. Blass and N. Hindman, *Sums of ultrafilters and the Rudin-Keisler and Rudin-Frolik orders*, in General Topology and Applications, R. Shortt ed., Lecture Notes in Pure and Applied Math. **123** (1990), 59-70.
5. R. Ellis, *Lectures in topological dynamics*, Benjamin, New York, 1969
6. A. El-Mabhouh, J. Pym, and D. Strauss, *Subsemigroups of βN* , Topology and its Applications, to appear.
7. L. Gillman and M. Jerison, *Rings of continuous functions*, van Nostrand, Princeton, 1960.
8. N. Hindman, *The semigroup βN and its applications to number theory*, in The Analytical and Topological Theory of Semigroups, K. Hofmann, J. Lawson, and J. Pym, eds. Walter deGruyter & Co., Berlin, 1990, 347-360.
9. N. Hindman, *Minimal ideals and cancellation in βN* , Semigroup Forum **25** (1982), 291-310.
10. N. Hindman, *Sums equal to products in βN* , Semigroup Forum **21** (1980), 221-255.
11. N. Hindman, *Ultrafilters and combinatorial number theory*, in Number Theory Carbonale 1979, M. Nathanson, ed., Lecture Notes in Math. **751** (1979), 119-184.
12. N. Hindman, J. van Mill, and P. Simon, *Increasing chains of ideals and orbit closures in βZ* , Proc. Amer. Math. Soc. **114** (1992), 1167-1172.
13. N. Hindman and D. Strauss, *Nearly prime subsemigroups of βN* , manuscript.
14. W. Ruppert, *Compact semitopological semigroups: an intrinsic theory*, Lecture Notes in Math. **1079**, Springer-Verlag, Berlin, 1984.
15. D. Strauss, *Ideals and commutativity in βN* , Topology and its Applications, to appear.
16. D. Strauss, *N^* does not contain an algebraic and topological copy of βN* , J. London Math. Soc. (2) **46** (1992), 463-470.
17. D. Strauss, *Semigroup structures on βN* , Semigroup Forum **44** (1992), 238-244.