This paper was published in *J. Comb. Theory (Series A)* **118** (2011) 373-386. To the best of my knowledge, this is the final version as it was submitted to the publisher. -NH

Some Combinatorially defined Subsets of $\beta\mathbb{N}$ and their Relation to the Idempotents

Neil Hindman[1]

and

Dona Strauss

Abstract. Members of idempotents in $(\beta\mathbb{N},+)$, especially those in the smallest ideal, have strong combinatorial properties. And the closure Γ of the set of idempotents has a simple combinatorial description. We investigate here the relationships among several subsets of $\beta\mathbb{N}$ that have simple combinatorial descriptions, as well as the semigroups they generate and their closures.

1. Introduction

In 1974, the following theorem was proved.

1.1 Theorem. *Let $r \in \mathbb{N}$ and let $\mathbb{N} = \bigcup_{i=1}^{r} C_i$. There exist $i \in \{1, 2, \ldots, r\}$ and a sequence $\langle x_t \rangle_{t=1}^{\infty}$ such that $FS(\langle x_t \rangle_{t=1}^{\infty}) \subseteq C_i$, where*

$$FS(\langle x_t \rangle_{t=1}^{\infty}) = \{\textstyle\sum_{t \in F} x_t : F \in \mathcal{P}_f(\mathbb{N})\}$$

and $\mathcal{P}_f(\mathbb{N})$ is the set of finite nonempty subsets of $\mathbb{N}$.

Proof. [8]. □

The proof given in [8] was elementary but very complicated. Another, less complicated, elementary proof was found by Baumgartner [2]. Subsequently a much simpler proof was found by F. Galvin and S. Glazer. (See the notes to Chapter 5 of [11] for an account of the discovery of this proof.) The crux of this simpler proof is that ordinary addition on $\mathbb{N}$ can be extended to its Stone-Čech compactification $\beta\mathbb{N}$ so that $(\beta\mathbb{N},+)$ becomes a right topological semigroup (meaning that for each $p \in \beta\mathbb{N}$, the function $\rho_p : \beta\mathbb{N} \to \beta\mathbb{N}$ is continuous, where $\rho_p(q) = q + p$) with $\mathbb{N}$ contained in its topological center (meaning that for each $x \in \mathbb{N}$, the function $\lambda_x : \beta\mathbb{N} \to \beta\mathbb{N}$ is continuous, where $\lambda_x(q) = x + q$). Any compact Hausdorff right topological semigroup has idempotents [5, Corollary 2.10]. And idempotents are intimately related to finite sums sets.

[1] This author acknowledges support received from the National Science Foundation via Grant DMS-0852512.

1.2 Theorem. *Let $A \subseteq \mathbb{N}$. There is an idempotent $p \in c\ell A$ if and only if there is a sequence $\langle x_t \rangle_{t=1}^{\infty}$ in $\mathbb{N}$ with $FS(\langle x_t \rangle_{t=1}^{\infty}) \subseteq A$.*

Proof. [11, Theorem 5.12]. □

We take the points of $\beta\mathbb{N}$ to be the ultrafilters on $\mathbb{N}$, identifying the principal ultrafilters with the points of $\mathbb{N}$. Given $A \subseteq \mathbb{N}$ and $p \in \beta\mathbb{N}$, one has that $p \in c\ell A = \overline{A}$ if and only if $A \in p$. If $p, q \in \beta\mathbb{N}$ and $A \subseteq \mathbb{N}$, one has that $A \in p + q$ if and only if $\{x \in \mathbb{N} : -x + A \in q\} \in p$. The set $\{\overline{A} : A \subseteq \mathbb{N}\}$ is a basis for the open sets and a basis for the closed sets of $\beta\mathbb{N}$.

1.3 Definition. $\Gamma = \{p \in \beta\mathbb{N} : (\forall A \in p)(\exists \langle x_t \rangle_{t=1}^{\infty})(FS(\langle x_t \rangle_{t=1}^{\infty}) \subseteq A\}$.

As an immediate consequence of Theorem 1.2 one has that $\Gamma = c\ell\{p \in \beta\mathbb{N} : p+p = p\}$.

In [7], H. Furstenberg defined *central* subsets of $\mathbb{N}$ in terms of some notions from topological dynamics and proved the *Central Sets Theorem*. Given sets A and B, we let ${}^{A}B$ be the set of functions from A to B.

1.4 Theorem. *Let A be a central subset of $\mathbb{N}$ and let $F \in \mathcal{P}_f({}^{\mathbb{N}}\mathbb{Z})$. There exist a sequence $\langle a_n \rangle_{n=1}^{\infty}$ in $\mathbb{N}$ and a sequence $\langle H_n \rangle_{n=1}^{\infty}$ in $\mathcal{P}_f(\mathbb{N})$ such that*
(1) for each $n \in \mathbb{N}$, $\max H_n < \min H_{n+1}$ and
(2) for each $K \in \mathcal{P}_f(\mathbb{N})$ and each $f \in F$, $\sum_{n \in K} \big(a_n + \sum_{t \in H_n} f(t)\big) \in A$.

Proof. [7, Proposition 8.21]. □

Central subsets of $\mathbb{N}$ have remarkably strong combinatorial properties. For example, they contain solutions to any partition regular system of homogenous linear equations in $\mathbb{N}$. See [11, Chapters 14 and 15] for many more examples.

Any compact Hausdorff right topological semigroup S has important algebraic properties. A non-empty subset V of S is a left ideal if $SV \subseteq V$, a right ideal if $VS \subseteq V$, and an ideal if it is both a left and a right ideal. Every left ideal of S contains a minimal left ideal, and every right ideal of S contains a minimal right ideal. S has a smallest two sided ideal $K(S)$, which is the union of all the minimal left ideals of S and the union of all the minimal right ideals of S. The intersection of any minimal left ideal and any minimal right ideal of S is a group. In particular, it contains an idempotent and so the set $E(S)$ of idempotents of S is non-empty. An idempotent in S is called minimal if it is in $K(S)$, and this is equivalent to being minimal for the ordering defined on idempotents by stating that $p \leq q$ if $p + q = q + p = p$. Proofs of these statements can be found in [11, Theorems 1.38 and 1.51].

In [3] it was shown, with the assistance of B. Weiss, that a subset A of $\mathbb{N}$ is central if and only if it is a member of a minimal idempotent. (This equivalence was subsequently extended to arbitrary semigroups by H. Shi and H. Yang [14].)

We thus see that members of idempotents have strong combinatorial properties, and members of idempotents in $K(\beta\mathbb{N})$ have even stronger combinatorial properties. We shall be interested in this paper in several subsets of $\beta\mathbb{N}$ that are defined by at least superficially weaker combinatorial properties. We mention one of these first, because of its relationship to an open problem of relatively long standing.

1.5 Definition. Let $k \in \mathbb{N} \setminus \{1\}$.

(a) Given a sequence $\langle x_t \rangle_{t=1}^{\infty}$ in $\mathbb{N}$,
$FS_{\leq k}(\langle x_t \rangle_{t=1}^{\infty}) = \{\sum_{t \in F} x_t : F \in \mathcal{P}_f(\mathbb{N}) \text{ and } |F| \leq k\}$.

(b) $\mathcal{F}_k = \{p \in \beta\mathbb{N} : (\forall A \in p)(\exists \langle x_t \rangle_{t=1}^{\infty})(FS_{\leq k}(\langle x_t \rangle_{t=1}^{\infty}) \subseteq A)\}$.

1.6 Definition.

(a) For $n \in \mathbb{N}$, the *support* of n, $\mathrm{supp}(n)$, is the finite nonempty subset of $\omega = \{0, 1, 2, \ldots\}$ such that $n = \sum_{t \in \mathrm{supp}(n)} 2^t$.

(b) For $x, y \in \mathbb{N}$, $x << y$ if and only if $\max\ \mathrm{supp}(x) < \min\ \mathrm{supp}(y)$.

We see that members of $\mathcal{F}_k(\mathbb{N})$ for $k \geq 2$ enjoy a property which might, at first sight, seem to be stronger than their defining property.

1.7 Lemma. *Let $k \in \mathbb{N} \setminus \{1\}$, let $p \in \mathcal{F}_k(\mathbb{N})$, and let $A \in p$. Then there exists a sequence $\langle x_t \rangle_{t=1}^{\infty}$ such that $FS_{\leq k}(\langle x_t \rangle_{t=1}^{\infty}) \subseteq A$ and for each $t \in \mathbb{N}$, $x_t << x_{t+1}$.*

Proof. Let $B_0 = \{x \in A : \min\ \mathrm{supp}(x) \text{ is even}\}$ and let $B_1 = \{x \in A : \min\ \mathrm{supp}(x) \text{ is odd}\}$. Pick $i \in \{0, 1\}$ such that $B_i \in p$ and pick a sequence $\langle x_t \rangle_{t=1}^{\infty}$ such that $FS_{\leq k}(\langle x_t \rangle_{t=1}^{\infty}) \subseteq B_i$. We note that no three of these have the same minimum of their supports. Indeed, suppose that $n < m < r$ and $\min\ \mathrm{supp}(x_n) = \min\ \mathrm{supp}(x_m) = \min\ \mathrm{supp}(x_r) = k$. Then some two have $k + 1$ in their support or some two do not have $k + 1$ in their support. Say these two are x_m and x_n. Then $\min\ \mathrm{supp}(x_n + x_m) = k + 1$ so $x_n + x_m \notin B_i$.

Since no three terms have the same minimum of their supports, we can choose a subsequence $\langle y_t \rangle_{t=1}^{\infty}$ of $\langle x_t \rangle_{t=1}^{\infty}$ such that for each n, $y_n << y_{n+1}$. □

A favorite problem of I. Leader is whether there is a proof that whenever $r \in \mathbb{N}$ and $\mathbb{N} = \bigcup_{i=1}^{r} C_i$, there must exist $i \in \{1, 2, \ldots, r\}$ and $\langle x_t \rangle_{t=1}^{\infty}$ such that $FS_{\leq 2}(\langle x_t \rangle_{t=1}^{\infty}) \subseteq$

C_i that does not also prove the Finite Sums Theorem (Theorem 1.1). The question is asked in this form in [9, Question 12].

If one wants to be picky, the answer to the question phrased in this fashion is "yes". For example, one may take an idempotent p in $\beta\mathbb{N}$ and let $\mathbb{N} = \bigcup_{i=1}^{r} C_i$. Then pick $i \in \{1, 2, \ldots, r\}$ such that $C_i \in p$ and let $B = \{x \in C_i : -x + C_i \in p\}$. Then $B \in p$ so pick $x_1 \in B$ and inductively pick $x_{n+1} \in B \cap \bigcap_{k=1}^{n} (-x_k + C_i)$. Then $FS_{\leq 2}(\langle x_t \rangle_{t=1}^{\infty}) \subseteq C_i$ and one certainly does not know that $FS(\langle x_t \rangle_{t=1}^{\infty}) \subseteq C_i$. However, since $C_i \in p$ one does in fact have by Theorem 1.2 that there does exist $\langle y_t \rangle_{t=1}^{\infty}$ with $FS(\langle y_t \rangle_{t=1}^{\infty}) \subseteq C_i$.

In [4, Question 8.1] Leader rephrased the question as follows to make it more precise.

1.8 Question. *Does there exist a set $A \subseteq \mathbb{N}$ such that whenever $r \in \mathbb{N}$ and $A = \bigcup_{i=1}^{r} C_i$, there must exist $i \in \{1, 2, \ldots, r\}$ and $\langle x_t \rangle_{t=1}^{\infty}$ such that $FS_{\leq 2}(\langle x_t \rangle_{t=1}^{\infty}) \subseteq C_i$, but there does not exist $\langle y_t \rangle_{t=1}^{\infty}$ with $FS(\langle y_t \rangle_{t=1}^{\infty}) \subseteq A$?*

An affirmative answer to Question 1.8 is equivalent to the assertion that $\Gamma \neq \mathcal{F}_2$. On the one hand, if $p \in \mathcal{F}_2 \setminus \Gamma$ and $A \in p$ such that there is no $\langle y_t \rangle_{t=1}^{\infty}$ with $FS(\langle y_t \rangle_{t=1}^{\infty}) \subseteq A$, then A provides an affirmative answer to Question 1.8. The other implication is an immediate consequence of the following lemma, which we shall be using frequently.

1.9 Lemma. *Let $\mathcal{A}$ be a set of subsets of $\mathbb{N}$ such that whenever $A \in \mathcal{A}$ and $A \subseteq B \subseteq \mathbb{N}$, one has $B \in \mathcal{A}$ and let $\mathcal{R} = \{A \subseteq \mathbb{N} :$ whenever $r \in \mathbb{N}$ and $A = \bigcup_{i=1}^{r} C_i$, there exists $i \in \{1, 2, \ldots, r\}$ such that $C_i \in \mathcal{A}\}$. Given any $A \in \mathcal{R}$, there exists $p \in \beta\mathbb{N}$ such that $A \in p$ and $p \subseteq \mathcal{A}$.*

Proof. One notes that if $A \in \mathcal{R}$, $r \in \mathbb{N}$, and $A = \bigcup_{i=1}^{r} C_i$, then some $C_i \in \mathcal{R}$. Thus [11, Theorem 3.11] applies. $\square$

We pause to point out that Question 1.8 makes sense in any semigroup $(S, +)$. (If the operation is not commutative one needs to specify the order of the sums in $FS_{\leq 2}(\langle x_t \rangle_{t=1}^{\infty})$ and $FS(\langle x_t \rangle_{t=1}^{\infty})$. Standardly they would be taken in increasing order.) We shall see in Theorem 1.11 that a negative answer to Question 1.8 would have to involve special properties of the semigroup $(\mathbb{N}, +)$.

Recall the infinite version of Ramsey's Theorem. Given a set X and a cardinal k, $[X]^k = \{A \subseteq X : |A| = k\}$.

1.10 Theorem. *Let $k, r \in \mathbb{N}$. If $|X| = \omega$ and $[X]^k = \bigcup_{i=1}^{r} D_i$, then there exists $i \in \{1, 2, \ldots, r\}$ and $B \in [X]^{\omega}$ such that $[B]^k \subseteq D_i$.*

Proof. [13]. $\square$

1.11 Theorem. *Let $(G,+)$ be a direct sum of infinitely many copies of $\mathbb{Z}_2$. There exists a set $A \subseteq G$ such that whenever $r \in \mathbb{N}$ and $A = \bigcup_{i=1}^r C_i$, there must exist $i \in \{1, 2, \ldots, r\}$ and $\langle x_t \rangle_{t=1}^\infty$ such that $FS_{\leq 2}(\langle x_t \rangle_{t=1}^\infty) \subseteq C_i$, but there does not exist $\langle y_t \rangle_{t=1}^\infty$ with $FS(\langle y_t \rangle_{t=1}^\infty) \subseteq A$.*

Proof. Viewing G as a vector space over $\mathbb{Z}_2$, choose a sequence $\langle e_n \rangle_{n=1}^\infty$ of linearly independent members of G. Let $A = \{e_n + e_m : n, m \in \mathbb{N} \text{ and } n \neq m\}$. There do not exist $x_1, x_2, x_3 \in A$ such that $FS(\langle x_t \rangle_{t=1}^3) \subseteq A$.

Let $r \in \mathbb{N}$ and let $A = \bigcup_{i=1}^r C_i$. For $i \in \{1, 2, \ldots, r\}$, let $D_i = \big\{\{n, m\} \in [\mathbb{N}]^2 : e_n + e_m \in C_i\big\}$. Pick by Ramsey's Theorem some $i \in \{1, 2, \ldots, r\}$ and $B \in [\mathbb{N}]^\omega$ such that $[B]^2 \subseteq D_i$. Then $\{e_n + e_m : \{n, m\} \in [B]^2\} \subseteq C_i$. Pick $k \in B$ and enumerate $B \setminus \{k\}$ as $\langle n_t \rangle_{t=1}^\infty$. For $t \in \mathbb{N}$, let $x_t = e_k + e_{n_t}$. Then $FS_{\leq 2}(\langle x_t \rangle_{t=1}^\infty) \subseteq C_i$. $\square$

Defining $\mathcal{F}_2(S) = \{p \in \beta S : (\forall A \in p)(\exists \langle x_t \rangle_{t=1}^\infty)(FS_{\leq 2}(\langle x_t \rangle_{t=1}^\infty) \subseteq A)\}$ and $\Gamma(S) = \{p \in \beta S : (\forall A \in p)(\exists \langle x_t \rangle_{t=1}^\infty)(FS(\langle x_t \rangle_{t=1}^\infty) \subseteq A)\}$, one sees as a consequence of Theorem 1.11 that, if S is an infinite group of index 2, then $\mathcal{F}_2(S) \neq \Gamma(S)$.

Since we cannot answer Question 1.8, we do not know whether for some $k \in \mathbb{N} \setminus \{1\}$ $\mathcal{F}_{k+1} \neq \mathcal{F}_k$, nor do we know whether this holds for all such k.

We now introduce the other subsets of $\beta\mathbb{N}$ with which we shall be concerned. Given a finite sequence $\langle x_t \rangle_{t=1}^k$, $FS(\langle x_t \rangle_{t=1}^k) = \big\{\sum_{t \in F} x_t : \emptyset \neq F \subseteq \{1, 2, \ldots, k\}\big\}$.

1.12 Definition. Let $k \in \mathbb{N} \setminus \{1\}$.

$$P_k = \{p \in \beta\mathbb{N} : (\forall A \in p)(\exists \langle x_t \rangle_{t=1}^k)(FS(\langle x_t \rangle_{t=1}^k) \subseteq A)\}.$$

$$S_k = \{p \in \beta\mathbb{N} : (\forall A \in p)(\exists \langle x_t \rangle_{t=1}^k) \\ (x_1 << x_2 << \ldots << x_k \text{ and } FS(\langle x_t \rangle_{t=1}^k) \subseteq A)\}.$$

$$M_k = \{p \in \beta\mathbb{N} : (\forall A \in p)(\exists \langle x_t \rangle_{t=1}^k)(\exists \langle y_t \rangle_{t=1}^\infty) \\ (FS(\langle x_t \rangle_{t=1}^k) + FS(\langle y_t \rangle_{t=1}^\infty) \subseteq A)\}.$$

1.13 Definition.

(a) $E = \{p \in \beta\mathbb{N} : p + p = p\}$.
(b) $EK = \{p \in K(\beta\mathbb{N}) : p + p = p\}$.
(c) $P = \bigcap_{k=2}^\infty P_k$.
(d) $M = \bigcap_{k=2}^\infty M_k$.
(e) $S = \bigcap_{k=2}^\infty S_k$.

1.14 Lemma. *Let $k \in \mathbb{N} \setminus \{1\}$. Then M_k and P_k are subsemigroups of $\beta\mathbb{N}$.*

Proof. That P_k is a semigroup was proved in [10, Lemma 2.3]. (What we are calling P_k here was called S_k there.)

To see that M_k is a semigroup, let $p, q \in M_k$ and let $A \in p + q$. Let

$$B = \{x \in \mathbb{N} : -x + A \in q\} .$$

Then $B \in p$, so pick $\langle x_t \rangle_{t=1}^k$ and $\langle y_t \rangle_{t=1}^\infty$ such that $FS(\langle x_t \rangle_{t=1}^k) + FS(\langle y_t \rangle_{t=1}^\infty) \subseteq B$. In particular $FS(\langle x_t + y_t \rangle_{t=1}^k) \subseteq B$. Let $C = \bigcap\{-v + A : v \in FS(\langle x_t + y_t \rangle_{t=1}^k)\}$. Then $C \in q$ so pick $\langle w_t \rangle_{t=1}^k$ and $\langle z_t \rangle_{t=1}^\infty$ such that $FS(\langle w_t \rangle_{t=1}^k) + FS(\langle z_t \rangle_{t=1}^\infty) \subseteq C$. Then $FS(\langle x_t + y_t + w_t \rangle_{t=1}^k) + FS(\langle z_t \rangle_{t=1}^\infty) \subseteq A$. $\square$

As a consequence of Lemma 1.14 one also has that P and M are subsemigroups of $\beta\mathbb{N}$. It is a consequence of results of Section 3 that none of the other sets defined in Definitions 1.12 and 1.13 is a semigroup.

1.15 Definition. Given $H \subseteq \beta\mathbb{N}$, $\big\langle H \big\rangle$ is the subsemigroup generated by H.

It is immediate from the definitions that all of the objects defined in Definitions 1.12 and 1.13 except E and EK are closed.

In Section 2 we shall investigate the relationships that hold among $\mathcal{F}_k$, S_k, P_k, and M_k for various values of k. Except for our already confessed ignorance about whether any or all $\mathcal{F}_k = \Gamma$, we shall determine precisely which inclusions hold, namely only the trivial ones.

In Section 3 we shall investigate the relationships that hold among EK, E, Γ, $\mathcal{F}_2$, S_2, P_2, S, M, and P, and the semigroups they generate and the closures of those generated semigroups. Again, with the same exceptions regarding $\mathcal{F}_2$ we will show that the only inclusions that hold among them are the trivial ones.

2. Sets determined by finitely many sums

By virtue of Lemma 1.7 we have that for each $k \in \mathbb{N} \setminus \{1\}$, $\mathcal{F}_k \subseteq S_k$. All of the other inclusions in the following diagram (wherein the fact that $A \subseteq B$ is indicated by an arrow from A to B) are trivial.

$$
\begin{tikzcd}
 & & P_2 & \\
 & S_2 \arrow[ur] & P_3 \arrow[u] & M_2 \arrow[ul] \\
\mathcal{F}_2 \arrow[ur] & S_3 \arrow[u] \arrow[ur] & P_4 \arrow[u] & M_3 \arrow[u] \arrow[ul] \\
\mathcal{F}_3 \arrow[u] \arrow[ur] & S_4 \arrow[u] \arrow[ur] & P_5 \arrow[u] & M_4 \arrow[u] \arrow[ul] \\
\vdots & \vdots & \vdots & \vdots
\end{tikzcd}
$$

Figure 1

Trivially Γ is a subset of each of the sets in Figure 1 We do not know whether $\mathcal{F}_k = \Gamma$ for some or all $k \in \mathbb{N} \setminus \{1\}$. We show in this section that, with that glaring gap, the only inclusions that hold among the sets in Figure 1 are those that follow from the indicated inclusions.

Recall the finite version of Ramsey's Theorem.

2.1 Theorem. *Let $k, r, n \in \mathbb{N}$. There exists $m \in \mathbb{N}$ such that if $|X| = m$ and $[X]^k = \bigcup_{i=1}^r D_i$, there exists $i \in \{1, 2, \ldots, r\}$ and $B \in [X]^n$ such that $[B]^k \subseteq D_i$.*

Proof. [13]. $\square$

The following theorem lies behind many of the facts of this section. We let $FU(\langle F_t \rangle_{t=1}^k) = \big\{ \bigcup_{t \in H} F_t : \emptyset \neq H \subseteq \{1, 2, \ldots, k\} \big\}$.

2.2 Theorem. *(Nešetřil and Rödl). Let $r, k \in \mathbb{N}$ There is a finite set $\mathcal{S}$ of finite nonempty sets such that:*

(1) Whenever $\mathcal{S} = \bigcup_{i=1}^r \mathcal{D}_i$, there exist $i \in \{1, 2, \ldots, r\}$ and pairwise disjoint $F_1, F_2, \ldots, F_k$ in $\mathcal{S}$ with $FU(\langle F_t \rangle_{t=1}^k) \subseteq \mathcal{D}_i$ and

(2) there do not exist pairwise disjoint $F_1, F_2, \ldots, F_{k+1}$ in $\mathcal{S}$ with $FU(\langle F_t \rangle_{t=1}^{k+1}) \subseteq \mathcal{S}$.

Proof. [12, Theorem 1.1]. (Or see [6, p. 126].) (The fact that $\mathcal{S}$ and the members of $\mathcal{S}$ are finite is not stated, but follows from the proof.) $\square$

Properly interpreted, the proof of [12, Theorem 1.1] produces the following family for $k = 2$ and arbitrary r. For $u, v \in \mathbb{N}$ with $u < v$, let $A_{u,v} = \{u, u+1, \ldots, v-1\}$. By Theorem 2.1, pick $m \in \mathbb{N}$ such that whenever $|X| = m$ and $[X]^2 = \bigcup_{i=1}^r D_i$, there exists $i \in \{1, 2\}$ and $B \in [X]^3$ such that $[B]^2 \subseteq D_i$. Let $\mathcal{S} = \{A_{u,v} : 1 \leq u < v \leq m\}$. Given that $\mathcal{S} = \bigcup_{i=1}^r \mathcal{D}_i$, for each $i \in \{1, 2, \ldots, r\}$, let $D_i = \big\{\{u, v\} : 1 \leq u < v \leq$

m and $A_{u,v} \in \mathcal{D}_i\big\}$. Pick $i \in \{1,2,\dots,r\}$ and u,v,w with $1 \le u < v < w \le m$ such that $\big\{\{u,v\},\{u,w\},\{v,w\}\big\} \subseteq D_i$. Then $\{A_{u,v}, A_{u,w}, A_{v,w}\} \subseteq \mathcal{D}_i$. Thus if $F_1 = A_{u,v}$ and $F_2 = A_{v,w}$ one has that $FU(\langle F_t\rangle_{t=1}^2) \subseteq \mathcal{D}_i$ so (1) holds. Further, if $u,v,w,z \in \{1,2,\dots,m\}$, $u < v$, $w < z$, $u \le w$, $A_{u,v} \in \mathcal{S}$, $A_{w,z} \in \mathcal{S}$, and $A_{u,v} \cup A_{w,z} \in \mathcal{S}$, we must have that $v = w$. Consequently, (2) holds.

The family in the paragraph above has the additional property that $\max F_1 < \min F_2$. Unfortunately, the proof of [12, Theorem 1.1] does not produce such increasing sets for $k > 2$. We are grateful to Imre Leader for providing us with the proof of Theorem 2.5 below in which increasing sets are guaranteed.

2.3 Lemma. *Let $k, r \in \mathbb{N}$. There exists $S \in \mathcal{P}_f(\mathbb{N})$ such that*

(1) whenever $S = \bigcup_{i=1}^r C_i$, there exist $i \in \{1,2,\dots,r\}$ and a sequence $\langle x_t\rangle_{t=1}^k$ such that $FS(\langle x_t\rangle_{t=1}^k) \subseteq C_i$ and

(2) there does not exist a sequence $\langle x_t\rangle_{t=1}^{k+1}$ such that $FS(\langle x_t\rangle_{t=1}^{k+1}) \subseteq S$.

Proof. Pick a finite set $\mathcal{S}$ as guaranteed for k and r by Theorem 2.2. We may presume that $\bigcup \mathcal{S} \subseteq \mathbb{N}$. Let $S = \{\sum_{t\in F} 3^t : F \in \mathcal{S}\}$.

To verify (1), let $S = \bigcup_{i=1}^r C_i$. For $i \in \{1,2,\dots,r\}$, let

$$\mathcal{D}_i = \{F \in \mathcal{S} : \textstyle\sum_{t\in F} 3^t \in C_i\}\,.$$

Pick $i \in \{1,2,\dots,r\}$ and pairwise disjoint $F_1, F_2, \dots, F_k$ in $\mathcal{S}$ such that $FU(\langle F_t\rangle_{t=1}^k) \subseteq \mathcal{D}_i$. For $n \in \{1,2,\dots,k\}$, let $x_n = \sum_{t\in F_n} 3^t$.

To verify (2), suppose that we have a sequence $\langle x_n\rangle_{n=1}^{k+1}$ such that $FS(\langle x_n\rangle_{n=1}^{k+1}) \subseteq S$. For $n \in \{1,2,\dots,k+1\}$ let F_n be the member of $\mathcal{S}$ such that $x_n = \sum_{t\in F_n} 3^t$. If $n \neq m$, then $F_n \cap F_m = \emptyset$ since otherwise $x_n + x_m$ would have a 2 in its ternary expansion, which none of the members of S have. Then $FU(\langle F_n\rangle_{n=1}^{k+1}) \subseteq \mathcal{S}$, a contradiction. $\square$

2.4 Lemma. *Let $m, r \in \mathbb{N}$ and let $S \in \mathcal{P}_f(\mathbb{N})$. There exists $X \in \mathcal{P}_f(\mathbb{N})$ such that whenever $\mathcal{P}_f(X) = \bigcup_{i=1}^r \mathcal{D}_i$, there exists $Y \in [X]^m$ and for each $x \in S$ there exists $i(x) \in \{1,2,\dots,r\}$ such that $[Y]^x \subseteq \mathcal{D}_{i(x)}$.*

Proof. We proceed by induction on $|S|$, the case $|S| = 1$ being an immediate consequence of Theorem 2.1. Now assume that $|S| > 1$ and the result holds for smaller sets. Pick $z \in S$ and pick by Theorem 2.1, $n \in \mathbb{N}$ such that if $|W| = n$ and $[W]^z = \bigcup_{i=1}^r \mathcal{D}_i$, there exist $Y \in [W]^m$ and $i(z) \in \{1,2,\dots,r\}$ such that $[Y]^z \subseteq \mathcal{D}_{i(z)}$.

Pick X as guaranteed by the induction hypothesis for n, r, and $S \setminus \{z\}$. Let $\mathcal{P}_f(X) = \bigcup_{i=1}^r \mathcal{D}_i$ and pick $W \in [X]^n$ and for each $x \in S \setminus \{z\}$ pick $i(x) \in \{1,2,\dots,r\}$

such that $[W]^x \subseteq \mathcal{D}_{i(x)}$. Pick $Y \in [W]^m$ and $i(z) \in \{1,2,\ldots,r\}$ such that $[Y]^z \subseteq \mathcal{D}_{i(z)}$. $\square$

2.5 Theorem. *Let $r,k \in \mathbb{N}$ There is a finite set $\mathcal{S}$ of finite nonempty subsets of $\mathbb{N}$ such that:*

(1) Whenever $\mathcal{S} = \bigcup_{i=1}^r \mathcal{D}_i$, there exist $i \in \{1,2,\ldots,r\}$ and $F_1, F_2, \ldots, F_k$ in $\mathcal{S}$ such that for each $t \in \{1,2,\ldots,k-1\}$, $\max F_t < \min F_{t+1}$, and $FU(\langle F_t\rangle_{t=1}^k) \subseteq \mathcal{D}_i$ and

(2) there do not exist pairwise disjoint $F_1, F_2, \ldots, F_{k+1}$ in $\mathcal{S}$ with $FU(\langle F_t\rangle_{t=1}^{k+1}) \subseteq \mathcal{S}$.

Proof. Let S be as guaranteed by Lemmma 2.3 and let $m = k \cdot \max S$. Pick X as guaranteed by Lemma 2.4 for m and r. Let $\mathcal{S} = \{B \subseteq X : |B| \in S\}$.

To verify (1), let $\mathcal{S} = \bigcup_{i=1}^r \mathcal{D}_i$. For $i \in \{2,3,\ldots,r\}$ let $\mathcal{E}_i = \mathcal{D}_i$ and let $\mathcal{E}_1 = \mathcal{P}_f(X) \setminus \bigcup_{i=2}^r \mathcal{D}_i$. Pick $Y \in [X]^m$ and for each $x \in S$ pick $i(x) \in \{1,2,\ldots,r\}$ such that $[Y]^x \subseteq \mathcal{E}_{i(x)}$. For $i \in \{1,2,\ldots,r\}$, let $C_i = \{x \in S : i(x) = i\}$. Pick $i \in \{1,2,\ldots,r\}$ and $\langle x_t\rangle_{t=1}^k$ such that $FS(\langle x_t\rangle_{t=1}^k) \subseteq C_i$.

Enumerate Y in order as $\langle y_t\rangle_{t=1}^m$. Let $F_1 = \{y_1, y_2, \ldots, y_{x_1}\}$. For $t \in \{2,3,\ldots,k\}$, let $F_t = \{y_s : \sum_{j=1}^{t-1} x_j < s \leq \sum_{j=1}^t x_j\}$. Then for each $t \in \{1,2,\ldots,k-1\}$, $\max F_t < \min F_{t+1}$ and for each $t \in \{1,2,\ldots,k\}$, $|F_t| = x_t$. To see that $FU(\langle F_t\rangle_{t=1}^k) \subseteq \mathcal{D}_i$, let $\emptyset \neq H \subseteq \{1,2,\ldots,k\}$. Let $z = \sum_{t\in H} x_t$. Then $\bigcup_{t\in H} F_t \in [Y]^z$ and $i(z) = i$ so $\bigcup_{t\in H} F_t \in \mathcal{E}_i$. Since $z \in S$, $\bigcup_{t\in H} F_t \in \mathcal{S}$ so $\bigcup_{t\in H} F_t \in \mathcal{D}_i$.

To verify (2), suppose we have pairwise disjoint $F_1, F_2, \ldots, F_{k+1}$ in $\mathcal{S}$ such that $FU(\langle F_t\rangle_{t=1}^{k+1}) \subseteq \mathcal{S}$. For each $t \in \{1,2,\ldots,k+1\}$, let $x_t = |F_t|$. Then $FS(\langle x_t\rangle_{t=1}^{k+1}) \subseteq S$, a contradiction. $\square$

The proof of the following corollary can be taken nearly verbatim from the proof of [10, Corollary 3.8], so we omit it here. (Much of that argument can be taken from the proof of Theorem 2.8 also.)

2.6 Corollary. *Let $k \in \mathbb{N} \setminus \{1\}$. There is a set $A \subseteq \mathbb{N}$ such that*

(1) Whenever $r \in \mathbb{N}$ and $A = \bigcup_{i=1}^r C_i$ there exist $i \in \{1,2,\ldots,r\}$ and a sequence $\langle x_t\rangle_{t=1}^k$ such that $x_1 << x_2 << \ldots << x_k$ and $FS(\langle x_t\rangle_{t=1}^k) \subseteq C_i$ and

(2) there does not exist a sequence $\langle x_t\rangle_{t=1}^{k+1}$ with $FS(\langle x_t\rangle_{t=1}^{k+1}) \subseteq A$.

2.7 Theorem. *For each $k \in \mathbb{N} \setminus \{1\}$, $S_k \setminus P_{k+1} \neq \emptyset$.*

Proof. Let

$$\mathcal{R} = \{A \subseteq \mathbb{N} : \text{whenever } r \in \mathbb{N} \text{ and } A = \textstyle\bigcup_{i=1}^{r} C_i\,,\ \text{there exist } i \in \{1,2,\ldots,r\} \text{ and } \langle x_t \rangle_{t=1}^{k} \text{ such that } x_1 << x_2 << \ldots << x_k \text{ and } FS(\langle x_t \rangle_{t=1}^{k}) \subseteq C_i\}\,.$$

Pick A as guaranteed by Corollary 2.6. By Lemma 1.9 pick $p \in S_k \cap c\ell A$. Since $A \in p$, $p \notin P_{k+1}$. □

2.8 Theorem. *Let $k \in \mathbb{N} \setminus \{1\}$. Then $M_k \setminus P_{k+1} \neq \emptyset$.*

Proof. For each $r \in \mathbb{N}$, pick by Theorem 2.2 a finite set $\mathcal{S}_r$ of finite nonempty subsets of $\mathbb{N}$ such that:

(1) Whenever $\mathcal{S}_r = \bigcup_{i=1}^{r} \mathcal{D}_i$, there exist $i \in \{1,2,\ldots,r\}$ and pairwise disjoint $F_1, F_2, \ldots, F_k$ in $\mathcal{S}_r$ with $FU(\langle F_t \rangle_{t=1}^{k}) \subseteq \mathcal{D}_i$;

(2) there do not exist pairwise disjoint $F_1, F_2, \ldots, F_{k+1}$ in $\mathcal{S}_r$ with $FU(\langle F_t \rangle_{t=1}^{k+1}) \subseteq \mathcal{S}_r$; and

(3) $r + \max \bigcup \mathcal{S}_r < \min \bigcup \mathcal{S}_{r+1}$.

For each r, let $a_r = \max \bigcup \mathcal{S}_r$ and let $B = \bigcup_{r=1}^{\infty} \{a_r + 1, a_r + 2, \ldots, a_r + r\}$. Let $D = \{\sum_{t \in F} 3^t + \sum_{t \in G} 3^t : G \in \mathcal{P}_f(B) \text{ and } (\exists r)(F \in \mathcal{S}_r)\}$. Let

$$\mathcal{R} = \{A \subseteq \mathbb{N} : \text{whenever } r \in \mathbb{N} \text{ and } A = \textstyle\bigcup_{i=1}^{r} C_i\,,\ \text{there exist } i \in \{1,2,\ldots,r\} \text{ and sequences} \langle x_t \rangle_{t=1}^{k} \text{ and } \langle y_t \rangle_{t=1}^{\infty} \text{ such that } FS(\langle x_t \rangle_{t=1}^{k}) + FS(\langle y_t \rangle_{t=1}^{\infty}) \subseteq C_i\}\,.$$

We claim that $D \in \mathcal{R}$ so that by Lemma 1.9, $\overline{D} \cap M_k \neq \emptyset$.

So let $r \in \mathbb{N}$ and let $D = \bigcup_{i=1}^{r} C_i$. By $|\mathcal{S}_r|$ repetitions of [11, Corollary 5.17] pick a sequence $\langle G_n \rangle_{n=1}^{\infty}$ in $\mathcal{P}_f(B)$ such that for each n, $\max G_n < \min G_{n+1}$, and for each $F \in \mathcal{S}_r$ there exists $i(F) \in \{1,2,\ldots,r\}$ such that

$$\{\textstyle\sum_{t \in F} 3^t + \sum_{t \in H} 3^t : H \in FU(\langle G_n \rangle_{n=1}^{\infty}\} \subseteq C_{i(F)}\,.$$

For $i \in \{1,2,\ldots,r\}$, let $\mathcal{D}_i = \{F \in \mathcal{S}_r : i(F) = i\}$ and pick $i \in \{1,2,\ldots,r\}$ and pairwise disjoint $F_1, F_2, \ldots, F_k \in \mathcal{S}_r$ such that $FU(\langle F_t \rangle_{t=1}^{k}) \subseteq \mathcal{D}_i$. For $t \in \{1,2,\ldots,k\}$, let $x_t = \sum_{n \in F_t} 3^n$ and for $t \in \mathbb{N}$, let $y_t = \sum_{n \in G_t} 3^n$. Then $FS(\langle x_t \rangle_{t=1}^{k}) + FS(\langle y_t \rangle_{t=1}^{\infty}) \subseteq C_i$. We thus have $\overline{D} \cap M_k \neq \emptyset$.

Suppose now that $\overline{D} \cap P_{k+1} \neq \emptyset$ and pick $\langle x_n \rangle_{n=1}^{k+1}$ such that $FS(\langle x_n \rangle_{n=1}^{k+1}) \subseteq D$. For each $n \in \{1,2,\ldots,k+1\}$, pick $r(n) \in \mathbb{N}$, $F_n \in \mathcal{S}_{r(n)}$, and $G_n \in \mathcal{P}_f(B)$ such that $x_n = \sum_{t \in F_n} 3^t + \sum_{t \in G_n} 3^t$. Note that if $n \neq l$, then $(F_n \cup G_n) \cap (F_l \cup G_l) = \emptyset$ because $x_n + x_l \in D$ and so has all 1's in its ternary expansion. Since any member of D has ternary support meeting exactly one $\mathcal{S}_r$, we have that there is some r such that $r(n) = r$ for all $n \in \{1, 2, \ldots, k+1\}$. But now, $FU(\langle F_n\rangle_{n=1}^{k+1}) \subseteq \mathcal{S}_r$, a contradiction. $\square$

We extend the notion of a *sum subsystem* defined in [11, Definition 5.13] for infinite sequences in the obvious way to apply to finite sequences.

2.9 Definition. Let $n, k \in \mathbb{N}$ and let $\langle y_t\rangle_{t=1}^n$ be a sequence in $\mathbb{N}$. Then $\langle x_t\rangle_{t=1}^k$ is a *sum subsystem* of $\langle y_t\rangle_{t=1}^n$ if and only if there exists a sequence $\langle F_t\rangle_{t=1}^k$ in $\mathcal{P}_f(\{1, 2, \ldots, n\})$ such that for each $t \in \{1, 2, \ldots, k\}$, $x_t = \sum_{s\in F_t} y_s$ and for each $t \in \{1, 2, \ldots, k-1\}$, $\max F_t < \min F_{t+1}$.

2.10 Lemma. *Let $k, r \in \mathbb{N}$. There exists $n \in \mathbb{N}$ such that whenever $\langle y_t\rangle_{t=1}^n$ is a sequence in $\mathbb{N}$ and $FS(\langle y_t\rangle_{t=1}^n) = \bigcup_{i=1}^r C_i$, there exist $i \in \{1, 2, \ldots, r\}$ and a sum subsystem $\langle x_t\rangle_{t=1}^k$ of $\langle y_t\rangle_{t=1}^n$ such that $FS(\langle x_t\rangle_{t=1}^k) \subseteq C_i$.*

Proof. This follows from [11, Corollary 5.15] by a standard compactness argument. (See [11, Section 5.5].) $\square$

2.11 Theorem. *Let $D = \big\{x \in \mathbb{N} : (\exists n \in \mathbb{N})(\mathrm{supp}(x) \subseteq \{2^n+1, 2^n+2, \ldots, 2^n+n\}\big\}$. Then $\overline{D} \cap S \neq \emptyset$, $\overline{D} \cap M_2 = \emptyset$, $\overline{D} \cap \mathcal{F}_2 = \emptyset$, and $\overline{D} \cap (\beta\mathbb{N} + \mathcal{F}_2) = \emptyset$. In particular, $S \setminus M_2 \neq \emptyset$ and $S \setminus c\ell\langle\mathcal{F}_2\rangle \neq \emptyset$.*

Proof. To see that $\overline{D} \cap S \neq \emptyset$, it suffices to let $k \in \mathbb{N}$ and show that $\overline{D} \cap S_k \neq \emptyset$. (For then, $\{\overline{D} \cap S_k : k \in \mathbb{N}\}$ is a collection of closed subsets of $\beta\mathbb{N}$ with the finite intersection property.) So let $k \in \mathbb{N}$. Let

$$\begin{aligned}\mathcal{R} = \{A \subseteq \mathbb{N} :\ & \text{whenever } r \in \mathbb{N} \text{ and } A = \textstyle\bigcup_{i=1}^r C_i\,,\\ & \text{there exist } i \in \{1, 2, \ldots, r\} \text{ and } \langle x_t\rangle_{t=1}^k \text{ such that}\\ & x_1 << x_2 << \ldots << x_k \text{ and } FS(\langle x_t\rangle_{t=1}^k) \subseteq C_i\}\,.\end{aligned}$$

To see that $\overline{D} \cap S_k \neq \emptyset$, it suffices by Lemma 1.9 to show that $D \in \mathcal{R}$. So let $r \in \mathbb{N}$ and let $D = \bigcup_{i=1}^r C_i$.

Pick by Lemma 2.10, $n \in \mathbb{N}$ such that whenever $\langle y_t\rangle_{t=1}^n$ is a sequence in $\mathbb{N}$ and $FS(\langle y_t\rangle_{t=1}^n) = \bigcup_{i=1}^r E_i$, there exist $i \in \{1, 2, \ldots, r\}$ and a sum subsystem $\langle x_t\rangle_{t=1}^k$ of $\langle y_t\rangle_{t=1}^n$ such that $FS(\langle x_t\rangle_{t=1}^k) \subseteq E_i$. For $t \in \{1, 2, \ldots, n\}$ let $y_t = 2^{2^n+t}$. Pick $i \in \{1, 2, \ldots, r\}$ and a sum subsystem $\langle x_t\rangle_{t=1}^k$ of $\langle y_t\rangle_{t=1}^n$ such that $FS(\langle x_t\rangle_{t=1}^k) \subseteq C_i$. Since $\langle x_t\rangle_{t=1}^k$ is a sum subsystem of $\langle y_t\rangle_{t=1}^n$ we have that $x_1 << x_2 << \ldots << x_k$.

To see that $\overline{D} \cap M_2 = \emptyset$ suppose instead we have x_1, x_2, and $\langle y_t\rangle_{t=1}^\infty$ such that $FS(\langle x_t\rangle_{t=1}^2) + FS(\langle y_t\rangle_{t=1}^\infty) \subseteq D$. By passing to a sum subsystem, we may presume that for each $t \in \mathbb{N}$, $y_t << y_{t+1}$. Pick n such that $\operatorname{supp}(x_1+y_1) \subseteq \{2^n+1, 2^n+2, \ldots, 2^n+n\}$ and pick t such that $\min\ \operatorname{supp}(y_t) > 2^n+n$. Then $x_1+y_1+y_t \notin D$, a contradiction.

To see that $\overline{D} \cap \mathcal{F}_2 = \emptyset$, suppose instead that $\overline{D} \cap \mathcal{F}_2 \neq \emptyset$ and pick by Lemma 1.7 a sequence $\langle x_t \rangle_{t=1}^{\infty}$ such that $FS_{\leq 2}(\langle x_t \rangle_{t=1}^{\infty}) \subseteq D$ and for each $t \in \mathbb{N}$, $x_t << x_{t+1}$. Pick n such that $\operatorname{supp}(x_1) \subseteq \{2^n+1, 2^n+2, \ldots, 2^n+n\}$ and pick t such that $\min\ \operatorname{supp}(x_t) > 2^n+n$. Then $x_1+x_t \notin D$, a contradiction.

To see that $\overline{D} \cap (\beta\mathbb{N} + \mathcal{F}_2) = \emptyset$, suppose instead we have $p \in \beta\mathbb{N}$ and $q \in \mathcal{F}_2$ such that $D \in p+q$. Then $\{z \in \mathbb{N} : -z+D \in q\} \in p$ so pick $z \in \mathbb{N}$ such that $-z+D \in q$. Pick by Lemma 1.7 a sequence $\langle x_t \rangle_{t=1}^{\infty}$ such that $FS_{\leq 2}(\langle x_t \rangle_{t=1}^{\infty}) \subseteq -z+D$ and for each $t \in \mathbb{N}$, $x_t << x_{t+1}$. Pick n such that $\max\ \operatorname{supp}(z) < 2^n$ and pick t such that $\min\ \operatorname{supp}(x_t) > 2^n$. Then $x_t \notin -z+D$, a contradiction.

Since $\langle \mathcal{F}_2 \rangle \subseteq \mathcal{F}_2 \cup (\beta\mathbb{N} + \mathcal{F}_2)$ we have that $S \setminus c\ell\langle \mathcal{F}_2 \rangle \neq \emptyset$. $\square$

A subsemigroup known to contain much of the algebraic structure of $\beta\mathbb{N}$ (and with copies arising in many contexts) is $\mathbb{H}$. (See [11, Sections 6.1 and 7.2].)

2.12 Definition. $\mathbb{H} = \bigcap_{n=1}^{\infty} \overline{2^n\mathbb{N}}$.

2.13 Lemma. $P_2 \subseteq \mathbb{H}$.

Proof. Let $p \in P_2$ and suppose that $p \notin \mathbb{H}$. Pick the least $n \in \mathbb{N}$ such that $2^n\mathbb{N} \notin p$. Then $2^n\mathbb{N} + 2^{n-1} \in p$. Pick x and y such that $\{x, y, x+y\} \subseteq 2^n\mathbb{N} + 2^{n-1}$. then $\min\ \operatorname{supp}(x) = \min\ \operatorname{supp}(y) = n-1$ and so $\min\ \operatorname{supp}(x+y) > n-1$, a contradiction. $\square$

2.14 Theorem. *There is a set $D \subseteq \mathbb{N}$ such that $\overline{D} \cap M \neq \emptyset$, $\overline{D} \cap S_2 = \emptyset$, and $\overline{D} \cap (\mathbb{H}+\mathbb{H}) = \emptyset$. In particular, $M \setminus c\ell\langle S_2 \rangle \neq \emptyset$.*

Proof. For $n \in \mathbb{N}$, let $A_n = \{2^n+1, 2^n+2, \ldots, 2^n+n\}$ and let $\langle B_n \rangle_{n=1}^{\infty}$ be a sequence of pairwise disjoint infinite subsets of $\mathbb{N} \setminus \bigcup_{n=1}^{\infty} A_n$ with $\min B_n > 2^n+n$ for each n. Let $D = \{x+y : (\exists n \in \mathbb{N})(\operatorname{supp}(x) \subseteq A_n \text{ and } \operatorname{supp}(y) \subseteq B_n)\}$.

To see that $\overline{D} \cap M \neq \emptyset$ we let $k \in \mathbb{N}$ and show that $\overline{D} \cap M_k \neq \emptyset$. Let

$$\begin{aligned}\mathcal{R} = \{A \subseteq \mathbb{N} : \ & \text{whenever } r \in \mathbb{N} \text{ and } A = \textstyle\bigcup_{i=1}^{r} C_i\,, \\ & \text{there exist } i \in \{1,2,\ldots,r\} \text{ and sequences} \langle x_t \rangle_{t=1}^{k} \text{ and } \langle y_t \rangle_{t=1}^{\infty} \\ & \text{such that } FS(\langle x_t \rangle_{t=1}^{k}) + FS(\langle y_t \rangle_{t=1}^{\infty}) \subseteq C_i\}\,.\end{aligned}$$

By Lemma 1.9 it suffices to show that $D \in \mathcal{R}$. So let $r \in \mathbb{N}$ and let $D = \bigcup_{i=1}^{r} C_i$. Pick by Lemma 2.10 some $n \in \mathbb{N}$ such that given any sequence $\langle z_t \rangle_{t=1}^{n}$, if $FS(\langle z_t \rangle_{t=1}^{n}) = \bigcup_{i=1}^{r} E_i$, there exist $i \in \{1, 2, \ldots, r\}$ and $\langle x_t \rangle_{t=1}^k$ such that $FS(\langle x_t \rangle_{t=1}^k) \subseteq E_i$. For $t \in \{1, 2, \ldots, n\}$, let $z_t = 2^{2^n + t}$.

By $2^n - 1$ repeated applications of [11, Corollary 5.15] choose a sum subsystem $\langle y_t \rangle_{t=1}^\infty$ of $\langle 2^t \rangle_{t \in B_n}$ such that for each $w \in FS(\langle z_t \rangle_{t=1}^n)$ there exists $i(w) \in \{1, 2, \ldots, r\}$ such that $w + FS(\langle y_t \rangle_{t=1}^\infty) \subseteq C_{i(w)}$. For $i \in \{1, 2, \ldots, r\}$, let

$$E_i = \{w \in FS(\langle z_t \rangle_{t=1}^n) : i(w) = i\}.$$

Pick $i \in \{1, 2, \ldots, r\}$ and $\langle x_t \rangle_{t=1}^k$ such that $FS(\langle x_t \rangle_{t=1}^k) \subseteq E_i$. Then $FS(\langle x_t \rangle_{t=1}^k) + FS(\langle y_t \rangle_{t=1}^\infty) \subseteq C_i$.

To see that $\overline{D} \cap S_2 = \emptyset$, suppose instead that we have $u \ll v$ such that $\{u, v, u + v\} \subseteq D$. Then there exist n, m, x_1, x_2, y_1, and y_2 such that $u = x_1 + y_1$, $v = x_2 + y_2$, $\mathrm{supp}(x_1) \subseteq A_n$, $\mathrm{supp}(y_1) \subseteq B_n$, $\mathrm{supp}(x_2) \subseteq A_m$, and $\mathrm{supp}(y_2) \subseteq B_m$. Since $\mathrm{supp}(u + v) \cap A_n \neq \emptyset$ and $\mathrm{supp}(u + v) \cap A_m \neq \emptyset$, we have that $n = m$. But now $\max\ \mathrm{supp}(u) > 2^n + n \geq \min\ \mathrm{supp}(v)$, a contradiction.

To see that $\overline{D} \cap (\mathbb{H} + \mathbb{H}) = \emptyset$ suppose instead that we have $p, q \in \mathbb{H}$ such that $D \in p + q$. Then $\{w \in \mathbb{N} : -w + D \in q\} \in p$. Pick w_1 such that $-w_1 + D \in q$ and pick n such that $2^n > \max\ \mathrm{supp}(w_1)$. Pick w_2 such that $\min\ \mathrm{supp}(w_2) > 2^n$ and $-w_2 + D \in q$. Pick $z \in (-w_1 + D) \cap (-w_2 + D)$ such that $\min\ \mathrm{supp}(z) > \max\ \mathrm{supp}(w_2)$. Then $\min\ \mathrm{supp}(w_1 + z) \in A_k$ for some $k < n$ and $\min\ \mathrm{supp}(w_2 + z) \in A_l$ for some $l \geq n$. But then $\max\ \mathrm{supp}(z) \in B_k \cap B_l$, a contradiction.

By Lemma 2.13, we have $\langle S_2 \rangle \subseteq S_2 \cup (\mathbb{H} + \mathbb{H})$, so $M \setminus c\ell\langle S_2 \rangle \neq \emptyset$. $\square$

Recall that we do not know that $\mathcal{F}_2 \neq \Gamma$. If $\mathcal{F}_2 = \Gamma$, then $\mathcal{F}_2 \subseteq S_k \cap P_k \cap M_k$ for each $k \in \mathbb{N} \setminus \{1\}$.

2.15 Theorem. *For any $k, l \in \mathbb{N} \setminus \{1\}$, if A is any one of S_k, P_k, or M_k and B is any one of $\mathcal{F}_l$, S_l, P_l, or M_l, then $A \subseteq B$ if and only if the inclusion follows from the inclusions shown in Figure* 1.

Proof. By Theorem 2.7 we have that each $S_k \setminus P_{k+1} \neq \emptyset$. By Theorem 2.8 we have that each $M_k \setminus P_{k+1} \neq \emptyset$. By Theorem 2.11 we have for each k and l that $S_k \setminus M_l \neq \emptyset$ and $S_k \setminus \mathcal{F}_l \neq \emptyset$. By Theorem 2.14 we have for each k and l that $M_k \setminus S_l \neq \emptyset$ and $M_k \setminus \mathcal{F}_l \neq \emptyset$. $\square$

The next result is not related to any of our inclusions, but we feel it provides an interesting relationship.

2.16 Theorem. *Let $k \in \mathbb{N}$. Then $P_k + \Gamma \subsetneq M_k$.*

Proof. Since $P_k \subseteq \mathbb{H}$ by Lemma 2.13 and $\Gamma \subseteq \mathbb{H}$, we have by Theorem 2.14 that $M_k \setminus (P_k + \Gamma) \neq \emptyset$.

To see that $P_k + \Gamma \subseteq M_k$, let $p \in P_k$, let $q \in \Gamma$, and let $A \in p + q$. Then $\{x \in \mathbb{N} : -x + A \in q\} \in p$ so pick $\langle x_t \rangle_{t=1}^k$ such that $FS(\langle x_t \rangle_{t=1}^k) \subseteq \{x \in \mathbb{N} : -x + A \in q\}$. Then $\bigcap\{-z + A : z \in FS(\langle x_t \rangle_{t=1}^k)\} \in q$ so pick $\langle y_t \rangle_{t=1}^\infty$ such that $FS(\langle y_t \rangle_{t=1}^\infty) \subseteq \bigcap\{-z + A : z \in FS(\langle x_t \rangle_{t=1}^k)\}$. $\square$

3. The generated semigroups and their closures

All of the inclusions indicated in the following diagram are trivial. We begin this section by showing that no inclusion holds among the sets in Figure 2 unless it follows from the diagramed ones.

$$\begin{tikzcd}
 & P & & \\
c\ell\langle S\rangle \arrow[ur] & M \arrow[u] & & \\
\langle S\rangle \arrow[u] & c\ell\langle \Gamma\rangle \arrow[ul] \arrow[u] & & \\
S \arrow[u] & \langle \Gamma\rangle \arrow[ul] \arrow[u] & c\ell\langle E\rangle \arrow[ul] & \\
 & \Gamma \arrow[ul] \arrow[u] \arrow[ur] & \langle E\rangle \arrow[ul] \arrow[u] & c\ell\langle EK\rangle \arrow[ul] \\
 & & E \arrow[ul] \arrow[u] & \langle EK\rangle \arrow[ul] \arrow[u] \\
 & & & EK \arrow[ul] \arrow[u]
\end{tikzcd}$$

Figure 2

The following lemma is certainly known by a number of people, but we cannot find a convenient reference.

3.1 Lemma. *Let $q \in EK$. Then for each $n \in \mathbb{N}$, $nq \in EK$ where nq is the product computed in $(\beta\mathbb{N}, \cdot)$.*

Proof. The mapping $x \mapsto nx$ from $\beta\mathbb{N}$ to itself is an injective homomorphism by [11, Lemma 13.1 and Exercise 3.4.1]. So it maps idempotents to idempotents and preserves their order. And by [11, Lemma 6.6] all idempotents are in the range of this map. It therefore maps minimal idempotents to minimal idempotents. $\square$

3.2 Theorem. $c\ell\big\langle EK\big\rangle \setminus \big\langle S_2\big\rangle \neq \emptyset$.

Proof. We define functions f and g from $\mathbb{N}$ to ω by $f(n) = \min\big(\mathrm{supp}(n)\big)$ and $g(n) = \max\big(\mathrm{supp}(n)\big)$. As usual, $\widetilde{f}$ and $\widetilde{g}$ will denote the extensions of f and g respectively to continuous functions from $\beta\mathbb{N}$ to $\beta\omega$.

Let $\langle A_n\rangle_{n=1}^{\infty}$ be a sequence of pairwise disjoint infinite subsets of $2\mathbb{N}-1$. It follows from [11, Lemma 6.8] that, for any $x \in \beta\mathbb{N}$ and any $y \in \mathbb{H}$, $\widetilde{f}(x+y) = \widetilde{f}(x)$ and $\widetilde{g}(x+y) = \widetilde{g}(y)$. So, for each $n \in \mathbb{N}$, $\{x \in \mathbb{H} : \widetilde{f}(x) \in \overline{A}_n\}$ and $\{x \in \mathbb{H} : \widetilde{g}(x) \in \overline{A}_n\}$ are respectively right and left ideals of $\mathbb{H}$. We can therefore choose for each n a minimal idempotent p_n of $\mathbb{H}$ in the intersection of these two sets. Since $\mathbb{H}$ contains all the idempotents of $\beta\mathbb{N}$ [11, Lemma 6.8], $\mathbb{H} \cap K(\beta\mathbb{N}) \neq \emptyset$ and is therefore an ideal of $\mathbb{H}$ so $K(\mathbb{H}) \subseteq K(\beta\mathbb{N})$. So each $p_n \in EK$. By Lemma 3.1 we also have $2p_n \in EK$. Let q be a point of accumulation of the sequence $\langle p_n + 2p_n\rangle_{n=1}^{\infty}$. Then $q \in c\ell\big\langle EK\big\rangle$. We shall show that $q \notin \big\langle S_2\big\rangle$.

We shall first show that $q \notin \mathbb{H}+\mathbb{H}$. Suppose, on the contrary, that $q = u+v$ for some $u, v \in \mathbb{H}$. We claim that there is at most one value of $n \in \mathbb{N}$ for which $p_n + 2p_n \in \beta\mathbb{N}+v$. To see this, observe that $g(2n) = g(n)+1$ for every $n \in \mathbb{N}$ and hence, by continuity, $\widetilde{g}(2x) = \widetilde{g}(x)+1$ for every $x \in \beta\mathbb{N}$. Also, for all $x \in \beta\mathbb{N}+v$ we have that $\widetilde{g}(x) = \widetilde{g}(v)$. Thus, if $p_n+2p_n \in \beta\mathbb{N}+v$, we have that $\widetilde{g}(v) = \widetilde{g}(p_n+2p_n) = \widetilde{g}(2p_n) = \widetilde{g}(p_n)+1$. Since the mapping $n \mapsto \widetilde{g}(p_n)$ is injective, the claim is established. Now let $M = \{n \in \mathbb{N} : p_n+2p_n \notin \beta\mathbb{N}+v\}$, so that $|\mathbb{N} \setminus M| \leq 1$. Now $q \in c\ell(\mathbb{N}+v)$ and $q \in c\ell(\{p_n+2p_n : n \in M\})$. By [11, Theorem 3.40], this implies that $\{p_n+2p_n : n \in M\} \cap (\beta\mathbb{N}+v) \neq \emptyset$ or $c\ell(\{p_n+2p_n : n \in M\}) \cap (\mathbb{N}+v) \neq \emptyset$. However, the first possibility does not hold, by definition of M, and the second does not hold because $\mathbb{N}+\mathbb{H}$ does not meet $\mathbb{H}$. This contradiction shows that $q \notin \mathbb{H}+\mathbb{H}$. So $q \notin \big\langle S_2\big\rangle \setminus S_2$.

We shall now show that $q \notin S_2$, and this will complete the proof. For $n \in \mathbb{N}$, we define $h(n) \in \mathbb{Z}_2$ as follows: if $\mathrm{supp}(n)$ is written as $\{i_1, i_2, \cdots, i_m\}$ in increasing order, $h(n)$ is the number mod 2 of values of k in $\{1, 2, \cdots, m-1\}$ for which i_k is odd and i_{k+1} is even. Then $\widetilde{h} : \beta\mathbb{N} \to \mathbb{Z}_2$ denotes the continuous extension of h. We observe that, for any $n_1, n_2 \in \mathbb{N}$ for which $n_1 << n_2$, $h(n_1+n_2) = h(n_1)+h(n_2)$ except in the case in which $g(n_1)$ is odd and $f(n_2)$ is even, and that, in this case, $h(n_1+n_2) = h(n_1)+h(n_2)+1$. We claim that $\widetilde{h}(p_n) = 0$ for every $n \in \mathbb{N}$. To see this, let $\widetilde{h}(p_n) = a$ and let $D = \{r \in \mathbb{N} : h(r) = a\,,\ f(r) \in A_n$, and $g(r) \in A_n\}$. Then $D \in p_n$. So we can choose $n_1, n_2 \in D$ for which $n_1 << n_2$ and $n_1 + n_2 \in D$. This implies that $a = a + a$ and hence that $a = 0$. Similarly one can show that $\widetilde{h}(2p_n) = 0$ and $\widetilde{h}(p_n + 2p_n) = 1$. So $\widetilde{h}(q) = 1$. Now let $E = \{r \in \mathbb{N} : h(r) = 1\,,\ f(r)$ is odd and $g(r)$ is even$\}$. Then $E \in q$. However, if $r_1, r_2 \in E$ and $r_1 << r_2$, then $h(r_1 + r_2) = 0$ and so $r_1 + r_2 \notin E$. So $q \notin S_2$. $\square$

3.3 Theorem. *There exist p and q in EK such that $p + q \notin S_2$. In particular $\langle EK \rangle \setminus S_2 \neq \emptyset$.*

Proof. Let S be the semigroup of [11, Example 2.13]. (This is the 8 element semigroup generated by e and f and determined by the relations $ee = e$, $ff = f$, $efefe = e$, and $fefef = f$.) As can be quickly verified S is simple. That is, $K(S) = S$.

Define $h : \mathbb{N} \to S$ as follows:

(1) for $k \in \omega$, $h(2^k) = \begin{cases} e & \text{if } k \text{ is even} \\ f & \text{if } k \text{ is odd;} \end{cases}$

(2) for $F \in \mathcal{P}_f(\omega)$ with $|F| > 1$, $h(\sum_{t \in F} 2^t) = \prod_{t \in F} h(2^t)$.

Let $\widetilde{h} : \beta\mathbb{N} \to S$ be the continuous extension of h. By [11, Theorem 4.21] the restriction of $\widetilde{h}$ to $\mathbb{H}$ is a homomorphism, and it is easy to verify that $\widetilde{h}[\mathbb{H}] = S$. By [11, Exercise 1.7.3], $\widetilde{h}[K(\mathbb{H})] = K(S) = S$. Pick idempotents $p, q \in K(\mathbb{H})$ such that $\widetilde{h}(p) = e$ and $\widetilde{h}(q) = f$. Since all of the idempotents of $\beta\mathbb{N}$ are in $\mathbb{H}$, $\mathbb{H} \cap K(\beta\mathbb{N}) \neq \emptyset$ and so $\mathbb{H} \cap K(\beta\mathbb{N})$ is an ideal of $\beta\mathbb{N}$ and thus $K(\mathbb{H}) \subseteq \mathbb{H} \cap K(\beta\mathbb{N})$. (In fact, equality holds by [11, Theorem 1.65], but we don't need that now.) Therefore $p, q \in EK$.

We have that $\widetilde{h}(p+q) = ef$ so $h^{-1}[\{ef\}] \in p+q$. Suppose that we have $x << y$ such that $\{x, y, x+y\} \subseteq h^{-1}[\{ef\}]$. Since $x << y$ we have $h(x+y) = h(x)h(y) = efef \neq ef$, a contradiction. $\square$

3.4 Theorem. *There exists $p \in \Gamma$ such that $p+p \notin c\ell(\mathbb{H}+E)$. In particular, $\Gamma \setminus \langle E \rangle \neq \emptyset$ and $\langle \Gamma \rangle \setminus c\ell\langle E \rangle \neq \emptyset$.*

Proof. Let $\langle A_n \rangle_{n=1}^{\infty}$ partition ω into infinite sets and for each $n \in \mathbb{N}$ let $C_n = \{x \in \mathbb{N} : \operatorname{supp}(x) \subseteq A_n\}$. Then $C_n = FS(\langle 2^t \rangle_{t \in A_n})$ so pick by [11, Lemma 5.11] an idempotent $q_n \in \overline{C_n}$. Let p be a cluster point of the sequence $\langle q_n \rangle_{n=1}^{\infty}$. Then $p \in c\ell E = \Gamma$.

Suppose that $p+p \in c\ell(\mathbb{H}+E)$. Let $D = \{z \in \mathbb{N} :$ there exist $m < n$ in $\mathbb{N}$, $x \in C_m$, and $y \in C_n$ such that $x << y$ and $z = x + y\}$. Note that $z \in D$ if and only if for some $m < n$ the support of z consists of members of A_m followed by members of A_n.

We claim that $D \in p+p$. To see this, we show that $\bigcup_{n=1}^{\infty} C_n \subseteq \{x \in \mathbb{N} : -x+D \in p\}$ which suffices since $\bigcup_{n=1}^{\infty} C_n \in p$. So let $x \in \bigcup_{n=1}^{\infty} C_n$ and pick $m \in \mathbb{N}$ such that $x \in C_m$.

Then $\bigcup_{n=m+1}^{\infty} C_n \in p$ and $\bigcup_{n=m+1}^{\infty} C_n \subseteq -x + D$.

For each $n \in \mathbb{N}$, let $V_n = \mathbb{H} \cap \overline{\{x \in \mathbb{N} : \max\ \mathrm{supp}(x) \in A_n\}}$. We claim that $p + p \in c\ell(\bigcup_{n=1}^{\infty} V_n)$. To see this let $B \in p + p$. Since $p + p \in c\ell(\mathbb{H} + E)$, pick $v \in \mathbb{H}$ and $u \in E$ such that $v + u \in \overline{B \cap D}$. We shall show that there is some $n \in \mathbb{N}$ such that $C_n \in u$, from which it follows easily that $v + u \in V_n$ and thus $\overline{B} \cap V_n \neq \emptyset$. To this end pick $x \in \mathbb{N}$ such that $-x + D \in u$. Let $k = 1 + \max\ \mathrm{supp}(x)$. We can't have $|\{n \in \mathbb{N} : \mathrm{supp}(x) \cap A_n \neq \emptyset\}| \geq 3$ since then $\mathbb{N}2^k \cap (-x + D) = \emptyset$ while $\mathbb{N}2^k \cap (-x + D) \in u$.

Assume first that $\{n \in \mathbb{N} : \mathrm{supp}(x) \cap A_n \neq \emptyset\} = \{m, n\}$ for some $m < n$. Then $\mathbb{N}2^k \cap (-x + D) \subseteq C_n$ so $C_n \in u$.

Now assume that $\{n \in \mathbb{N} : \mathrm{supp}(x) \cap A_n \neq \emptyset\} = \{m\}$. Since $u + u = u$, pick $y \in \mathbb{N}2^k \cap (-x + D)$ such that $-y + (-x + D) \in u$. Let $l = 1 + \max\ \mathrm{supp}(y)$. Since $x + y \in D$ we have some $n > m$ such that the support of $x + y$ consists of some members of A_m followed by some members of A_n. Then $\mathbb{N}2^l \cap (-(x + y) + D) \subseteq C_n$ so again $C_n \in u$.

We have now established our claim that $p + p \in c\ell(\bigcup_{n=1}^{\infty} V_n)$. Also, $p + p \in \beta\mathbb{N} + p = c\ell(\mathbb{N} + p)$ so by [11, Theorem 3.40] either $(\mathbb{N} + p) \cap c\ell(\bigcup_{n=1}^{\infty} V_n) \neq \emptyset$ or $\bigcup_{n=1}^{\infty} V_n \cap (\beta\mathbb{N} + p) \neq \emptyset$. One can't have $(\mathbb{N} + p) \cap c\ell(\bigcup_{n=1}^{\infty} V_n) \neq \emptyset$ because if $x \in \mathbb{N}$, then $x + p \notin \mathbb{H}$ while $c\ell(\bigcup_{n=1}^{\infty} V_n) \subseteq \mathbb{H}$. So there exist some $q \in \beta\mathbb{N}$ and $m \in \mathbb{N}$ such that $q + p \in V_m$. Let $M = \{x \in \mathbb{N} : \max\ \mathrm{supp}(x) \in A_m\}$. Since $q + p \in V_m$, we have that $M \in p$. (We know that $M \in q + p$ so pick $x \in \mathbb{N}$ such that $-x + M \in p$. Let $k = 1 + \max\ \mathrm{supp}(x)$. Then $\mathbb{N}2^k \cap (-x + M) \subseteq M$.) But also $\bigcup_{n=m+1}^{\infty} C_m \in p$ and $M \cap \bigcup_{n=m+1}^{\infty} C_m = \emptyset$, a contradiction. $\square$

Our final preliminary result is very simple.

3.5 Theorem. *$E \setminus c\ell K(\beta\mathbb{N}) \neq \emptyset$. In particular, $E \setminus c\ell\langle EK\rangle \neq \emptyset$.*

Proof. Pick by [11, Lemma 5.11] an idempotent $p \in \overline{FS(\langle 2^{2n}\rangle_{n=1}^{\infty})}$. By [1, Corollary 4.2], $FS(\langle 2^{2n}\rangle_{n=1}^{\infty})$ is not piecewise syndetic so by [11, Corollary 4.41], $p \notin c\ell K(\beta\mathbb{N})$. For the "in particular" assertion note that $K(\beta\mathbb{N})$ is a semigroup so $\langle EK\rangle \subseteq K(\beta\mathbb{N})$. $\square$

3.6 Theorem. *Let A and B each be any one of EK, $\langle EK\rangle$, $c\ell\langle EK\rangle$, E, $\langle E\rangle$, $c\ell\langle E\rangle$, Γ, $\langle\Gamma\rangle$, $c\ell\langle\Gamma\rangle$, S, $\langle S\rangle$, $c\ell\langle S\rangle$, M, or P. Then $A \subseteq B$ if and only if the inclusion follows from the inclusions shown in Figure 2.*

Proof. The necessary examples are contained in Theorems 2.11, 2.14, 3.2, 3.3, 3.4, and 3.5. For example, the fact that $S \setminus \langle EK\rangle \neq \emptyset$ follows from the fact in Theorem 3.5 that $E \setminus c\ell\langle EK\rangle \neq \emptyset$ because $E \subseteq S$ and $\langle EK\rangle \subseteq c\ell\langle EK\rangle$. $\square$

Finally, consider the following diagram. (If we had three dimensional paper, we could combine it with Figure 2.) Again, all of the indicated inclusions are trivial.

$$
\begin{tikzcd}
 & P_2 & & & \\
c\ell\langle S_2\rangle \arrow[ur] & & P \arrow[ul] & & \\
\langle S_2\rangle \arrow[u] & c\ell\langle \mathcal{F}_2\rangle \arrow[ul] \arrow[uu] & M \arrow[u] & & \\
S_2 \arrow[u] & \langle \mathcal{F}_2\rangle \arrow[ul] \arrow[u] & c\ell\langle \Gamma\rangle \arrow[ul, "*"] \arrow[u] & & \\
 & \mathcal{F}_2 \arrow[ul] \arrow[u] & \langle \Gamma\rangle \arrow[ul, "*"] \arrow[u] & c\ell\langle E\rangle \arrow[ul] & \\
 & & \Gamma \arrow[ul, "*"] \arrow[u] \arrow[ur] & \langle E\rangle \arrow[ul] \arrow[u] & c\ell\langle EK\rangle \arrow[ul] \\
 & & & E \arrow[ul] \arrow[u] & \langle EK\rangle \arrow[ul] \arrow[u] \\
 & & & & EK \arrow[ul] \arrow[u]
\end{tikzcd}
$$

Figure 3

We have already obtained all of the necessary results to show that no inclusion among the sets in Figure 3 holds unless it is forced by the indicated inclusions, except that we don't know whether any or all of the starred inclusions is reversible.

3.7 Theorem. *Let A be any one of EK, $\langle EK\rangle$, $c\ell\langle EK\rangle$, E, $\langle E\rangle$, $c\ell\langle E\rangle$, Γ, $\langle\Gamma\rangle$, $c\ell\langle\Gamma\rangle$, S_2, $\langle S_2\rangle$, $c\ell\langle S_2\rangle$, M, P, or P_2. Let B be any of those sets or $\mathcal{F}_2$, $\langle\mathcal{F}_2\rangle$, or $c\ell\langle\mathcal{F}_2\rangle$. Then $A \subseteq B$ if and only if the inclusion follows from the inclusions shown in Figure* 2.

Proof. The necessary examples are contained in Theorems 2.7, 2.11, 2.14, 3.2, 3.3, 3.4, and 3.5. □

References

[1] C. Adams, N. Hindman, and D. Strauss, *Largeness of the set of finite products in a semigroup*, Semigroup Forum **76** (2008), 276-296.

[2] J. Baumgartner, *A short proof of Hindman's Theorem*, J. Comb. Theory (Series A) **17** (1974), 384-386.

[3] V. Bergelson and N. Hindman, *Nonmetrizable topological dynamics and Ramsey Theory*, Trans. Amer. Math. Soc. **320** (1990), 293-320.

[4] V. Bergelson and B. Rothschild, *A selection of open problems*, Topology and its Applications, to appear.

[5] R. Ellis, *Lectures on topological dynamics*, Benjamin, New York, 1969.

[6] P. Frankl, R. Graham, and V. Rödl, *Induced restricted Ramsey theorems for spaces*, J. Comb. Theory (Series A) **44** (1987), 120-128.

[7] H. Furstenberg, *Recurrence in ergodic theory and combinatorical number theory*, Princeton University Press, Princeton, 1981.

[8] N. Hindman, *Finite sums from sequences within cells of a partition of* $\mathbb{N}$, J. Comb. Theory (Series A) **17** (1974), 1-11.

[9] N. Hindman, I. Leader, and D. Strauss, *Open problems in partition regularity* Comb. Prob. and Comp. **12** (2003), 571-583.

[10] N. Hindman and D. Strauss, *Compact subsemigroups of* $(\beta\mathbb{N},+)$ *containing the idempotents*, Proc. Edinburgh Math. Soc. **39** (1996), 291-307.

[11] N. Hindman and D. Strauss, *Algebra in the Stone-Čech compactification: theory and applications*, de Gruyter, Berlin, 1998.

[12] J. Nešetřil and V. Rödl, *Finite union theorem with restrictions*, Graphs and Comb. **2** (1986), 357-361.

[13] F. Ramsey, *On a problem of formal logic*, Proc. London Math. Soc. **30** (1930), 264-286.

[14] H. Shi and H. Yang, *Nonmetrizable topological dynamical characterization of central sets*, Fund. Math. **150** (1996), 1-9.

Neil Hindman
Department of Mathematics
Howard University
Washington, DC 20059
USA
nhindman@aol.com

Dona Strauss
Department of Pure Mathematics
University of Leeds
Leeds LS2 9J2
UK
d.strauss@hull.ac.uk