Combining extensions of the Hales-Jewett Theorem with Ramsey Theory in other structures

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Abstract

The Hales-Jewett Theorem states that given any finite nonempty set $\mathbb{A}$ and any finite coloring of the free semigroup S over the alphabet $\mathbb{A}$ there is a *variable word* over $\mathbb{A}$ all of whose instances are the same color. This theorem has some extensions involving several distinct variables occurring in the variable word. We show that, when combined with a sufficiently well behaved homomorphism, the relevant variable word simultaneously satisfies a Ramsey-Theoretic conclusion in the other structure. As an example we show that if τ is the homomorphism from the set of variable words into the natural numbers which associates to each variable word w the number of occurrences of the variable in w, then given any finite coloring of S and any infinite sequence of natural numbers, there is a variable word w whose instances are monochromatic and $\tau(w)$ is a sum of distinct members of the given sequence.

Our methods rely on the algebraic structure of the Stone-Čech compactification of S and the other semigroups that we consider. We show for example that if τ is as in the paragraph above, there is a compact subsemigroup P of $\beta\mathbb{N}$ which contains all of the idempotents of $\beta\mathbb{N}$ such that, given any $p \in P$, any $A \in p$, and any finite coloring of S, there is a variable word w whose instances are monochromatic and $\tau(w) \in A$.

We end with a new short algebraic proof of an infinitary extension of the Graham-Rothschild Parameter Sets Theorem.

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1 Introduction

Given a nonempty set A (or *alphabet*) we let A^+ be the set of all finite words $w = a_1a_2\cdots a_n$ with $n \geq 1$ and $a_i \in A$. The quantity n is called the *length* of w and denoted $|w|$. The set A^+ is naturally a semigroup under the operation of concatenation of words, known as the *free semigroup* over A. For each $u \in A^+$ and $a \in A$, we let $|u|_a$ be the number of occurrences of a in u. For $u, w \in A^+$ we say u and w are Abelian equivalent, and write $u \sim_{Ab} w$, whenever $|u|_a = |w|_a$ for all $a \in A$. As is customary, we will identify the elements of A with the length one words over A.

Throughout this paper we will let $\mathbb{A}$ be a nonempty set, let $S_0 = \mathbb{A}^+$ be the free semigroup over $\mathbb{A}$, and let v (a *variable*) be a letter not belonging to $\mathbb{A}$. By a *variable word* over $\mathbb{A}$ we mean a word w over $\mathbb{A} \cup \{v\}$ with $|w|_v \geq 1$. We let S_1 be the set of variable words over $\mathbb{A}$. If $w \in S_1$ and $a \in \mathbb{A}$, then $w(a) \in S_0$ is the result of replacing each occurrence of v by a. For example if $\mathbb{A} = \{a, b, c\}$ and $w = avbvva$, then $w(a) = aabaaa$ while $w(c) = acbcca$. A *finite coloring* of a set X is a function from X to a finite set. A subset A of X is *monochromatic* if the function is constant on A.

Theorem 1.1 (A. Hales and R. Jewett)**.** *Assume that $\mathbb{A}$ is finite. For each finite coloring of S_0 there exists a variable word w such that $\{w(a) : a \in \mathbb{A}\}$ is monochromatic.*

Proof. [5, Theorem 1]. □

Some extensions of the Hales-Jewett Theorem, including for example Theorem 1.3 or the Graham-Rothschild Parameter Sets Theorem [4], involve the notion of *n-variable words*.

Definition 1.2. Let $n \in \mathbb{N}$ and $v_1, v_2, \ldots, v_n$ be distinct variables which are not members of $\mathbb{A}$.

(a) An *n-variable word* over $\mathbb{A}$ is a word w over $\mathbb{A} \cup \{v_1, v_2, \ldots, v_n\}$ such that $|w|_{v_i} \geq 1$ for each $i \in \{1, 2, \ldots, n\}$.

(b) If w is an n-variable word over $\mathbb{A}$ and $\vec{x} = (x_1, x_2, \ldots, x_n)$, then $w(\vec{x})$ is the result of replacing each occurrence of v_i in w by x_i for each $i \in \{1, 2, \ldots, n\}$.

(c) If w is an n-variable word over $\mathbb{A}$ and $u = l_1l_2\cdots l_n$ is a length n word, then $w(u)$ is the result of replacing each occurrence of v_i in w by l_i for each $i \in \{1, 2, \ldots, n\}$.

(d) A *strong n-variable word* is an n-variable word such that for each $i \in \{1, 2, \ldots, n-1\}$, the first occurrence of v_i precedes the first occurrence of v_{i+1}.

(e) S_n is the set of n-variable words over $\mathbb{A}$ and $\widetilde{S}_n$ is the set of strong n-variable words over $\mathbb{A}$.

(f) $\widetilde{S}_0 = S_0$.

(g) If $m \in \omega = \mathbb{N} \cup \{0\}$ and $m < n$, then $\widetilde{S}\binom{n}{m}$ is the set of $u \in \widetilde{S}_m$ such that $|u| = n$.

(h) If $m \in \omega$ and $m < n$, then $S\binom{n}{m}$ is the set of $u \in S_m$ such that $|u| = n$.

The notation above does not reflect the dependence on the alphabet $\mathbb{A}$.

We note that if $m, n \in \omega$ and $m < n$, then for each $w \in \widetilde{S}_n$ and each $u \in \widetilde{S}\binom{n}{m}$, the word $w(u)$ belongs to $\widetilde{S}_m$.

The following is a first simple example of a multivariable extension of the Hales-Jewett Theorem:

Theorem 1.3. *Assume that $\mathbb{A}$ is finite. Let S_0 be finitely colored and let $n \in \mathbb{N}$. There exists $w \in S_n$ such that $\{w(\vec{x}) : \vec{x} \in \mathbb{A}^n\}$ is monochromatic.*

Theorem 1.3 follows immediately from Theorem 1.1 applied to the alphabet $\mathbb{A}^n$, replacing each occurrence of v_1 in the variable word over $\mathbb{A}^n$ by $v_1 v_2 \cdots v_n$. It is also a consequence of Theorem 2.10, which constitutes one of the main results of this paper. (See the paragraph immediately following Theorem 5.1.) Theorem 1.3 also follows directly from Theorem 1.5 later in this section which we regard as an algebraic extension of Theorem 1.3.

It is natural to ask the following question. *Assume that $\mathbb{A}$ is finite. Let S_∞ be the set of infinite words over $\mathbb{A} \cup \{v_i : i \in \mathbb{N}\}$ in which each v_i occurs and assume that $\mathbb{A}^{\mathbb{N}}$ is finitely colored. Must there exist $w \in S_\infty$ such that $\{w(\vec{x}) : \vec{x} \in \mathbb{A}^{\mathbb{N}}\}$ is monochromatic, where $w(\vec{x})$ has the obvious meaning?* As long as $|\mathbb{A}| \geq 2$, the answer is easily seen to be "no", using a standard diagonalization argument: One has that $|\mathbb{A}^{\mathbb{N}}| = |S_\infty| = \mathfrak{c}$, so one may inductively color two elements of $\mathbb{A}^{\mathbb{N}}$ for each $w \in S_\infty$ so that there exist $\vec{x}$ and $\vec{y}$ in $\mathbb{A}^{\mathbb{N}}$ with the color of $w(\vec{x})$ and $w(\vec{y})$ different. (When one gets to w, fewer than $\mathfrak{c}$ things have been colored and there are $\mathfrak{c}$ distinct values of $w(\vec{x})$ possible.)

The following simplified version of the Graham-Rothschild Parameter Sets Theorem constitutes yet another fundamental multivariable extension of the Hales-Jewett Theorem. It was shown in [2, Theorem 5.1] that the full version as stated in [4] can be easily derived from the version stated here.

Theorem 1.4 (R. Graham and B. Rothschild)**.** *Assume that $\mathbb{A}$ is finite. Let $m, n \in \omega$ with $m < n$ and let $\widetilde{S}_m$ be finitely colored. There exists $w \in \widetilde{S}_n$ such that $\{w(u) : u \in \widetilde{S}\binom{n}{m}\}$ is monochromatic.*

Proof. [4, Section 7]. □

After identifying the elements of $\mathbb{A}$ with the length 1 words over $\mathbb{A}$, one sees that Theorem 1.1 is exactly the $m = 0$ and $n = 1$ case of Theorem 1.4. Notice also that Theorem 1.3 is actually equivalent to Theorem 1.4 in the special case of $m = 0$. In fact if $w \in S_n$, σ is a permutation of $\{1, 2, \ldots, n\}$ and u is the result of replacing each v_i in w by $v_{\sigma(i)}$ for each $i \in \{1, 2, \ldots, n\}$, then $\{u(\vec{x}) : \vec{x} \in \mathbb{A}^n\} = \{w(\vec{x}) : \vec{x} \in \mathbb{A}^n\}$. In this paper we shall be mostly concerned with cases of Theorem 1.4 with $m = 0$ and arbitrary $n \in \mathbb{N}$. (We are not concerned with $m > 0$ because the natural versions of our main theorems are not valid for $m > 0$. We shall discuss this point at the end of Section 2.) Accordingly, from this point on until Section 6 we will not be concerned with the order of occurrence of the variables.

In contrast to Theorem 1.3, the Graham-Rothschild Parameter Sets Theorem does not appear to be deducible directly from the Hales-Jewett Theorem; at least we know of no such proof. In Section 6 we present a new purely algebraic proof of an infinitary extension of Theorem 1.4.

Our main results in this paper deal with obtaining n-variable words satisfying the Hales-Jewett Theorem and simultaneously relating to Ramsey-Theoretic results in some relevant semigroup. The paper is organized as follows:

In Section 2 we present our main theorems relating S_n with other structures. In Section 3 we determine precisely which homomorphisms from S_n to $(\mathbb{N}, +)$ satisfy the hypotheses of our main theorem of Section 2, namely Theorem 2.10.

As consequences of the results of Section 2 we establish that for $k \in \mathbb{N}$, the set of points $(p_1, p_2, \ldots, p_k) \in (\beta\mathbb{N})^k$ with the property that whenever $B_i \in p_i$ for $i \in \{1, 2, \ldots, k\}$, the k-tuple $(B_1, B_2, \ldots, B_k)$ satisfies the conclusions of one of those theorems, is a compact subsemigroup of $(\beta\mathbb{N})^k$ containing the idempotents of $(\beta\mathbb{N})^k$ (or the minimal idempotents, depending on the theorem). The details of these results will be presented in Section 4.

In Section 5 we restrict our attention to versions of the Hales-Jewett Theorem. Letting $R_n = \{p \in \beta S_0 : (\forall B \in p)(\exists w \in S_n)(\{w(\vec{x}) : \vec{x} \in \mathbb{A}^n\} \subseteq B)\}$. We show that each R_n is a compact ideal of βS_0, that $R_{n+1} \subsetneq R_n$ for each $n \in \mathbb{N}$, and that $c\ell K(\beta S_0) \subsetneq \bigcap_{n=1}^{\infty} R_n$.

In Section 6 we present a new fully algebraic proof of an infinitary extension of the Graham-Rothschild Parameter Sets Theorem. This new proof is a significant simplification of the original.

The statements and proofs of the results in this paper use strongly the algebraic structure of the Stone-Čech compactification of a discrete semigroup. We now present a brief description of this structure. For more details or for any unfamiliar facts encountered in this paper, we refer the reader to [6, Part I]. All topological spaces considered herein are assumed to be Hausdorff.

Let S be a semigroup. For each $s \in S$, $\rho_s : S \to S$ and $\lambda_s : S \to S$ are defined by $\rho_s(x) = xs$ and $\lambda_s(x) = sx$. If S is also a topological space, S is said to be *right topological* if the map ρ_s is continuous for every $s \in S$. In this case, the set of elements $s \in S$ for which λ_s is continuous, is called the *topological center* of S.

The assumption that S is compact and right topological has powerful algebraic implications. S has a smallest two sided ideal $K(S)$ which is the union of all of the minimal right ideals, as well as the union of all of the minimal left ideals. The intersection of any minimal left ideal and any minimal right ideal is a group. In particular, S has idempotents. Any left ideal of S contains a minimal left ideal of S, and any right ideal of S contains a minimal right ideal of S. So the intersection of any left ideal of S and any right ideal of S contains an idempotent. An idempotent in S is said to be *minimal* if it is in $K(S)$. This is equivalent to being minimal in the ordering of idempotents defined by $p \leq q$ if $pq = qp = p$. If q is any idempotent in S, there is a minimal idempotent $p \in S$ for which $p \leq q$.

Given a discrete semigroup $(T, \cdot)$, let $\beta T = \{p : p \text{ is an ultrafilter on } T\}$. We identify the principal ultrafilter $e(x) = \{A \subseteq T : x \in A\}$ with the point $x \in T$ and thereby pretend that $T \subseteq \beta T$. A base for the topology of βT consists of the clopen sets $\overline{A}$ for all $A \subseteq T$, where $\overline{A} = \{p \in \beta T : A \in p\}$. The operation $\cdot$ on T extends to an operation on βT, also denoted by $\cdot$ making $(\beta T, \cdot)$ a right topological semigroup with T contained in its topological center. So, given $p, q \in \beta T$, $p \cdot q = \lim\limits_{s \to p} \lim\limits_{t \to q} s \cdot t$, where s and t denote elements of T. If $A \subseteq T$, $A \in p \cdot q$ if and only if $\{x \in T : x^{-1}A \in q\} \in p$, where $x^{-1}A = \{y \in T : x \cdot y \in A\}$. If $(T, +)$ is a commutative discrete semigroup, we will use $+$ for the semigroup operation on βT, even though βT is likely to be far from commutative. In this case, we have that $A \in p + q$ if and only if $\{x \in T : -x + A \in q\} \in p$, where $-x + A = \{y \in T : x + y \in A\}$.

A set $D \subseteq T$ is *piecewise syndetic* if and only if $D \in p$ for some $p \in K(\beta T)$ and is *central* if and only $D \in p$ for some idempotent $p \in K(\beta T)$. We will also need the following equivalent characterization of piecewise syndetic sets: D is piecewise syndetic if and only if there exists a finite subset G of T with the property that for every finite subset F of T there exists $x \in T$ such that $Fx \subseteq \bigcup_{t \in G} t^{-1}D$. (See [6, Theorem 4.40].) Given a sequence $\langle x_n \rangle_{n=1}^{\infty}$ and $m \in \mathbb{N}$, we set $FP(\langle x_n \rangle_{n=m}^{\infty}) = \{\prod_{t \in F} x_t : F \in \mathcal{P}_f(\mathbb{N}) \text{ and } \min F \geq m\}$, where $\mathcal{P}_f(\mathbb{N})$ is the set of finite nonempty subsets of $\mathbb{N}$ and the products are computed in increasing order of indices. Then $\bigcap_{m=1}^{\infty} \overline{FP(\langle x_n \rangle_{n=m}^{\infty})}$ is a compact semigroup so there is an idempotent p with $FP(\langle x_n \rangle_{n=m}^{\infty}) \in p$ for every m. (See [6, Lemma 5.11].) If the operation is denoted by $+$, we write $FS(\langle x_n \rangle_{n=m}^{\infty}) = \{\sum_{t \in F} x_t : F \in \mathcal{P}_f(\mathbb{N}) \text{ and } \min F \geq m\}$. Given an idempotent p and $B \in p$ let $B^\star(p) = \{x \in B : x^{-1}B \in p\}$. Then $B^\star(p) \in p$ and for each $x \in B^\star(p)$, one has that $x^{-1}B^\star(p) \in p$. (See [6, Lemma 4.14]). If there is no risk of confusion, we will sometimes write $B^\star$ for $B^\star(p)$.

If γ is a function from the discrete semigroup T to a compact space C, then

γ has a continuous extension from βT to C, which we will also denote by γ. If $\gamma : T \to W$, where W is discrete, we will view the continuous extension as taking βT to βW, unless we state otherwise. If $\gamma : T \to C$ is a homomorphism from T into a compact right topological semigroup C, with $\gamma[T]$ contained in the topological center of C, then the continuous extension $\gamma : \beta T \to C$ is a homomorphism by [6, Corollary 4.22].

We end this section with a few simple illustrations of how the algebraic structure described above may be applied to derive simple algebraic proofs of some of the results discussed earlier including for instance the Hales-Jewett Theorem. We begin with the following theorem whose proof is based on an argument due to Andreas Blass which first appeared in [1].

Theorem 1.5. *Let T be a semigroup and let S be a subsemigroup of T. Let F be a nonempty set of homomorphisms mapping T to S which are equal to the identity on S.*

(1) *Let p be a minimal idempotent in βS. Let q be an idempotent in βT for which $q \leq p$. Then $\nu(q) = p$ for every $\nu \in F$.*

(2) *For any finite subset F_0 of F and any central subset D of S, there is a central subset Q of T such that, for every $t \in Q$, $\{\nu(t) : \nu \in F_0\} \subseteq D$.*

(3) *For any finite subset F_0 of F and any finite coloring of S, there is a central subset Q of T such that, for every $t \in Q$, $\{\nu(t) : \nu \in F_0\}$ is monochromatic.*

Proof. (1) For each $\nu \in F$, $\nu(q) \leq \nu(p) = p$ and so, since $\nu(q) \in \beta S$, $\nu(q) = p$.

(2) Pick a minimal idempotent $p \in \beta S$ such that $D \in p$. By [6, Theorem 1.60], pick a minimal idempotent $q \in \beta T$ such that $q \leq p$. Then $\nu(q) = p$ for every $\nu \in F_0$. Hence, if $Q = \bigcap_{\nu \in F_0} \nu^{-1}[D]$, then $Q \in q$.

(3) Pick a minimal idempotent $p \in \beta S$ and let D be a monochromatic member of p. □

We note that the above theorem provides an algebraic proof of Theorem 1.3 and hence of the Hales-Jewett Theorem. In fact, put $S = S_0$, $T = S_0 \cup S_n$ and $F = \{h_{\vec{x}} : \vec{x} \in \mathbb{A}^n\}$, where $h_{\vec{x}}(w) = \begin{cases} w(\vec{x}) & \text{if } w \in S_n \\ w & \text{if } w \in S_0 \,. \end{cases}$ Then by Theorem 1.5 we deduce that for any finite coloring of S_0 there exists a central subset Q of T such that for every $w \in Q$, $\{w(\vec{x}) : \vec{x} \in \mathbb{A}^n\}$ is monochromatic. Pick $q \in K(\beta T)$ with $Q \in q$. Then since S_n is an ideal of T it follows that $S_n \in q$. So for any $w \in S_n \cap Q$ we have $\{w(\vec{x}) : \vec{x} \in \mathbb{A}^n\}$ is monochromatic.

We conclude this section with two additional simple corollaries of Theorem 1.5 that will not be needed in the rest of the paper.

Corollary 1.6. *Let T be a semigroup and let S be a subsemigroup of T. Let F be a finite nonempty set of homomorphisms mapping T to S which are equal to* *the identity on S. Let D be a piecewise syndetic subset of S. Then $\bigcap_{\nu\in F}\nu^{-1}[D]$ is a piecewise syndetic subset of T.*

Proof. By [6, Theorem 4.43], we may pick $s\in S$ for which $s^{-1}D$ is a central subset of S. We can choose a minimal idempotent p in βS for which $s^{-1}D\in p$, and we can then choose a minimal idempotent q in βT for which $q\leq p$, by [6, Theorem 1.60]. By Theorem 1.5(1), $\nu(q)=p$ for every $\nu\in F$. Hence, if $Q=\bigcap_{\nu\in F}\nu^{-1}[s^{-1}D]$, then $Q\in q$. Now sQ is a piecewise syndetic subset of T, because $sQ\in sq$ and $sq\in K(\beta T)$. We claim that $sQ\subseteq\bigcap_{\nu\in F}\nu^{-1}[D]$. In fact, let $x\in sQ$, pick $t\in Q$ such that $x=st$, and let $\nu\in F$. Then $\nu(x)=\nu(st)=s\nu(t)\in s(s^{-1}D)\subseteq D$. □

In the following corollary, we use the abbreviated notation $P^\star$ for $P^\star(p)$ for P belonging to an idempotent p.

Corollary 1.7. *Let T be a semigroup and let S be a subsemigroup of T. Let F be a finite nonempty set of homomorphisms from T onto S which are equal to the identity on S. Let p be a minimal idempotent in βS and let $P\in p$. Let q be a minimal idempotent of βT for which $q\leq p$ and let $Q=\bigcap_{\nu\in F}\nu^{-1}[P^\star]$. Then $Q\in q$. There is an infinite sequence $\langle w_n\rangle_{n=1}^\infty$ of elements of Q such that for each $H\in\mathcal{P}_f(\mathbb{N})$ and each $\varphi:H\to F$, $\prod_{t\in H}\varphi(t)(w_t)\in P^\star$, where the product is computed in increasing order of indices.*

Proof. Choose $w_1\in Q$. Let $m\in\mathbb{N}$ and assume we have chosen $\langle w_t\rangle_{t=1}^m$ in Q such that whenever $\emptyset\neq H\subseteq\{1,2,\ldots,m\}$ and $\varphi:H\to F$, $\prod_{t\in H}\varphi(t)(w_t)\in P^\star$. Note that this hypothesis is satisfied for $m=1$. Let

$$E=\{\textstyle\prod_{t\in H}\varphi(t)(w_t):\emptyset\neq H\subseteq\{1,2,\ldots,m\}\text{ and }\varphi:H\to F\}\,.$$

Then $E\subseteq P^\star$. Let $R=P^\star\cap\bigcap_{y\in E}y^{-1}P^\star$. Then $R\in p$ so $\bigcap_{\nu\in F}\nu^{-1}[R]\in q$. Pick $w_{m+1}\in\bigcap_{\nu\in F}\nu^{-1}[R]$ and note that $w_{m+1}\in Q$.

To verify the hypothesis let $\emptyset\neq H\subseteq\{1,2,\ldots,m+1\}$ and let $\varphi:H\to F$. If $m+1\notin H$, the conclusion holds by assumption, so assume that $m+1\in H$. If $H=\{m+1\}$, then $w_{m+1}\in\varphi(m+1)^{-1}[P^\star]$, so assume that $\{m+1\}\subsetneq H$ and let $G=H\setminus\{m+1\}$. Let $y=\prod_{t\in G}\varphi(t)(w_t)$. Then $w_{m+1}\in\varphi(m+1)^{-1}[y^{-1}P^\star]$ so $\prod_{t\in H}\varphi(t)(w_t)=y\varphi(m+1)(w_{m+1})\in P^\star$. □

2 Combining structures

Throughout this section, and up until Section 6, $\mathbb{A}$ is a fixed non-empty finite alphabet. Most of the results in this paper involve families of well behaved homomorphisms between certain semigroups:

Definition 2.1. Let $n\in\mathbb{N}$ and let $\nu:S_n\to S_0$ be a homomorphism. We shall say that ν is *S_0-preserving* if $\nu(uw)=u\nu(w)$ and $\nu(wu)=\nu(w)u$ for every $u\in S_0$ and every $w\in S_n$.

Note that if $\vec{x} \in \mathbb{A}^n$, then the function $h_{\vec{x}} : S_n \to S_0$ defined by $h_{\vec{x}}(w) = w(\vec{x})$ is an S_0-preserving homomorphism. Also, the function $\delta : S_n \to S_0$ which simply deletes all occurrences of variables is an S_0-preserving homomorphism. As another example, assume that $n \geq 2$ and define $\mu : S_n \to S_n$ where $\mu(w)$ is obtained from w by replacing each occurrence of v_2 by $v_1 v_2$. Given $\vec{x} \in \mathbb{A}^n$, $h_{\vec{x}} \circ \mu$ is an S_0-preserving homomorphism which cannot be obtained by composing those of the kind mentioned previously; in fact $|h_{\vec{x}} \circ \mu(w)| > |w|$ for each $w \in S_n$.

Definition 2.2. Let S, T, and R be semigroups such that $S \cup T$ is a semigroup and T is an ideal of $S \cup T$. Then a homomorphism $\tau : T \to R$ is said to be *S-independent* if, for every $w \in T$ and every $u \in S$, $\tau(uw) = \tau(w) = \tau(wu)$.

In most cases, the above definition will be applied to the case $S = S_0$ and $T = S_n$ for some $n \in \mathbb{N}$. We shall see later in Lemma 3.3 that if $n \in \mathbb{N}$, R is a cancellative commutative semigroup, and $\tau : S_n \to R$ is an S_0-independent homomorphism, then $\tau(w) = \tau(w')$ whenever $|w|_{v_i} = |w'|_{v_i}$ for each $i \in \{1, 2, \ldots, n\}$. For reasons which will be made clear in Section 3, we will primarily be concerned with S_0-independent homomorphisms from S_n to $(\mathbb{N}, +)$ of the form $\tau(w) = |w|_{v_i}$ for some $i \in \{1, 2, \ldots, n\}$.

Lemma 2.3. *Let S and T be semigroups such that $S \cup T$ is a semigroup and T is an ideal of $S \cup T$. Let $\phi : T \to C$ be an S-independent homomorphism from T into the topological center of a compact right topological semigroup C. Then ϕ extends to a continuous homomorphism from βT into C, which we shall also denote by ϕ. For every $q \in \beta T$ and every $p \in \beta S$, $\phi(q) = \phi(pq) = \phi(qp)$.*

Proof. The fact that ϕ extends to a continuous homomorphism is [6, Corollary 4.22]. Let $p \in \beta S$ and $q \in \beta T$ be given. In the following expressions let s and t denote members of S and T respectively. Since ϕ is continuous on βT and since both pq and qp are in βT by [6, Corollary 4.18], we have that

$$\phi(pq) = \phi(\lim_{s \to p} \lim_{t \to q} st) = \lim_{s \to p} \lim_{t \to q} \phi(st) = \lim_{t \to q} \phi(t) = \phi(q)$$

and similarly

$$\phi(qp) = \phi(\lim_{t \to q} \lim_{s \to p} ts) = \lim_{t \to q} \lim_{s \to p} \phi(ts) = \lim_{t \to q} \phi(t) = \phi(q).$$

□

Theorem 2.4. *Let S and T be semigroups such that $S \cup T$ is a semigroup and T is an ideal of $S \cup T$. Let $\phi : T \to C$ be an S-independent homomorphism from T into a compact right topological semigroup C with $\phi[T]$ contained in the topological center of C and denote also by ϕ its continuous extension to βT. Let F be a finite nonempty set of homomorphisms from $S \cup T$ into S which are each equal to the identity on S, and let D be a piecewise syndetic subset of S. Let p be an idempotent in $\phi[\beta T]$, and let U be a neighborhood of p in C. There exists $w \in T$ such that $\phi(w) \in U$ and $\nu(w) \in D$ for every $\nu \in F$.*

Proof. Since D is piecewise syndetic in S, pick by [6, Theorem 4.43] some $s \in S$ such that $s^{-1}D$ is central in S and pick a minimal idempotent $r \in \beta S$ such that $s^{-1}D \in r$.

Let $V = \phi^{-1}[\{p\}]$. Since ϕ is a continuous homomorphism from βT to C, V is a compact subsemigroup of βT. By Lemma 2.3, Vr is a left ideal of V and rV is a right ideal of V. Pick an idempotent $q \in Vr \cap rV$ and note that $q \leq r$ in βT. By Theorem 1.5(1), $\nu(q) = r$ for every $\nu \in F$.

Since $s^{-1}D \in r$ we have that for each $\nu \in F$, $\nu^{-1}[s^{-1}D] \in q$. Since U is a neighborhood of p, pick $R \in q$ such that $\phi[\overline{R}] \subseteq U$. Pick $w \in R \cap \bigcap_{\nu \in F} \nu^{-1}[s^{-1}D]$. Then $\phi(sw) = \phi(w) \in U$ and for $\nu \in F$, $\nu(w) \in s^{-1}D$ so $\nu(sw) = s\nu(w) \in D$. □

We obtain the first result that was stated in the abstract as a corollary to Theorem 2.4.

Corollary 2.5. *Define $\tau : S_1 \to \mathbb{N}$ by $\tau(w) = |w|_{v_1}$, let S_0 be finitely colored, and let $\langle x_n \rangle_{n=1}^\infty$ be a sequence in $\mathbb{N}$. There exists $w \in S_1$ such that $\{w(a) : a \in \mathbb{A}\}$ is monochromatic and $\tau(w) \in FS(\langle x_n \rangle_{n=1}^\infty)$.*

Proof. Let $S = S_0$, let $T = S_1$, and let $C = \beta\mathbb{N}$. Then $\tau[S_1]$ is contained in the topological center of C. Denote also by τ the continuous extension taking βS_1 to $\beta\mathbb{N}$. Given $a \in \mathbb{A}$, define $f_a : S_0 \cup S_1 \to S_0$ by

$$f_a(w) = \begin{cases} w(a) & \text{if } w \in S_1 \\ w & \text{if } s \in S_0\,, \end{cases}$$

and let $F = \{f_a : a \in \mathbb{A}\}$. Then F is a finite nonempty set of homomorphisms from $S_0 \cup S_1$ into S_0 which are each equal to the identity on S_0. Pick by [6, Lemma 5.11] an idempotent $p \in \beta\mathbb{N}$ such that $FS(\langle x_n \rangle_{n=1}^\infty) \in p$. Pick any $q \in K(\beta S_0)$ and pick $D \in q$ which is monochromatic. Note that $\tau[S_1] = \mathbb{N}$ so by [6, Exercise 3.4.1], $\tau[\beta S_1] = \beta\mathbb{N}$. Therefore, $p \in \tau[\beta S_1]$. Consequently, Theorem 2.4 applies with $U = \overline{FS(\langle x_n \rangle_{n=1}^\infty)}$. □

Corollary 2.6. *Let $n \in \mathbb{N}$. Let $\phi : S_n \to C$ be an S_0-independent homomorphism from S_n into a compact right topological semigroup C with $\phi[S_n]$ contained in the topological center of C and denote also by ϕ the continuous extension to βS_n. Let F be a finite nonempty set of S_0-preserving homomorphisms from S_n into S_0, let D be a piecewise syndetic subset of S_0, let p be an idempotent in $\phi[\beta S_n]$, and let U be a neighborhood of p in C. There exists $w \in S_n$ such that $\phi(w) \in U$ and $\nu(w) \in D$ for every $\nu \in F$.*

Proof. Let $S = S_0$, let $T = S_n$, and for $\nu \in F$, extend ν to $S_0 \cup S_n$ by defining ν to be the identity on S_0. Then Theorem 2.4 applies. □

Lemma 2.7. *Let $(T, \cdot)$ be a discrete semigroup and let $m, n \in \mathbb{N}$. Let $\phi : S_n \to \times_{i=1}^m T$ be an S_0-independent homomorphism. Then ϕ extends to a* *continuous S_0-independent homomorphism $\phi : \beta S_n \to \times_{i=1}^m \beta T$. Moreover if $\vec{p} = (p_1, p_2, \ldots, p_m)$ is an idempotent in $\times_{i=1}^m \beta T$ with the property that whenever $B_i \in p_i$ for each $i \in \{1, 2, \ldots, m\}$, there exists $w \in S_n$ such that $\phi(w) \in \times_{i=1}^m B_i$, then $\vec{p} \in \phi[\beta S_n]$.*

Proof. Let $C = \times_{i=1}^m \beta T$. Regarding ϕ as an S_0-independent homomorphism from S_n into the right topological semigroup C, we see that $\phi[S_n]$ is contained in $\times_{i=1}^m T$ which in turn is contained in the topological center of C by [6, Theorem 2.22]. Hence by [6, Corollary 4.22], ϕ extends to a continuous homomorphism from βS_n into C. To see that the extension is S_0-independent, let $u \in S_0$ and let $p \in \beta S_n$. Then, letting s denote a member of S_n, we have

$$\phi(up) = \phi(\lim_{s \to p} us) = \lim_{s \to p} \phi(us) = \lim_{s \to p} \phi(s) = \phi(\lim_{s \to p} s) = \phi(p)$$

and similarly, $\phi(pu) = \phi(p)$.

Now assume that $\vec{p} = (p_1, p_2, \ldots, p_m)$ is an idempotent in $\times_{i=1}^m \beta T$ and whenever $B_i \in p_i$ for each $i \in \{1, 2, \ldots, m\}$, there exists $w \in S_n$ such that $\phi(w) \in \times_{i=1}^m B_i$, To see that $\vec{p} \in \phi[\beta S_n]$ let $(B_1, B_2, \ldots, B_m) \in \times_{i=1}^m p_i$, and let

$$G_{(B_1, \ldots, B_m)} = \{w \in S_n : \phi(w) \in \times_{i=1}^m B_i\}.$$

Then by assumption, $\mathcal{G} = \{G_{(B_1, \ldots, B_m)} : (B_1, B_2, \ldots, B_m) \in \times_{i=1}^m p_i\}$ has the finite intersection property so one may pick $q \in \beta S_n$ such that $\mathcal{G} \subseteq q$. Then $\vec{p} = \phi(q) \in \phi[\beta S_n]$. □

Theorem 2.8. *Let $(T, \cdot)$ be a discrete semigroup and let $m, n \in \mathbb{N}$. Let $\vec{p} = (p_1, p_2, \ldots, p_m)$ be an idempotent in $\times_{i=1}^m \beta T$. For $i \in \{1, 2, \ldots, m\}$ let τ_i be an S_0-independent homomorphism from S_n to T. Assume that whenever $B_i \in p_i$ for each $i \in \{1, 2, \ldots, m\}$, there exists $w \in S_n$ such that $\big(\tau_1(w), \tau_2(w), \ldots, \tau_m(w)\big) \in \times_{i=1}^m B_i$. Let D be a piecewise syndetic subset of S_0 and let F be a finite nonempty set of S_0-preserving homomorphisms from S_n to S_0. Then whenever $B_i \in p_i$ for each $i \in \{1, 2, \ldots, m\}$, there exists $w \in S_n$ such that $\nu(w) \in D$ for each $\nu \in F$ and for each $i \in \{1, 2, \ldots, m\}$, $\tau_i(w) \in B_i$.*

Proof. Define $\phi : S_n \to \times_{i=1}^m T$ by

$$\phi(w) = \big(\tau_1(w), \tau_2(w), \ldots, \tau_m(w)\big).$$

Then ϕ is an S_0-independent homomorphism and hence by Lemma 2.7, ϕ extends to a continuous S_0-independent homomorphism $\phi : \beta S_n \to \times_{i=1}^m \beta T$ and $\vec{p} \in \phi[\beta S_n]$. The result now follows from Corollary 2.6. □

Corollary 2.9. *Let $k, n \in \mathbb{N}$ with $k < n$ and let T be the set of words over $\{v_1, v_2, \ldots, v_k\}$ in which v_i occurs for each $i \in \{1, 2, \ldots, k\}$. Given $w \in S_n$ let $\tau(w)$ be obtained from w by deleting all occurrences of elements of $\mathbb{A}$ as well as all occurrences of v_i for $k < i \leq n$ deleted. Let $\langle y_t \rangle_{t=1}^\infty$ be a sequence in T,* *let F be a finite nonempty set of S_0-preserving homomorphisms from S_n to S_0, and let D be a piecewise syndetic subset of S_0. There exists $w \in S_n$ such that $\nu(w) \in D$ for all $\nu \in F$ and $\tau(w) \in FP(\langle y_t \rangle_{t=1}^{\infty})$.*

Proof. As noted in the introduction, we can pick an idempotent $p \in \beta T$ such that $FP(\langle y_t \rangle_{t=1}^{\infty}) \in p$. Since τ is an S_0-independent homomorphism from S_n onto T, Theorem 2.8 applies with $m = 1$. □

Theorem 2.10 involves a matrix with entries from $\mathbb{Q}$ or $\mathbb{Z}$. In order to ensure that matrix multiplication is distributive, we assume that T is commutative and write the operation as $+$.

Theorem 2.10. *Let $(T, +)$ be a commutative semigroup, let $k, m, n \in \mathbb{N}$, and let M be a $k \times m$ matrix. If T is not cancellative assume that the entries of M come from ω. If T is isomorphic to a subsemigroup of a direct sum of copies of $(\mathbb{Q}, +)$ (so that multiplication by members of $\mathbb{Q}$ makes sense), assume that the entries of M come from $\mathbb{Q}$. Otherwise assume that the entries of M come from $\mathbb{Z}$. For $i \in \{1, 2, \ldots, m\}$ let τ_i be an S_0-independent homomorphism from S_n to T. Define a function ψ on S_n by $\psi(w) = \begin{pmatrix} \tau_1(w) \\ \tau_2(w) \\ \vdots \\ \tau_m(w) \end{pmatrix}$. Let $\vec{p} = (p_1, p_2, \ldots, p_k)$ be an idempotent in $\times_{i=1}^{k} \beta T$ with the property that whenever $B_i \in p_i$ for each $i \in \{1, 2, \ldots, k\}$, there exists $\vec{z} \in \psi[S_n]$ such that $M\vec{z} \in \times_{i=1}^{k} B_i$. Let F be a finite nonempty set of S_0-preserving homomorphisms from S_n to S_0 and let D be a piecewise syndetic subset of S_0. Then whenever $B_i \in p_i$ for each $i \in \{1, 2, \ldots, k\}$, there exists $w \in S_n$ such that $\nu(w) \in D$ for every $\nu \in F$ and $M\psi(w) \in \times_{i=1}^{k} B_i$.*

Proof. If T is not cancellative, let $G = T$. If T is isomorphic to a subsemigroup of $\bigoplus_{i \in I} \mathbb{Q}$ for some set I, assume that $T \subseteq \bigoplus_{i \in I} \mathbb{Q}$ and let $G = \bigoplus_{i \in I} \mathbb{Q}$. Otherwise let G be the group of differences of T. In each case we define an S_0-independent homomorphism $\phi : S_n \to \times_{j=1}^{k} G$. by $\phi(w) = M\psi(w)$. Let $C = \times_{j=1}^{k} \beta G$. Then by Lemma 2.7, ϕ extends to an S_0-independent homomorphism $\phi : \beta S_n \to C$ and $\vec{p} \in \phi[\beta S_n]$. The rest now follows from Corollary 2.6. □

Definition 2.11. Let $n \in \mathbb{N}$ and let $j \in \{1, 2, \ldots, n\}$. Define $\mu_j : S_n \to \mathbb{N}$ by $\mu_j(w) = |w|_{v_j}$.

The following corollary provides sufficient conditions for applying Theorem 2.10.

Corollary 2.12. *Let $m, n \in \mathbb{N}$ with $m \leq n$. Let M be an $m \times m$ lower triangular matrix with rational entries. Assume that the entries on the diagonal are positive and the entries below the diagonal are negative or zero. Let $\vec{p} = (p_1, p_2, \ldots, p_m)$*

be an idempotent in $\times_{i=1}^m \beta\mathbb{N}$. *For* $i \in \{1,2,\ldots,m\}$ *let* $\tau_i = \sum_{j=1}^n \alpha_{i,j}\mu_j$ *where each* $\alpha_{i,j} \in \mathbb{Q}$. *Assume that for each* $i \in \{1,2,\ldots,m\}$ *we can choose* $t(i) \in \{1,2,\ldots,n\}$ *such that* $\alpha_{i,t(i)} > 0$, *if* $l \in \{1,2,\ldots,m\}$ *and* $l > i$, *then* $\alpha_{i,t(l)} = 0$, *and if* $l \in \{1,2,\ldots,m\}$ *and* $l < i$, *then* $\alpha_{i,t(l)} \leq 0$. *Then each* τ_i *is an* S_0*-independent homomorphism from* S_n *to* $\mathbb{Q}$. *Let* F *be a nonempty finite set of* S_0*-preserving homomorphisms from* S_n *to* S_0 *and let* D *be a piecewise syndetic subset of* S_0. *Whenever* $B_i \in p_i$ *for each* $i \in \{1,2,\ldots,m\}$, *there exists* $w \in S_n$ *such that* $\nu(w) \in D$ *for each* $\nu \in F$ *and*

$$M\begin{pmatrix}\tau_1(w)\\ \tau_2(w)\\ \vdots\\ \tau_m(w)\end{pmatrix} \in \times_{i=1}^m B_i\,.$$

Proof. Define $\psi : S_n \to \mathbb{Q}^m$ by $\psi(w) = \begin{pmatrix}\tau_1(w)\\ \vdots\\ \tau_m(w)\end{pmatrix}$. We wish to apply Theorem 2.10 with $T = \mathbb{Q}$. For this we need to show that whenever $B_i \in p_i$ for $i \in \{1,2,\ldots,m\}$, there exists $\vec{z} \in \psi[S_n]$ such that $M\vec{z} \in \times_{i=1}^m B_i$. So let $B_i \in p_i$ for $i \in \{1,2,\ldots,m\}$.

We show first that for each $r \in \mathbb{N}$, there exists $\vec{z} \in (r\mathbb{N})^m$ such that $M\vec{z} \in \times_{i=1}^m B_i$, so let $r \in \mathbb{N}$ be given. Note that M^{-1} is lower triangular with positive diagonal entries and nonnegative entries below the diagonal. (Probably the easiest way to see this is to solve the system of equations $M\vec{z} = \vec{x}$ by back substitution. Alternatively we may write $M = D(I+N)$ where D is diagonal with positive entries and N is a strictly lower triangular matrix (all of whose non-zero entries are negative) verifying $N^m = O$. Setting $x = -N$ in $1 - x^m = (1-x)(1 + x + x^2 + \cdots x^{m-1})$ gives $(I+N)^{-1} = I + \sum_{j=1}^{m-1}(-1)^j N^j$. Hence $(I+N)^{-1}$ is lower triangular with 1s along the diagonal and nonnegative entries below the diagonal. Multiplying $(I+N)^{-1}$ by D^{-1} on the right gives the desired result.) Let $c \in \mathbb{N}$ be such that all entries of cM^{-1} are nonnegative integers. By [6, Lemma 6.6] $rc\mathbb{N} \in p_i$ for each $i \in \{1,2,\ldots,m\}$ so pick $x_i \in B_i \cap rc\mathbb{N}$. Letting $\vec{z} = M^{-1}\vec{x}$ one has that $\vec{z} \in (r\mathbb{N})^m$ and $M\vec{z} \in \times_{i=1}^m B_i$.

Now assume we have chosen $t(i)$ for $i \in \{1,2,\ldots,m\}$ as in the statement of the corollary. Pick $d \in \mathbb{N}$ such that $d\alpha_{i,j} \in \mathbb{Z}$ for each $i \in \{1,2,\ldots,m\}$ and each $j \in \{1,2,\ldots,n\}$ and let $\delta_{i,j} = d\alpha_{i,j}$. Let

$$J = \{1,2,\ldots,n\} \setminus \{t(1), t(2), \ldots, t(m)\}\,.$$

Let $s = \prod_{i=1}^m \delta_{i,t(i)}$ and pick $r \in \mathbb{N}$ such that s divides r and

$$r > \max\left\{s \textstyle\sum_{j\in J} |\delta_{i,j}| : i \in \{1,2,\ldots,m\}\right\}.$$

Pick $\vec{z} \in (r\mathbb{N})^m$ such that $M\vec{z} \in \times_{i=1}^m B_i$. We shall produce $w \in S_n$ such that $\psi(w) = \vec{z}$ by determining $\mu_j(w)$ for each $j \in \{1,2,\ldots,n\}$. (To be definite, we then let $w = \prod_{j=1}^n v_j^{\mu_j(w)}$.)

For $j \in J$, let $\mu_j(w) = s$. Let

$$\mu_{t(1)}(w) = d\frac{z_1}{\delta_{1,t(1)}} - \textstyle\sum_{j\in J}\delta_{1,j}\frac{s}{\delta_{1,t(1)}}$$

and note that $\prod_{l=2}^{m}\delta_{l,t(l)}$ divides $\mu_{t(1)}(w)$ and by the choice of r, $\mu_{t(1)}(w) > 0$, as is, of course, required. Now let $k \in \{2, 3, \ldots, m\}$ and assume that for each $i \in \{1, 2, \ldots, k-1\}$, we have chosen $\mu_{t(i)}(w) \in \mathbb{N}$ such that $\sum_{l=i+1}^{m}\delta_{l,t(l)}$ divides $\mu_{t(i)}(w)$. Then let

$$\mu_{t(k)}(w) = d\frac{z_k}{\delta_{k,t(k)}} - \textstyle\sum_{i=1}^{k-1}\frac{\delta_{k,t(i)}}{\delta_{k,t(k)}}\mu_{t(i)}(w) - \textstyle\sum_{j\in J}s\frac{\delta_{k,j}}{\delta_{k,t(k)}}\,.$$

Then $\mu_{t(k)}(w) \geq \frac{1}{\delta_{k,t(k)}}(dz_k - \sum_{j\in J}s\delta_{k,j}) > 0$ and, if $k < m$, then $\sum_{l=k+1}^{m}\delta_{l,t(l)}$ divides $\mu_{t(k)}(w)$.

It is now a routine matter to verify that for $k \in \{1, 2, \ldots, m\}$, $\tau_k(w) = \sum_{i=1}^{k}\alpha_{k,t(i)}\mu_{t(i)}(w) + \sum_{j\in J}\alpha_{k,j}\mu_j(w) = z_k$. □

The sufficient conditions in Corollary 2.12 on the coefficients $\alpha_{i,j}$ of the homomorphisms τ_i apply to all lower triangular matrices with positive diagonal entries and entries below the diagonal less than or equal to zero. A complete solution to the problem of which matrices and which S_0-independent homomorphisms satisfy the hypotheses of Theorem 2.10 seems quite difficult. The following simple example illustrates that one cannot get necessary and sufficient conditions on the coefficients of the homomorphisms τ_i valid for all lower triangular matrices with positive diagonal entries and entries below the diagonal less than or equal to zero.

Theorem 2.13. *Let $M = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$, let $N = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, let $\tau_1 = 2\mu_1 + \mu_2$ and let $\tau_2 = \mu_1 + 2\mu_2$.*

(1) If p_1 and p_2 are any idempotents in $\beta\mathbb{N}$, $B_1 \in p_1$, and $B_2 \in p_2$, F is a finite set of S_0-preserving homomorphisms from S_2 to S_0, and D is a piecewise syndetic subset of S_0, then there exists $w \in S_2$ such that $M\begin{pmatrix} \tau_1(w) \\ \tau_2(w) \end{pmatrix} \in \times_{i=1}^{2}B_i$ and $\nu(w) \in D$ for each $\nu \in F$.

(2) There exist idempotents p_1 and p_2 in $\beta\mathbb{N}$ and sets $B_1 \in p_1$ and $B_2 \in p_2$ for which there does not exist $w \in S_2$ such that $N\begin{pmatrix} \tau_1(w) \\ \tau_2(w) \end{pmatrix} \in \times_{i=1}^{2}B_i$.

Proof. (1) Let p_1 and p_2 be idempotents in $\beta\mathbb{N}$, and let $B_1 \in p_1$ and $B_2 \in p_2$ be given. By Theorem 2.8, it suffices to show that there exists $w \in S_2$ such that $M\begin{pmatrix} \tau_1(w) \\ \tau_2(w) \end{pmatrix} \in \times_{i=1}^{2}B_i$ By [6, Lemma 6.6], $3\mathbb{N} \in p_1$ and $3\mathbb{N} \in p_2$. Pick $z_2 \in B_2 \cap 3\mathbb{N}$ and pick $z_1 > z_2$ in $B_1 \cap 3\mathbb{N}$. Let $k_1 = \frac{1}{3}z_1 - \frac{1}{3}z_2$ and let $k_2 =$

$\frac{1}{3}z_1 - \frac{1}{3}z_2$. Let $w = v_1^{k_1}v_2^{k_2}$ so that $\mu_1(w) = \frac{1}{3}z_1 - \frac{1}{3}z_2$ and $\mu_2(w) = \frac{1}{3}z_1 + \frac{2}{3}z_2$. Then $\tau_1(w) = z_1$, $\tau_2(w) = z_1 + z_2$, and $M\begin{pmatrix} \tau_1(w) \\ \tau_2(w) \end{pmatrix} = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$.

(2) Let $B_1 = FS(\langle 2^{4n}\rangle_{n=1}^{\infty})$ and let $B_2 = FS(\langle 2^{4n+2}\rangle_{n=1}^{\infty})$. By [6, Lemma 5.11] pick idempotents p_1 and p_2 in $\beta\mathbb{N}$ such that $B_1 \in p_1$ and $B_2 \in p_2$. Suppose we have some $w \in S_2$ and elements $z_1 \in B_1$ and $z_2 \in B_2$ such that $N\begin{pmatrix} \tau_1(w) \\ \tau_2(w) \end{pmatrix} = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$. Then $2z_1 - z_2 = 3\mu_1(w) > 0$ and $2z_2 - z_1 = 3\mu_2(w) > 0$ so $z_2 < 2z_1$ and $z_1 < 2z_2$. Pick $F, G \in \mathcal{P}_f(\mathbb{N})$ such that $z_1 = \sum_{t\in F} 2^{4t}$ and $z_2 = \sum_{t\in G} 2^{4t+2}$. Let $m = \max F$ and let $k = \max G$. Then $2^{4m} \leq z_1 < 2^{4m+1}$ and $2^{4k+2} \leq z_2 < 2^{4k+3}$. Then $2^{4m+2} > 2z_1 > z_2 \geq 2^{4k+2}$ so $m \geq k+1$. Also $2^{4k+4} > 2z_2 > z_1 \geq 2^{4m} \geq 2^{4k+4}$, a contradiction. □

A $k \times m$ matrix M is *image partition regular over* $\mathbb{N}$ if and only if, whenever $\mathbb{N}$ is finitely colored, there is some $\vec{z} \in \mathbb{N}^m$ such that the entries of $M\vec{z}$ are monochromatic. This class includes all triangular (upper or lower) matrices with rational entries and positive diagonal entries. See [6, Theorem 15.24] for half an alphabet of characterizations of matrices that are image partition regular over $\mathbb{N}$.

Corollary 2.14. *Let $k, m, n \in \mathbb{N}$ with $m \leq n$. Let M be a $k \times m$ matrix with rational entries which is image partition regular over $\mathbb{N}$. Let p be a minimal idempotent in $\beta\mathbb{N}$ and let $\widehat{p} = (p, p, \ldots, p) \in \times_{i=1}^{k}\beta\mathbb{N}$. Let σ be an injection from $\{1, 2, \ldots, m\}$ to $\{1, 2, \ldots, n\}$. For $i \in \{1, 2, \ldots, m\}$ let $\tau_i = \mu_{\sigma(i)}$. Let F be a nonempty finite set of S_0-preserving homomorphisms from S_n to S_0 and let D be a piecewise syndetic subset of S_0. Then whenever $B \in p$ there exists $w \in S_n$ such that $\nu(w) \in D$ for each $\nu \in F$ and*

$$M\begin{pmatrix} \tau_1(w) \\ \tau_2(w) \\ \vdots \\ \tau_m(w) \end{pmatrix} \in \times_{i=1}^{k} B\,.$$

Proof. We note that the mapping

$$w \mapsto \big(\tau_1(w), \tau_2(w), \ldots, \tau_m(w)\big)$$

defines an S_0-independent homomorphism from S_n onto $\mathbb{N}^m$. So in order to apply Theorem 2.10, we must verify that whenever $B_i \in p$ for each $i \in \{1, 2, \ldots, k\}$, there exists $\vec{z} \in \mathbb{N}^m$ such that $M\vec{z} \in \times_{i=1}^{k} B_i$. We then pick $w \in S_n$ such that $\tau_i(w) = z_i$ for each $i \in \{1, 2, \ldots, m\}$, which one may do because σ is injective.

Now $\bigcap_{i=1}^{k} B_i \in p$ so $B = \bigcap_{i=1}^{k} B_i$ is central in $\mathbb{N}$. By [6, Theorem 15.24(h)] there exists $\vec{z} \in \mathbb{N}^m$ such that $M\vec{z} \in B^k$. □

Corollary 2.14 applies to a much larger class of matrices than Corollary 2.12, but is more restrictive in that the same minimal idempotent must occur in each coordinate. Suppose we have a $k \times m$ matrix M which is image partition regular over $\mathbb{N}$. If we knew that whenever $B_1, B_2, \ldots, B_k$ are central subsets of $\mathbb{N}$, there exist $\vec{z} \in \mathbb{N}^m$ with $M\vec{z} \in \times_{i=1}^{k} B_i$, then in Corollary 2.14 we could allow $\vec{p} = (p_1, p_2, \ldots, p_k)$ to be an arbitrary minimal idempotent in $\times_{i=1}^{k} \beta\mathbb{N}$. We shall see now that this fails.

Theorem 2.15. *Let* $M = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$. *Then M is image partition regular over $\mathbb{N}$. For $x \in \mathbb{N}$ let $\phi(x) = \max\{t \in \omega : 2^t \leq x\}$ and for $i \in \{0, 1, 2, 3\}$ let $B_i = \{x \in \mathbb{N} : \phi(x) \equiv i \pmod 4\}$. Then B_0 and B_2 are central and there do not exist x and $y \in \mathbb{N}$ such that* $M \begin{pmatrix} x \\ y \end{pmatrix} \in B_0 \times B_2$.

Proof. By [6, Theorem 15.5] M is image partition regular over $\mathbb{N}$. Since $\mathbb{N} = \bigcup_{i=0}^{3} B_i$ some B_i is central. But then, by [6, Lemma 15.23.2], each B_i is central. Suppose we have some $x, y \in \mathbb{N}$ such that $M \begin{pmatrix} x \\ y \end{pmatrix} \in B_0 \times B_2$. Let $n = \phi(x + y)$. Then $2^n \leq x + y < 2^{n+1}$ so $y < 2^{n+1} - x$ and thus $2y < 2^{n+2} - 2x$ so $x + 2y < 2^{n+2} - x < 2^{n+2}$ and thus $\phi(x + 2y) \in \{n, n + 1\}$. □

Note also that Corollary 2.14 is more restrictive than Corollary 2.12 in that the idempotent p is required to be minimal. It is well known and easy to see that $FS(\langle 2^{2t} \rangle_{t=1}^{\infty})$ does not contain any three term arithmetic progression. Consequently, if

$$M = \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{pmatrix},$$

then the assumption in Corollary 2.14 that the idempotent p is minimal cannot be deleted.

We remarked in the introduction that we are not concerned with the instances of Theorem 1.4 with $m > 0$ because the natural versions of our results in this section are not valid. The results in this section apply to all piecewise syndetic subsets of S_0. In particular, they apply to central subsets. It was shown in [2, Theorem 3.6] that, given $a \in \mathbb{A}$, there is a central set $D \in S_1$ such that there is no $w \in S_2$ with $\{w(av_1), w(v_1 a)\} \subseteq D$.

3 Homomorphisms satisfying our hypotheses

In Corollary 2.12 we produced S_0-independent homomorphisms from S_n to $\mathbb{Q}$ as linear combinations of the functions μ_i with coeffients from $\mathbb{Q}$. We shall see in Corollary 3.5 that if T is commutative and cancellative, then the only S_0-independent homomorphisms $\varphi : S_n \to T$ are of the form $\varphi(w) = \sum_{i=1}^{n} \mu_i(w)\cdot a_i$ where each a_i is in the group of differences of T.

In Corollary 2.14 we used S_0-independent homomorphisms $\tau_i = \mu_{\sigma(i)}$ from S_n to $\mathbb{N}$ and the surjection $w \mapsto \big(\tau_1(w), \tau_2(w), \ldots, \tau_m(w)\big)$ from S_n onto $\mathbb{N}^m$. We show in Corollary 3.8 that if $T = \mathbb{N}$, then these are essentially the only choices for τ_i satisfying the hypotheses of Theorem 2.10.

Recall that throughout this section $\mathbb{A}$ is a fixed nonempty finite alphabet.

Definition 3.1. Let $n \in \mathbb{N}$. For $w \in S_n$, let $w' \in \{v_1, v_2, \ldots, v_n\}^+$ be obtained from w by deleting all occurrences in w of letters belonging to $\mathbb{A}$.

Lemma 3.2. *Fix $n \in \mathbb{N}$. Let $(T, +)$ be a cancellative semigroup and let $\varphi : S_n \to T$ be an S_0-independent homomorphism. Then $\varphi(w) = \varphi(w')$ for all $w \in S_n$.*

Proof. It suffices to show that if $w_1, w_2 \in S_n$ and $u \in S_0$, then $\varphi(w_1 u w_2) = \varphi(w_1 w_2)$. Let $v = v_1 v_2 \cdots v_n$. On one hand $\varphi(v w_1 u w_2 v) = \varphi(v) + \varphi(w_1 u w_2) + \varphi(v)$, and on the other hand $\varphi(v w_1 u w_2 v) = \varphi(v w_1 u) + \varphi(w_2 v) = \varphi(v w_1) + \varphi(w_2 v) = \varphi(v w_1 w_2 v) = \varphi(v) + \varphi(w_1 w_2) + \varphi(v)$. The result now follows. $\square$

Lemma 3.3. *Fix $n \in \mathbb{N}$. Let $(T, +)$ be a cancellative and commutative semigroup and let $\varphi : S_n \to T$ be an S_0-independent homomorphism. For each $w_1, w_2 \in S_n$ we have $\varphi(w_1) = \varphi(w_2)$ whenever $w_1' \sim_{Ab} w_2'$.*

Proof. Assume $w_1, w_2 \in S_n$ and $w_1' \sim_{Ab} w_2'$. Let $m = |w_1'| = |w_2'|$. We show that $\varphi(w_1') = \varphi(w_2')$ which in turn implies that $\varphi(w_1) = \varphi(w_2)$ by Lemma 3.2. The result is immediate in case $n = 1$ for in this case $w_1' = w_2' = v_1^m$. So let us assume that $n \geq 2$ in which case $m \geq 2$. Since the symmetric group on m-letters is generated by the 2-cycle $(1, 2)$ and the m-cycle $(1, 2, \ldots, m)$ it suffices to show

(i) If $x, w \in (\mathbb{A} \cup \{v_1, v_2, \ldots, v_n\})^+$ and $xw \in S_n$, then $wx \in S_n$ and $\varphi(wx) = \varphi(xw)$.

(ii) Let ε be the empty word. If $x, y \in (\mathbb{A} \cup \{v_1, v_2, \ldots, v_n\})^+$, $w \in (\mathbb{A} \cup \{v_1, v_2, \ldots, v_n\})^+ \cup \{\varepsilon\}$ and $xyw \in S_n$, then $yxw \in S_n$ and $\varphi(yxw) = \varphi(xyw)$.

Then given $l_1, l_2, \ldots, l_m \in \mathbb{A} \cup \{v_1, v_2, \ldots, v_n\}$ by (i) we have $\varphi(l_1 l_2 \cdots l_m) = \varphi(l_2 l_3 \cdots l_m l_1)$ and by (ii) we have $\varphi(l_1 l_2 l_3 \cdots l_m) = \varphi(l_2 l_1 l_3 \cdots l_m)$.

To establish (i), we have $\varphi(xw) + \varphi(xwx) = \varphi(xwxwx) = \varphi(xwx) + \varphi(wx)$, whence $\varphi(xw) = \varphi(wx)$. Note that we are using here that T is commutative. For (ii), let $v = v_1 v_2 \cdots v_n$. Then, using (i) twice, $\varphi(v) + \varphi(xyw) + \varphi(v) = \varphi(vxywv) = \varphi(vx) + \varphi(ywv) = \varphi(xv) + \varphi(ywv) = \varphi(xvywv) = \varphi(xvy) + \varphi(wv) = \varphi(vyx) + \varphi(wv) = \varphi(vyxwv) = \varphi(v) + \varphi(yxw) + \varphi(v)$. The result now follows. $\square$

We remark that Lemma 3.3 does not hold in general if T is not commutative. For example, consider the homomorphism $\varphi : S_3 \to S_2$ where $\varphi(w)$ is the word in S_2 obtained from w by deleting all occurrences of the variable v_3 in addition to all letters belonging to $\mathbb{A}$. Then S_2 is cancellative and φ is an S_0-independent homomorphism. However, $\varphi(v_1v_2v_3) = v_1v_2 \neq v_2v_1 = \varphi(v_2v_1v_3)$ yet $v_1v_2v_3 \sim_{Ab} v_2v_1v_3$.

Theorem 3.4. *Fix $n \in \mathbb{N}$. Let $(T, +)$ be a cancellative and commutative semigroup and let $\varphi : S_n \to T$ be an S_0-independent homomorphism. Then there exists a homomorphism $f : \mathbb{N}^n \to T$ such that $\varphi(w) = f\big(\mu_1(w), \mu_2(w), \ldots, \mu_n(w)\big)$ for all $w \in S_n$.*

Proof. Define $f : \mathbb{N}^n \to T$ by $f(x_1, x_2, \ldots, x_n) = \varphi(v_1^{x_1} v_2^{x_2} \cdots v_n^{x_n})$. By Lemma 3.3, f is as required. □

Corollary 3.5. *Let $n \in \mathbb{N}$, let $(T, +)$ be a commutative and cancellative semigroup, let G be the group of differences of T, and let φ be an S_0-independent homomorphism from S_n to T. There exist $a_1, a_2, \ldots, a_n$ in G such that for each $w \in S_n$, $\varphi(w) = \sum_{i=1}^n \mu_i(w) \cdot a_i$.*

Proof. Pick a homomorphism $f : \mathbb{N}^n \to T$ as guaranteed by Theorem 3.4. For $j \in \{1, 2, \ldots, n\}$, define $\vec{z}^{\,[j]} \in \mathbb{N}^n$ by, for $i \in \{1, 2, \ldots, n\}$,

$$z_i^{[j]} = \begin{cases} 2 & \text{if } i = j \\ 1 & \text{if } i \neq j\,. \end{cases} \tag{1}$$

Let $\widehat{1} = (1, 1, \ldots, 1) \in \mathbb{N}^n$. Let $c = f(\widehat{1})$ and for $j \in \{1, 2, \ldots, n\}$, let $a_j = f(\vec{z}^{\,[j]}) - c$. Then

$$(n+1) \cdot c = f(n+1, n+1, \ldots, n+1) = \sum_{j=1}^n f(\vec{z}^{\,[j]}) = (\sum_{j=1}^n a_j) + n \cdot c$$

so $c = \sum_{j=1}^n a_j$.

We claim that $f(x_1, x_2, \ldots, x_n) = \sum_{j=1}^n x_j \cdot a_j$ for all $(x_1, x_2, \ldots, x_n) \in \mathbb{N}^n$. To see this we proceed by induction on $\sum_{j=1}^n x_j$. If $\sum_{j=1}^n x_j = n$ then $(x_1, x_2, \ldots, x_n) = \widehat{1}$ whence

$$f(x_1, x_2, \ldots, x_n) = c = \textstyle\sum_{j=1}^n a_j = \sum_{j=1}^n 1 \cdot a_j\,.$$

Next let $N \geq n$ and suppose that $f(x_1, x_2, \ldots, x_n) = \sum_{j=1}^n x_j \cdot a_j$ for all $(x_1, x_2, \ldots, x_n) \in \mathbb{N}^n$ with $\sum_{j=1}^n x_j \leq N$. Let $(x_1, x_2, \ldots, x_n) \in \mathbb{N}^n$ be such that $\sum_{j=1}^n x_j = N + 1$. Pick $j \in \{1, 2, \ldots, n\}$ such that $x_j \geq 2$. Then

$$f(x_1, x_2, \ldots, x_n) + f(\widehat{1}) = f(x_1, \ldots, x_{j-1}, x_j - 1, x_{j+1}, \ldots, x_n) + f(\vec{z}^{\,[j]}).$$

Since $f(\vec{z}^{\,[j]}) - f(\widehat{1}) = a_j$ it follows by our induction hypothesis that

$$f(x_1,x_2,\ldots,x_n) = \sum_{i=1}^{j-1} x_i \cdot a_i + (x_j-1)\cdot a_j + \sum_{i=j+1}^{n} x_i \cdot a_i + a_j = \sum_{i=1}^{n} x_i \cdot a_i \,.$$

Consequently, for all $w \in S_n$,

$$\varphi(w) = f\big(\mu_1(w), \mu_2(w), \ldots, \mu_n(w)\big) = \sum_{j=1}^{n} \mu_j(w) \cdot a_j \,.$$

□

In the proof of the next lemma, we shall use the fact that if $n \in \mathbb{N}$, $f : \mathbb{N}^n \to \mathbb{N}$ is a homomorphism, $\vec{x}, \vec{y}^{\,[1]}, \vec{y}^{\,[2]}, \ldots, \vec{y}^{\,[n]} \in \mathbb{N}^n$, $\alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{Z}$, and $\vec{x} = \sum_{i=1}^{n} \alpha_i \vec{y}^{\,[i]}$, then

$$f(\vec{x}) = \sum_{i=1}^{n} \alpha_i f(\vec{y}^{\,[i]}) \,.$$

We note that if $\alpha_i \leq 0$, then f is not defined at $\alpha_i \vec{y}^{\,[i]}$. So to verify the above equality, let $I = \{i \in \{1,2,\ldots,n\} : \alpha_i < 0\}$ and let $J = \{i \in \{1,2,\ldots,n\} : \alpha_i > 0\}$. Then

$$\vec{x} + \sum_{i\in I} (-\alpha_i) \vec{y}^{\,[i]} = \sum_{i \in J} \alpha_i \vec{y}^{\,[i]}$$

so

$$f(\vec{x}) + \sum_{i\in I} (-\alpha_i) f(\vec{y}^{\,[i]}) = \sum_{i \in J} \alpha_i f(\vec{y}^{\,[i]})$$

so

$$f(\vec{x}) = \sum_{i\in I \cup J} \alpha_i f(\vec{y}^{\,[i]}) = \sum_{i=1}^{n} \alpha_i f(\vec{y}^{\,[i]}) \,.$$

Lemma 3.6. *Let $n \in \mathbb{N}$ and $f : \mathbb{N}^n \to \mathbb{N}$ be a surjective homomorphism. Then there exists $i \in \{1,2,\ldots,n\}$ such that $f(\vec{x}) = x_i$ for each $\vec{x} = (x_1,x_2,\ldots,x_n) \in \mathbb{N}^n$, i.e., f is the projection onto the i'th coordinate.*

Proof. We begin by showing that $f(\widehat{1}\,) = 1$ where $\widehat{1} = (1,1,\ldots,1)$. Since f is surjective, it suffices to show that $f(\vec{x}) \geq f(\widehat{1}\,)$ for each $\vec{x} = (x_1,x_2,\ldots,x_n) \in \mathbb{N}^n$. For each $r \in \mathbb{N}$ we have that

$$r\vec{x} = (r-1)\widehat{1} + \big(1 + r(x_1-1), 1 + r(x_2-1), \ldots, 1 + r(x_n-1)\big) \,.$$

It follows that $rf(\vec{x}) = f(r\vec{x}) > f\big((r-1)\widehat{1}\,\big) = (r-1)f(\widehat{1}\,)$ or equivalently that $r\big(f(\vec{x}) - f(\widehat{1})\big) > -f(\widehat{1}\,)$. As r is arbitrary we deduce that $f(\vec{x}) - f(\widehat{1}\,) \geq 0$ as claimed.

For each $j \in \{1,2,\ldots,n\}$ let $\vec{z}^{\,[j]} = (z_1^{[j]}, z_2^{[j]}, \ldots, z_n^{[j]}) \in \mathbb{N}^n$ be as in (1). As $\sum_{j=1}^{n} \vec{z}^{\,[j]} = (n+1)\widehat{1}$ we have $\sum_{j=1}^{n} f(\vec{z}^{\,[j]}) = f\big((n+1)\widehat{1}\big) = n+1$. It follows that there exists a unique $k \in \{1,2,\ldots,n\}$ such that $f(\vec{z}^{\,[k]}) = 2$ and $f(\vec{z}^{\,[j]}) = 1$ for all $j \neq k$. Without loss of generality, we may assume that $f(\vec{z}^{\,[1]}) = 2$ and $f(\vec{z}^{\,[j]}) = 1$ for all $j \in \{2,3,\ldots n\}$.

Let $\vec{x} = (x_1,x_2,\ldots,x_n) \in \mathbb{N}^n$. We will show that $f(\vec{x}) = x_1$. We first note that

$$(n+1)\vec{x} = \sum_{i=1}^{n} \big(nx_i - \sum_{\substack{j=1 \\ j\neq i}}^{n} x_j\big)\vec{z}^{\,[i]}.$$

Therefore

$$(n+1)f(\vec{x}) = (nx_1 - \sum_{j=2}^n x_j)\cdot 2 + \sum_{i=2}^n (nx_i - \sum_{\substack{j=1\\ j\neq i}}^n x_j)\cdot 1$$

and thus $(n+1)f(\vec{x}) = (n+1)x_1$. □

Corollary 3.7. *Let $n, m \in \mathbb{N}$. For $i \in \{1,2,\ldots,n\}$ and $j \in \{1,2,\ldots,m\}$ let $\pi_i : \mathbb{N}^n \to \mathbb{N}$ and $\pi'_j : \mathbb{N}^m \to \mathbb{N}$ denote the projections onto the i'th and j'th coordinates respectively. Assume that $f : \mathbb{N}^n \to \mathbb{N}^m$ is a surjective homomorphism. For $i \in \{1,2,\ldots,m\}$, let $f_i = \pi'_i \circ f$. Then there is an injection $\sigma : \{1,2,\ldots,m\} \to \{1,2,\ldots,n\}$ such that for each $i \in \{1,2,\ldots,m\}$, $f_i = \pi_{\sigma(i)}$. In particular $m \le n$.*

Proof. By hypothesis each $f_i : \mathbb{N}^n \to \mathbb{N}$ is a surjective homomorphism. Thus by Lemma 3.6, there exists a mapping $\sigma : \{1,2,\ldots,m\} \to \{1,2,\ldots,n\}$ such that $f_i(\vec{x}) = \pi_{\sigma(i)}(\vec{x}) = x_{\sigma(i)}$ for each $\vec{x} \in \mathbb{N}^n$. But as f is surjective, it follows that σ is injective. □

Corollary 3.8. *Let $n, m \in \mathbb{N}$. For each $i \in \{1,2,\ldots,m\}$ let $\tau_i : S_n \to \mathbb{N}$ be an S_0-independent homomorphism. If the mapping $w \mapsto \big(\tau_1(w), \tau_2(w), \ldots, \tau_m(w)\big)$ takes S_n onto $\mathbb{N}^m$, then there exists an injection $\sigma : \{1,2,\ldots,m\} \to \{1,2,\ldots,n\}$ such that $\tau_i = \mu_{\sigma(i)}$ for each $i \in \{1,2,\ldots,m\}$. In particular we must have $m \le n$.*

Proof. By Theorem 3.4, for each $i \in \{1,2,\ldots,m\}$, pick a homomorphism $f_i : \mathbb{N}^n \to \mathbb{N}$ such that $\tau_i(w) = f_i\big(\mu_1(w), \mu_2(w), \ldots, \mu_n(w)\big)$ for each $w \in S_n$. Define $f : \mathbb{N}^n \to \mathbb{N}^m$ by $f(\vec{x}) = \big(f_1(\vec{x}), f_2(\vec{x}), \ldots, f_m(\vec{x})\big)$. We claim that f is surjective, so let $\vec{y} \in \mathbb{N}^m$ be given and pick $w \in S_n$ such that $\big(\tau_1(w), \tau_2(w), \ldots, \tau_m(w)\big) = \vec{y}$. For $j \in \{1,2,\ldots,n\}$, let $x_j = |w|_{v_j}$. Then $f(\vec{x}) = \vec{y}$.

By Corollary 3.7, pick an injection $\sigma : \{1,2,\ldots,m\} \to \{1,2,\ldots,n\}$ such that for each $i \in \{1,2,\ldots,m\}$, $f_i = \pi_{\sigma(i)}$. Let $w \in S_n$ be given and let $\vec{x} = (|w|_{v_1}, |w|_{v_2}, \ldots, |w|_{v_n})$, Then for $i \in \{1,2,\ldots,m\}$, $\tau_i(w) = f_i(\vec{x}) = x_{\sigma(i)} = |w|_{v_{\sigma(i)}}$. □

4 Compact subsemigroups of $(\beta\mathbb{N})^k$

Theorem 4.1. *Let $n \in \mathbb{N}$, let C be a compact right topological semigroup, let $\phi : S_n \to C$ be an S_0-independent homomorphism for which $\phi[S_n]$ is contained in the topological center of C, denote also by ϕ the continuous extension from βS to C, and let F be a finite nonempty set of S_0-preserving homomorphisms from S_n into S_0. Let*

$$P = \big\{p \in \phi[\beta S_n] : \begin{array}{l}\textit{for every neighborhood } U \textit{ of } p\\ \textit{and every piecewise syndetic subset } D \textit{ of } S_0\\ (\exists w \in S_n)\big(\phi(w) \in U \textit{ and } (\forall \nu \in F)(\nu(w) \in D)\big)\big\}.\end{array}$$

Then P is a compact subsemigroup of C containing all the idempotents of $\phi[\beta S_n]$.

Proof. It is clear that P is compact. By Corollary 2.6, P contains all the idempotents in $\phi[\beta S_n]$. To see that P is a subsemigroup of C, let $p, q \in P$. Let U be an open neighborhood of pq and let D be a piecewise syndetic subset of S_0. By [6, Theorem 4.43] pick $s \in S_0$ such that $s^{-1}D$ is central and pick a minimal idempotent r in βS_0 such that $s^{-1}D \in r$. Pick a neighborhood V of p such that $\rho_q[V] \subseteq U$. Since $(s^{-1}D)^\star \in r$, it is piecewise syndetic so pick $w \in S_n$ such that $\phi(w) \in V$ and $\nu(w) \in (s^{-1}D)^\star$ for each $\nu \in F$.

Then $\phi(w)q \in U$ and $\phi(w)$ is in the topological center of C so pick a neighborhood Q of q such that $\lambda_{\phi(w)}[Q] \subseteq U$. For each $\nu \in F$, $\nu(w)^{-1}(s^{-1}D)^\star \in r$. Let $E = \bigcap_{\nu \in F} \nu(w)^{-1}(s^{-1}D)^\star$. Then $E \in r$ so E is piecewise syndetic in S_0. Pick $u \in S_n$ such that $\phi(u) \in Q$ and $\nu(u) \in E$ for each $\nu \in F$. Then $\phi(swu) = \phi(w)\phi(u) \in U$ and for each $\nu \in F$, $\nu(swu) = s\nu(w)\nu(u) \in D$. □

In the next results we focus on $(\beta\mathbb{N})^k$ and S_0-independent homomorphisms from S_n onto $\mathbb{N}^m$, so by Corollary 3.8 we may assume that we have $m \leq n$ and are dealing with S_0-independent homomorphisms τ_i from S_n to $\mathbb{N}$ defined by $\tau_i(w) = |w|_{v_{\sigma(i)}}$ for some injection $\sigma : \{1, 2, \ldots, m\} \to \{1, 2, \ldots, n\}$.

Definition 4.2. Let $k, m, n \in \mathbb{N}$ with $m \leq n$, let M be a $k \times m$ matrix with entries from $\mathbb{Q}$, let F be a finite nonempty set of S_0-preserving homomorphisms from S_n to S_0, and let σ be an injection from $\{1, 2, \ldots, m\}$ to $\{1, 2, \ldots, n\}$.

$$P_{M,F,\sigma} = \{\vec{p} \in \times_{i=1}^k \beta\mathbb{N} : \begin{array}{l} \text{whenever } D \text{ is a piecewise syndetic subset of } S_0 \\ \text{and for all } i \in \{1, 2, \ldots, k\}\,,\ B_i \in p_i, \text{ there exists} \\ w \in S_n \text{ such that } (\forall \nu \in F)(\nu(w) \in D) \text{ and} \\ M \begin{pmatrix} \mu_{\sigma(1)}(w) \\ \vdots \\ \mu_{\sigma(m)}(w) \end{pmatrix} \in \times_{i=1}^k B_i\} \end{array}$$

Recall that for $\vec{x} \in \mathbb{A}^n$ we have defined the S_0-preserving homomorphism $h_{\vec{x}} : S_n \to S_0$ by $h_{\vec{x}}(w) = w(\vec{x})$. We are particularly interested in the set $\{h_{\vec{x}} : \vec{x} \in \mathbb{A}^n\}$ because of the relationship with the Hales-Jewett Theorem. We see now that if $F = \{h_{\vec{x}} : \vec{x} \in \mathbb{A}^n\}$, then $P_{M,F,\sigma}$ does not depend on σ. We keep σ in the notation because there are S_0-preserving homomorphisms which are not of the form $h_{\vec{x}}$.

Theorem 4.3. *Let $k, m, n \in \mathbb{N}$ with $m \leq n$, let M be a $k \times m$ matrix with entries from $\mathbb{Q}$, let $F = \{h_{\vec{x}} : \vec{x} \in \mathbb{A}^n\}$, and let σ and η be injections from $\{1, 2, \ldots, m\}$ to $\{1, 2, \ldots, n\}$. Then $P_{M,F,\sigma} = P_{M,F,\eta}$.*

Proof. It suffices to show that $P_{M,F,\sigma} \subseteq P_{M,F,\eta}$, so let $\vec{p} \in P_{M,F,\sigma}$. To see that $\vec{p} \in P_{M,F,\eta}$, let D be a piecewise syndetic subset of S_0 and for $i \in$

$\{1,2,\ldots,k\}$, let $B_i \in p_i$. Pick $w \in S_n$ such that for all $\vec{x} \in \mathbb{A}^n$, $h_{\vec{x}}(w) \in D$ and $M\begin{pmatrix}\mu_{\sigma(1)}(w)\\ \vdots \\ \mu_{\sigma(m)}(w)\end{pmatrix} \in \times_{i=1}^k B_i$.

Define $\delta : \{\sigma(1),\sigma(2),\ldots,\sigma(m)\} \to \{1,2,\ldots,n\}$ by, for $i \in \{1,2,\ldots,m\}$, $\delta\big(\sigma(i)\big) = \eta(i)$ and extend δ to a permutation of $\{1,2,\ldots,n\}$. Define $w' \in S_n$ by $w' = w(v_{\delta(1)}v_{\delta(2)}\cdots v_{\delta(n)})$. Then for $j \in \{1,2,\ldots,n\}$, $\mu_j(w) = \mu_{\delta(j)}(w')$ so for $i \in \{1,2,\ldots,m\}$, $\mu_{\sigma(i)}(w) = \mu_{\delta(\sigma(i))}(w') = \mu_{\eta(i)}(w')$ and thus

$$M\begin{pmatrix}\mu_{\eta(1)}(w')\\ \vdots \\ \mu_{\eta(m)}(w')\end{pmatrix} = M\begin{pmatrix}\mu_{\sigma(1)}(w)\\ \vdots \\ \mu_{\sigma(m)}(w)\end{pmatrix} \in \times_{i=1}^k B_i\,.$$

Now let $\vec{x} \in \mathbb{A}^n$ be given and define $\vec{z} \in \mathbb{A}^n$ by, for $i \in \{1,2,\ldots,n\}$, $z_i = x_{\delta(i)}$. Then $h_{\vec{x}}(w') = h_{\vec{z}}(w) \in D$. □

If one lets $C = (\beta\mathbb{N})^k$ and defines ϕ on S_n by $\phi(w) = M\begin{pmatrix}\mu_{\sigma(1)}(w)\\ \vdots \\ \mu_{\sigma(m)}(w)\end{pmatrix}$, one may not be able to invoke Theorem 4.1 to conclude that $P_{M,F,\sigma}$ is a semigroup because ϕ may not take S_n to C. Consider, for example, $M = \begin{pmatrix}1\\ -1\end{pmatrix}$.

Lemma 4.4. *Let $k,m,n \in \mathbb{N}$ with $m \le n$, let M be a $k \times m$ matrix with entries from $\mathbb{Q}$, and let F be a finite nonempty set of S_0-preserving homomorphisms from S_n to S_0. Let σ be an injection from $\{1,2,\ldots,m\}$ to $\{1,2,\ldots,n\}$. If $P_{M,F,\sigma} \neq \emptyset$, then $P_{M,F,\sigma}$ is a compact subsemigroup of $(\beta\mathbb{N})^k$.*

Proof. Assume that $P_{M,F,\sigma} \neq \emptyset$. We begin by showing that $P_{M,F,\sigma}$ is compact. Let $\vec{p} = (p_1,p_2,\ldots,p_k) \in (\beta\mathbb{N})^k \setminus P_{M,F,\sigma}$ and pick piecewise syndetic $D \subseteq S_0$ and $B_i \in p_i$ for each $i \in \{1,2,\ldots,k\}$ such that there is no $w \in S_n$ with $\nu(w) \in D$ for all $\nu \in F$ and $M\begin{pmatrix}\mu_{\sigma(1)}(w)\\ \vdots \\ \mu_{\sigma(m)}(w)\end{pmatrix} \in \times_{i=1}^k B_i$; then $\times_{i=1}^k \overline{B_i}$ is a neighborhood of $\vec{p}$ which misses $P_{M,F,\sigma}$ so $P_{M,F,\sigma}$ is closed and hence compact.

To see that $P_{M,F,\sigma}$ is a semigroup, let $\vec{p},\vec{q} \in P_{M,F,\sigma}$. Let D be a piecewise syndetic subset of S_0 and for each $i \in \{1,2,\ldots,k\}$, let $B_i \in p_i + q_i$. By [6, Theorem 4.43], pick $s \in S_0$ such that $s^{-1}D$ is central in S_0 and pick a minimal idempotent $r \in \beta S_0$ such that $s^{-1}D \in r$. For each $i \in \{1,2,\ldots,k\}$, let $C_i = \{x \in \mathbb{N} : -x + B_i \in q_i\}$ and note that $C_i \in p_i$. Then as $(s^{-1}D)^\star \in r$, we deduce that $(s^{-1}D)^\star$ is central and hence in particular piecewise syndetic. Since $\vec{p} \in P_{M,F,\sigma}$, pick $w \in S_n$ such that $\nu(w) \in (s^{-1}D)^\star$ for all $\nu \in F$ and

$M\begin{pmatrix}\mu_{\sigma(1)}(w)\\ \vdots\\ \mu_{\sigma(m)}(w)\end{pmatrix} = \vec{z} \in \times_{i=1}^{k} C_i$. Let $G = \bigcap_{\nu\in F} \nu(w)^{-1}(s^{-1}D)^{\star}$. Then $G \in r$ so G is piecewise syndetic in S_0. Also $\vec{q} \in P_{M,F,\sigma}$ and for each $i \in \{1, 2, \ldots, k\}$, $-z_i + B_i \in q_i$ so pick $u \in S_n$ such that $\nu(u) \in G$ for each $\nu \in F$ and $M\begin{pmatrix}\mu_{\sigma(1)}(u)\\ \vdots\\ \mu_{\sigma(m)}(u)\end{pmatrix} = \vec{y} \in \times_{i=1}^{k}(-z_i + B_i)$.

Given $\nu \in F$, $\nu(wu) = \nu(w)\nu(u) \in s^{-1}D$ so $\nu(swu) = s\nu(wu) \in D$. Finally $M\begin{pmatrix}\mu_{\sigma(1)}(swu)\\ \vdots\\ \mu_{\sigma(m)}(swu)\end{pmatrix} = M\begin{pmatrix}\mu_{\sigma(1)}(wu)\\ \vdots\\ \mu_{\sigma(m)}(wu)\end{pmatrix} = M\begin{pmatrix}\mu_{\sigma(1)}(w)+\mu_{\sigma(1)}(u)\\ \vdots\\ \mu_{\sigma(m)}(w)+\mu_{\sigma(m)}(u)\end{pmatrix} =$ $\vec{z} + \vec{y} \in \times_{i=1}^{k} B_i$. □

Theorem 4.5. *Let $m, n \in \mathbb{N}$ with $m \leq n$. Let M be an $m \times m$ lower triangular matrix with rational entries. Assume that the entries on the diagonal are positive and the entries below the diagonal are negative or zero. Let F be a finite nonempty set of S_0-preserving homomorphisms from S_n to S_0. Let σ be an injection from $\{1, 2, \ldots, m\}$ to $\{1, 2, \ldots, n\}$. Then $P_{M,F,\sigma}$ is a compact subsemigroup of $(\beta\mathbb{N})^m$ containing the idempotents of $(\beta\mathbb{N})^m$.*

Proof. Let $k = m$. By Corollary 2.12, $P_{M,F,\sigma}$ contains the idempotents of $(\beta\mathbb{N})^k$ so in particular $P_{M,F,\sigma} \neq \emptyset$. The result now follows by Lemma 4.4. □

Theorem 4.6. *Let $k, m, n \in \mathbb{N}$ with $m \leq n$. Let M be a $k\times m$ matrix with rational entries which is image partition regular over $\mathbb{N}$. Let F be a finite nonempty set of S_0-preserving homomorphisms from S_n to S_0. Let σ be an injection from $\{1, 2, \ldots, m\}$ to $\{1, 2, \ldots, n\}$. Then $P_{M,F,\sigma}$ is a compact subsemigroup of $(\beta\mathbb{N})^k$ containing $\{(p, p, \ldots, p) \in (\beta\mathbb{N})^k : p$ is a minimal idempotent of $\beta\mathbb{N}\}$.*

Proof. By Corollary 2.14, $P_{M,F,\sigma}$ contains $\{(p, p, \ldots, p) \in (\beta\mathbb{N})^k : p$ is a minimal idempotent of $\beta\mathbb{N}\}$ so Lemma 4.4 applies. □

If $M = \begin{pmatrix}1 & 1\\ 1 & 2\end{pmatrix}$ and F is a finite nonempty set of S_0-preserving homomorphisms from S_n to S_0, then by Theorem 4.6 we have that $P_{M,F,\sigma}$ contains $\{(p, p) : p$ is a minimal idempotent of $\beta\mathbb{N}\}$ but by Theorem 2.15, $P_{M,F,\sigma}$ does not contain $\{(p_1, p_2) : p_1$ and p_2 are minimal idempotents of $\beta\mathbb{N}\}$.

Given a finite coloring of a semigroup, at least one of the color classes must be piecewise syndetic, so results concluding that piecewise syndetic sets have a certain property guarantee the corresponding conclusion for finite colorings. We see now a situation where the conclusions are equivalent – a fact that has interesting consequences for both versions.

Theorem 4.7. *Let $n \in \mathbb{N}$, let τ be an S_0-independent homomorphism from S_n to $\mathbb{N}$, and let $B \subseteq \mathbb{N}$. The following statements are equivalent.*

(a) Whenever S_0 is finitely colored, there exists $w \in S_n$ such that $\{w(\vec{x}) : x \in \mathbb{A}^n\}$ is monochromatic and $\tau(w) \in B$.

(b) Whenever D is a piecewise syndetic subset of S_0, there exists $w \in S_n$ such that $\{w(\vec{x}) : \vec{x} \in \mathbb{A}^n\} \subseteq D$ and $\tau(w) \in B$.

Proof. It is trivial that (b) implies (a), so assume that (a) holds and let D be a piecewise syndetic subset of S_0. Note that for each $r \in \mathbb{N}$, there is some $m \in \mathbb{N}$ such that whenever the length m words in S_0 are r-colored, there is some $w \in S_n$ of length m such that $\{w(\vec{x}) : \vec{x} \in \mathbb{A}^n\}$ is monochromatic and $\tau(w) \in B$. (If there is a bad r-coloring φ_m of the length m words for each m then $\bigcup_{m=1}^{\infty} \varphi_m$ is a bad r-coloring of S_0.)

Since D is piecewise syndetic, pick finite nonempty $G \subseteq S_0$ such that for every finite nonempty subset H of S_0 there exists $s \in S_0$ with $Hs \subseteq \bigcup_{t \in G} t^{-1}D$. Let $r = |G|$ and pick $m \in \mathbb{N}$ such that whenever the length m words in S_0 are r-colored, there is some $w \in S_n$ such that $\{w(\vec{x}) : \vec{x} \in \mathbb{A}^n\}$ is monochromatic and $\tau(w) \in B$. Let H be the set of length m words in S_0 and pick $s \in S_0$ such that $Hs \subseteq \bigcup_{t \in G} t^{-1}D$. For $u \in H$ pick $\varphi(u) \in G$ such that $us \in \varphi(u)^{-1}G$. Pick $w \in S_n$ of length m and $t \in G$ such that for all $\vec{x} \in \mathbb{A}^n$, $\varphi\big(w(\vec{x})\big) = t$ and $\tau(w) \in B$. Let $w' = tws$. Then for $\vec{x} \in \mathbb{A}^n$, $w'(\vec{x}) = t\big(w(\vec{x})\big)s \in D$ and $\tau(w') = \tau(w) \in B$. □

If $n = 1$, the following corollary yields the statement in the second paragraph of the abstract.

Corollary 4.8. *Let $n \in \mathbb{N}$, let τ be an S_0-independent homomorphism from S_n onto $\mathbb{N}$, and let $Q = \{p \in \beta\mathbb{N}$: whenever S_0 is finitely colored and $B \in p$, there exists $w \in S_n$ such that $\{w(\vec{x}) : \vec{x} \in \mathbb{A}^n\}$ is monochromatic and $\tau(w) \in B\}$. Then Q is a compact subsemigroup of $\beta\mathbb{N}$ containing all of the idempotents.*

Proof. Let $k = m = 1$, let $M = (1)$, and let $F = \{h_{\vec{x}} : \vec{x} \in \mathbb{A}^n\}$. By Corollary 3.8, pick $\sigma(1) \in \{1, 2, \ldots, n\}$ such that $\tau = \mu_{\sigma(1)}$. By Theorem 4.5, $P_{M,F,\sigma}$ is a compact subsemigroup of $\beta\mathbb{N}$ containing all of the idempotents and by Theorem 4.7, $Q = P_{M,F,\sigma}$. □

Recall that a set of sets $\mathcal{B}$ is said to be partition regular if whenever $\mathcal{F}$ is a finite set of sets and $\bigcup \mathcal{F} \in \mathcal{B}$, there exist $A \in \mathcal{F}$ and $B \in \mathcal{B}$ such that $B \subseteq A$.

Corollary 4.9. *Let $n \in \mathbb{N}$ and let τ be an S_0-independent homomorphism from S_n to $\mathbb{N}$. Let $\mathcal{B} = \{B \subseteq \mathbb{N}$: whenever D is a piecewise syndetic subset of S_0, there exists $w \in S_n$ such that $\{w(\vec{x}) : \vec{x} \in \mathbb{A}^n\} \subseteq D$ and $\tau(w) \in B\}$. Then $\mathcal{B}$ is partition regular.*

Proof. By Theorem 4.7, $\mathcal{B} = \{B \subseteq \mathbb{N}$: whenever S_0 is finitely colored, there exists $w \in S_n$ such that $\{w(\vec{x}) : x \in \mathbb{A}^n\}$ is monochromatic and $\tau(w) \in B\}$. It is routine to show that if $k \in \mathbb{N}$, $B_i \subseteq \mathbb{N}$ for each $i \in \{1, 2, \ldots, k\}$, and $\bigcup_{i=1}^k B_i$ has the property that whenever S_0 is finitely colored, there exists $w \in S_n$ such that $\{w(\vec{x}) : x \in \mathbb{A}^n\}$ is monochromatic and $\tau(w) \in \bigcup_{i=1}^k B_i$, then some $B_i \in \mathcal{B}$. □

Since the intersection of any collection of compact semigroups having the finite intersection property is a compact semigroup, it follows that there exists a smallest compact subsemigroup of $(\beta\mathbb{N})^k$ containing the idempotents of $(\beta\mathbb{N})^k$.

Question 4.10. *Let $k \in \mathbb{N}$, let M be the $k \times k$ identity matrix, and let σ be the identity function on $\{1, 2, \ldots, k\}$.*

(a) If $F = \{h_{\vec{x}} : \vec{x} \in \mathbb{A}^k\}$, is $P_{M,F,\sigma}$ the smallest compact subsemigroup of $(\beta\mathbb{N})^k$ containing the idempotents of $(\beta\mathbb{N})^k$?

(b) If not, does there exist a finite nonempty set F of S_0-preserving homomorphisms such that $P_{M,F,\sigma}$ is the smallest compact subsemigroup of $(\beta\mathbb{N})^k$ containing the idempotents of $(\beta\mathbb{N})^k$?

Question 4.11. *Let $k \in \mathbb{N}$ and let M and N be $k \times k$ lower triangular matrices with positive diagonal entries and nonpositive entries below the diagonal. Do there exist a finite nonempty set F of S_0-preserving homomorphisms from S_k to S_0 and a permutation σ of $\{1, 2, \ldots, k\}$ such that $P_{M,F,\sigma} \neq P_{N,F,\sigma}$?*

Because of Question 4.10, we are interested in the smallest compact subsemigroup of $(\beta\mathbb{N})^k$ containing the idempotents of $(\beta\mathbb{N})^k$.

Given a compact right topological semigroup T, we let $E(T)$ be the set of idempotents in T. If I is a set and for each $i \in I$, T_i is a compact right topological semigroup, then $E(\times_{i\in I} T_i) = \times_{i\in I} E(T_i)$ because the operation in $\times_{i\in I} T_i$ is coordinatewise. Also by [6, Theorem 2.23] $K(\times_{i\in I} T_i) = \times_{i\in I} K(T_i)$ so that $E\big(K(\times_{i\in I} T_i)\big) = \times_{i\in I} E\big(K(T_i)\big)$.

Definition 4.12. Let T be a compact right topological semigroup and let $A \subseteq T$. Then $J_T(A)$ is the smallest compact subsemigroup of T containing A.

We next show that $J_{(\beta\mathbb{N})^k}\big(E((\beta\mathbb{N})^k)\big) = \big(J_{\beta\mathbb{N}}(E(\beta\mathbb{N}))\big)^k$ for $k \in \mathbb{N}$ and that a similar result applies to the minimal idempotents. Notice that in general $J_{T_1\times T_2}(A_1 \times A_2) \subseteq J_{T_1}(A_1) \times J_{T_2}(A_2)$. But equality need not always hold even in the case that $T_1 = T_2$ and $A_1 = A_2$. For example, let A^+ be the free semigroup on the alphabet $A = \{a, b\}$, and $T = \beta A^+$. Then, identifying the letters of A with the length one words so that A is a subset of T, we have $J_T(A) \times J_T(A) = T \times T$ while $J_{T\times T}(A \times A) = c\ell_{T\times T}\{(u, w) \in A^+ \times A^+ : |u| = |w|\}$.

Theorem 4.13. *Let T_1 and T_2 be compact right topological semigroups and for $i \in \{1, 2\}$ let A_i be a nonempty subset of T_i with $A_i \subseteq \{ab : a, b \in A_i\}$. Then $J_{T_1\times T_2}(A_1 \times A_2) = J_{T_1}(A_1) \times J_{T_2}(A_2)$.*

Proof. As $J_{T_1}(A_1) \times J_{T_2}(A_2)$ is a compact subsemigroup of $T_1 \times T_2$ containing $A_1 \times A_2$ we have immediately that $J_{T_1 \times T_2}(A_1 \times A_2) \subseteq J_{T_1}(A_1) \times J_{T_2}(A_2)$. So it remains to show that $J_{T_1}(A_1) \times J_{T_2}(A_2) \subseteq J_{T_1 \times T_2}(A_1 \times A_2)$. Let $Y = \{q \in J_{T_2}(A_2) : (p, q) \in J_{T_1 \times T_2}(A_1 \times A_2) \text{ for all } p \in A_1\}$. Then Y is compact and $A_2 \subseteq Y$. Further, let $q_1, q_2 \in Y$ and $p \in A_1$, and write $p = p_1 p_2$ with $p_1, p_2 \in A_1$. Then $(p_1, q_1), (p_2, q_2) \in J_{T_1 \times T_2}(A_1 \times A_2)$ and hence $(p_1, q_1)(p_2, q_2) = (p, q_1 q_2) \in J_{T_1 \times T_2}(A_1 \times A_2)$. Thus Y is a compact subsemigroup of $J_{T_2}(A_2)$ containing A_2 so $Y = J_{T_2}(A_2)$.

Now let $X = \{x \in J_{T_1}(A_1) : \{x\} \times J_{T_2}(A_2) \subseteq J_{T_1 \times T_2}(A_1 \times A_2)\}$. Then X is compact and if $p \in A_1$, then $\{p\} \times J_{T_2}(A_2) = \{p\} \times Y \subseteq J_{T_1 \times T_2}(A_1 \times A_2)$, so $A_1 \subseteq X$. We next claim that X is a semigroup. In fact, let $x_1, x_2 \in X$ and set $Z = \{z \in J_{T_2}(A_2) : (x_1 x_2, z) \in J_{T_1 \times T_2}(A_1 \times A_2)\}$. Then Z is compact. Let $q \in A_2$ and write $q = q_1 q_2$ with $q_1, q_2 \in A_2$. Then $(x_1, q_1), (x_2, q_2) \in J_{T_1 \times T_2}(A_1 \times A_2)$ and hence $(x_1 x_2, q) \in J_{T_1 \times T_2}(A_1 \times A_2)$. Thus Z contains A_2. Finally, let $z_1, z_2 \in Z$. Then since $z_1, z_2 \in J_{T_2}(A_2)$ and $x_1, x_2 \in X$ we deduce that $(x_1, z_1), (x_2, z_2) \in J_{T_1 \times T_2}(A_1 \times A_2)$ implying that $(x_1 x_2, z_1 z_2) \in J_{T_1 \times T_2}(A_1 \times A_2)$ and hence $z_1 z_2 \in Z$. Thus Z is a compact subsemigroup of $J_{T_2}(A_2)$ containing A_2 and hence $Z = J_{T_2}(A_2)$ from which it follows that $x_1 x_2 \in X$. Having shown that X is compact subsemigroup of $J_{T_1}(A_1)$ containing A_1 we deduce that $X = J_{T_1}(A_1)$. In conclusion, $J_{T_1}(A_1) \times J_{T_2}(A_2) = X \times J_{T_2}(A_2) \subseteq J_{T_1 \times T_2}(A_1 \times A_2)$ as required. □

Notice in particular that if for $i \in \{1, 2\}$, A_i is a nonempty subset of $E(T_i)$, then $A_i \subseteq \{ab : a, b \in A_i\}$, so $J_{T_1 \times T_2}(A_1 \times A_2) = J_{T_1}(A_1) \times J_{T_2}(A_2)$.

Corollary 4.14. *Let $k \in \mathbb{N}$. The smallest compact subsemigroup of $(\beta\mathbb{N})^k$ containing the idempotents of $(\beta\mathbb{N})^k$ is $\big(J_{\beta\mathbb{N}}\big(E(\beta\mathbb{N})\big)\big)^k$. The smallest compact subsemigroup of $(\beta\mathbb{N})^k$ containing the minimal idempotents of $(\beta\mathbb{N})^k$ is $\big(J_{\beta\mathbb{N}}\big(E(K(\beta\mathbb{N}))\big)\big)^k$.*

Proof. By Theorem 4.13 and induction,

$$J_{(\beta\mathbb{N})^k}((E(\beta\mathbb{N}))^k) = \big(J_{\beta\mathbb{N}}\big(E(K(\beta\mathbb{N}))\big)\big)^k$$

and we already observed that the set of idempotents of $(\beta\mathbb{N})^k$ is $\big(E(\beta\mathbb{N})\big)^k$. The second conclusion is essentially the same. □

We note now that the version of Theorem 4.13 for infinite products is also valid.

Theorem 4.15. *Let I be an infinite set. For each $i \in I$, let T_i be a compact right topological semigroup and let A_i be a nonempty subset of T_i such that $A_i \subseteq \{ab : a, b \in A_i\}$. Then $J_{\times_{i \in I} T_i}(\times_{i \in I} A_i) = \times_{i \in I} J_{T_i}(A_i)$.*

Proof. Let $Y = \times_{i\in I} T_i$. For each $i \in I$, choose $e_i \in A_i$. Given $F \in \mathcal{P}_f(I)$, let $Y_F = \times_{i\in F} T_i$, let $Z_F = \times_{i\in I\setminus F} T_i$, let

$$X_F = \{\vec{x} \in \times_{i\in I} J_{T_i}(A_i) : (\forall i \in I \setminus F)(x_i = e_i)\}\,,$$

and let $B_F = \{\vec{x} \in \times_{i\in I} A_i : (\forall i \in I \setminus F)(x_i = e_i)\}$.

We shall show that for each $F \in \mathcal{P}_f(I)$, $X_F \subseteq J_Y(\times_{i\in I} A_i)$. Let $F \in \mathcal{P}_f(I)$ be given. Now X_F is topologically and algebraically isomorphic to

$$\times_{i\in F} J_{T_i}(A_i) \times \times_{i\in I\setminus F}\{e_i\}\,,$$

B_F is topologically and algebraically isomorphic to $\times_{i\in F} A_i \times \times_{i\in I\setminus F}\{e_i\}$, and $\times_{i\in I\setminus F}\{e_i\} \subseteq J_{Z_F}(\times_{i\in I\setminus F}\{e_i\})$. So using Theorem 4.13 we have

$$\begin{aligned} X_F &\approx \times_{i\in F} J_{T_i}(A_i) \times \times_{i\in I\setminus F}\{e_i\} \\ &\subseteq J_{Y_F}(\times_{i\in F} A_i) \times J_{Z_F}(\times_{i\in I\setminus F}\{e_i\}) \\ &= J_{Y_F\times Z_F}(\times_{i\in F} A_i \times \times_{i\in I\setminus F}\{e_i\}) \\ &\approx J_Y(B_F) \\ &\subseteq J_Y(\times_{i\in I} A_i)\,. \end{aligned}$$

Next we claim that $\times_{i\in I} J_{T_i}(A_i) \subseteq c\ell_Y \bigcup_{F\in\mathcal{P}_f(I)} X_F$. To see this, let $\vec{z} \in \times_{i\in I} J_{T_i}(A_i)$ and let U be a neighborhood of $\vec{z}$ in Y. Pick $F \in \mathcal{P}_f(I)$ and for each $i \in F$, pick a neighborhood V_i of z_i in T_i such that $\bigcap_{i\in F} \pi_i^{-1}[V_i] \subseteq U$. Define $\vec{x} \in Y$ by $x_i = \begin{cases} z_i & \text{if } i \in F \\ e_i & \text{if } i \in I \setminus F\,. \end{cases}$ Then $\vec{x} \in U \cap X_F$. Therefore $\times_{i\in I} J_{T_i}(A_i) \subseteq J_Y(\times_{i\in I} A_i)$. Since $\times_{i\in I} J_{T_i}(A_i)$ is a compact semigroup containing $\times_{i\in I} A_i$, the reverse inclusion is immediate. □

The curious reader may wonder what the situation is with respect to the smallest semigroup containing a given set. Given a semigroup T and a nonempty subset A of T, let $J'_T(A)$ be the smallest subsemigroup of T containing A, that is the set of all finite products of members of A in any order allowing repetition. If T_1 and T_2 are any semigroups and A_1 and A_2 are nonempty subsets of T_1 and T_2 respectively such that $A_i \subseteq \{ab : a, b \in A_i\}$ for $i \in \{1, 2\}$, then $J'_{T_1\times T_2}(A_1 \times A_2) = J'_{T_1}(A_1) \times J'_{T_2}(A_2)$. This follows from the proof of Theorem 4.13 by omitting all references to the topology.

However, the analogue of Theorem 4.15 need not hold. To see this, let T be the set of words over the alphabet $\{a_n : n \in \mathbb{N}\}$ that have no adjacent occurrences of any one letter. Given $u, w \in T$, then $u \cdot w$ is ordinary concatenation unless $u = xa_n$ and $w = a_n y$ for some $n \in \mathbb{N}$ and some $x, y \in T \cup \{\emptyset\}$, in which case $u \cdot w = xa_n y$. Let A be the set of idempotents in T, that is A is the set of length one words. Then $J'_T(A) = T$ but $\{\vec{x} \in \times_{n=1}^{\infty} T : \{|x_n| : n \in \mathbb{N}\}$ is bounded$\}$ is a proper subsemigroup of $\times_{n=1}^{\infty} T$ containing the idempotents.

5 Compact ideals of $(\beta S)^k$

In this section we deal with results related to the Hales-Jewett Theorem and its extensions by themselves. The first result here is motivated by the following theorem characterizing image partition regular matrices.

Theorem 5.1. *Let $k, m \in \mathbb{N}$ and let M be a $k \times m$ matrix with entries from $\mathbb{Q}$. The following statements are equivalent.*

(a) M is image partition regular over $\mathbb{N}$.

(b) For every central subset D of $\mathbb{N}$, there exists $\vec{x} \in \mathbb{N}^m$ such that $M\vec{x} \in D^k$.

(c) For every central subset D of $\mathbb{N}$, $\{\vec{x} \in \mathbb{N}^m : M\vec{x} \in D^k\}$ is central in $\mathbb{N}^m$.

Proof. These are statements (a), (h), and (i) of [6, Theorem 15.24]. $\square$

As a consequence of Theorem 2.10 (with $k = m = 1$, $T = \mathbb{N}$, $M = (1)$, $\tau = \mu_1$, $F = \{h_{\vec{x}} : \vec{x} \in \mathbb{A}^n\}$, and p any idempotent in $\beta\mathbb{N}$) we have that whenever D is piecewise syndetic in S_0 and $n \in \mathbb{N}$, there exists $w \in S_n$ such that $\{w(\vec{x}) : \vec{x} \in \mathbb{A}^n\} \subseteq D$. Since whenever S_0 is finitely colored, one color class must be piecewise syndetic, we see that Theorem 1.3 follows. And, since central sets are piecewise syndetic, we have that whenever D is central in S_0 and $n \in \mathbb{N}$, there exists $w \in S_n$ such that $\{w(\vec{x}) : \vec{x} \in \mathbb{A}^n\} \subseteq D$.

Theorem 5.2. *Let $n \in \mathbb{N}$ and let D be a central subset of S_0. Let F be a finite nonempty set of S_0-preserving homomorphisms from S_n into S_0. Then $\{w \in S_n : (\forall \nu \in F)(\nu(w) \in D)\}$ is central in S_n.*

Proof. Let $T = S_n \cup S_0$ and extend each $\nu \in F$ to all of T by defining ν to be the identity on S_0. By Theorem 1.5(2), pick a central subset Q of T such that for each $t \in Q$, $\{\nu(t) : \nu \in F\} \subseteq D\}$. Since S_n is an ideal of T, $Q \cap S_n$ is central in S_n and $Q \cap S_n \subseteq \{w \in S_n : (\forall \nu \in F)(\nu(w) \in D)\}$. $\square$

We conclude this section by investigating ideals related to the extensions of the Hales-Jewett Theorem.

Definition 5.3. For $n \in \mathbb{N}$,

$$R_n = \{p \in \beta S_0 : (\forall B \in p)(\exists w \in S_n)(\{w(\vec{x}) : \vec{x} \in \mathbb{A}^n\} \subseteq B)\}.$$

There are numerous ways to use known results to show that each $R_n \neq \emptyset$. From the point of view of this paper, it is probably easiest to invoke Theorem 2.10 as discussed above.

Theorem 5.4. *Let $n \in \mathbb{N}$. Then R_n is a compact two sided ideal of βS_0.*

Proof. We have that $R_n \neq \emptyset$ and it is trivially compact. Let $p \in R_n$ and let $q \in \beta S_0$. To see that R_n is a left ideal, let $B \in qp$. Pick $u \in S_0$ such that $u^{-1}B \in p$ and pick $w \in S_n$ such that $\{w(\vec{x}) : \vec{x} \in \mathbb{A}^n\} \subseteq u^{-1}B$. Then $uw \in S_n$ and $\{(uw)(\vec{x}) : \vec{x} \in \mathbb{A}^n\} \subseteq B$.

To see that R_n is a right ideal, let $B \in pq$. Pick $w \in S_n$ such that $\{w(\vec{x}) : \vec{x} \in \mathbb{A}^n\} \subseteq \{u \in S_0 : u^{-1}B \in q\}$. Pick $u \in \bigcap_{\vec{x} \in \mathbb{A}^n} w(\vec{x})^{-1}B$. Then $wu \in S_n$ and $\{(wu)(\vec{x}) : \vec{x} \in \mathbb{A}^n\} \subseteq B$. □

Theorem 5.5. *Let $n \in \mathbb{N}$. Then $R_{n+1} \subseteq R_n$.*

Proof. Let $p \in R_{n+1}$ and let $B \in p$. Pick $w \in S_{n+1}$ such that $\{w(\vec{x}) : \vec{x} \in \mathbb{A}^{n+1}\} \subseteq B$. Define $u \in S_n$ by $u = w(v_1, v_2, \ldots, v_n, v_n)$. Then given $\vec{x} \in \mathbb{A}^n$, $u(\vec{x}) = w(x_1, x_2, \ldots, x_n, x_n) \in B$. □

Lemma 5.6. *For each $r, n \in \mathbb{N}$ there exists $m \in \mathbb{N}$ such that for all $k \geq m$, if $S\binom{k}{0}$ is r-colored, there exists $w \in S\binom{k}{n}$ such that $\{w(\vec{x}) : \vec{x} \in \mathbb{A}^n\}$ is monochromatic.*

Proof. Let $r, n \in \mathbb{N}$. By Theorem 1.3, whenever S_0 is r-colored, there exists $w \in S_n$ such that $\{w(\vec{x}) : \vec{x} \in \mathbb{A}^n\}$ is monochromatic. As in the proof of Theorem 4.7, pick $m \in \mathbb{N}$ such that whenever $S\binom{m}{0}$ is r-colored, there exists $w \in S\binom{m}{n}$ such that $\{w(\vec{x}) : \vec{x} \in \mathbb{A}^n\}$ is monochromatic. Let $k > m$ and pick $c \in \mathbb{A}$. Let $\varphi : S\binom{k}{0} \to \{1, 2, \ldots, r\}$ and define $\psi : S\binom{m}{0} \to \{1, 2, \ldots, r\}$ by $\psi(u) = \varphi(uc^{k-m})$. Pick $w \in S\binom{m}{n}$ such that ψ is constant on $\{w(\vec{x}) : \vec{x} \in \mathbb{A}^n\}$. Define $u \in S\binom{k}{n}$ by $u = wc^{k-m}$. Then φ is constant on $\{u(\vec{x}) : \vec{x} \in \mathbb{A}^n\}$. □

Theorem 5.7. *$c\ell K(\beta S_0) \subsetneq \bigcap_{n=1}^{\infty} R_n$.*

Proof. That $c\ell K(\beta S_0) \subseteq \bigcap_{n=1}^{\infty} R_n$ is an immediate consequence of Theorem 5.4.

Let $B = \bigcup_{k=1}^{\infty} S\binom{k!}{0}$. We claim first that B is not piecewise syndetic, so that $\overline{B} \cap c\ell K(\beta S_0) = \emptyset$. We need to show that there is no $G \in \mathcal{P}_f(S_0)$ such that for all $F \in \mathcal{P}_f(S_0)$ there exists $x \in S_0$ such that $Fx \subseteq \bigcup_{t \in G} t^{-1}B$. Suppose we have such G and let $m = \max\{|t| : t \in G\}$, let $r = m!$, pick $b \in \mathbb{A}$, and let $F = \{b^r, b^{2r}\}$. Pick $t, s \in G$ and $x \in S_0$ such that $tb^r x \in B$ and $sb^{2r}x \in B$. Then $|tb^r x| = n!$ for some $n > m$ and $|sb^{2r}x| = k!$ for some k. Now $k! = |sb^{2r}x| = |sb^r| + n! - |t| > n!$ so $k! \geq (n+1)!$ so $|sb^r| + n! - |t| \geq (n+1)!$. Thus $n \cdot n! = (n+1)! - n! < |sb^r| = |s| + r \leq m + m! < n + n!$, a contradiction.

Now let $n \in \mathbb{N}$. We will show that $\overline{B} \cap R_n \neq \emptyset$. Let $\mathcal{R} = \{D \subseteq S_0 :$ whenever D is finitely colored, there exists $w \in S_n$ such that $\{w(\vec{x}) : \vec{x} \in \mathbb{A}^n\}$ is monochromatic$\}$. Notice that $\mathcal{R}$ is partition regular. It suffices to show that $B \in \mathcal{R}$, for then by [6, Theorem 3.11] there exists $p \in \beta S_0$ such that $B \in p$ and $p \subseteq \mathcal{R}$ so that $p \in \overline{B} \cap R_n$. So let $r \in \mathbb{N}$ and let $\varphi : B \to \{1, 2, \ldots, r\}$. Pick m as guaranteed by Lemma 5.6 for r and n. The φ is an r-coloring of $S\binom{m!}{0}$ so pick $w \in S\binom{m!}{n}$ such that $\{w(\vec{x}) : \vec{x} \in \mathbb{A}^n\}$ is monochromatic.

Since $\{\overline{B} \cap R_n : n \in \mathbb{N}\}$ is a collection of closed sets with the finite intersection property, we have that $\overline{B} \cap \bigcap_{n=1}^{\infty} R_n \neq \emptyset$. □

Theorem 5.8 (Deuber, Prömel, Rothschild, and Voigt)**.** *Let $n, r \in \mathbb{N}$. There exist $m \in \mathbb{N}$ and $C_{n,r} \subseteq S\binom{m}{0}$ such that*

(1) there does not exist $w \in S\binom{m}{n+1}$ with $\{w(\vec{x}) : \vec{x} \in \mathbb{A}^{n+1}\} \subseteq C_{n,r}$ and

(2) whenever $C_{n,r}$ is r-colored, there exists $w \in S\binom{m}{n}$ such that $\{w(\vec{x}) : \vec{x} \in \mathbb{A}^n\}$ is monochromatic.

Proof. This is the "main theorem" of [3]. □

Theorem 5.9. *Let $n \in \mathbb{N}$. Then $R_{n+1} \subsetneq R_n$.*

Proof. For each $r \in \mathbb{N}$ pick $m(r)$ and $C_{n,r}$ as guaranteed for r and n by Theorem 5.8. Choose an increasing sequence $\langle r_i \rangle_{i=1}^{\infty}$ such that the sequence $\langle m(r_i) \rangle_{i=1}^{\infty}$ is strictly increasing and let $D_i = C_{n,r_i}$ for each i. Let $E = \bigcup_{i=1}^{\infty} D_i$. There does not exist $w \in S_{n+1}$ such that $\{w(\vec{x}) : \vec{x} \in \mathbb{A}^{n+1}\} \subseteq E$ because any such w would have to have length $m(r_i)$ for some i, and then one would have $\{w(\vec{x}) : \vec{x} \in \mathbb{A}^{n+1}\} \subseteq C_{n,r_i}$. Thus $\overline{E} \cap R_{n+1} = \emptyset$.

As in the proof of Theorem 5.7, let $\mathcal{R} = \{D \subseteq S_0 :$ whenever D is finitely colored, there exists $w \in S_n$ such that $\{w(\vec{x}) : \vec{x} \in \mathbb{A}^n\}$ is monochromatic$\}$. It suffices to show that $E \in \mathcal{R}$ so let $k \in \mathbb{N}$ and let $\varphi : E \to \{1, 2, \ldots, k\}$. Pick i such that $r_i \geq k$. Then $\varphi_{|D_i} : D_i \to \{1, 2, \ldots, r_i\}$ so pick $w \in S\binom{m(r_i)}{n}$ such that φ is constant on $\{w(\vec{x}) : \vec{x} \in \mathbb{A}^n\}$. □

6 A simpler proof of an infinitary extension

We set out in this section to provide a proof of [2, Theorem 2.12] applied to the simpler description of n-variable words which we have been using. As defined in this paper, what is called the set of n-variable words in [2], is what we call the set of strong n-variable words where we take $D = E = \{e\}$ in [2], take the function T_e to be the identity, and let $v_n = (e, \nu_n)$ for $n \in \mathbb{N}$. As we remarked earlier, in [2, Theorem 5.1] it was shown that the version of the Graham-Rothschild that we are using here is sufficient to derive the full original version as used in [2] and [4]. Using that simplified notion, Corollary 6.13 implies [2, Theorem 2.12] and has a vastly simpler proof.

The first few results apply to an arbitrary nonempty alphabet $\mathbb{A}$. For the results of this section, except for Corollary 6.14, we do not need to assume that $\mathbb{A}$ is finite.

Definition 6.1. For $n \in \mathbb{N}$, T_n is the free semigroup over $\mathbb{A} \cup \{v_1, v_2, \ldots, v_n\}$. Also we set $T_0 = S_0$ and $T = \bigcup_{i\in\omega} T_i$.

Note that for $n \in \mathbb{N}$, S_n is a proper subset of T_n and that $T_n \subseteq T_{n+1}$.

For $u = l_1 l_2 \cdots l_m \in T$ with $|u| = m$ we define $h_u : T \to T$ by stating that $h_u(w)$ is the result of replacing each occurrence of v_i in w by l_i for $i \in \{1, 2, \ldots, m\}$. (Thus, if $w \in S_m$, $h_u(w) = w(u)$ as defined in Definition 1.2.) Denote also by h_u the continuous extension of h_u taking βT to βT. Observe that, if $u \in \widetilde{S}\binom{m}{k}$, then $h_u[T_m] \subseteq T_k$.

Definition 6.2. For $\alpha \in \mathbb{N} \cup \{\omega\}$, a reductive sequence of height α over $\mathbb{A}$ is a sequence of minimal idempotents $\langle p_t \rangle_{t<\alpha}$ with $p_t \in E\big(K(\beta S_t)\big)$ such that for each $i, j \in \omega$ with $0 \le j < i < \alpha$ one has $p_i \le p_j$ and $h_u(p_i) = p_j$ for each $u \in \widetilde{S}\binom{i}{j}$.

Lemma 6.3. *Let $i \in \omega$. Then $K(\beta S_i) = K(\beta T_i)$.*

Proof. We have that S_i is an ideal of T_i so by [6, Corollary 4.18] βS_i is an ideal of βT_i. Therefore $K(\beta T_i) \subseteq \beta S_i$ so that by [6, Theorem 1.65] $K(\beta S_i) = K(\beta T_i)$. □

Lemma 6.4. *Let $k, m \in \omega$ with $k < m$ and let $p \in E(\beta S_k)$. There exists $q \in E\big(K(\beta S_m)\big)$ such that $q < p$.*

Proof. We have $\beta S_m \cup \beta S_k \subseteq T_m$. Pick $q \in E\big(K(\beta T_m)\big)$ such that $q \le p$. By Lemma 6.3, $q \in K(\beta S_m)$ and since $\beta S_k \cap \beta S_m = \emptyset$, $q \neq p$. □

Lemma 6.5. *Let $\alpha \in \mathbb{N} \cup \{\omega\}$ and let $\langle p_t \rangle_{t<\alpha}$ be a reductive sequence of height α. For each $t < \alpha$, $p_t \in E\big(K(\beta \widetilde{S}_t)\big)$.*

Proof. Since each p_t is an idempotent, it suffices to show that $p_t \in K(\beta \widetilde{S}_t)$. Given $t < \alpha$, $\widetilde{S}_t$ is a right ideal of S_t so $\beta \widetilde{S}_t$ is a right ideal of βS_t and thus $K(\beta \widetilde{S}_t) \cap K(\beta S_t) \neq \emptyset$ so that by [6, Theorem 1.65], $K(\beta \widetilde{S}_t) = \beta \widetilde{S}_t \cap K(\beta S_t)$.

Thus it suffices to show that each $p_t \in \beta\widetilde{S}_t$. We proceed by induction on t. For $t = 0$, we have $p_0 \in \beta S_0 = \beta\widetilde{S}_0$. Now assume that $t+1 < \alpha$ and $p_t \in \beta\widetilde{S}_t$. We need to show that $\widetilde{S}_{t+1} \in p_{t+1}$. We begin by observing that if $w \in \widetilde{S}_t$ then $S_{t+1} \subseteq w^{-1}\widetilde{S}_{t+1}$ from which it follows that $\widetilde{S}_t \subseteq \{w \in T_{t+1} : w^{-1}\widetilde{S}_{t+1} \in p_{t+1}\}$. Now since $\widetilde{S}_t \in p_t$ we have that $\{w \in T_{t+1} : w^{-1}\widetilde{S}_{t+1} \in p_{t+1}\} \in p_t$ or equivalently that $\widetilde{S}_{t+1} \in p_t p_{t+1}$. The result now follows from the fact that $p_{t+1} \leq p_t$ and hence in particular $p_{t+1} = p_t p_{t+1}$. □

We now introduce some new notation. We fix a nonempty (possibly infinite) alphabet $\mathbb{A}$ together with an infinite sequence of symbols $\{x_1, x_2, x_3, \ldots\}$ each of which is not a member of $\mathbb{A} \cup \{v_i : i \in \mathbb{N}\}$. We let $\mathbb{A}^{(0)} = \mathbb{A}$ and for $m \in \mathbb{N}$, we let $\mathbb{A}^{(m)} = \mathbb{A} \cup \{x_1, x_2, \ldots, x_m\}$. For each $m \in \omega$ we let $S^{(m)}$ denote the free semigroup over $\mathbb{A}^{(m)}$. For each $i \in \mathbb{N}$ we let $S_i^{(m)}$ denote the set of all i-variable words over the alphabet $\mathbb{A}^{(m)}$ and let $\widetilde{S}_i^{(m)}$ denote the set of all strong i-variable words over $\mathbb{A}^{(m)}$. $T_i^{(j)}$ will denote the free semigroup of all words over the alphabet $\mathbb{A}^{(j)} \cup \{v_1, v_2, \ldots, v_i\}$. Also, for each $j \in \omega$, let $S_0^{(j)} = \widetilde{S}_0^{(j)} = T_0^{(j)} = S^{(j)}$, and let $T^{(j)} = \bigcup_{i=1}^{\infty} T_i^{(j)}$. Then $T^{(j)}$ is the set of all words over $\mathbb{A}^{(j)} \cup \{v_i : i \in \mathbb{N}\}$.

To each $u = u_1 u_2 \cdots u_m \in T^{(j)}$ with $|u| = m$ we associate a morphism $h_u : \bigcup_{j\in\omega} T^{(j)} \to \bigcup_{j\in\omega} T^{(j)}$ where for each $w \in \bigcup_{j\in\omega} T^{(j)}$, $h_u(w)$ is obtained from w by replacing each occurrence of v_i in w by u_i for each $i \in \{1, 2, \ldots, m\}$. We also denote by h_u its continuous extension taking $\beta(\bigcup_{j\in\omega} T^{(j)})$ to $\beta(\bigcup_{j\in\omega} T^{(j)})$. Also, for each $i, j \in \mathbb{N}$ we define the morphism, $\tau_i^{(j)} : T^{(j)} \to T^{(j-1)}$ where $\tau_i^{(j)}(w)$ is the word obtained from w by replacing every occurrence of x_j by v_i and leaving all other symbols unchanged. We also denote by $\tau_i^{(j)}$ the continuous extension of $\tau_i^{(j)}$ taking $\beta T^{(j)}$ to $\beta T^{(j-1)}$. Note that $\tau_i^{(j)}[T_{i-1}^{(j)}] = T_i^{(j-1)}$ and the restriction of $\tau_i^{(j)}$ to $T_{i-1}^{(j)}$ is an isomorphism onto $T_i^{(j-1)}$. Consequently the restriction of $\tau_i^{(j)}$ to $\beta T_{i-1}^{(j)}$ is an isomorphism onto $\beta T_i^{(j-1)}$.

Lemma 6.6. *Let $m \in \omega$, let $i \in \mathbb{N} \setminus \{1\}$, and assume that $p_{i-1}^{(m)} \in \beta S_{i-1}^{(m)}$ and $p_{i-1}^{(m+1)} \in K(\beta S_{i-1}^{(m+1)})$. Then*

$$G_i^{(m)} = p_{i-1}^{(m)} \tau_i^{(m+1)}(p_{i-1}^{(m+1)}) \beta S_i^{(m)} \cap \beta S_i^{(m)} \tau_i^{(m+1)}(p_{i-1}^{(m+1)}) p_{i-1}^{(m)}$$

is a group contained in $K(\beta S_i^{(m)})$.

Proof. We will show that $G_i^{(m)}$ is the intersection of a minimal right ideal and a minimal left ideal of $\beta S_i^{(m)}$ and hence by [6, Theorem 1.61], $G_i^{(m)}$ is a group contained in $K(\beta S_i^{(m)})$. Notice first that $\beta S_{i-1}^{(m)} \cup \beta S_i^{(m)} \subseteq \beta T_i^{(m)}$ so that the products $p_{i-1}^{(m)} \tau_i^{(m+1)}(p_{i-1}^{(m+1)})$ and $\tau_i^{(m+1)}(p_{i-1}^{(m+1)}) p_{i-1}^{(m)}$ are computed in $\beta T_i^{(m)}$.

Since $\tau_i^{(m+1)}$ is an isomorphism on $\beta T_{i-1}^{(m+1)}$, we have $\tau_i^{(m+1)}[K(\beta T_{i-1}^{(m+1)})] = K(\beta T_i^{(m)})$. Since $p_{i-1}^{(m+1)} \in K(\beta S_{i-1}^{(m+1)})$ and $K(\beta S_{i-1}^{(m+1)}) = K(\beta T_{i-1}^{(m+1)})$

(Lemma 6.3 with the underlying alphabet taken to be $\mathbb{A}^{(m+1)}$), it follows that $\tau_i^{(m+1)}(p_{i-1}^{(m+1)}) \in K(\beta T_i^{(m)})$. Since $p_{i-1}^{(m)} \in \beta T_{i-1}^{(m)} \subseteq \beta T_i^{(m)}$, we have that $p_{i-1}^{(m)}\tau_i^{(m+1)}(p_{i-1}^{(m+1)}) \in K(\beta T_i^{(m)}) = K(\beta S_i^{(m)})$ so that $p_{i-1}^{(m)}\tau_i^{(m+1)}(p_{i-1}^{(m+1)})\beta S_i^{(m)}$ is a minimal right ideal of $\beta S_i^{(m)}$. Similarly $\beta S_i^{(m)}\tau_i^{(m+1)}(p_{i-1}^{(m+1)})p_{i-1}^{(m)}$ is a minimal left ideal of $\beta S_i^{(m)}$. □

Definition 6.7. For $\alpha \in \mathbb{N}\cup\{\omega\}$, a *reductive array of height* α over $\mathbb{A}$ is an $\alpha\times\omega$ array of minimal idempotents $\langle p_i^{(m)}\rangle_{i<\alpha}^{m<\omega}$ with $p_i^{(m)} \in E\big(K(\beta S_i^{(m)})\big)$ satisfying the following conditions:

1. For each $m \in \omega$ the sequence $\langle p_i^{(m)}\rangle_{i<\alpha}$ is a reductive sequence of height α over $\mathbb{A}^{(m)}$.
2. $\tau_1^{(m)}(p_0^{(m)}) = p_1^{(m-1)}$ for each $m \in \mathbb{N}$.
3. For each $m \in \omega$ and $i < \alpha$ with $i \geq 2$, $p_i^{(m)}$ is the identity of the group

$$G_i^{(m)} = p_{i-1}^{(m)}\tau_i^{(m+1)}(p_{i-1}^{(m+1)})\beta S_i^{(m)} \cap \beta S_i^{(m)}\tau_i^{(m+1)}(p_{i-1}^{(m+1)})p_{i-1}^{(m)}\,.$$

Definition 6.8. Let $i, j, m \in \omega$, with $j < i$. $X_{i,j}^{(m)}$ will denote the set of words in $\widetilde{S}^{(m)}\binom{i}{j}$ in which v_j occurs only as the last letter.

Lemma 6.9. *Let $m, n \in \mathbb{N}$ and let $\vec{p} = \langle p_0, p_1, p_2, \ldots, p_n\rangle$ be a reductive sequence of height $n+1$ over $\mathbb{A}^{(m)}$. Let p_{n+1} be a minimal idempotent in $\beta S_{n+1}^{(m)}$ for which $p_{n+1} \leq p_n$. Let $j \in \omega$ with $j \leq n$ and let $u = u_1u_2\cdots u_{n+1} \in \widetilde{S}^{(m)}\binom{n+1}{j}$.*

(1) If $u \notin X_{n+1,j}^{(m)}$, then $h_u(p_{n+1}) = p_j$.

(2) If $u \in X_{n+1,j}^{(m)}$, then for $w \in T_n^{(m+1)}$, $h_u\big(\tau_{n+1}^{(m+1)}(w)\big) = \tau_j^{(m+1)}\big(h_u(w)\big)$.

Proof. (1) Assume $u \notin X_{n+1,j}^{(m)}$. We have that $h_u(p_{n+1})$ and $h_u(p_n)$ are both idempotents in $\beta S_j^{(m)}$ and $h_u(p_{n+1}) \leq h_u(p_n)$ because h_u is a homomorphism. Assume first that $j < n$ and let $s = u_1u_2\cdots u_n$. Then $s \in \widetilde{S}^{(m)}\binom{n}{j}$ so $h_s(p_n) = p_j$ and since h_s and h_u agree on $S_n^{(m)}$, $h_u(p_n) = p_j$. If $j = n$, then $u = v_1v_2\cdots v_nu_{n+1}$ so h_u is the identity on $S_n^{(m)}$ and again $h_u(p_n) = p_j$. Consequently, $h_u(p_{n+1}) \leq p_j$ and p_j is minimal in $\beta S_j^{(m)}$ so $h_u(p_{n+1}) = p_j$.

(2) It suffices to show that $h_u\big(\tau_{n+1}^{(m+1)}(l)\big) = \tau_j^{(m+1)}\big(h_u(l)\big)$ for each $l \in \mathbb{A}^{(m+1)}\cup\{v_1, v_2, \ldots, v_n\}$. Now $h_u\big(\tau_{n+1}^{(m+1)}(x_{m+1})\big) = h_u(v_{n+1}) = u_{n+1} = v_j$ and $\tau_j^{(m+1)}\big(h_u(x_{m+1})\big) = \tau_j^{(m+1)}(x_{m+1}) = v_j$. If $l \in \mathbb{A}^{(m)}$, then both sides leave

l fixed. Finally if $i \in \{1, 2, \ldots, n\}$, then $h_u\big(\tau_{n+1}^{(m+1)}(v_i)\big) = h_u(v_i) = u_i$ and $\tau_j^{(m+1)}\big(h_u(v_i)\big) = \tau_{n+1}^{(m+1)}(u_i) = u_i$ because $u_i \neq x_{m+1}$. □

Lemma 6.10. *Let $q \in E\big(K(\beta S_0)\big)$ and let $r \in E\big(K(\beta S_1)\big)$ such that $r < q$. There is a reductive array $\langle p_i^{(m)} \rangle_{i<2}^{m<\omega}$ of height 2 over $\mathbb{A}$ such that $p_0^{(0)} = q$ and $p_1^{(0)} = r$.*

Proof. Let $p_0^{(0)} = q$ and $p_1^{(0)} = r$. Let $m \in \mathbb{N}$ and assume that we have chosen $\langle p_i^{(t)} \rangle_{i<2}^{t<m}$ such that for each $t < m$ and each $i \in \{0, 1\}$, $p_i^{(t)} \in E\big(K(\beta S_i^{(t)})\big)$ and $p_1^{(t)} < p_0^{(t)}$. By Lemma 6.3, $p_1^{(m-1)} \in K(\beta T_1^{(m-1)})$. Since $\tau_1^{(m)}$ is an isomorphism from $\beta T_0^{(m)}$ onto $\beta T_1^{(m-1)}$, we may let $p_0^{(m)}$ be the unique member of $E\big(K(\beta T_0^{(m)})\big)$ such that $\tau_1^{(m)}(p_0^{(m)}) = p_1^{(m-1)}$. By Lemma 6.4, we may pick $p_1^{(m)} \in E\big(K(\beta S_1^{(m)})\big)$ such that $p_1^{(m)} < p_0^{(m)}$.

We need to show that for each $u \in \widetilde{S}^{(m)}\binom{1}{0}$, $h_u(p_1^{(m)}) = p_0^{(m)}$ so let $u \in \widetilde{S}^{(m)}\binom{1}{0}$. Now $p_0^{(m)}$ and $p_1^{(m)}$ are in $\beta T_1^{(m)}$, $p_1^{(m)} \leq p_0^{(m)}$, and h_u is a homomorphism so $h_u(p_1^{(m)}) \leq h_u(p_0^{(m)})$. Since h_u is the identity on $\beta T_0^{(m)}$, $h_u(p_0^{(m)}) = p_0^{(m)}$ so $h_u(p_1^{(m)}) \leq p_0^{(m)}$. Since $p_0^{(m)}$ is minimal in $\beta T_0^{(m)}$, $h_u(p_1^{(m)}) = p_0^{(m)}$ as required. □

Theorem 6.11. *Let $n \in \mathbb{N}$ and assume that $\langle p_i^{(m)} \rangle_{i<n+1}^{m<\omega}$ is a reductive array of height $n+1$ over $\mathbb{A}$. There exist unique $p_{n+1}^{(m)}$ for each $m < \omega$ such that $\langle p_i^{(m)} \rangle_{i<n+2}^{m<\omega}$ is a reductive array of height $n+2$ over $\mathbb{A}$.*

Proof. For each $m < \omega$, let $p_{n+1}^{(m)}$ be the identity of the group $G_{n+1}^{(m)}$. This is required by Definition 6.7(3), so the uniqueness is satisfied. Let $m < \omega$ be given. We need to show that $\langle p_i^{(m)} \rangle_{i<n+2}$ is a reductive sequence over $\mathbb{A}^{(m)}$. Since $G_{n+1}^{(m)} \subseteq p_n^{(m)} \beta S_{n+1}^{(m)} \cap \beta S_{n+1}^{(m)} p_n^{(m)}$ we have that $p_{n+1}^{(m)} \leq p_n^{(m)}$. And by Lemma 6.6 we have that $p_{n+1}^{(m)} \in E\big(K(\beta S_{n+1}^{(m)})\big)$.

Now let $0 \leq j < i < n+2$ and let $u \in \widetilde{S}^{(m)}\binom{i}{j}$. We need to show that $h_u(p_i^{(m)}) = p_j^{(m)}$. If $i < n+1$, this holds by assumption, so assume that $i = n+1$ so that $u \in \widetilde{S}^{(m)}\binom{n+1}{j}$. If $j = 0$, then h_u is the identity on $\beta S_0^{(m)}$ so $h_u(p_{n+1}^{(m)}) \leq h_u(p_0^{(m)}) = p_0^{(m)}$ and $h_u(p_{n+1}^{(m)}) \in \beta S_0^{(m)}$ so $h_u(p_{n+1}^{(m)}) = p_0^{(m)}$. So assume that $j \geq 1$. If $u \notin X_{n+1,j}^{(m)}$, then by Lemma 6.9, $h_u(p_{n+1}^{(m)}) = p_j^{(m)}$.

So we assume that $u = u_1 u_2 \cdots u_{n+1} \in X_{n+1,j}^{(m)}$ and let $s = u_1 u_2 \cdots u_n$.

Then we have that $s \in \widetilde{S}^{(m)}\binom{n}{j-1} \subseteq \widetilde{S}^{(m+1)}\binom{n}{j-1}$ and hence $h_u(p_n^{(m)}) = h_s(p_n^{(m)}) = p_{j-1}^{(m)}$ and $h_u(p_n^{(m+1)}) = h_s(p_n^{(m+1)}) = p_{j-1}^{(m+1)}$.

Combined with Lemma 6.9, we have that

$$h_u\big(\tau_{n+1}^{(m+1)}(p_n^{(m+1)})\big) = \tau_j^{(m+1)}\big(h_u(p_n^{(m+1)})\big) = \tau_j^{(m+1)}\big(p_{j-1}^{(m+1)}\big)\,.$$

So as $p_{n+1}^{(m)} \in G_{n+1}^{(m)} = p_n^{(m)}\tau_{n+1}^{(m+1)}(p_n^{(m+1)})\beta S_{n+1}^{(m)} \cap \beta S_{n+1}^{(m)}\tau_{n+1}^{(m+1)}(p_n^{(m+1)})p_n^{(m)}$ we deduce that $h_u(p_{n+1}^{(m)}) \in p_{j-1}^{(m)}\tau_j^{(m+1)}(p_{j-1}^{(m+1)})\beta S_j^{(m)} \cap \beta S_j^{(m)}\tau_j^{(m+1)}(p_{j-1}^{(m+1)})p_{j-1}^{(m)}$. If $j \geq 2$, this says that $h_u(p_{n+1}^{(m)})$ is an idempotent in $G_j^{(m)}$ and $p_j^{(m)}$ is the identity of $G_j^{(m)}$ so $h_u(p_{n+1}^{(m)}) = p_j^{(m)}$ as required.

Finally, assume that $j = 1$. Then

$$\begin{aligned} h_u(p_{n+1}^{(m)}) &\in p_0^{(m)}\tau_1^{(m+1)}(p_0^{(m+1)})\beta S_1^{(m)} \cap \beta S_1^{(m)}\tau_1^{(m+1)}(p_0^{(m+1)})p_0^{(m)} \\ &= p_0^{(m)}p_1^{(m)}\beta S_1^{(m)} \cap \beta S_1^{(m)}p_1^{(m)}p_0^{(m)} \\ &= p_1^{(m)}\beta S_1^{(m)} \cap \beta S_1^{(m)}p_1^{(m)}\,. \end{aligned}$$

Since $p_1^{(m)}$ is minimal in $\beta S_1^{(m)}$, $p_1^{(m)}\beta S_1^{(m)} \cap \beta S_1^{(m)}p_1^{(m)}$ is a group with identity $p_1^{(m)}$, so $h_u(p_{n+1}^{(m)}) = p_1^{(m)}$. □

Combining Lemma 6.10 and Theorem 6.11 we obtain:

Corollary 6.12. *For each $p \in E\big(K(\beta S_0)\big)$ there is a reductive array $\langle p_i^{(m)}\rangle_{i<\omega}^{m<\omega}$ of height ω such that $p_0^{(0)} = p$. Morever $p_1^{(0)}$ may be taken to be any minimal idempotent of βS_1 such that $p_1^{(0)} \leq p$.*

Corollary 6.13. *Let p be a minimal idempotent in βS_0. There is a sequence $\langle p_n\rangle_{n=0}^{\infty}$ such that*

(1) $p_0 = p$;

(2) for each $n \in \mathbb{N}$, p_n is a minimal idempotent of $\beta\widetilde{S}_n$;

(3) for each $n \in \mathbb{N}$, $p_n \leq p_{n-1}$;

(4) for each $n \in \mathbb{N}$, each $j \in \{0, 1, \ldots, n-1\}$, and each $u \in \widetilde{S}\binom{n}{j}$, $h_u(p_n) = p_j$.

Further, p_1 can be any minimal idempotent of βS_1 such that $p_1 \leq p_0$.

Proof. Let $\langle p_i^{(m)}\rangle_{i<\omega}^{m<\omega}$ be as guaranteed by Corollary 6.12 and for each $i < \omega$ let $p_i = p_i^{(0)}$. By Lemma 6.5 each $p_n \in E\big(K(\beta\widetilde{S}_n)\big)$. □

For several stronger combinatorial consequences of Corollary 6.13, see Sections 3 and 4 of [2].

To derive the following extension of Theorem 1.4, we need to restrict to a finite alphabet.

Corollary 6.14. *Assume that $\mathbb{A}$ is finite and for each $m < \omega$, let φ_m be a finite coloring of $\widetilde{S}_m$. For each $m < \omega$, there exists a central subset C_m of $\widetilde{S}_m$ such that*

(1) φ_m is constant on C_m and

(2) whenever $n \in \mathbb{N}$, the set of all $w \in \widetilde{S}_n$ such that for each $m < n$, $\{w(u) : u \in \widetilde{S}\binom{n}{m}\} \subseteq C_m$ is central in $\widetilde{S}_n$

Proof. Pick $\langle p_m \rangle_{m<\omega}$ as guaranteed by Corollary 6.13 and for each $m < \omega$ pick $C_m \in p_m$ with $C_m \subseteq \widetilde{S}_m$ such that φ_m is constant on C_m. Let $n \in \mathbb{N}$ be given. For each $m < n$ and each $u \in \widetilde{S}\binom{n}{m}$, $h_u(p_n) = p_m$. Let $D = \bigcap_{m<n} \bigcap (h_u^{-1}[C_m] : u \in \widetilde{S}\binom{n}{m})$. Then $D \in p_n$. □

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