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Compact Subsemigroups of $(\beta\mathbb{N}, +)$ Containing the Idempotents

Neil Hindman[1]

and

Dona Strauss

Abstract. The space $\beta\mathbb{N}$ is the Stone-Čech compactification of the discrete space of positive integers. The set of elements of $\beta\mathbb{N}$ which are in the kernel of every continuous homomorphism from $\beta\mathbb{N}$ to a topological group is a compact semigroup containing the idempotents. At first glance it would seem a good candidate for the smallest such semigroup. We produce an infinite nested sequence of smaller such semigroups, all defined naturally in terms of addition on $\mathbb{N}$.

1. **Introduction**. Given a discrete semigroup $(S,\cdot)$ the operation can be extended to the Stone-Čech compactification βS of S so that $(\beta S,\cdot)$ is a compact right topological semigroup. (See [12] for an elementary construction of this extension, with the caution that there βS is left rather than right topological.) As a compact right topological semigroup βS has idempotents [6, Corollary 2.10]. The existence of these idempotents, especially idempotents in the smallest ideal of βS, has important combinatorial consequences (See [11] and [15], for example.)

Of special interest are the semigroups $(\mathbb{N},+)$ and $(\mathbb{N},\cdot)$, where $\mathbb{N}$ is the set of positive integers. Let $E = \{p \in \beta\mathbb{N} : p + p = p\}$ and let $\Gamma = c\ell E$. It turns out that Γ is a right ideal of $(\beta\mathbb{N},\cdot)$. This fact provided the first (and for a long time only) proof of the following result: If $\mathbb{N}$ is partitioned into finitely many cells, then there exist sequences $\langle x_n\rangle_{n=1}^{\infty}$ and $\langle y_n\rangle_{n=1}^{\infty}$ such that $FS(\langle x_n\rangle_{n=1}^{\infty}) \cup FP(\langle y_n\rangle_{n=1}^{\infty})$ is contained in one cell of the partition [9, Theorem 2.6]. (Here $FS(\langle x_n\rangle_{n=1}^{\infty}) = \{\sum_{n\in F} x_n : F$ is a finite nonempty subset of $\mathbb{N}\}$ and $FP(\langle y_n\rangle_{n=1}^{\infty}) = \{\Pi_{n\in F} y_n : F$ is a finite nonempty subset of $\mathbb{N}\})$.

It is an intriguing fact that Γ is defined additively, is a right ideal, in particular a subsemigroup, of $(\beta\mathbb{N},\cdot)$, and yet is not a subsemigroup of $(\beta\mathbb{N},+)$. In fact there exist idempotents p and q in $(\beta\mathbb{N},+)$ such that $p+q \notin \Gamma$. (See Section 3 for the easy proof of this latter assertion.) An intriguing and potentially useful problem then arises: Characterize the smallest compact subsemigroup of $(\beta\mathbb{N},+)$ which contains the set E of idempotents.

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We take the points of $\beta\mathbb{N}$ to be the ultrafilters on $\mathbb{N}$. The reader is referred to [12] for background material. We will often use the fact that $A \in p+q$ if and only if $\{x \in \mathbb{N} : A - x \in q\} \in p$, where $A - x = \{y \in \mathbb{N} : y + x \in A\}$. (And similarly $A \in p \cdot q$ if and only if $\{x \in \mathbb{N} : A/x \in q\} \in p$, where $A/x = \{y \in \mathbb{N} : y \cdot x \in A\}$.)

Homomorphisms to other algebraic structures are a useful tool for investigating the algebraic structure of $\beta\mathbb{N}$. For example, such homomorphisms were used in [13] to show that the maximal groups in the smallest ideal of $(\beta\mathbb{N}, +)$ contain copies of the free group on 2^c generators. Now given any continuous homomorphism from $(\beta\mathbb{N}, +)$ to a compact topological group the kernel necessarily contains E. (It also must contain any element of finite order [1, Corollary 2.3]. Whether any such exist besides the idempotents is a difficult open problem.)

Let C be the intersection of the kernels of all continuous homomorphisms from $(\beta\mathbb{N}, +)$ to arbitrary compact topological groups. (We use "C" for kernel because K standardly represents the smallest ideal.) Then C is a compact semigroup containing E and at first glance seems like a good candidate for the smallest such. This turns out to fail badly, as we shall see.

The set $\Gamma = c\ell E$ can be characterized as follows [11, Lemma 2.3(a)]: Let $p \in \beta\mathbb{N}$. Then $p \in \Gamma$ if and only if for every $A \in p$ there is a sequence $\langle x_n \rangle_{n=1}^{\infty}$ with $FS(\langle x_n \rangle_{n=1}^{\infty}) \subseteq A$. In a similar fashion we define sets $S_n \subseteq \beta\mathbb{N}$ for each $n \in \mathbb{N}\backslash\{1\}$ as follows: Let $p \in \beta\mathbb{N}$. Then $p \in S_n$ if and only if for each $A \in p$, there is a sequence $\langle x_t \rangle_{t=1}^{n}$ with $FS(\langle x_t \rangle_{t=1}^{n}) \subseteq A$. (Given an index set J, $FS(\langle x_i \rangle_{i \in J}) = \{\sum_{i \in F} x_i : F$ is a finite nonempty subset of $J.\}$.) In a similar vein define T and M by agreeing that, given $p \in \beta\mathbb{N}$, $p \in T$ if and only if whenever $A \in p$, there exist some a and some $\langle y_t \rangle_{t=1}^{\infty}$ with $a + FS(\langle y_t \rangle_{t=1}^{\infty}) \subseteq A$ and that $p \in M$ if and only if whenever $A \in p$ and $n \in \mathbb{N}$, there exist $\langle x_t \rangle_{t=1}^{n}$ and $\langle y_t \rangle_{t=1}^{\infty}$ such that $FS(\langle x_t \rangle_{t=1}^{n}) + FS(\langle y_t \rangle_{t=1}^{\infty}) \subseteq A$. It will be shown in Theorem 2.4 that T is the smallest closed left ideal of $(\beta\mathbb{N}, +)$ containing the idempotents.

Let I be the semigroup generated by the set E of idempotents and let S_I be the smallest compact subsemigroup of $(\beta\mathbb{N}, +)$ containing E. In Section 2 we investigate each of the objects defined above, show that all (except Γ and $c\ell I$) are semigroups and show that the following pattern of inclusions holds:

$$\begin{array}{l} \Gamma \subseteq \\ \qquad c\ell I \subseteq S_I \subseteq M \subseteq T \cap \bigcap_{n=2}^{\infty} S_n \subseteq \bigcap_{n=2}^{\infty} S_n \subseteq \ldots S_3 \subseteq S_2 \subseteq C. \\ I \subseteq \end{array}$$

In Section 3 we show that $\Gamma\backslash I \neq \emptyset$, $I\backslash\Gamma \neq \emptyset$, $T\backslash \bigcap_{n=1}^{\infty} S_n \neq \emptyset$, $\bigcap_{n=1}^{\infty} S_n\backslash T \neq \emptyset$, and that all but one of the inclusions displayed above (including "...") is proper. (We have been unable to decide whether $M = T \cap \bigcap_{n=2}^{\infty} S_n$.) In Section 4 we present relationships between these sets and other structures.

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We conclude this introduction by displaying some results which we will utilise later.

1.1 **Lemma**. *(a) Let $p \in E$ and let $A \in p$. There exists $\langle x_n \rangle_{n=1}^{\infty}$ such that $FS(\langle x_n \rangle_{n=1}^{\infty}) \subseteq A$.*

(b) Let $\langle x_n \rangle_{n=1}^{\infty}$ be a sequence in $\mathbb{N}$. There exist $p \in E$ such that for all $m \in \mathbb{N}$, $FS(\langle x_n \rangle_{n=m}^{\infty}) \in p$.

Proof. (a) This is what is show in the Galvin-Glazer proof of the Finite Sum Theorem. See [5, Theorem 10.3] or [12].

(b) [10, Lemma 2.4 and Theorem 2.5]. $\square$

1.2 **Lemma**. *Let n and r be in $\mathbb{N}$. There is some $m \in \mathbb{N}$ such that whenever $\langle y_t \rangle_{t=1}^{m}$ is a sequence in $\mathbb{N}$ and $D_1, D_2, \ldots, D_r$ are subsets of $\mathbb{N}$ with $FS(\langle y_t \rangle_{t=1}^{m}) \subseteq \bigcup_{i=1}^{r} D_i$, there exist $i \in \{1, 2, \ldots, r\}$ and $\langle x_t \rangle_{t=1}^{n}$ with $FS(\langle x_t \rangle_{t=1}^{n}) \subseteq D_i$.*

Proof. By the finite version of the Finite Unions Theorem [8, p. 82] pick $m \in \mathbb{N}$ such that whenever the finite nonempty subsets of $\{1, 2, \ldots, m\}$ are covered by r cells, there will exist pairwise disjoint $B_1, B_2, \ldots, B_n$ with all sets of the form $\bigcup_{t \in F} B_t$ in the same cell of the cover (for $\emptyset \neq F \subseteq \{1, 2, \ldots, n\}$).

Next let $\langle y_t \rangle_{t=1}^{m}$ and $\langle D_i \rangle_{i=1}^{r}$ be given with $FS(\langle y_t \rangle_{t=1}^{m}) \subseteq \bigcup_{i=1}^{r} D_i$. For each $i \in \{1, 2, \ldots, r\}$, let $H_i = \{F \subseteq \{1, 2, \ldots, m\} : F \neq \emptyset \text{ and } \sum_{t \in F} y_t \in D_i\}$. Pick $i \in \{1, 2, \ldots, r\}$ and pairwise disjoint $B_1, B_2, \ldots, B_n$ with $\bigcup_{j \in F} B_j \in H_i$ whenever $\emptyset \neq F \subseteq \{1, 2, \ldots, n\}$. Let $x_j = \sum_{t \in B_j} y_t$ for $j \in \{1, 2, \ldots, n\}$. Then given $\emptyset \neq F \subseteq \{1, 2, \ldots, n\}$, $\sum_{j \in F} x_j = \sum_{j \in F} \sum_{t \in B_j} y_t$. Since $\bigcup_{j \in F} B_j \in H_i$ one has that $\sum_{j \in F} x_t \in D_i$. $\square$

The following lemma is apparently originally due to Frolík.

1.3 **Lemma**. *Let X and Y be σ-compact subsets of $\beta\mathbb{N}$. If $c\ell X \cap c\ell Y \neq \emptyset$, then $X \cap c\ell Y \neq \emptyset$ or $Y \cap c\ell X \neq \emptyset$.*

Proof. See [14, Lemma 1.1]. $\square$

2. **Inclusions among semigroups containing the idempotents**.

We begin by displaying the definitions of the objects we are studying. Recall that $E = \{p \in \beta\mathbb{N} : p + p = p\}$.

2.1 **Definition**. (a) $C = \{p \in \beta\mathbb{N}$: for any compact topological group G and any continuous homomorphism φ from $(\beta\mathbb{N}, +)$ to G, $\varphi(p)$ is the identity of $G\}$.

(b) For $n \in \mathbb{N}\backslash\{1\}$, $S_n = \{p \in \beta\mathbb{N}$: for all $A \in p$ there exists $\langle x_t \rangle_{t=1}^n$ such that $FS(\langle x_t \rangle_{t=1}^n) \subseteq A\}$.

(c) $T = \{p \in \beta\mathbb{N}$: for all $A \in p$ there exist $a \in \mathbb{N}$ and $\langle y_t \rangle_{t=1}^\infty$ such that $a + FS(\langle y_t \rangle_{t=1}^\infty) \subseteq A\}$.

(d) $M = \{p \in \beta\mathbb{N}$: for all $A \in p$ and all $n \in \mathbb{N}$ there exist $\langle x_t \rangle_{t=1}^n$ and $\langle y_t \rangle_{t=1}^\infty$ such that $FS(\langle x_t \rangle_{t=1}^n) + FS(\langle y_t \rangle_{t=1}^\infty) \subseteq A\}$.

(e) $S_I = \bigcap\{S : S$ is a compact subsemigroup of $(\beta\mathbb{N}, +)$ and $E \subseteq S\}$.

(f) $I = \bigcap\{S : S$ is a semigroup of $(\beta\mathbb{N}, +)$ and $E \subseteq S\}$.

(g) $\Gamma = \{p \in \beta\mathbb{N}$: for all $A \in p$ there exists $\langle y_t \rangle_{t=1}^\infty$ such that $FS(\langle y_t \rangle_{t=1}^\infty) \subseteq A\}$.

2.2 **Lemma**. *Each of the objects defined in Definition 2.1 contains E and all except I are compact.*

Proof. The idempotents are contained in Γ by Lemma 1.1(a). Clearly Γ is contained in each of M, T, and S_n (for $n \in \mathbb{N}\backslash\{1\}$). The idempotents are contained in I and S_I by definition and are contained in C by elementary algebra.

That S_I and C are compact follows from elementary topology. The others all have definitions which begin "for all $A \in p$" (and refer no more to p). If a point p is not in the specified set it has a member A failing the definition. Then $c\ell A$ is a neighborhood of p missing the specified set. $\square$

We will see in the next section that I is not closed when we show that the inclusion $I \subseteq c\ell I$ is proper.

2.3 **Lemma**. *Each of the objects defined in Definition 2.1 except Γ is a semigroup.*

Proof. That C, I, and S_I are semigroups follows by elementary algebra.

Let $n \in \mathbb{N}\backslash\{1\}$ and let $p, q \in S_n$. To see that $p + q \in S_n$, let $A \in p + q$. Then $\{x \in \mathbb{N} : A - x \in q\} \in p$ so pick $\langle x_t \rangle_{t=1}^n$ such that $FS(\langle x_t \rangle_{t=1}^n) \subseteq \{x \in \mathbb{N} : A - x \in q\}$. Now $FS(\langle x_t \rangle_{t=1}^n)$ is finite so if $B = \bigcap\{A - a : a \in FS(\langle x_t \rangle_{t=1}^n)\}$ we have $B \in q$. Pick $\langle y_t \rangle_{t=1}^n$ such that $FS(\langle y_t \rangle_{t=1}^n) \subseteq B$. We claim $FS(\langle x_t + y_t \rangle_{t=1}^n) \subseteq A$. To see this let $\emptyset \neq F \subseteq \{1, 2, \ldots, n\}$. Then $\sum_{t \in F} y_t \in B \subseteq A - \sum_{t \in F} x_t$ so $\sum_{t \in F}(x_t + y_t) \in A$.

That T is a semigroup follows from the fact that it is a left ideal which we will present in Theorem 2.4. To see that M is a semigroup, let $p, q \in M$ and let $A \in p + q$. Let $B = \{x \in \mathbb{N} : A - x \in q\}$. Then $B \in p$ so pick $\langle x_t \rangle_{t=1}^n$ and $\langle y_t \rangle_{t=1}^\infty$ such that $FS(\langle x_t \rangle_{t=1}^n) + FS(\langle y_t \rangle_{t=1}^\infty) \subseteq B$. In particular $FS(\langle x_t + y_t \rangle_{t=1}^n) \subseteq B$. Let

$D = \cap\{A - a : a \in FS(\langle x_t + y_t\rangle_{t=1}^n)\}$. Then $D \in q$ so pick $\langle z_t\rangle_{t=1}^n$ and $\langle w_t\rangle_{t=1}^\infty$ such that $FS(\langle z_t\rangle_{t=1}^n) + FS(\langle w_t\rangle_{t=1}^\infty) \subseteq D$. Then $FS(\langle z_t + x_t + y_t\rangle_{t=1}^n) + FS(\langle w_t\rangle_{t=1}^\infty) \subseteq A$. $\Box$

We shall see in Theorem 2.11 that Γ is not a semigroup.

2.4 **Theorem**. *T is the smallest closed left ideal of $(\beta\mathbb{N}, +)$ which contains the idempotents and $T = c\ell\bigcup\{\beta\mathbb{N} + p : p \in E\} = c\ell\bigcup\{\mathbb{N} + p : p \in E\}$.*

Proof. By Lemma 2.2 T is closed and contains the idempotents. To see that S is a left ideal let $p \in \beta\mathbb{N}$ and $q \in T$. Let $A \in p + q$. Then $\{x \in \mathbb{N} : A - x \in q\} \in p$ so pick x such that $A - x \in q$. Pick a and $\langle y_t\rangle_{t=1}^\infty$ such that $a + FS(\langle y_t\rangle_{t=1}^\infty) \subseteq A - x$. Then $x + a + FS(\langle y_t\rangle_{t=1}^\infty) \subseteq A$.

As a closed left ideal containing the idempotents, $T \supseteq c\ell\bigcup\{\beta\mathbb{N} + p : p \in E\}$. To complete the proof, we show $T \subseteq c\ell\bigcup\{\beta\mathbb{N} + p : p \in E\}$. To this end let $q \in T$ and let $A \in q$. Pick a and $\langle y_t\rangle_{t=1}^\infty$ such that $a + FS(\langle y_t\rangle_{t=1}^\infty) \subseteq A$. Pick by Lemma 1.1 $p \in E$ with $FS(\langle y_t\rangle_{t=1}^\infty) \in p$. Then $A \in a + p$ so $(c\ell A) \cap (\mathbb{N} + p) \neq \emptyset$. $\Box$

2.5 **Theorem**. *(a) $\Gamma \subseteq c\ell I$*
(b) $c\ell I \subseteq S_I$
(c) $S_I \subseteq M$
(d) $M \subseteq T \cap \bigcap_{n=2}^\infty S_n$
(e) For each $n \in \mathbb{N}\backslash\{1\}$, $S_{n+1} \subseteq S_n$.
(f) $S_2 \subseteq C$.

Proof. Statements (b), (d), and (e) are trivial and (c) follows immediately from the fact that M is a compact subsemigroup of $\beta\mathbb{N}$ containing the idempotents. By [11, Lemma 2.3], $\Gamma = c\ell E$ so (a) holds.

To verify (f), let $p \in S_2$ and let φ be a continuous homomorphism from $(\beta\mathbb{N}, +)$ to a topological group $(G, +)$ with identity 0. Suppose that $\varphi(p) = a \neq 0$. Then $a \neq a + a$ so pick a neighborhood V of a such that $V \cap (V + V) = \emptyset$. Pick $A \in p$ such that $\varphi[c\ell A] \subseteq V$ and pick x_1 and x_2 with $\{x_1, x_2, x_1 + x_2\} \subseteq A$. Then $\varphi(x_1 + x_2) \in V \cap (V + V)$, a contradiction. $\Box$

The following simple result allows us to tell when a set A has closure intersecting various of our special semigroups. For example, it tells us that for $A \subseteq \mathbb{N}$ and $n \in \mathbb{N}\backslash\{1\}$, $c\ell A \cap S_n \neq \emptyset$ if and only if whenever F is a finite partition of A there exist $B \in F$ and $\langle x_t\rangle_{t=1}^n$ with $FS(\langle x_t\rangle_{t=1}^n) \subseteq B$. (Let $\mathcal{G} = \{FS(\langle x_t\rangle_{t=1}^n) : \langle x_t\rangle_{t=1}^n$ is an n-term sequence in $\mathbb{N}\}$. Then $S_n = \{p \in \beta\mathbb{N} :$ for each $A \in p$ there exists $G \in \mathcal{G}$ with $G \subseteq A\}$.)

2.6 **Theorem**. *Let X be a discrete space, let $A \subseteq X$, and let $\mathcal{G} \subseteq \mathcal{P}(X)$. The following statements are equivalent.*

(a) There exists $p \in c\ell A$ such that for every $B \in p$ there exists $G \in \mathcal{G}$ with $G \subseteq B$.

(b) Whenever $\mathcal{F}$ is a finite partition of A there exist $B \in \mathcal{F}$ and $G \in \mathcal{G}$ with $G \subseteq B$.

(c) When $\mathcal{F}$ is finite and $\bigcup \mathcal{F} = A$, there exist $B \in \mathcal{F}$ and $G \in \mathcal{G}$ with $G \subseteq B$.

Proof. That (a) implies (b) and (b) implies (c) is trivial.

To see that (c) implies (a), it suffices to show that $\{A\} \cup \{\mathbb{N} \backslash B : B \subseteq \mathbb{N}$ and for all $G \in \mathcal{G}$, $G \backslash B \neq \emptyset\}$ has the finite intersection property, since any ultrafilter p extending this family is as required by (a). But a failure of the finite intersection property would make $A = \bigcup \mathcal{F}$ where $\mathcal{F}$ is finite and for each $B \in \mathcal{F}$, one has no $G \in \mathcal{G}$ with $G \subseteq B$, contradicting (c).▯

2.7 **Theorem**. *Let $A \subseteq \mathbb{N}$. Then $(c\ell A) \cap \bigcap_{n=2}^{\infty} S_n \neq \emptyset$ if and only if for every $n \in \mathbb{N}$ there exists $\langle x_t \rangle_{t=1}^{n}$ with $FS(\langle x_t \rangle_{t=1}^{n}) \subseteq A$.*

Proof. The necessity is an immediate consequence of Theorem 2.6.

Sufficiency. We have by Lemma 2.2 and Theorem 2.5 that $\{(c\ell A) \cap S_n : n \in \mathbb{N} \backslash \{1\}\}$ is a nested collection of closed sets so it suffices to show that $(c\ell A) \cap S_n \neq \emptyset$ for each $n \in \mathbb{N} \backslash \{1\}$. To this end let $n \in \mathbb{N} \backslash \{1\}$ and let $\mathcal{F}$ be a finite partition of A. Let $r = |\mathcal{F}|$ and pick m as guaranteed by Lemma 1.2 for n and r. Pick $\langle y_t \rangle_{t=1}^{m}$ with $FS(\langle y_t \rangle_{t=1}^{m}) \subseteq A$. By Lemma 1.2 pick $B \in \mathcal{F}$ and $\langle x_t \rangle_{t=1}^{n}$ with $FS(\langle x_t \rangle_{t=1}^{n}) \subseteq B$. ▯

The following notion, used to characterize members of C, is of independent interest.

2.8 **Definition**. Let $A \subseteq \mathbb{N}$. Then A is a *rational approximation set* if and only if whenever F is a finite nonempty subset of $\mathbb{R}$ and $\epsilon > 0$, there exists some $n \in A$ such that for each $x \in F$ there exists $m \in \mathbb{Z}$ with $|x - m/n| < \epsilon/n$.

2.9 **Lemma**. *Let $p \in \beta\mathbb{N}$. The following statements are equivalent.*

(a) $p \in C$;

(b) for each $A \in p$, A is a rational approximation set;

(c) for each $A \in p$, each $x \in \mathbb{R}$, and each $\epsilon > 0$ there exist $n \in A$ and $m \in \mathbb{Z}$ with $|x - m/n| < \epsilon/n$.

Proof. To see that (a) implies (b), let $A \in p$ and let finite nonempty $F \subseteq \mathbb{R}$ be given. Write $F = \{x_1, x_2, \ldots, x_k\}$. We view the circle group $\mathbb{T}$ as $\mathbb{R}/\mathbb{Z}$, denoting by $[x]$ the equivalence class $x + \mathbb{Z}$. Define $h : \mathbb{N} \longrightarrow \bigtimes_{i=1}^{k} \mathbb{T}$ by $h(n) = ([nx_1], [nx_2], \ldots, [nx_k])$. Then h is a homomorphism so the continuous extension $h^{\beta} : \beta\mathbb{N} \longrightarrow \bigtimes_{i=1}^{k} \mathbb{T}$ is a homomorphism, as was observed by Milnes [17]. Since $p \in C$, $h^{\beta}(p) = ([0], [0], \ldots, [0])$ so pick

$B \in p$ such that $h^\beta[c\ell B] \subseteq \{([y_1],[y_2],\ldots,[y_k]) : \text{for each } i \in \{1,2,\ldots,k\},\ -\epsilon < y_i < \epsilon\}$. Pick $n \in B \cap A$. Since $n \in B$, pick for each $i \in \{1,2,\ldots,k\}$ some y_i with $-\epsilon < y_i < \epsilon$ such that $[nx_i] = [y_i]$. Given $i \in \{1,2,\ldots,k\}$, pick $m_i \in \mathbb{Z}$ such that $nx_i = y_i + m_i$ then $-\epsilon < nx_i - m_i < \epsilon$ so $|x_i - m_i/n| < \epsilon/n$.

That (b) implies (c) is trivial.

To see that (c) implies (a), observe that it suffices to show that given any continuous homomorphism $\varphi : \beta\mathbb{N} \longrightarrow \mathbb{T}$ one has $\varphi(p) = [0]$. (See for example the introduction to [1].) So let such φ be given and pick $x \in \mathbb{R}$ with $[x] = \varphi(1)$. Suppose that $\varphi(p) \neq [0]$ and pick $\epsilon > 0$ such that $\varphi(p) \notin \{[y] : -\epsilon \leq y \leq \epsilon\}$. Pick $A \in p$ such that $\varphi[c\ell A] \cap \{[y] : -\epsilon \leq y \leq \epsilon\} = \emptyset$. Pick $n \in A$ and $m \in \mathbb{Z}$ such that $|x - m/n| < \epsilon/n$ and let $y = nx - m$. Then $\varphi(n) = [y]$ and $-\epsilon < y < \epsilon$, a contradiction. $\Box$

2.10 **Theorem**. *Let $A \subseteq \mathbb{N}$. Then $c\ell A \cap C \neq \emptyset$ if and only if A is a rational approximation set.*

Proof. Necessity. Pick $p \in c\ell A \cap C$. By Lemma 2.9, A is a rational approximation set.

Sufficiency. Let $\mathcal{G} = \{B \subseteq \mathbb{N} : B$ is a rational approximation set.$\}$ It is an easy consequence of the definition of rational approximation sets that whenever $\mathcal{F}$ is a finite partition of A, one has $\mathcal{F} \cap \mathcal{G} \neq \emptyset$. Thus by Theorem 2.6 there is some $p \in c\ell A$ such that for every $B \in p$ there is some $G \in \mathcal{G}$ with $G \subseteq B$ (and hence $B \in \mathcal{G}$). Then by Lemma 2.9 $p \in C$. $\Box$

2.11 **Theorem**. Γ *is not a semigroup. In fact $(E+E)\backslash\Gamma \neq \emptyset$.*

Proof. Pick by Lemma 1.1(b) idempotents p and q such that $FS(\langle 2^{2t}\rangle_{t=m}^\infty) \in p$ and $FS(\langle 2^{2t+1}\rangle_{t=m}^\infty) \in q$ for each $m \in \mathbb{N}$. Let $A = \{\sum_{t\in F} 2^{2t} + \sum_{t\in G} 2^{2t+1} : F$ and G are finite nonempty subsets of $\mathbb{N}$ and $\max F < \min G\}$. We claim that $A \in p+q$. To see this it suffices to show that $FS(\langle 2^{2t}\rangle_{t=1}^\infty) \subseteq \{x \in \mathbb{N} : A - x \in q\}$ so let F be a finite nonempty subset of $\mathbb{N}$ and let $m = \max F + 1$. Then $FS(\langle 2^{2t+1}\rangle_{t=m}^\infty) \subseteq A - \sum_{t\in F} 2^{2t}$ so $A - \sum_{t\in F} 2^{2t} \in q$.

Now suppose $p + q \in \Gamma$. Then pick a sequence $\langle y_t\rangle_{t=1}^\infty$ with $FS(\langle y_t\rangle_{t=1}^\infty) \subseteq A$. Pick F_1 and G_1 with $\max F_1 < \min G_1$ such that $y_1 = \sum_{t\in F_1} 2^{2t} + \sum_{t\in G_1} 2^{2t+1}$. Let $m = \max G_1 + 1$. Pick nonempty $H \subseteq \mathbb{N}\backslash\{1\}$ such that 2^{2m} divides $\sum_{t\in H} y_t$. (Take any 2^m elements with all y_t in the same congruence class mod 2^{2m}.) Pick F_2 and G_2 with $\max F_2 < \min G_2$ such that $\sum_{t\in H} y_t = \sum_{t\in F_2} 2^{2t} + \sum_{t\in G_2} 2^{2t+1}$. Since 2^{2m} divides $\sum_{t\in H} y_t$ we have $\min F_2 \geq m$. Thus $y_1 + \sum_{t\in H} y_t = \sum_{t\in F_1} 2^{2t} + \sum_{t\in G_1} 2^{2t+1} +$

$\sum_{t\in F_2} 2^{2t} + \sum_{t\in G_2} 2^{2t+1}$ where $\max F_1 < \min G_1 < \max G_1 < \min F_2 < \max F_2 < \min G_2$ so by uniqueness of binary expansion, $y_1 + \sum_{t\in H} y_t \notin A$, a contradiction. $\Box$

Our proof that $c\ell I$ is not a semigroup is in some respects similar to the proof that Γ is not a semigroup. However, instead of the binary expansion of integers we use the factorial expansion, $x = \sum_{t\in F} a_t \cdot t!$ where each $a_t \in \{1, 2, \ldots, t\}$. In the proof we also utilize in an incidental fashion the semigroup $(\beta\mathbb{N}, \cdot)$.

2.12 **Theorem**. *$c\ell I$ is not a semigroup. In fact $(E + \Gamma)\backslash c\ell I \neq \emptyset$.*

Proof. Since $\Gamma \subseteq c\ell I$, the second statement implies the first. Let $A = \{\sum_{n\in F} n! + \sum_{n\in G} k \cdot n! : F$ and G are finite nonempty subsets of $\mathbb{N}$ and $\max F < \min G$ and $k \in \mathbb{N}$ and $k \leq \min G\}$. Define $g : \mathbb{N} \longrightarrow \mathbb{N}$ by $g(x) = a_\ell$ where $x = \sum_{t\in F} a_t \cdot t!$, each $a_t \in \{1, 2, \ldots, t\}$, and $\ell = \max F$. That is $g(x)$ is the leftmost nonzero digit in the factorial expansion of x. Denote also by g its continuous extension from $\beta\mathbb{N}$ to $\beta\mathbb{N}$.

We claim that:

(1) $$\text{If } q \in \bigcap_{n=1}^{\infty} c\ell(\mathbb{N}n), \text{ then } g(p+q) = g(q) \text{ for all } p \in \beta\mathbb{N}.$$

To see this, suppose instead there is some $B \subseteq \mathbb{N}$ with $g(p + q) \in c\ell B$ and $g(q) \in c\ell(\mathbb{N}\backslash B)$. Pick $C \in p + q$ and $D \in q$ with $g[c\ell C] \subseteq c\ell B$ and $g[c\ell D] \subseteq c\ell(\mathbb{N}\backslash B)$. Since $C \in p + q$ pick $x \in \mathbb{N}$ with $C - x \in q$. Pick $y \in (C - x) \cap D \cap \mathbb{N}x!$. Then $y + x \in C$ so $g(y + x) \in B$. But $g(y + x) = g(y) \in \mathbb{N}\backslash B$, a contradiction.

Next we claim:

(2) $$\text{If } q \in E \text{ and } c\ell A \cap (\beta\mathbb{N} + q) \neq \emptyset, \text{ then } g(q) \in \mathbb{N}.$$

To see this suppose that $g(q) \notin \mathbb{N}$, so that for each k, $D_k = \{m \in \mathbb{N} : g(m) > k\} \in q$. Pick $p \in \beta\mathbb{N}$ with $p + q \in c\ell A$. Let $B = \{m + n : m, n \in \mathbb{N}$ and $g(n) > g(m) > 1$ and $n \in \mathbb{N}m!\}$. We show that $B \in p + q$ which will be a contradiction since $B \cap A = \emptyset$. We claim in fact that for all $x \in \mathbb{N}$, $B - x \in q$. For this, since $q = q + q$, it suffices to show that $(\mathbb{N}x!) \cap D_1 \subseteq \{m \in \mathbb{N} : (B - x) - m \in q\}$ so let $m \in (\mathbb{N}x!) \cap D_1$. Then $D_{g(m)} \cap \mathbb{N}m! \subseteq (B - x) - m$ (since $g(m + x) = g(m)$) so $(B - x) - m \in q$.

Next we claim:

(3) $$\text{If } p \in c\ell(FS(\langle n! \rangle_{n=1}^{\infty})) \cap \bigcap_{n=1}^{\infty} c\ell(\mathbb{N}n) \text{ and } r \in \beta\mathbb{N}, \text{ then } g(r \cdot p) = r.$$

To see this it suffices to show that for all $n \in \mathbb{N}$, $g(n \cdot p) = n$, so let $n \in \mathbb{N}$ be given. Let $B = \{\sum_{t\in F} t! : F$ is a finite nonempty subset of $\mathbb{N}$ and $\min F \geq n\}$. Then $B \in p$ so $n \cdot B \in n \cdot p$ and $g[n \cdot B] = \{n\}$. Now by Lemma 1.1 pick $p \in E \cap c\ell FS(\langle n! \rangle_{n=1}^{\infty})$ and let $r \in \beta\mathbb{N}\backslash\mathbb{N}$. Now for each $x \in \mathbb{N}$, $x \cdot p \in E$ so $r \cdot p \in \Gamma$. Let $s = p + r \cdot p$. We show that

$s \notin c\ell I$. Suppose instead that $s \in c\ell I$. Observe that $A \in s$. Indeed $FS(\langle n! \rangle_{n=1}^{\infty}) \subseteq \{x \in \mathbb{N} : A - x \in r \cdot p\}$. (Given $\sum_{n \in F} n!$ one sees that $\mathbb{N} \backslash \{1\} \subseteq \{k : (A - \sum_{n \in F} n!)/k \in p\}$ by noting that $\{\sum_{n \in G} n! : \min G > \max F$ and $\min G \geq k\} \subseteq (A - \sum_{n \in F} n!)/k$.)

We claim that $s \in c\ell \bigcup_{k=1}^{\infty}(I \cap g^{-1}[\{k\}])$. To see this, let $B \in s$. Since $s \in c\ell I$, we have $c\ell(A \cap B) \cap I \neq \emptyset$ so we may pick $\ell \in \mathbb{N}$ and $q_1, q_2, \ldots, q_\ell \in E$ with $q_1 + q_2 + \ldots + q_\ell \in c\ell(A \cap B)$. We may presume $\ell \geq 2$. Now by (2) we have $g(q_\ell) \in \mathbb{N}$. Let $k = g(q_\ell)$. By (1), $g(q_1 + q_2 + \ldots + q_\ell) = k$ so $c\ell B \cap (I \cap g^{-1}[\{k\}]) \neq \emptyset$.

Now also $s \in c\ell(\mathbb{N} + r \cdot p)$ so $c\ell(\mathbb{N} + r \cdot p) \cap c\ell \bigcup_{k=1}^{\infty} c\ell(I \cap g^{-1}[\{k\}]) \neq \emptyset$ so by Lemma 1.3 either one has some $n \in \mathbb{N}$ with $n + r \cdot p \in c\ell \bigcup_{k=1}^{\infty} c\ell(I \cap g^{-1}[\{k\}]) \subseteq \bigcap_{m=1}^{\infty} c\ell(\mathbb{N}m))$ or one has some $q \in \beta\mathbb{N}$ and some $k \in \mathbb{N}$ with $q + r \cdot p \in c\ell(I \cap g^{-1}[\{k\}]) \subseteq g^{-1}[\{k\}]$. The first possibility would imply that $n \in \bigcap_{m=1}^{\infty} c\ell(\mathbb{N}m)$. The second would imply that $g(q + r \cdot p) = k$ while by (1) and (3) $g(q + r \cdot p) = g(r \cdot p) = r \notin \mathbb{N}$. $\square$

3. **The inclusions are proper**. We show in this section that the objects mentioned in Theorem 2.5 are all distinct (except that we have been unable to determine whether $M = T \cap \bigcap_{n=2}^{\infty} S_n$). We proceed from the left in the inclusion diagram from the introduction.

3.1 **Theorem**. *$I \backslash \Gamma \neq \emptyset$ and $\Gamma \backslash I \neq \emptyset$.*

Proof. That $I \backslash \Gamma \neq \emptyset$ follows from Theorem 2.11. That $\Gamma \backslash I \neq \emptyset$ follows from Theorem 2.12 since $E + I \subseteq I$. $\square$

In the following theorem (and the rest of this section) the inclusions hold by Theorem 2.5 (or are completely trivial). We concentrate on establishing the inequalities.

3.2 **Theorem**. *$\Gamma \subsetneqq c\ell I$, $I \subsetneqq c\ell I$, and $c\ell I \subsetneqq S_I$.*

Proof. That $\Gamma \neq c\ell I$ follows from the fact from Theorem 3.1 that $I \backslash \Gamma \neq \emptyset$. The remaining two conclusions follow from the fact (Theorem 2.12) that $c\ell I$ is not a semigroup. $\square$

We produce in the following lemma another closed subsemigroup of $\beta\mathbb{N}$ containing the idempotents. It was not included in those discussed in Section 2 because its definition is less natural than those defined there. When we write $\sum_{t \in F} a_t \cdot t!$, we shall assume F is finite and nonempty and each $a_t \in \{1, 2, \ldots, t\}$.

3.3 **Lemma**. *Let $B = \{\sum_{t \in F} a_t \cdot t! :$ (1) F is a finite nonempty subset of $\mathbb{N}$; (2) for each $t \in F$, $a_t \in \{1, 2, \ldots, t\}$; (3) there exists $t \in F$ such that $a_t > 1$; and (4) whenever $n, t \in F$ with $t < n$ either $a_t = a_n = 1$ or $a_t > a_n\}$. Then $(\bigcap_{n=1}^{\infty} c\ell\mathbb{N}n) \backslash c\ell B$ is a closed subsemigroup of $(\beta\mathbb{N}, +)$ containing the idempotents.*

Proof. To see that it is a semigroup, let $p, q \in (\bigcap_{n=1}^{\infty} c\ell \mathbb{N}n) \backslash c\ell B$. Then $p + q \in \bigcap_{n=1}^{\infty} c\ell \mathbb{N}n$ so we only need to show that $\mathbb{N} \backslash B \in p + q$. To this end and we let $x \in \mathbb{N} \backslash B$ and show that $(\mathbb{N} \backslash B) - x \in q$. Write $x = \sum_{t \in F} a_t \cdot t!$ and let $m = \max F + 1$. We show that $\mathbb{N}m! \subseteq (\mathbb{N} \backslash B) - x$, so let $y \in \mathbb{N}m!$ and write $y = \sum_{t \in G} b_t \cdot t!$ and note that $\min G \geq m$.

Now $x \notin B$. Assume first that for all $t \in F$, $a_t = 1$. If for all $t \in G$, $b_t = 1$, we have $y + x \notin B$ so assume for some $n \in G$, $b_n > 1$. Pick any $t \in F$. Then $t < n$ and $a_t = 1 < b_n$ so again $y + x \notin B$. Now assume we have $t < n$ in F with $a_t \leq a_n$ and it is not the case that $a_t = a_n = 1$. Then directly we have $y + x$ fails to satisfy (4) of the definition so $y + x \notin B$.

Now let $p \in E$. Then $p \in \bigcap_{n=1}^{\infty} c\ell \mathbb{N}n$ so we show that $\mathbb{N} \backslash B \in p$. Suppose instead that $B \in p$ and let $D = \{\sum_{t \in F} t! : F$ is a finite nonempty subset of $\mathbb{N}\}$. Then $D \subseteq \mathbb{N} \backslash B$ so if $D \in p$ we are done. Assume $D \notin p$.

Assume that for some $k \geq 2$, $\{\sum_{t \in F} a_t \cdot t! : \min F \geq k$ and $\{a_t : t \in F\} \subseteq \{1, 2, \ldots, k\}\} \in p$. Since $p = p + p + ... + p$ (k times) and $p \in \bigcap_{n=1}^{\infty} c\ell \mathbb{N}n$ we have that $\{\sum_{t \in F} a_t \cdot t! : |F| \geq k\} \in p$. Let $E = B \cap \{\sum_{t \in F} a_t \cdot t! : \min F \geq k$ and $|F| \geq k$ and $\{a_t : t \in F\} \subseteq \{1, 2, \ldots, k\}\}$. Then $E \in p$ so pick $x \in E$ such that $E - x \in p$. Write $x = \sum_{t \in F} a_t \cdot t!$ and let $m = \max F + 1$. Pick $y \in \mathbb{N}m! \cap (E - x)$, and write $y = \sum_{t \in G} b_t \cdot t!$. Since $x \in B$ and $|F| \geq k$ and each $a_t \leq k$, there is some $t \in F$ with $a_t = 1$. Since $y \in E$, $y \in B$ so $y \notin D$ so there is some $n \in G$ with $b_n > 1$. But then $t < n$ and $a_t < b_n$ so $y + x \notin B$ so $y + x \notin E$ a contradiction.

Thus it must be the case that for all $k \in \mathbb{N}$, $E_k = \{\sum_{t \in F} a_t \cdot t! : \{a_t : t \in F\} \backslash \{1, 2, \ldots, k\} \neq \emptyset\} \in p$. Since $B \in p$, pick x such that $B - x \in p$ and write $x = \sum_{t \in F} a_t \cdot t!$. Let $k = \max\{a_t : t \in F\}$ and let $m = \max F + 1$. Pick $y \in \mathbb{N}m! \cap E_k \cap (B - x)$ and write $y = \sum_{t \in G} b_t \cdot t!$. Pick $n \in G$ such that $b_n > k$ and pick any $t \in F$. Then $t < n$ and $b_n > a_t$ so $y + x \notin B$, a contradiction. $\square$

3.4 **Theorem**. $S_I \subsetneq M$.

Proof. Let B be as in Lemma 3.3 and let $H = \{\sum_{t \in F} a_t \cdot t! :$ whenever $n, t \in F$ with $t < n$ one has $a_t > a_n\}$. Observe that given any $n \in \mathbb{N}$ there exists $\langle x_t \rangle_{t=1}^{n}$ with $FS(\langle x_t \rangle_{t=1}^{n}) \subseteq H$. (For example let $x_t = (n + 1 - t) \cdot (n + t)!$.) Thus by Theorem 2.7 we may pick $p \in c\ell H \cap \bigcap_{n=2}^{\infty} S_n$. By Lemma 1.1 pick $q = q + q \in \bigcap_{m=1}^{\infty} c\ell(FS(\langle t! \rangle_{t=m}^{\infty}))$. We claim that $p + q \in M \cap c\ell B$ (so that by Lemma 3.3, $p + q \in M \backslash S_I$).

To see that $p + q \in M$, let $A \in p + q$ and let $n \in \mathbb{N}$ be given. Since $\{x \in \mathbb{N} : A - x \in q\} \in p$ and $p \in S_n$, pick $\langle x_t \rangle_{t=1}^{n}$ with $FS(\langle x_t \rangle_{t=1}^{n}) \subseteq \{x \in \mathbb{N} : A - x \in q\}$. Let $D = \bigcap\{A - z : z \in FS(\langle x_t \rangle_{t=1}^{n})\}$. Since $D \in q = q + q$ pick $\langle y_t \rangle_{t=1}^{\infty}$ with $FS(\langle y_t \rangle_{t=1}^{\infty}) \subseteq D$.

Then $FS(\langle x_t\rangle_{t=1}^n)+FS(\langle y_t\rangle_{t=1}^\infty)\subseteq A$.

To see that $B\in p+q$ we show that $H\subseteq\{x\in\mathbb{N}: B-x\in q\}$. So let $x\in H$ and write $x=\sum_{t\in F}a_t\cdot t!$. Let $m=\max F+1$. Then $FS(\langle t!\rangle_{t=m}^\infty)\subseteq B-x$ so $B-x\in q$. $\Box$

As we have remarked, we do not know whether $M=T\cap\bigcap_{n=2}^\infty S_n$. It is trivial that $T\backslash\bigcap_{n=2}^\infty S_n\neq\emptyset$, indeed that $T\backslash S_2\neq\emptyset$. In fact by Theorem 2.5 $S_2\subseteq C$ and trivially $C\subseteq\bigcap_{n=1}^\infty c\ell(\mathbb{N}n)$ while, given any idempotent p we have by Theorem 2.4 that $1+p\in c\ell(\mathbb{N}2+1)\cap T$. This suggests replacing T by $T\cap\bigcap_{n=1}^\infty c\ell(\mathbb{N}n)$.

3.5 **Theorem**. *$\bigcap_{n=2}^\infty S_n\backslash T\neq\emptyset$, $(T\cap\bigcap_{n=1}^\infty c\ell(\mathbb{N}n))\backslash S_2\neq\emptyset$, and $T\cap\bigcap_{n=2}^\infty S_n\subsetneqq\bigcap_{n=2}^\infty S_n$.*

Proof. For the first statement, let $A=\bigcup_{n=1}^\infty FS(\langle 2^{2^n+i}\rangle_{i=1}^n)$. By Theorem 2.7, $(c\ell A)\cap\bigcap_{n=2}^\infty S_n\neq\emptyset$. It is easy to see however that one cannot get any $t\in\mathbb{N}$ and any sequence $\langle x_n\rangle_{n=1}^\infty$ with $t+FS(\langle x_n\rangle_{n=1}^\infty)\subseteq A$ (since all elements of A have binary expansions with support restricted to a small segment of $\mathbb{N}$). Thus $(c\ell A)\cap T=\emptyset$.

Now let $B=\bigcup_{k=4}^\infty(2\cdot(k!)+FS(\langle n!\rangle_{n=k+1}^\infty))$, so that B consists of all numbers whose rightmost nonzero factorial digit is a 2, occurring at position 4 or above and all other nonzero digits are 1. Then there do not exist $x,y\in B$ with $x+y\in B$. (Given $x,y\in B$ either the rightmost digit of $x+y$ is 4 or there are two digits in the expansion of $x+y$ which are greater than 1.) Thus $(c\ell B)\cap S_2=\emptyset$.

Now pick by Lemma 1.1 $p=p+p$ with $p\in\bigcap_{m=1}^\infty c\ell(FS(\langle n!\rangle_{n=m}^\infty))$ and pick $q\in\beta\mathbb{N}\backslash\mathbb{N}$ with $\{2\cdot(k!):k\in\mathbb{N}\}\in q$. Then $p,q\in\bigcap_{n=1}^\infty c\ell(\mathbb{N}n)$ so $q+p\in\bigcap_{n=1}^\infty c\ell(\mathbb{N}n)$. By Theorem 2.4, $q+p\in T$. Since $\{2\cdot(k!):k\in\mathbb{N}\text{ and }k\geq 4\}\subseteq\{x\in\mathbb{N}:B-x\in p\}$, one has $q+p\in c\ell B$.

The last conclusion of the theorem follows from the first. $\Box$

The following result is a special case of Theorem 3.9, but its proof is much simpler so we present it separately.

3.6 **Theorem**. *$S_3\subsetneqq S_2$.*

Proof. Let $A=\{2^{2m}-2^{2n}:m,n\in\mathbb{N}\text{ and }m>n\}$. It is easy to see that one cannot get any $x_1,x_2,x_3\in A$ with $\{x_1+x_2,\,x_1+x_3,\,x_2+x_3\}\subseteq A$. Thus $(c\ell A)\cap S_3=\emptyset$. To see that $(c\ell A)\cap S_2\neq\emptyset$ we use Theorem 2.6. So let $\mathcal{F}$ be a finite partition of A. For each $F\in\mathcal{F}$, let $B(F)=\{\{n,m\}:n,m\in\mathbb{N}\text{ and }m>n\text{ and }2^{2m}-2^{2n}\in F\}$. By Ramsey's Theorem [8, p. 7] pick $F\in\mathcal{F}$ and $n<m<r$ in $\mathbb{N}$ with $\{\{n,m\},\{n,r\},\{m,r\}\}\subseteq B(F)$. Let $x_1=2^{2m}-2^{2n}$ and $x_2=2^{2r}-2^{2m}$. Then $x_1+x_2=2^{2r}-2^{2n}$ so $\{x_1,x_2,x_1+x_2\}\subseteq F$. $\Box$

For our proof of Theorem 3.9 we need the following result. Given a sequence $\langle F_t\rangle_{t=1}^n$ of sets we write $FU(\langle F_t\rangle_{t=1}^n) = \{\bigcup_{t\in G} F_t : G$ is a (finite) nonempty subset of $\{1,2,\ldots,n\}\}$.

3.7 **Theorem**. (Nešetřil and Rödl). *Let $r, n \in \mathbb{N}$. There is a finite set $\mathcal{S}$ of finite nonempty sets such that:*

(a) Whenever $\mathcal{S} = \bigcup_{i=1}^r \mathcal{B}_i$, there exist $i \in \{1,2,\ldots,r\}$ and pairwise disjoint $F_1, F_2, \ldots, F_n$ in $\mathcal{S}$ with $FU(\langle F_t\rangle_{t=1}^n) \subseteq \mathcal{B}_i$ and

(b) there do not exist pairwise disjoint $F_1, F_2, \ldots, F_{n+1}$ in $\mathcal{S}$ with $FU(\langle F_t\rangle_{t=1}^{n+1}) \subseteq \mathcal{S}$.

Proof. [18, Theorem 1.1]. (Or see [7].) (The fact that $\mathcal{S}$ and the members of $\mathcal{S}$ are finite is not stated, but follows from the proof.) $\square$

The following corollary is not stated in [7] or [18], and we feel it is interesting in its own right.

3.8 **Corollary**. *Let $n \in \mathbb{N}\backslash\{1\}$. There is a set $A \subseteq \mathbb{N}$ such that*

(a) Whenever $\mathcal{F}$ is a finite partition of A there exist $B \in \mathcal{F}$ and $\langle x_t\rangle_{t=1}^n$ in $\mathbb{N}$ with $FS(\langle x_t\rangle_{t=1}^n) \subseteq B$ and

(b) there does not exist $\langle x_t\rangle_{t=1}^{n+1}$ with $FS(\langle x_t\rangle_{t=1}^{n+1}) \subseteq A$.

Proof. Pick by Theorem 3.7 a sequence $\langle \mathcal{S}_r\rangle_{r=1}^\infty$ such that

(i) for each $r \in \mathbb{N}$, $\mathcal{S}_r$ is a finite set of finite nonempty subsets of $\mathbb{N}$ and $\max(\bigcup \mathcal{S}_r) < \min(\bigcup \mathcal{S}_{r+1})$;

(ii) for each $r \in \mathbb{N}$, whenever $\mathcal{S}_r = \bigcup_{i=1}^r \mathcal{B}_i$, there exist $i \in \{1,2,\ldots,r\}$ and pairwise disjoint $F_1, F_2, \ldots, F_n$ in $\mathcal{S}_r$ with $FU(\langle F_t\rangle_{t=1}^n) \subseteq \mathcal{B}_i$ and

(iii) for each $r \in \mathbb{N}$, there do not exist pairwise disjoint $F_1, F_2, \ldots, F_{n+1}$ with $FU(\langle F_t\rangle_{t=1}^{n+1}) \subseteq \mathcal{S}_r$.

Let $\mathcal{S} = \bigcup_{r=1}^\infty \mathcal{S}_r$. Then

(iv) whenever $\mathcal{F}$ is a finite partition of $\mathcal{S}$, there exist $\mathcal{B} \in \mathcal{F}$ and pairwise disjoint $F_1, F_2, ..., F_n$ in $\mathcal{S}$ with $FU(\langle F_t\rangle_{t=1}^n) \subseteq \mathcal{B}$, and

(v) there do not exist pairwise disjoint $F_1, F_2, \ldots, F_{n+1}$ in $\mathcal{S}$with $FU(\langle F_t\rangle_{t=1}^{n+1}) \subseteq \mathcal{S}$.

Indeed, (iv) is immediate since if $r = |F|$ one has $\mathcal{S}_r \subseteq \mathcal{S}$. To verify (v), suppose we have pairwise disjoint $F_1, F_2, \ldots, F_{n+1}$ in $\mathcal{S}$ with $FU(\langle F_t\rangle_{t=1}^{n+1}) \subseteq \mathcal{S}$. Observe that, given any $r \in \mathbb{N}$ and any $G \in \mathcal{S}$, $G \in \mathcal{S}_r$ if and only if $\min(\bigcup \mathcal{S}_r) \leq \min G$ and $\max G \leq \max(\bigcup \mathcal{S}_r)$. Pick $r \in \mathbb{N}$ with $F_1 \in \mathcal{S}_r$. If any $F_t \notin \mathcal{S}_r$ we have by the above observation that $F_1 \cup F_t \notin \mathcal{S}$. Thus each $F_t \in \mathcal{S}_r$ so, again using the observation, $FU(\langle F_t\rangle_{t=1}^{n+1}) \subseteq \mathcal{S}_r$, contradicting (iii).

Now let $A = \{\sum_{t\in F} 3^t : F \in \mathcal{S}\}$. Given a finite partition $\mathcal{F}$ of A and $B \in \mathcal{F}$, let $\mathcal{G}(B) = \{F \in \mathcal{S} : \sum_{t\in F} 3^t \in B\}$. Then $\{\mathcal{G}(B) : B \in \mathcal{F}\}$ is a finite partition of $\mathcal{S}$ so by (iv), pick $B \in \mathcal{F}$ and pairwise disjoint $F_1, F_2, \ldots, F_n$ in $\mathcal{S}$ with $FU(\langle F_t\rangle_{t=1}^n) \subseteq \mathcal{G}(B)$. For $t \in \{1, 2, \ldots, n\}$, let $x_t = \sum_{i\in F_t} 3^i$. Then $FS(\langle x_t\rangle_{t=1}^n) \subseteq B$, so (a) holds.

To verify (b), suppose we have $x_1, x_2, \ldots, x_{n+1}$ in $\mathbb{N}$ with $FS(\langle x_t\rangle_{t=1}^{n+1}) \subseteq A$. For each $t \in \{1, 2, \ldots, n+1\}$, pick F_t such that $x_t = \sum_{i\in F_t} 3^i$. We claim that the sets $F_1, F_2, \ldots, F_{n+1}$ are pairwise disjoint (so that $FU(\langle F_t\rangle_{t=1}^{n+1}) \subseteq \mathcal{S}$, contradicting (v)). Suppose instead we have $t \neq s$ with $F_t \cap F_s \neq \emptyset$. Then $x_t + x_s = \sum_{i\in F_t\Delta F_s} 3^i + \sum_{i\in F_t\cap F_s} 2\cdot 3^i$. But $x_t + x_s \in A$ so for some G, $x_t + x_s = \sum_{i\in G} 3^i$, contradicting the uniqueness of ternary expansions. $\Box$

3.9 **Theorem**. *Let $n \in \mathbb{N}\backslash\{1\}$. Then $S_{n+1} \subsetneq S_n$.*

Proof. Pick A as guaranteed by Corollary 3.8. By (b), $(c\ell A) \cap S_{n+1} = \emptyset$ while by (a) and Theorem 2.7, $(c\ell A) \cap S_n \neq \emptyset$. $\Box$

Now we need to show that $C \neq S_2$. We will utilize $\beta\mathbb{Z}$. We brush aside the distinction between ultrafilters on $\mathbb{Z}$ with $\mathbb{N}$ as a member and ultrafilters on $\mathbb{N}$, and thus pretend that $\beta\mathbb{N} \subseteq \beta\mathbb{Z}$. Given $p \in \beta\mathbb{N}$ we let $-p = \{-A : A \in p\}$ and note that $-p \in \beta\mathbb{Z}$. (But be cautioned that unless $p \in \mathbb{N}$, $-p + p \neq 0$; in fact $\beta\mathbb{N}\backslash\mathbb{N}$ is a left ideal of $\beta\mathbb{Z}$ so if $p \in \beta\mathbb{N}\backslash\mathbb{N}$ then also $-p + p \in \beta\mathbb{N}\backslash\mathbb{N}$.)

3.10 **Lemma**. *Let φ be a homomorphism from $\beta\mathbb{Z}$ to the circle group $\mathbb{T}$ and let $p \in \beta\mathbb{N}$. Then $\varphi(-p) = -\varphi(p)$.*

Proof. Note that the function $f : \beta\mathbb{N} \longrightarrow \beta\mathbb{Z}$ defined by $f(p) = -p$ is continuous. For all $n \in \mathbb{N}$, $\varphi(-n) = -\varphi(n)$ (since $\varphi|_{\mathbb{Z}}$ is a group homomorphism). Thus $\varphi \circ f$ and $-\varphi$ are continuous functions agreeing on $\mathbb{N}$, hence on $\beta\mathbb{N}$. $\Box$

3.11 **Lemma**. *Let $\langle x_n\rangle_{n=1}^\infty$ be any increasing sequence in $\mathbb{N}$ and let A and B be infinite subsets of $\mathbb{N}$. Let $D = \{x_n + x_m - x_r - x_s : n > m+3 > m > r+3 > r > s+3$ and $n, s \in A$ and $m, r \in B\}$ and let $p, q \in \beta\mathbb{N}\backslash\mathbb{N}$ with $\{x_n : n \in A\} \in p$ and $\{x_n : n \in B\} \in q$. Then $D \in -p + -q + q + p$ and $-p + -q + q + p \in C$. In particular D is a rational approximation set.*

Proof. To see that $-p + -q + q + p \in C$ it suffices (as is well known and explained in the introduction to [1]) to let φ be a homomorphism from $\beta\mathbb{N}$ to $\mathbb{T}$ and show that $\varphi(-p + -q + q + p) = [0]$. To this end let such φ be given. Define $\tau : \mathbb{Z} \longrightarrow \mathbb{T}$ by $\tau(0) = [0]$, and $\tau(n) = \varphi(n)$ and $\tau(-n) = -\varphi(n)$ for $n \in \mathbb{N}$. Then the continuous extension τ^β of τ to $\beta\mathbb{Z}$ is a homomorphism and τ^β agrees with φ on $\beta\mathbb{N}$. Thus, using Lemma 3.10, we have $\varphi(-p+-q+q+p) = \tau(-p+-q+q+p) = \tau(-p)+\tau(-q)+\tau(q)+\tau(p) = -\tau(p)-\tau(q)+\tau(q)+\tau(p) = [0]$.

It is completely routine to verify that $D \in -p+-q+q+p$. The "in particular" conclusion follows from Lemma 2.9. $\square$

3.12 **Lemma**. *Let $\langle x_n\rangle_{n=1}^{\infty}$ be a sequence in $\mathbb{N}$ such that for each $n \in \mathbb{N}$, $x_{n+1} \geq 2x_n$. Let A and B be disjoint infinite subsets of $\mathbb{N}$ such that for some $i,j \in \{0,1,2\}$ $A \subseteq \mathbb{N}3+i$ and $B \subseteq \mathbb{N}3+j$. Let $D = \{x_n + x_m - x_r - x_s : n > m+3 > m > r+3 > r > s+3$ and $n,s \in A$ and $m,r \in B\}$. There do not exist $a,b \in D$ with $a+b \in D$.*

Proof. Suppose we have $a,b \in D$ with $a+b \in D$ and pick $n_1 > m_1+3 > m_1 > r_1+3 > r_1 > s_1+3$, $n_2 > m_2+3 > m_2 > r_2+3 > r_2 > s_2+3$, and $n_3 > m_3+3 > m_3 > r_3+3 > r_3 > s_3+3$ such that $a = x_{n_1}+x_{m_1}-x_{r_1}-x_{s_1}$, $b = x_{n_2}+x_{m_2}-x_{r_2}-x_{s_2}$, and $a+b = x_{n_3}+x_{m_3}-x_{r_3}-x_{s_3}$ and $\{n_1,n_2,n_3,s_1,s_2,s_3\} \subseteq A$ and $\{m_1,m_2,m_3,r_1,r_2,r_3\} \subseteq B$. Then we have

$$(*) \qquad x_{n_1}+x_{m_1}+x_{n_2}+x_{m_2}+x_{r_3}+x_{s_3} = x_{n_3}+x_{m_3}+x_{r_1}+x_{s_1}+x_{r_2}+x_{s_2}.$$

We may assume without loss of generality that $n_1 \geq n_2$. We claim first that $n_1 = n_3$. Suppose $n_1 < n_3$. Then since $n_1, n_3 \in \mathbb{N}3+i$, the left hand side of (*) is at most $x_{n_3-3}+x_{n_3-6}+x_{n_3-3}+x_{n_3-6}+x_{n_3-6}+x_{n_3-9} \leq x_{n_3-2}+x_{n_3-5}+x_{n_3-6}+x_{n_3-9} < x_{n_3}$, a contradiction. (Observe that for each n, $x_{n+1} > \sum_{t=1}^{n} x_t$.) Similarly if we had $n_3 < n_1$ we would have that the right hand side of (*) is at most $x_{n_1-3}+x_{n_1-6}+x_{n_1-6}+x_{n_1-9}+x_{n_1-6}+x_{n_1-9} \leq x_{n_1-3}+x_{n_1-5}+x_{n_1-8}+x_{n_1-6} < x_{n_1}$. Thus $n_1 = n_3$ so we have

$$(**) \qquad x_{m_1}+x_{n_2}+x_{m_2}+x_{r_3}+x_{s_3} = x_{m_3}+x_{r_1}+x_{s_1}+x_{r_2}+x_{s_2}.$$

Now $n_2 \in A$ and $m_1 \in B$ so $n_2 \neq m_1$. We claim that $n_2 < m_1$ so suppose instead that $n_2 > m_1$. If $m_3 < n_2$ we have (since $n_2 > m_1 > r_1+3$) that the right hand side of (**) is at most $x_{n_2-1}+x_{n_2-4}+x_{n_2-7}+x_{n_2-6}+x_{n_2-9} < x_{n_2}$, a contradiction. If $m_3 > n_2$ ($> m_1$) we have that the left hand side of (**) is at most $x_{m_3-3}+x_{m_3-1}+x_{m_3-4}+x_{m_3-3}+x_{m_3-6} \leq x_{m_3-1}+x_{m_3-2}+x_{m_3-4}+x_{m_3-6} < x_{m_3}$, a contradiction. Thus $n_2 < m_1$ as claimed.

Now we claim $m_3 = m_1$. Suppose first that $m_3 < m_1$. Then the right hand side of (**) is at most $x_{m_1-3}+x_{m_1-3}+x_{m_1-6}+x_{m_1-7}+x_{m_1-10} < x_{m_1}$, a contradiction. Similarly if $m_1 < m_3$ one has the left hand side of (**) is at most $x_{m_3-3}+x_{m_3-4}+x_{m_3-7}+x_{m_3-3}+x_{m_3-6} < x_{m_3}$, a contradiction. Thus $m_3 = m_1$ so we have

$$(***) \qquad x_{n_2}+x_{m_2}+x_{r_3}+x_{s_3} = x_{r_1}+x_{s_1}+x_{r_2}+x_{s_2}.$$

Now we claim that $n_2 < r_1$. Suppose not. Then since $n_2 \in A$ and $r_1 \in B$ we have $n_2 > r_1$ so the right hand side of (***) is at most $x_{n_2-1} + x_{n_2-4} + x_{n_2-6} + x_{n_2-9} < x_{n_2}$, a contradiction. Thus $n_2 < r_1$ as claimed.

Next we claim $r_3 = r_1$. If $r_3 < r_1$ we have the left hand side of (***) is at most $x_{r_1-1} + x_{r_1-4} + x_{r_1-3} + x_{r_1-6} < x_{r_1}$, a contradiction. If $r_3 > r_1$ $(> n_2)$ we have the right hand side of (***) is at most $x_{r_3-3} + x_{r_3-6} + x_{r_3-10} + x_{r_3-13} < x_{r_3}$. Thus $r_3 = r_1$ so we have

$$(****) \qquad x_{n_2} + x_{m_2} + x_{s_3} = x_{s_1} + x_{r_2} + x_{s_2}.$$

Continuing in this fashion we see that if $n_2 = s_3$ then also $n_2 = s_1$ so that $x_{m_2} + x_{s_3} = x_{r_2} + x_{s_2}$ and hence that $m_2 = r_2$ which is a contradiction.

Thus one must have $n_2 \neq s_3$, and hence that $|\{n_2, m_2, s_3\}| = 3$. Now if $s_1 = s_2$ one has $s_1 = s_2 < r_2 < n_2$ so the right hand side of (****) is at most $x_{n_2-9}+x_{n_2-6}+x_{n_2-9} < x_{n_2}$, a contradiction. Thus $s_1 \neq s_2$ so $|\{s_1, r_2, s_2\}| = 3$. Since $x_{n+1} > \sum_{t=1}^{n} x_t$ for each n, expressions in $FS(\langle x_t \rangle_{t=1}^{\infty})$ are unique. Thus from (****) we have $\{n_2, m_2, s_3\} = \{s_1, r_2, s_2\}$ so that $\{m_2\} = \{n_2, m_2, s_3\} \cap B = \{s_1, r_2, s_2\} \cap B = \{r_2\}$ while $r_2 < m_2$. This contradiction completes the proof. $\Box$

3.13 **Theorem**. *Let $\langle x_n \rangle_{n=1}^{\infty}$ be a sequence in $\mathbb{N}$ such that for each n, $x_{n+1} \geq 2x_n$. Let A and B be disjoint infinite subsets of $\mathbb{N}$ and let $p, q \in \beta\mathbb{N}\backslash\mathbb{N}$ such that $\{x_n : n \in A\} \in p$ and $\{x_n : n \in B\} \in q$. Then $-p + -q + q + p \in C\backslash S_2$.*

Proof. Pick $i, j \in \{0, 1, 2\}$ such that $\mathbb{N}3+i \in p$ and $\mathbb{N}3+j \in q$. Let $A' = A\cap(\mathbb{N}3+i)$ and $B' = B\cap(\mathbb{N}3+j)$. Let $D = \{x_n+x_m-x_r-x_s : n > m+3 > m > r+3 > r > s+3$ and $n, s \in A'$ and $m, r \in B'\}$. By Lemma 3.11, $D \in -p+-q+q \in p$ and $-p+-q+q+p \in C$. By Lemma 3.12 $-p + -q + q + p \notin S_2$. $\Box$

It is natural to ask whether in lieu of $-p + -q + q + p$ above one might be able to get by with $-p + p$ for some suitable p. We conclude this section by showing that this is not possible.

3.14 **Theorem**. *Let $p \in \beta\mathbb{N}\backslash\mathbb{N}$. Then $-p + p \in S_2$.*

Proof. Let $A \in -p + p$. Then $\{x \in \mathbb{Z} : A - x \in p\} \in -p$ so $B = \{x \in \mathbb{N} : A + x \in p\} \in p$. Pick $x_1 \in B$, pick $x_2 \in B \cap (A + x_1)$, pick $x_3 \in (A + x_1) \cap (A + x_2)$. Let $y = x_2 - x_1$ and let $z = x_3 - x_2$. Then $y, z \in A$ and $y + z = x_3 - x_1 \in A$. $\Box$

4. **Connections with other structures**. The interaction of the operations $+$ and $\cdot$ on $\beta\mathbb{N}$ has been a very useful combinatorial tool. (See [3] for an example where this interaction is utilized several times in succession.)

Recall that, given p and q in $\beta\mathbb{N}$ and $A \subseteq \mathbb{N}$, $A \in p \cdot q$ if and only if $\{x \in \mathbb{N} : A/x \in q\} \in p$ where $A/x = \{y \in \mathbb{N} : y \cdot x \in A\}$.

It is not generally true that for $n \in \mathbb{N}$ and $p \in \beta\mathbb{N}$ one has $n \cdot p = p + p + ... + p$ (n-times). (For example one sees easily that if $n \neq 1$ then $n \cdot p \neq p$ while if $p = p + p$, then $p = p + p + ... + p$ (n-times).) On the other hand we do have the following lemma. Recall that, given $p \in \beta\mathbb{N}$ and a sequence $\langle x_n \rangle_{n=1}^{\infty}$ in a topological space X, one has $p\text{-}\lim_{n\in\mathbb{N}} x_n = y$ if and only if for each neighbourhood U of y, $\{n \in \mathbb{N} : x_n \in U\} \in p$.

4.1 **Lemma**. *Let $(G, +)$ be a compact topological group, let $\varphi : \beta\mathbb{N} \longrightarrow G$ be a continuous homomorphism, let $p \in \beta\mathbb{N}$, and let $n \in \mathbb{N}$. Then $\varphi(n \cdot p) = n \cdot \varphi(p)$, where $n \cdot \varphi(p) = \varphi(p) + ... + \varphi(p)$ (n-times).*

Proof. Recall that the function λ_n defined by $\lambda_n(p) = n \cdot p$ is continuous since $n \in \mathbb{N}$. Recall further that by the joint continuity of addition in G, we have $n \cdot p\text{-}\lim_{m\in\mathbb{N}} \varphi(m) = p\text{-}\lim_{m\in\mathbb{N}} n \cdot \varphi(m)$. Thus we have $\varphi(n \cdot p) = \varphi(n \cdot p\text{-}\lim_{m\in\mathbb{N}} m) = p\text{-}\lim_{m\in\mathbb{N}} \varphi(n \cdot m) = p\text{-}\lim_{m\in\mathbb{N}} n \cdot \varphi(m) = n \cdot p\text{-}\lim_{m\in\mathbb{N}} \varphi(m) = n \cdot \varphi(p\text{-}\lim_{m\in\mathbb{N}} m) = n \cdot \varphi(p)$.$\square$

4.2 **Theorem**. *C is a two sided ideal of* $(\beta\mathbb{N}, \cdot)$.

Proof. Let G be a compact topological group with identity 0 and let $\varphi : \beta\mathbb{N} \longrightarrow G$ be a homomorphism. Let $p \in C$ and let $q \in \beta\mathbb{N}$. Pick nets $\langle x_\eta \rangle_{\eta\in D}$ and $\langle y_\tau \rangle_{\tau\in E}$ in $\mathbb{N}$ converging to p and q respectively.

Then $\varphi(q \cdot p) = \varphi((\lim_{\tau\in E}\ y_\tau) \cdot p) = \lim_{\tau\in E}\ \varphi(y_\tau \cdot p) = \lim_{\tau\in E}\ (y_\tau \cdot \varphi(p)) = \lim_{\tau\in E}\ (y_\tau \cdot 0) = 0$.

Now let $z = \varphi(q)$ and define $\tau : \mathbb{N} \longrightarrow G$ by $\tau(n) = n \cdot z$. Then the continuous extension $\tau^\beta : \beta\mathbb{N} \longrightarrow G$ is a homomorphism. Thus $\varphi(p \cdot q) = \varphi((\lim_{\eta\in D}\ x_\eta) \cdot q) = \lim_{\eta\in D}\ \varphi(x_\eta \cdot q) = \lim_{\eta\in D}\ (x_\eta \cdot \varphi(q)) = \lim_{\eta\in D}\ \tau(x_\eta) = \lim_{\eta\in D}\ \tau^\beta(x_\eta) = \tau^\beta(\lim_{\eta\in D}\ x_\eta) = \tau^\beta(p) = 0$.$\square$

4.3 **Theorem**. *For each $n \in \mathbb{N}\backslash\{1\}$, S_n is a two sided ideal of* $(\beta\mathbb{N}, \cdot)$.

Proof. Let $n \in \mathbb{N}$, let $p \in S_n$ and let $q \in \beta\mathbb{N}$.

To see that $q \cdot p \in S_n$, let $A \in q \cdot p$ and pick $y \in \mathbb{N}$ such that $A/y \in p$. Pick $\langle x_t \rangle_{t=1}^{n}$ with $FS(\langle x_t \rangle_{t=1}^{n}) \subseteq A/y$. Then $FS(\langle y \cdot x_t \rangle_{t=1}^{n}) \subseteq A$.

To see that $p \cdot q \in S_n$, let $A \in p \cdot q$ and pick $\langle x_t \rangle_{t=1}^{n}$ such that $FS(\langle x_t \rangle_{t=1}^{n}) \subseteq \{y \in \mathbb{N} : A/z \in q)$. Pick $y \in \bigcap\{A/z : z \in FS(\langle x_t \rangle_{t=1}^{n})\}$. Then $FS(\langle y \cdot x_t \rangle_{t=1}^{n}) \subseteq A$. $\square$

In the process of our study of the semigroup C, we were led to the following result (and its fortuitous corollary). By a divisible sequence $\langle x_n \rangle_{n=1}^{\infty}$ in $\mathbb{N}$ we simply mean an increasing sequence with the property that each x_n divides x_{n+1}.

Recall that we are representing the circle group $\mathbb{T}$ as $\mathbb{R}/\mathbb{Z}$. By $\mathbb{T}^{\mathbb{T}}$ we mean the set of all functions from $\mathbb{T}$ to $\mathbb{T}$ with the product topology (= "topology of pointwise convergence").

4.4 **Theorem**. *Define $h : \mathbb{N} \longrightarrow \mathbb{T}^{\mathbb{T}}$ by $h(n)(\alpha) = n \cdot \alpha$ and let h^β be the continuous extension of h to $\beta\mathbb{N}$. Let $\langle x_n \rangle_{n=1}^\infty$ be any divisible sequence in $\mathbb{N}$. Then h^β is one-to-one on $c\ell\{x_n : n \in \mathbb{N}\}$.*

Proof. Let p and q be distinct elements of $c\ell\{x_n : n \in \mathbb{N}\}$. Pick disjoint A and B contained in $\mathbb{N}$ such that $\{x_n : n \in A\} \in p$ and $\{x_n : n \in B\} \in q$. Since $\{x_n : n \in \mathbb{N}\} = \bigcup_{i=0}^2 \{x_n : n \equiv i (\mathrm{mod} 3)\}$ we may presume we have some $i \in \{0, 1, 2\}$ such that for all $n, m \in A$, $n \equiv m (\mathrm{mod} 3)$. As a consequence, if $n, m \in A$ and $n < m$ then $n+3 \leq m$ so $x_m \geq x_{n+3} \geq 8 \cdot x_n$. Now let $t = \sum_{n \in A} \lfloor x_{n+1}/(2x_n) \rfloor / x_{n+1}$, where $\lfloor\ \rfloor$ denotes the greatest integer function. (Since $\langle x_n \rangle_{n=1}^\infty$ is a divisible sequence we have each $x_n \geq 2^{n-1}$ so $\lfloor x_{n+1}/(2x_n) \rfloor / x_{n+1} \leq 1/(2x_n) \leq 1/2^n$ so the series defining t converges (and $0 < t < 1$). As before write $[t] = t + \mathbb{Z}$. We show that $h^\beta(p)([t]) \neq h^\beta(q)([t])$.

Let $D = \{[s] : 1/3 \leq s \leq 4/7\}$ and let $E = \{[s] : 0 \leq s \leq 9/28\}$. Then D and E are disjoint closed subsets of T. We show that if $n \in A$ then $h(x_n)([t]) \in D$ and if $n \in B$ then $h(x_n)([t]) \in E$. As a consequence we will have that $h^\beta(p)([t]) \in D$ and $h^\beta(q)([t]) \in E$.

To this end we first observe that given any $n \in \mathbb{N}$, $\sum\{(\lfloor x_{k+1}/(2x_k) \rfloor / x_{k+1}) \cdot x_n : k \in A \text{ and } k \geq n+3\} \leq 1/14$. Indeed, given the first $k \in A$ with $k \geq n+3$ one has $(\lfloor x_{k+1}/(2x_k) \rfloor / x_{k+1}) \cdot x_n \leq x_n/(2x_k) \leq 1/16$. Given $k, m \in A$ with $m > k > n+3$, one has $x_m \geq x_{k+3} \geq 8 \cdot x_k$. Consequently $\sum\{(\lfloor x_{k+1}/(2x_k) \rfloor / x_{k+1}) \cdot x_n : k \in A$ and $k \geq n+3\} \leq (1/2) \sum_{k=1}^\infty 1/8^k = 1/14$.

Now let $n \in A$. Then $h(x_n)([t]) = x_n \cdot [t] = [x_n \cdot t]$. Now $x_n \cdot t = \sum\{(\lfloor x_{k+1}/(2x_k) \rfloor / x_{k+1}) \cdot x_n : k \in A \text{ and } k < n\} + (\lfloor x_{n+1}/(2x_n) \rfloor / x_{n+1}) \cdot x_n + \sum\{(\lfloor x_{k+1}/(2x_k) \rfloor / x_{k+1}) \cdot x_n : k \in A \text{ and } k \geq n+3\}$. The first of these sums is some integer ℓ and the last of these is at most $1/14$. Now consider the middle term. We have $(\lfloor x_{n+1}/(2x_n) \rfloor / x_{n+1}) \cdot x_n \leq 1/2$ and equality holds if x_{n+1}/x_n is even. If x_{n+1}/x_n is odd we have $x_{n+1} \geq 3x_n$ so $(\lfloor x_{n+1}/(2x_n) \rfloor / x_{n+1}) \cdot x_n = (x_{n+1}/(2x_n) - 1/2) \cdot x_n / x_{n+1} = 1/2 - 1/2 \cdot (x_n/x_{n+1}) \geq 1/2 - 1/6 = 1/3$. Thus $\ell + 1/3 \leq x_n \cdot t \leq \ell + 1/2 + 1/14$ so $[x_n \cdot t] \in D$ as required.

Finally let $n \in B$. Then $x_n \cdot t = \sum\{(\lfloor x_{k+1}/(2x_k) \rfloor / x_{k+1}) \cdot x_n : k \in A \text{ and } k < n\} + \sum\{(\lfloor x_{k+1}/(2x_k) \rfloor / x_{k+1}) \cdot x_n : k \in A \text{ and } n < k < n+3\} + \sum\{(\lfloor x_{k+1}/(2x_k) \rfloor / x_{k+1}) \cdot x_n : k \in A \text{ and } k \geq n+3\}$. Again the first sum is some integer ℓ and the last is at most 1/14. The middle sum has at most one term which is at most 1/4. Thus $\ell \leq x_n \cdot t \leq \ell + 1/4 + 1/14$ so $[x_n \cdot t] \in E$ as required. $\square$

We obtain as a corollary the following result communicated to us by Kenneth Berg. For extensions of this result see [2]. Recall that, given $f : \mathbb{T} \longrightarrow \mathbb{T}$, the enveloping semigroup of f is the closure in $\mathbb{T}^{\mathbb{T}}$ of $\{f^n : n \in \mathbb{N}\}$.

4.5 **Corollary**. *Define $f : \mathbb{T} \longrightarrow \mathbb{T}$ by $f(\alpha) = 2 \cdot \alpha$. Then the enveloping semigroup of f can be identified with $\beta\mathbb{N}$.*

Proof. Note that $f^n(\alpha) = 2^n \cdot \alpha$ so if h is defined as in Theorem 4.4, one has for each $n \in \mathbb{N}$, $h(2^n) = f^n$. Thus the enveloping semigroup of f is $h[c\ell\{2^n : n \in \mathbb{N}\}]$. Since h is one-to-one on this closure, it is a homeomorphism on $c\ell\{2^n : n \in \mathbb{N}\}$. $\square$

It was shown in [16] that if p is a right cancellable element of $\beta\mathbb{N}$, then every element of $c\ell\{p, p+p, p+p+p, \ldots\}$ is right cancellable. As a consequence, any such semigroup has a closure which misses the set of idempotents. We show next that one can get semigroups in $\beta\mathbb{N}$ whose closure is reasonably far removed from the idempotents. (In particular the closure cannot be a semigroup.)

4.6 **Theorem**. *Let $\langle x_n \rangle_{n=1}^{\infty}$ be any divisible sequence in $\mathbb{N}$ and let $p \in (c\ell\{x_n : n \in \mathbb{N}\})\backslash\mathbb{N}$. Then $c\ell\{p, p+p, p+p+p, \ldots\}) \cap T = \emptyset$.*

Proof. We may presume $x_1 = 1$. (If $x_1 > 1$, let $y_1 = 1$ and $y_{n+1} = x_n$ for $n \in \mathbb{N}$. Then $(c\ell\{y_n : n \in \mathbb{N}\})\backslash\mathbb{N} = (c\ell\{x_n : n \in \mathbb{N}\})\backslash\mathbb{N}$.) For each $n \in \mathbb{N}$ let $a_n = x_{n+1}/x_n$. Then each $m \in \mathbb{N}$ has a unique expression of the form $\sum_{t \in F} b_t \cdot x_t$ where for each $t \in F$, $b_t \in \{1, 2, \ldots, a_t - 1\}$. Further x_n divides m if and only if $\min F \geq n$. Given $m \in \mathbb{N}$, define $c(m) = |F|$ where $m = \sum_{t \in F} b_t \cdot x_t$ as above. Let $c^{\beta} : \beta\mathbb{N} \longrightarrow \beta\mathbb{N}$ be the continuous extension of c. Since c is constantly equal to 1 on $\{x_n : n \in \mathbb{N}\}$ we have $c^{\beta}(p) = 1$.

Let $X = (\bigcap_{n=1}^{\infty} c\ell(\mathbb{N}x_n)) \cap (\bigcap_{n=1}^{\infty} c\ell\{m \in \mathbb{N} : c(m) > n\})$. We observe that the idempotents are all in X. We have $C \subseteq \bigcap_{n=1}^{\infty} c\ell(\mathbb{N}x_n)$. To see that the idempotents are contained in $\bigcap_{n=1}^{\infty} c\ell\{m \in \mathbb{N} : c(m) > n\}$, let $e = e + e$ and suppose that for some n, $\{m \in \mathbb{N} : c(m) \leq n\} \in e$. Then, since e is an ultrafilter one has in fact that for some n, $\{m \in \mathbb{N} : c(m) = n\} \in e$. Let $A = \{m \in \mathbb{N} : c(m) = n\}$ and pick $m \in A$ such that $A - m \in e$. Pick t such that $x_t > m$ and pick $k \in \mathbb{N}x_t \cap (A - m)$. Then $c(k + m) = c(k) + c(m) > n$ so $k + m \notin A$, a contradiction.

Now suppose $(c\ell\{p, p+p, p+p+p, \ldots\}) \cap T \neq \emptyset$. By Theorem 2.4, $T = c\ell \bigcup\{\mathbb{N}+e : e \in \beta\mathbb{N}$ and $e + e = e\}$, so $T \subseteq c\ell(\bigcup_{n=1}^{\infty} n + X)$. Thus $c\ell\{p, p+p, p+p+p, \ldots\} \cap$

$c\ell(\bigcup_{n=1}^{\infty} n+X) \neq \emptyset$ so by Lemma 1.3 either $c\ell\{p, p+p, p+p+p, \ldots\} \cap (\bigcup_{n=1}^{\infty} n+X) \neq \emptyset$ or $\{p, p+p, p+p+p, \ldots\} \cap c\ell(\bigcup_{n=1}^{\infty} n+X) \neq \emptyset$. But $c\ell\{p, p+p, p+p+p, \ldots\} \subseteq \bigcap_{n=1}^{\infty} c\ell(\mathbb{N}x_n)$ and $\bigcap_{n=1}^{\infty} c\ell(\mathbb{N}x_n) \cap (\bigcup_{n=1}^{\infty} n + X) = \emptyset$. Thus we have some $q \in \{p, p+p, p+p+p, \ldots\} \cap c\ell(\bigcup_{n=1}^{\infty} n + X)$. Now $q = p + p + ... + p$ (m-times) so $c^{\beta}(q) = m$. Let $A = \{y \in \mathbb{N} : c(y) = m\}$. Then $A \in q$ so $c\ell A \cap (\bigcup_{n=1}^{\infty} n + X) \neq \emptyset$, so pick $n \in \mathbb{N}$ with $c\ell A \cap (n + X) \neq \emptyset$ and pick $r \in c\ell A \cap (n + X)$. Pick $k \in \mathbb{N}$ such that $x_k > n$. Now $r - n \in X \subseteq c\ell(\mathbb{N}x_k) \cap c\ell(\{y \in \mathbb{N} : c(y) > m\})$ so $\mathbb{N}x_k \cap \{y \in \mathbb{N} : c(y) > m\} \cap (A-n) \neq \emptyset$. Pick $y \in \mathbb{N}x_k \cap \{y \in \mathbb{N} : c(y) > m\} \cap (A - n)$. Since $y \in \mathbb{N}x_k$ and $x_k > n$ we have $c(y + n) = c(y) + c(n) > m$ so $y + n \notin A$, a contradiction. $\Box$

On the other hand, we see that no semigroup can get too far removed from the idempotents.

4.7 **Theorem**. *Let S be any subsemigroup of $\beta\mathbb{N}$. Then $(c\ell S) \cap \bigcap_{n=2}^{\infty} S_n \neq \emptyset$.*

Proof. Pick any $p \in S$. Define $\varphi : \mathbb{N} \longrightarrow \beta\mathbb{N}$ by $\varphi(n) = p + p + ... + p$ (n times) and let φ^{β} be the continuous extension to $\beta\mathbb{N}$. Note that $\varphi^{\beta} : \beta\mathbb{N} \longrightarrow \beta\mathbb{N}$ is a homomorphism. Pick any $q \in \bigcap_{n=2}^{\infty} S_n$. Then $\varphi^{\beta}(q) \in c\ell S$. We claim that $\varphi^{\beta}(q) \in \bigcap_{n=2}^{\infty} S_n$.

We show first that for any $A \in \varphi^{\beta}(q)$ and any $n \in \mathbb{N}\backslash\{1\}$, there exist $r_1, r_2, \ldots, r_n$ in $c\ell A$ that commute with each other with $FS(\langle r_t\rangle_{t=1}^{n}) \subseteq c\ell A$. (The fact that $r_1, r_2, \ldots, r_n$ commute with each other is not really relevant except that we do not need to spell out the order of the sums in $FS(\langle r_t\rangle_{t=1}^{n})$.) To see this let $A \in \varphi^{\beta}(q)$ and pick $B \in q$ such that $\varphi^{\beta}[c\ell B] \subseteq cl$ A. Now let $n \in \mathbb{N}\backslash\{1\}$ and (since $q \in S_n$) pick $x_1, x_2, \ldots, x_n$ in B with $FS(\langle x_t\rangle_{t=1}^{n}) \subseteq B$. For each $t \in \{1, 2, \ldots, n\}$, let $r_t = \varphi(x_t)$.

To complete the proof we show by induction on $n \in \mathbb{N}$ that given $A \subseteq \mathbb{N}$, if there exist commuting $r_1, r_2, \ldots, r_n$ with $FS(\langle r_t\rangle_{t=1}^{n}) \subseteq c\ell A$, then there exist $x_1, x_2, \ldots, x_n$ with $FS(\langle x_t\rangle_{t=1}^{n}) \subseteq A$. The case $n = 1$ is trivial, so let $n \in \mathbb{N}$ and assume the statement is true for n and let $r_1, r_2, \ldots, r_{n+1}$ be commuting elements of $c\ell A$ with $FS(\langle r_t\rangle_{t=1}^{n+1}) \subseteq c\ell A$. Let $D = \{x \in \mathbb{N} : A - x \in r_{n+1}\}$. Now given any nonempty $F \subseteq \{1, 2, \ldots, n\}$ we have $A \in \sum_{t\in F} r_t + r_{n+1}$ so $D \in \sum_{t\in F} r_t$. That is $FS(\langle r_t\rangle_{t=1}^{n}) \subseteq c\ell D$. Since also $FS(\langle r_t\rangle_{t=1}^{n}) \subseteq c\ell A$ we have $FS(\langle r_t\rangle_{t=1}^{n}) \subseteq c\ell(A \cap D)$ so by the induction hypothesis choose $\langle x_t\rangle_{t=1}^{n}$ with $FS(\langle x_t\rangle_{t=1}^{n}) \subseteq A \cap D$. Now $A \in r_{n+1}$ and for each nonempty $F \subseteq \{1, 2, \ldots, n\}$, $A - \sum_{t\in F} x_t \in r_{n+1}$ so pick $x_{n+1} \in A \cap \bigcap\{A - \sum_{t\in F} x_t : \emptyset \neq F \subseteq \{1, 2, \ldots, n\}\}$. Then $FS(\langle x_t\rangle_{t=1}^{n+1}) \subseteq A$. $\Box$

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