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CENTRAL SETS THEOREM FOR ARBITRARY ADEQUATE PARTIAL SEMIGROUPS

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ABSTRACT. We establish a Central Sets Theorem valid for arbitrary adequate partial semigroups. Except for the requirement that the sequences considered be *adequate*, it is identical to the currently most general version of the Central Sets Theorem for semigroups. We present an application to the partial semigroup of located words and obtain several related results.

1. Introduction

Since their introduction in 1981 in [3], central sets in semigroups have been shown to have significant combinatorial properties. And since, whenever a semigroup is partitioned into finitely many sets, one of those sets must be central, there are important consequences for Ramsey theory. For example, it is shown in [3] that if C is a central subset of $\mathbb{N}$, then C contains solutions to every partition regular system of homogeneous linear equations. See the survey [6] for many more examples of properties enjoyed by any central set. And see section 2 of this paper for definitions of this or any other unfamiliar notions discussed in this introduction.

Most of the desirable properties enjoyed by central sets are consequences of the Central Sets Theorem. The original version established in [3] is the following. (Given a set X, we write $\mathcal{P}_f(X)$ for the set of finite nonempty subsets of X.)

Theorem 1.1. *Let F be a finite set of sequences in $\mathbb{Z}$ and let C be central in $\mathbb{N}$. There exist a sequence $\langle a_n\rangle_{n=1}^{\infty}$ in $\mathbb{N}$ and a sequence $\langle H_n\rangle_{n=1}^{\infty}$ in $\mathcal{P}_f(\mathbb{N})$ such that*

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(1) *for each* $n \in \mathbb{N}$, $\max H_n < \min H_{n+1}$ *and*
(2) *for each* $K \in \mathcal{P}_f(\mathbb{N})$ *and each* $f \in F$, $\sum_{n\in K}\big(a_n + \sum_{t\in H_n} f(t)\big) \in C$.

In [9, Theorem 14.11 and Theorem 14.15], there are commutative and general versions of the Central Sets Theorem that apply to countably many sequences at a time. In [2, Theorem 2.2 and Corollary 3.10], commutative and general versions are proved that apply to all sequences at once.

What is currently the strongest version is from [11, section 3], which we state now. In this statement, we let $\mathcal{T}$ be the set of all sequences in S, and for $m \in \mathbb{N}$, we let $\mathcal{J}_m = \{t \in \mathbb{N}^m : t(1) < t(2) < \ldots < t(m)\}$.

Theorem 1.2. *Let* $(S,\cdot)$ *be a semigroup and let* C *be central in* S. *There exist functions* $m : \mathcal{P}_f(\mathcal{T}) \to \mathbb{N}$, $\alpha \in \times_{F\in\mathcal{P}_f(\mathcal{T})} S^{m(F)+1}$, *and* $\tau \in \times_{F\in\mathcal{P}_f(\mathcal{T})}\mathcal{J}_{m(F)}$ *such that*

(1) *if* $F, G \in \mathcal{P}_f(\mathcal{T})$ *and* $F \subsetneq G$, *then* $\tau(F)\big(m(F)\big) < \tau(G)(1)$ *and*
(2) *if* $n \in \mathbb{N}$, $G_1, G_2, \ldots, G_n \in \mathcal{P}_f(\mathcal{T})$, $G_1 \subsetneq G_2 \subsetneq \ldots \subsetneq G_n$, *and for each* $i \in \{1,2,\ldots,n\}$, $f_i \in G_i$, *then*

$$\prod_{i=1}^{n}\Big(\big(\prod_{j=1}^{m(G_i)} \alpha(G_i)(j)\cdot f_i\big(\tau(G_i)(j)\big)\big)\cdot \alpha(G_i)(m(G_i)+1)\Big) \in C\,.$$

In the event the semigroup is commutative, the statement of the Central Sets Theorem becomes much simpler.

Corollary 1.3. *Let* $(S,\cdot)$ *be a commutative semigroup and let* C *be central in* S. *There exist functions* $a : \mathcal{P}_f(\mathcal{T}) \to S$ *and* $H : \mathcal{P}_f(\mathcal{T}) \to \mathcal{P}_f(\mathbb{N})$ *such that*

(1) *if* $F, G \in \mathcal{P}_f(\mathcal{T})$ *and* $F \subsetneq G$, *then* $\max H(F) < \min H(G)$ *and*
(2) *if* $n \in \mathbb{N}$, $G_1, G_2, \ldots, G_n \in \mathcal{P}_f(\mathcal{T})$, $G_1 \subsetneq G_2 \subsetneq \ldots \subsetneq G_n$, *and for each* $i \in \{1,2,\ldots,n\}$, $f_i \in G_i$, *then*

$$\prod_{i=1}^{n}\big(a(G_i)\cdot \prod_{t\in H(G_i)} f_i(t)\big) \in C\,.$$

Proof. Let m, α, and τ be as in Theorem 1.2. For $F \in \mathcal{P}_f(\mathcal{T})$, let $a(F) = \prod_{j=1}^{m(F)+1}\alpha(F)(j)$ and let $H(F) = \{\tau(F)(j) : j \in \{1,2,\ldots,m(F)\}\}$. □

In [12], Jillian McLeod establishes a version of Theorem 1.1 valid for commutative adequate partial semigroups. In [13], Kendra Pleasant and in [5], Arpita Ghosh, independently but later, prove a version of Corollary 1.3 for commutative adequate partial semigroups.

In this paper, we show that Theorem 1.2 remains valid for arbitrary adequate partial semigroups, the only adjustment being that $\mathcal{T}$ is replaced by the set of all adequate sequences in S.

A substantial amount of algebraic background material is used, and we present that in section 2, including definitions of all of the notions discussed in this introduction. We present the proof of the main theorem in section 3. Section 4 has some results which we can only prove with the additional assumption that we are dealing with a countable adequate partial semigroup. Section 5 has an alternate proof, which itself uses a partial semigroup, that in any adequate partial semigroup, a piecewise syndetic set is a J-set.

2. Algebraic Background

A compact Hausdorff right topological semigroup is a semigroup $(T,\cdot)$ equipped with a compact Hausdorff topology with respect to which multiplication on the right by each element is continuous. Any compact Hausdorff right topological semigroup T has a smallest two-sided ideal, $K(T)$, which is the union of all of the minimal left ideals of T and is also the union of all of the minimal right ideals of T. If L is a minimal left ideal and R is a minimal right ideal, then $R\cap L$ is a group. In particular, any minimal left ideal has an idempotent and any minimal right ideal has an idempotent. There is a partial ordering of the idempotents of T defined by $e\leq f$ if and only if $e\cdot f=f\cdot e=e$. An idempotent is minimal with respect to this ordering if and only if it is a member of $K(T)$. Given any idempotent f in T, there is a minimal idempotent e in T such that $e\leq f$. (Proofs of the statements in this paragraph can be found in [10, Chapters 1 and 2].)

Given a semigroup $(S,\cdot)$, the operation can be uniquely extended to the Stone-Čech compactification βS of S with the discrete topology so that $(\beta S,\cdot)$ is a compact right topological semigroup with S contained in its topological center (meaning that multiplication on the left by each member of S is continuous). We take the points of βS to be the ultrafilters on S with the principal ultrafilters being identified with the points of S. Given $A\subseteq S$, $\overline{A}=c\ell_{\beta S}A=\{p\in\beta S: A\in p\}$, and $\overline{A}$ is clopen. A subset C of S is said to be *central* in S if and only if C is a member of a minimal idempotent in βS. (In [3], a different but equivalent definition of central is used.)

A *partial semigroup* is a pair $(S,*)$ where S is a nonempty set and $*$ is a binary operation defined on some, but not necessarily all, members of $S\times S$, and for all $a,b,c\in S$, $(a*b)*c=a*(b*c)$ in the sense that if either side is defined, so is the other and they are equal. Partial semigroups are introduced in [1] and used in the context of located words. (See Definition 3.8.) See the first two paragraphs of [8, section 2] for an explanation of why these are interesting objects. We deal with a subset of βS, where we again take S to have the discrete topology.

Definition 2.1. Let $(S,*)$ be a partial semigroup.

(a) If $a \in S$, then $\varphi(a) = \{b \in S : a * b \text{ is defined}\}$.
(b) If $F \in \mathcal{P}_f(S)$, then $\sigma(F) = \bigcap_{a\in F} \varphi(a) =$
$\{b \in S : (\forall a \in F)(a * b \text{ is defined})\}$.
(c) For $s \in S$ and $A \subseteq S$, $s^{-1}A = \{a \in \varphi(s) : s * a \in A\}$.
(d) S is *adequate* if and only if $\sigma(F) \neq \emptyset$ for each $F \in \mathcal{P}_f(S)$.
(e) $\delta S = \bigcap_{F\in\mathcal{P}_f(S)} c\ell_{\beta S}\sigma(F)$.
(f) If $q \in \delta S$ and $p \in \beta S$, then $p * q =$
$\{A \subseteq S : \{x \in S : x^{-1}A \in q\} \in p\}$.
(g) If $p \in \delta S$ and $A \in p$, then $A^\star = A^\star(p) = \{x \in A : x^{-1}A \in p\}$.

Notice that $\delta S \neq \emptyset$ if and only if S is adequate.

Most naturally arising partial semigroups are adequate. A glaring example of a nonadequate partial semigroup is the set of all finite matrices with entries from your favorite ring. If A is a 2×2 matrix and B is a 2×3 matrix, then $\sigma(\{A, B\}) = \emptyset$.

We now gather some basic facts about δS from [8].

Theorem 2.2. *Let $(S,*)$ be an adequate partial semigroup.*

(1) *If $q \in \delta S$ and $p \in \beta S$, then $p * q \in \beta S$.*
(2) *If $q \in \delta S$ and $\rho_q : \beta S \to \beta S$ is defined by $\rho_q(p) = p * q$, then ρ_q is continuous.*
(3) *$(\delta S, *)$ is a compact Hausdorff right topological semigroup.*
(4) *If $p \in \delta S$, $p * p = p$, $A \in p$, and $x \in A^\star$, then $x^{-1}A^\star \in p$.*

Proof. See [8, Lemmas 2.6, 2.7, 2.8, and 2.12 and Theorem 2.10] □

Since $(\delta S, *)$ is a compact Hausdorff right topological semigroup, it has a smallest ideal, $K(\delta S)$.

Definition 2.3. Let $(S,*)$ be an adequate partial semigroup and let $A \subseteq S$. Then A is *central* if and only if there is an idempotent $p \in K(\delta S)$ such that $A \in p$.

We write $\prod_{t\in F} f(t)$ to be the product in increasing order of indices. So, for example, $\prod_{t\in\{1,4,6\}} f(t) = f(1) * f(4) * f(6)$, and $\prod_{t\in\{1,4,6\}} f(t)$ is defined if and only if $f(1) * f(4) * f(6)$ is defined. When we write something like $\prod_{t\in H} f(t) \in A$, this statement includes the assertion that $\prod_{t\in H} f(t)$ is defined. Given a sequence f in S, we let $FP(\langle f(t)\rangle_{t=1}^\infty =$ $\{\prod_{t\in H} f(t) : H \in \mathcal{P}_f(\mathbb{N})\}$.

Definition 2.4. Let $(S,*)$ be a partial semigroup and let f be a sequence in S. Then f is *adequate* if and only if

(1) for each $H \in \mathcal{P}_f(\mathbb{N})$, $\prod_{t\in H} f(t)$ is defined and

(2) for each $F \in \mathcal{P}_f(S)$, there exists $m \in \mathbb{N}$ such that $FP(\langle f(t)\rangle_{t=m}^{\infty}) \subseteq \sigma(F)$.

Definition 2.5. Let $(S,*)$ be an adequate partial semigroup. Then $\mathcal{F} = \{f : f \text{ is an adequate sequence in } S\}$.

For the Central Sets Theorem for partial semigroups, we will restrict our attention to adequate sequences. Since the conclusions of the Central Sets Theorem involve products from the sequence, it is fairly obvious that we need requirement (1). The arguments used in the proofs demand requirement (2). In section 4, we will show that if S is countable, then adequate sequences abound. If S is uncountable, there may be no adequate sequences, in which case our results are trivial. On the other hand, there are uncountable noncommutative examples in which there are plenty of adequate sequences. And, of course, if S is a semigroup, then every sequence in S is adequate.

3. The Central Sets Theorem for Arbitrary Adequate Partial Semigroups

In this section, we show that Theorem 1.2 remains valid as written for an arbitrary adequate partial semigroup, provided only that the set $\mathcal{T}$ of all sequences in S is replaced by the set $\mathcal{F}$ of all adequate sequences in S.

Definition 3.1. Let $(S,*)$ be an adequate partial semigroup.

(a) A set $A \subseteq S$ is *piecewise syndetic* if and only if $\overline{A} \cap K(\delta S) \neq \emptyset$.
(b) A set $A \subseteq S$ is a *J-set* if and only if for all $F \in \mathcal{P}_f(\mathcal{F})$ and all $L \in \mathcal{P}_f(S)$, there exist $m \in \mathbb{N}$, $a \in S^{m+1}$, and $t \in \mathcal{J}_m$ such that for all $f \in F$, $\big(\prod_{i=1}^{m} a(i) * f\big(t(i)\big)\big) * a(m+1) \in A \cap \sigma(L)$.
(c) $J(S) = \{p \in \delta S : (\forall A \in p)(A \text{ is a } J\text{-set})\}$.

The proof of the next lemma is an adaptation of the proof of [10, Lemma 14.9] and uses an idea of H. Furstenberg and Y. Katznelson from [4]. For $m \in \mathbb{N}$, we define the operation $*$ on $\times_{i=1}^{m} S$ coordinatewise, specifying that $\vec{x} * \vec{y}$ is defined if and only if $x_i * y_i$ is defined for all $i \in \{1, 2, \ldots, m\}$.

Lemma 3.2. *Let $(S,*)$ be an adequate partial semigroup, let $m \in \mathbb{N}$, and let $D = \mathcal{P}_f(S) \times \mathbb{N}$ directed by $(W,n) \leq (W',n')$ if and only if $W \subseteq W'$ and $n \leq n'$. Let $\langle E_{W,n}\rangle_{(W,n)\in D}$ and $\langle I_{W,n}\rangle_{(W,n)\in D}$ be decreasing families of nonempty subsets of S^m such that*

(1) *for $(W,n) \in D$, $I_{W,n} \subseteq E_{W,n} \subseteq \times_{i=1}^{m} \sigma(W)$;*
(2) *for $(W,n) \in D$ and $\vec{x} \in I_{W,n}$, there exists $(M,l) \in D$ such that $\vec{x} * E_{M,l} \subseteq I_{W,n}$;*

(3) *for* $(W,n)\in D$ *and* $\vec{x}\in E_{W,n}\setminus I_{W,n}$*, there exists* $(M,l)\in D$ *such that* $\vec{x}*E_{M,l}\subseteq E_{W,n}$ *and* $\vec{x}*I_{M,l}\subseteq I_{W,n}$*; and*

(4) *for* $(W,n)\in D$ *and* $a\in\sigma(W)$, $\overline{a}\in E_{W,n}$*, where* $\overline{a}=(a,a,\ldots,a)\in S^m$.

Let $Y=\times_{i=1}^m\beta S$, $E=\bigcap_{(W,n)\in D}c\ell_Y E_{W,n}$*, and* $I=\bigcap_{(W,n)\in D}c\ell_Y I_{W,n}$. *Then* E *is a subsemigroup of* $(\delta S)^m$*,* I *is an ideal of* E*, and if* $p\in K(\delta S)$*, then* $\overline{p}=(p,p,\ldots,p)\in E\cap K\big((\delta S)^m\big)=K(E)\subseteq I$.

Proof. Given (W,n) and (M,l) in D, $\emptyset\neq I_{W\cup M,\max\{n,l\}}\subseteq I_{W,n}\cap I_{M,l}$ so $I\neq\emptyset$. Since each $E_{W,n}\subseteq\sigma(W)^m$, we have $E\subseteq(\delta S)^m$. (To see this, suppose that $\vec{q}\in E\setminus(\delta S)^m$ and pick $i\in\{1,2,\ldots,m\}$ such that $q_i\notin\delta S$. Pick $W\in\mathcal{P}_f(S)$ such that $\sigma(W)\notin q_i$. For $j\in\{1,2,\ldots,m\}\setminus\{i\}$, let $B_j=S$ and let $B_i=S\setminus\sigma(W)$. Then $\times_{j=1}^m\overline{B_j}$ is a neighborhood of $\vec{q}$ missing $E_{W,1}$.)

To establish that E is a subsemigroup of $(\delta S)^m$ and I is an ideal of E, let $\vec{p},\vec{q}\in E$ and let $(W,n)\in D$ be given. We show that $\vec{p}*\vec{q}\in E$ and if either $\vec{p}\in I$ or $\vec{q}\in I$, then $\vec{p}*\vec{q}\in I$.

To see that $\vec{p}*\vec{q}\in E$ with $\vec{p}*\vec{q}\in I$ if $\vec{p}\in I$ or $\vec{q}\in I$, let U be an open neighborhood of $\vec{p}*\vec{q}$ in Y. (At this stage, one might be tempted to invoke [10, Theorem 2.22], but we do not know that βS is a semigroup.) By Theorem 2.2(2), for each $i\in\{1,2,\ldots,m\}$, $\rho_{q_i}:\beta S\to\beta S$ is continuous and, consequently, $\rho_{\vec{q}}:Y\to Y$ is continuous. Pick a neighborhood V of $\vec{p}$ in Y such that $\rho_{\vec{q}}[V]\subseteq U$. Since $\vec{p}\in E$, pick $\vec{x}\in V\cap E_{W,n}$, choosing $\vec{x}\in I_{W,n}$ if $\vec{p}\in I$. Pick $(M,l)\in D$ as guaranteed by (2) or (3). Now $\vec{x}*\vec{q}\in U$. For each $i\in\{1,2,\ldots,m\}$, pick $A_i\in x_i*q_i$ such that $\times_{i=1}^m\overline{A_i}\subseteq U$. Given $i\in\{1,2,\ldots,m\}$, we have that $A_i\in x_i*q_i$ so, recalling that the principal ultrafilters on S are identified with the points of S, we have that $x_i^{-1}A_i\in q_i$. Therefore, $\times_{i=1}^m\overline{x_i^{-1}A_i}$ is a neighborhood of $\vec{q}$. Pick $\vec{y}\in E_{M,l}\cap\times_{i=1}^m\overline{x_i^{-1}A_i}$, choosing $\vec{y}\in I_{M,l}$ if $\vec{q}\in I$. Then $\vec{x}*\vec{y}\in U$. If $\vec{x}\in E_{W,n}\setminus I_{W,n}$, then by (3), $\vec{x}*\vec{y}\in E_{W,n}$ with $\vec{x}*\vec{y}\in I_{W,n}$ if $\vec{y}\in I_{M,l}$. If $\vec{x}\in I_{W,n}$, then by (2), $\vec{x}*\vec{y}\in I_{W,n}$.

We have thus established that E is a subsemigroup of $(\delta S)^m$ and I is an ideal of E. Now let $p\in K(\delta S)$. We claim that $\overline{p}\in E$. To see this, let U be an open neighborhood of $\overline{p}\in Y$ and let $(W,n)\in D$. For $i\in\{1,2,\ldots,m\}$, pick $A_i\in p$ such that $\times_{i=1}^m\overline{A_i}\subseteq U$. Since $p\in\delta S$, $\sigma(W)\in p$. Pick $a\in\sigma(W)\cap\times_{i=1}^m A_i$. Then $\overline{a}\in U\cap E_{W,n}$.

By [10, Theorem 2.23], $\overline{p}\in K\big((\delta S)^m\big)$, so $E\cap K\big((\delta S)^m\big)\neq\emptyset$, so by [10, Theorem 1.65], $K(E)=E\cap K\big((\delta S)^m\big)$. Since I is an ideal of E, $K(E)\subseteq I$. □

Theorem 3.3. *Let* $(S,*)$ *be an adequate partial semigroup and let* A *be a piecewise syndetic subset of* S*. Then* A *is a J-set. In fact, if* $r\in\mathbb{N}$*,*

$F \in \mathcal{P}_f(\mathcal{F})$, and $L \in \mathcal{P}_f(S)$, then there exist $m \in \mathbb{N}$, $a \in S^{m+1}$, and $t \in \mathcal{J}_m$ such that $t(1) > r$, and for all $f \in F$,

$$\big(\prod_{i=1}^{m} a(i) * f\big(t(i)\big)\big) * a(m+1) \in A \cap \sigma(L)\,.$$

Proof. Pick $p \in \overline{A} \cap K(\delta S)$. Let $r \in \mathbb{N}$, $L \in \mathcal{P}_f(S)$, and $F \in \mathcal{P}_f(\mathcal{F})$ be given. Enumerate F as $\{f_1, f_2, \ldots, f_k\}$. Let $D = \mathcal{P}_f(S) \times \mathbb{N}$ directed by $(W, n) \le (W', n')$ if and only if $W \subseteq W'$ and $n \le n'$. For $(W, n) \in D$, let

$$\begin{aligned}
&I_{W,n} = \\
&\big\{\big(\big(\prod_{i=1}^{m} a(i) * f_1\big(t(i)\big)\big) * a(m+1), \big(\prod_{i=1}^{m} a(i) * f_2\big(t(i)\big)\big) * a(m+1), \\
&\ldots, \big(\prod_{i=1}^{m} a(i) * f_k\big(t(i)\big)\big) * a(m+1)\big) : \\
&m \in \mathbb{N}\,,\ a \in S^{m+1}\,,\ t \in \mathcal{J}_m\,,\ t(1) > n, \text{ and for each } j \in \{1, 2, \ldots, k\}\,, \\
&\big(\prod_{i=1}^{m} a(i) * f_j\big(t(i)\big)\big) * a(m+1) \in \sigma(W)\big)\big\},
\end{aligned}$$

and let $E_{W,n} = I_{W,n} \cup \{(a, a, \ldots, a) \in S^k : a \in \sigma(W)\}$.

We first show that each $I_{W,n} \neq \emptyset$. To see this, pick $a(1) \in \sigma(W)$. Since each f_j is adequate, pick $q > n$ such that for each $j \in \{1, 2, \ldots, k\}$, $f_j(q) \in \sigma\big(W * a(1)\big)$. Then pick $a(2) \in \sigma\big(\bigcup_{j=1}^{k} W * a(1) * f_j(q)\big)$. Then $\big(a(1) * f_1(q) * a(2), \ldots, a(1) * f_k(q) * a(2)\big) \in I_{W,n}$.

Now we claim that $\langle E_{W,n}\rangle_{(W,n)\in D}$ and $\langle I_{W,n}\rangle_{(W,n)\in D}$ satisfy statements (1), (2), (3), and (4) of Lemma 3.2. Statements (1) and (4) are immediate. To verify statement (2), let $(W, n) \in D$ and let $\vec{x} \in I_{W,n}$. Pick $m \in \mathbb{N}$, $a \in S^{m+1}$, and $t \in \mathcal{J}_m$ such that $t(1) > n$ and for each $j \in \{1, 2, \ldots, k\}$, $x_j = \big(\prod_{i=1}^{m} a(i) * f_j\big(t(i)\big)\big) * a(m+1) \in \sigma(W)$. Let

$$M = \bigcup_{j=1}^{k} W * \big(\prod_{i=1}^{m} a(i) * f_j\big(t(i)\big)\big) * a(m+1)\big)$$

and let $l = t(m)$. Let $\vec{y} \in E_{M,l}$. If $\vec{y} = (b, b, \ldots, b) \in \sigma(M)^k$, then for each $j \in \{1, 2, \ldots, k\}$, coordinate j of $\vec{x} * \vec{y}$ is $\big(\prod_{i=1}^{m} a(i) * f_j\big(t(i)\big)\big) * a(m+1) * b$, so $\vec{x} * \vec{y} \in I_{W,n}$. Now assume we have $q \in \mathbb{N}$, $b \in S^{q+1}$, and $s \in \mathcal{J}_q$ such that $s(1) > l$ and for each $j \in \{1, 2, \ldots, k\}$, $y_j = \big(\prod_{i=1}^{q} b(i) * f_j\big(s(i)\big)\big) * b(q+1) \in \sigma(M)$. Define $c \in S^{m+q+1}$ and $u \in \mathcal{J}_{m+q}$ by

$$c(i) = \begin{cases} a(i) & \text{if } i < m \\ a(m) * b(1) & \text{if } i = m \\ b(i-m) & \text{if } i > m \end{cases} \quad u(i) = \begin{cases} t(i) & \text{if } i \le m \\ s(i-m) & \text{if } i > m\,. \end{cases}$$

Then for $j \in \{1, 2, \ldots, k\}$, coordinate j of $\vec{x} * \vec{y}$ is

$$\big(\prod_{i=1}^{m+q} c(i) * f_j\big(u(i)\big)\big) * c(m+q+1),$$

so $\vec{x} * \vec{y} \in I_{W,n}$.

To verify statement (3), let $(W, n) \in D$ and let $\vec{x} = (a, a, \ldots, a) \in E_{W,n} \setminus I_{W,n}$. Let $M = W * a$ and let $l = n$. If $\vec{y} = (b, b, \ldots, b) \in E_{M,l} \setminus I_{M,l}$, then $\vec{x} * \vec{y} = (a * b, a * b, \ldots, a * b) \in E_{W,n}$. So let $\vec{y} \in I_{M,l}$. Pick $m \in \mathbb{N}$,

$b \in S^{m+1}$, and $t \in \mathcal{J}_m$ such that $t(1) > l$ and for each $j \in \{1, 2, \ldots, k\}$, $y_j = \big(\prod_{i=1}^m b(i) * f_j\big(t(i)\big)\big) * b(m+1) \in \sigma(M)$. Define $c \in S^{m+1}$ by

$$c(i) = \begin{cases} a * b(1) & \text{if } i = 1 \\ b(i) & \text{if } i > 1\,. \end{cases}$$

Then for $j \in \{1, 2, \ldots, k\}$, coordinate j of $\vec{x} * \vec{y}$ is

$$\big(\prod_{i=1}^m c(i) * f_j\big(t(i)\big)\big) * c(m+1) \in \sigma(W),$$

so $\vec{x} * \vec{y} \in I_{W,n}$ as required.

Now by Lemma 3.2,

$$(p, p, \ldots, p) \in I = \bigcap_{(W,n) \in D} c\ell_Y I_{W,n} \text{ so } \times_{j=1}^k \overline{A} \cap I_{L,r} \neq \emptyset\,.$$

Pick $\vec{x} \in \times_{j=1}^k \overline{A} \cap I_{L,r}$. Pick $m \in \mathbb{N}$, $a \in S^{m+1}$, and $t \in \mathcal{J}_m$ such that $t(1) > r$ and, for $j \in \{1, 2, \ldots, k\}$, $x_j = \big(\prod_{i=1}^m a(i) * f_j\big(t(i)\big)\big) * a(m+1) \in A \cap \sigma(L)$. □

Corollary 3.4. *Let $(S, *)$ be an adequate partial semigroup. Then $J(S)$ is a compact two-sided ideal of δS.*

Proof. By Theorem 3.3, $K(\delta S) \subseteq J(S)$, so $J(S) \neq \emptyset$. Trivially, $J(S)$ is compact. To see that $J(S)$ is an ideal of δS, let $p \in J(S)$ and $q \in \delta S$. To see that $p * q \in J(S)$, let $A \in p * q$. Let $B = \{x \in S : x^{-1}A \in q\}$. Then $B \in p$, so B is a J-set. To see that A is a J-set, let $F \in \mathcal{P}_f(\mathcal{F})$ and let $L \in \mathcal{P}_f(S)$. Pick $m \in \mathbb{N}$, $a \in S^{m+1}$, and $t \in \mathcal{J}_m$ such that for each $f \in F$, $\big(\prod_{i=1}^m a(i) * f\big(t(i)\big)\big) * a(m+1) \in B \cap \sigma(L)$. Given $f \in F$, let $z_f = \big(\prod_{i=1}^m a(i) * f\big(t(i)\big)\big) * a(m+1)$. Let $Z = \{z_f : f \in F\}$ and let $W = L * Z$. Now $\bigcap_{f \in F} z_f^{-1}A \in q$ and $\sigma(W) \in q$, so pick $b \in \sigma(W) \cap \bigcap_{f \in F} z_f^{-1}A$. Then for $f \in F$, $z_f * b \in A \cap \sigma(L)$.

To see that $q * p \in J(S)$, let $A \in q * p$. Let $B = \{x \in S : x^{-1}A \in p\}$. To see that A is a J-set, let $F \in \mathcal{P}_f(\mathcal{F})$ and let $L \in \mathcal{P}_f(S)$. Now $B \in q$ and $\sigma(L) \in q$, so pick $b \in B \cap \sigma(L)$. Then $b^{-1}A$ is a J-set, so pick $m \in \mathbb{N}$, $a \in S^{m+1}$, and $t \in \mathcal{J}_m$ such that for each $f \in F$, $\big(\prod_{i=1}^m a(i) * f\big(t(i)\big)\big) * a(m+1) \in b^{-1}A \cap \sigma(L * b)$. Then, for $f \in F$, $b * \big(\prod_{i=1}^m a(i) * f\big(t(i)\big)\big) * a(m+1) \in A \cap \sigma(L)$. □

If A is piecewise syndetic, the conclusion of the following lemma is part of Theorem 3.3.

Lemma 3.5. *Let $(S, *)$ be an adequate partial semigroup and let A be a J-set in S. Let $r \in \mathbb{N}$, let $F \in \mathcal{P}_f(\mathcal{F})$, and let $L \in \mathcal{P}_f(S)$. There exist $m \in \mathbb{N}$, $a \in S^{m+1}$, and $t \in \mathcal{J}_m$ such that $t(1) > r$ and for all $f \in F$,*

$$\big(\prod_{i=1}^m a(i) * f\big(t(i)\big)\big) * a(m+1) \in A \cap \sigma(L)\,.$$

Proof. For $f \in F$, define $g_f : \mathbb{N} \to S$ by $g_f(n) = f(r+n)$. Then $\{g_f : f \in F\} \in \mathcal{P}_f(\mathcal{F})$, so pick $m \in \mathbb{N}$, $a \in S^{m+1}$, and $t \in \mathcal{J}_m$ such that for all $f \in F$, $\big(\prod_{i=1}^m a(i) * g_f\big(t(i)\big)\big) * a(m+1) \in A \cap \sigma(L)$. Define $s \in \mathcal{J}_m$ **by, for** $i \in \{1,2,\ldots,m\}$, $s(i) = r + t(i)$. Then for all $f \in F$, $\big(\prod_{i=1}^m a(i) * f\big(s(i)\big)\big) * a(m+1) \in A \cap \sigma(L)$. □

The following theorem is the Central Sets Theorem for adequate partial semigroups.

Theorem 3.6. *Let $(S,*)$ be an adequate partial semigroup and let C be a member of an idempotent in $J(S)$. There exist functions*

$$m : \mathcal{P}_f(\mathcal{F}) \to \mathbb{N}\,,\ \alpha \in \times_{F\in\mathcal{P}_f(\mathcal{F})} S^{m(F)+1},\ \text{and } \tau \in \times_{F\in\mathcal{P}_f(\mathcal{F})} \mathcal{J}_{m(F)}$$

such that

(1) *if $F, G \in \mathcal{P}_f(\mathcal{F})$ and $F \subsetneq G$, then $\tau(F)\big(m(F)\big) < \tau(G)(1)$, and*
(2) *if $n \in \mathbb{N}$, $G_1, G_2, \ldots, G_n \in \mathcal{P}_f(\mathcal{F})$, $G_1 \subsetneq G_2 \subsetneq \ldots \subsetneq G_n$, and for each $i \in \{1,2,\ldots,n\}$, $f_i \in G_i$, then*

$$\prod_{i=1}^n \Big(\big(\prod_{j=1}^{m(G_i)} \alpha(G_i)(j) * f_i\big(\tau(G_i)(j)\big)\big) * \alpha(G_i)(m(G_i)+1)\Big) \in C\,.$$

In particular, these conclusions hold if C is central in S.

Proof. Pick an idempotent $p \in J(S) \cap \overline{C}$ and let $C^\star = \{x \in S : x^{-1}C \in p\}$. By Theorem 2.2(4), if $x \in C^\star$, then $x^{-1}C^\star \in p$, and so $x^{-1}C^\star$ is a J-set.

We define $m(F)$, $\alpha(F)$, and $\tau(F)$ for $F \in \mathcal{P}_f(\mathcal{F})$ by induction on $|F|$, satisfying the following induction hypotheses:

(1) If $\emptyset \neq G \subsetneq F$, then $\tau(G)\big(m(G)\big) < \tau(F)(1)$ and
(2) if $n \in \mathbb{N}$, $G_1, G_2, \ldots, G_n \in \mathcal{P}_f(\mathcal{F})$, $G_1 \subsetneq G_2 \subsetneq \ldots \subsetneq G_n = F$, and for each $i \in \{1,2,\ldots,n\}$, $f_i \in G_i$, then

$$\prod_{i=1}^n \Big(\big(\prod_{j=1}^{m(G_i)} \alpha(G_i)(j) * f_i\big(\tau(G_i)(j)\big)\big) * \alpha(G_i)(m(G_i)+1)\Big) \in C^\star\,.$$

First assume that $F = \{f\}$. Pick any $d \in S$ and let $L = \{d\}$. (The set L does not enter into the argument at this stage.) Pick $m \in \mathbb{N}$, $a \in S^{m+1}$, and $t \in \mathcal{J}_m$ such that $\big(\prod_{i=1}^m a(i) * f\big(t(i)\big)\big) * a(m+1) \in C^\star$. Let $m(F) = m$, $\alpha(F) = a$, and $\tau(F) = t$. Hypothesis (1) is satisfied vacuously. For hypothesis (2), necessarily $n = 1$, so the conclusion is satisfied directly.

Now assume that $|F| > 1$ and that $m(G)$, $\alpha(G)$, and $\tau(G)$ have been defined for all nonempty proper subsets G of F. Let

$$L = \{\prod_{i=1}^k \big(\big(\prod_{j=1}^{m(G_i)} \alpha(G_i)(j) * f_i\big(\tau(G_i)(j)\big)\big) * \alpha(G_i)(m(G_i)+1)\big) : \\ k \in \mathbb{N}\,, \emptyset \neq G_1 \subsetneq \ldots \subsetneq G_k \subsetneq F, \text{ and for } i \in \{1,2,\ldots,k\}\,, f_i \in G_i\}\,.$$

By hypothesis (2), L is a finite subset of $C^\star$. Let $A = C^\star \cap \bigcap_{y\in L} y^{-1}C^\star$. Then $A \in p$, so A is a J-set.

Let $r = \max\{\tau(G)\big(m(G)\big) : \emptyset \neq G \subsetneq F\}$. By Lemma 3.5, pick $m \in \mathbb{N}$, $a \in S^{m+1}$, and $t \in \mathcal{J}_m$ such that $t(1) > r$, and for all $f \in F$,

$$\big(\prod_{i=1}^{m} a(i) * f\big(t(i)\big)\big) * a(m+1) \in A \cap \sigma(L)\,.$$

Define $m(F) = m$, $\alpha(F) = a$, and $\tau(F) = t$. Since $t(1) > r$, hypothesis (1) is satisfied.

To verify hypothesis (2), assume $n \in \mathbb{N}$, $\emptyset \neq G_1 \subsetneq G_2 \subsetneq \ldots \subsetneq G_n = F$, and for each $i \in \{1, 2, \ldots, n\}$, $f_i \in G_i$. Assume first that $n = 1$. Then the conclusion is satisfied because $\big(\prod_{i=1}^{m} a(i) * f_1\big(t(i)\big)\big) * a(m+1) \in C^\star$. Now assume that $n > 1$. Let

$$y = \prod_{i=1}^{n-1} \big(\prod_{j=1}^{m(G_i)} \alpha(G_i)(j) * f_i\big(\tau(G_i)(j)\big)\big) * \alpha(G_i)(m(G_i)+1)\,.$$

Then $y \in L$, so $\big(\prod_{j=1}^{m} a(j) * f_n\big(t(j)\big)\big) * a(m+1) \in y^{-1}C^\star$, so

$$\prod_{i=1}^{n} \big(\prod_{j=1}^{m(G_i)} \alpha(G_i)(j) * f_i\big(\tau(G_i)(j)\big)\big) * \alpha(G_i)(m(G_i)+1) \in C^\star,$$

as required.

The "in particular" conclusion holds because any idempotent in $K(\delta S)$ is a member of $J(S)$. □

As with semigroups, if the partial semigroup is commutative, the conclusion of the Central Sets Theorem is simpler.

Corollary 3.7. *Let $(S, *)$ be a commutative adequate partial semigroup and let C be a member of an idempotent in $J(S)$. There exist functions $\gamma : \mathcal{P}_f(\mathcal{F}) \to S$ and $H : \mathcal{P}_f(\mathcal{F}) \to \mathcal{P}_f(\mathbb{N})$ such that*

(1) *if $F, G \in \mathcal{P}_f(\mathcal{F})$ and $F \subsetneq G$, then $\max H(F) < \min H(G)$, and*
(2) *if $n \in \mathbb{N}$; $G_1, G_2, \ldots, G_n \in \mathcal{P}_f(\mathcal{F})$; $G_1 \subsetneq G_2 \subsetneq \ldots \subsetneq G_n$; and for each $i \in \{1, 2, \ldots, n\}$, $f_i \in G_i$, then $\prod_{i=1}^{n} \big(\gamma(G_i) * \prod_{t \in H(G_i)} f_i(t)\big) \in C$.*

Proof. Let m, α, and τ be as guaranteed by Theorem 3.6. For $F \in \mathcal{P}_f(\mathcal{F})$, let $\gamma(F) = \prod_{j=1}^{m(F)+1} \alpha(F)(j)$ and let $H(F) = \big\{\tau(F)(j) : j \in \{1, 2, \ldots, m(F)\}\big\}$. □

We present now an application of Theorem 3.6 to a partial semigroup of *located words*. If C is central, this is a consequence of [1, Theorem 4.1]. In the more general context, it is new. (We shall see in Theorem 4.12 that for the partial semigroup of located words, there is a member of an idempotent in $J(S)$ which is not central.)

Definition 3.8. Let Σ be a nonempty finite set.

(a) The *set of located words over* Σ is

$$S = \{f : \big(\exists H \in \mathcal{P}_f(\mathbb{N})\big)(f : H \to \Sigma)\}\,.$$

Define a partial semigroup operation $*$ on S by, for $f, g \in S$, $f * g = f \cup g$ defined if and only if $\max \operatorname{dom}(f) < \min \operatorname{dom}(g)$.

(b) The *set of located variable words over* Σ is the set of located words over $\Sigma \cup \{v\}$ in which v occurs, where v is a "variable" not in Σ.

(c) If f is a located variable word over Σ and $a \in \Sigma$, then $f\langle a\rangle$ is the located word over Σ defined by, for $t \in \operatorname{dom}(f)$,

$$f\langle a\rangle(t) = \begin{cases} f(t) & \text{if } f(t) \in \Sigma \\ a & \text{if } f(t) = v\,. \end{cases}$$

When we say that v occurs in f, we mean v is in the range of f.

Note that the set of located words over Σ is an adequate partial semigroup.

Corollary 3.9. *Let Σ be a nonempty finite set and let S be the partial semigroup of located words over Σ. Let C be a member of an idempotent in $J(S)$. There is a sequence $\langle w_n\rangle_{n=1}^\infty$ of located variable words over Σ such that for each $h \in S$, $\prod_{r\in\operatorname{dom}(h)} w_r\langle h(r)\rangle \in C$.*

Proof. Pick m, α, and τ as guaranteed by Theorem 3.6 for C. For $a \in \Sigma$, define $f_a \in \mathcal{F}$ by, for $k \in \mathbb{N}$, $f_a(k) = \{(k, a)\}$. Note that each f_a is an adequate sequence. Choose an injective sequence $\langle g_n\rangle_{n=1}^\infty$ in $\mathcal{F} \setminus \{f_a : a \in \Sigma\}$. (For example, one may fix $b \in \Sigma$ and define for $k \in \mathbb{N}$, $g_n(k) = \{(2k, b), (2k + 1, b)\}$.)

For each $n \in \mathbb{N}$, let $G_n = \{f_a : a \in \Sigma\} \cup \{g_1, g_2, \ldots, g_n\}$. (The point of introducing $\langle g_n\rangle_{n=1}^\infty$ is to guarantee that $G_1 \subsetneq G_2 \subsetneq \ldots$.) For $n \in \mathbb{N}$, define w_n by

$$w_n = \big(\textstyle\prod_{i=1}^{m(G_n)} \alpha(G_n)(i) * \{(\tau(G_n)(i), v)\}\big) * \alpha(G_n)(m(G_n) + 1)\,.$$

Note that $w_n\big(\tau(G_n)(1)\big) = v$, so v occurs in w_n. To see that each w_n is a located variable word, we need to note that the operations in the definition are all defined. To this end, fix $a \in \Sigma$ and $n \in \mathbb{N}$. Then $G_n \in \mathcal{P}_f(\mathcal{F})$ and $f_a \in G_n$, so by Theorem 3.6(2),

$$\big(\textstyle\prod_{i=1}^{m(G_n)} \alpha(G_n)(i) * f_a\big(\tau(G_n)(i)\big)\big) * \alpha(G_n)(m(G_n) + 1) \in C\,.$$

In particular, the operations in this expression are all defined. Given $i \in \{1, 2, \ldots, m(G_n)\}$,

$$\operatorname{dom}\big(f_a\big(\tau(G_n)(i)\big)\big) = \big\{\tau\big(G_n(i)\big)\big\} = \operatorname{dom}(\{(\tau(G_n)(i), v)\})\,.$$

So the corresponding operations in the definition of w_n are defined.

Now let $h \in S$ and let $H = \mathrm{dom}(h)$. Enumerate H in order as $\langle j(1), j(2), \ldots, j(k)\rangle$. Then $G_{j(1)} \subsetneq G_{j(2)} \subsetneq \ldots \subsetneq G_{j(k)}$ and for each $t \in \{1,2,\ldots,k\}$, $f_{h(j(t))} \in G_{j(t)}$, so

$$\begin{aligned}&\textstyle\prod_{r\in H} w_r\langle h(r)\rangle = \prod_{t=1}^{k} w_{j(t)}\langle h\big(j(t)\big)\rangle = \\ &\textstyle\prod_{t=1}^{k}\Big(\big(\prod_{i=1}^{m(G_{j(t)})} \alpha(G_{j(t)})(i) * \big\{\big(\tau(G_{j(t)})(i), h\big(j(t)\big)\big)\big\}\big) \\ &\qquad * \alpha(G_{j(t)})(m(G_{j(t)})+1)\Big) = \\ &\textstyle\prod_{t=1}^{k}\Big(\big(\prod_{i=1}^{m(G_{j(t)})} \alpha(G_{j(t)})(i) * f_{h(j(t))}\big(\tau(G_{j(t)})(i)\big)\big) \\ &\qquad * \alpha(G_{j(t)})(m(G_{j(t)})+1)\Big) \in C\,. \end{aligned}$$ □

We note that the proof of [1, Theorem 4.1] cannot be easily adjusted to prove Corollary 3.9 in its full generality because that proof used the fact that C was a member of a minimal idempotent.

4. Results for Countable Adequate Partial Semigroups

We present in this section results whose proofs require that we assume our adequate partial semigroup is countable.

Since the conclusion of the Central Sets Theorem for adequate partial semigroups involves adequate sequences, it is nice to know when we are guaranteed to have such objects.

Theorem 4.1. *Let $(S,*)$ be a countable adequate partial semigroup, let $n \in \mathbb{N}$, and let $\langle f(t)\rangle_{t=1}^{n}$ be any length n sequence in S such that $\prod_{t\in G} f(t)$ is defined whenever $\emptyset \neq G \subseteq \{1,2,\ldots,n\}$. Then f extends to an adequate sequence $\langle f(t)\rangle_{t=1}^{\infty}$. In particular, given any $a \in S$, there is an adequate sequence $\langle f(t)\rangle_{t=1}^{\infty}$ with $f(1) = a$.*

Proof. Enumerate S as $\langle s_t\rangle_{t=1}^{\infty}$ and, for $m \in \mathbb{N}$, let $F_m = \{s_1, s_2, \ldots, s_m\}$. Inductively, let $m \geq n$, and assume we have chosen $\langle f(t)\rangle_{t=1}^{m}$ such that $\prod_{t\in G} f(t)$ is defined whenever $\emptyset \neq G \subseteq \{1,2,\ldots,m\}$ and $\prod_{t\in G} f(t) \in \sigma(F_l)$ whenever $n < l \leq m$ and $l \in G \subseteq \{l, l+1, \ldots, m\}$. If $m = n$, pick $f(m+1) \in \sigma\big(F_n \cup \big\{\prod_{t\in G} f(t) : \emptyset \neq G \subseteq \{1,2,\ldots,n\}\big\}\big)$. Assume now $m > n$, and let $M = F_{m+1} \cup \big\{\prod_{t\in G} f(t) : \emptyset \neq G \subseteq \{1,2,\ldots,n\}\big\} \cup \bigcup_{l=n+1}^{m} \bigcup\{F_l * \prod_{t\in G} f(t) : l \in G \subseteq \{l, l+1, \ldots, m\}\}\big)$. Pick $f(m+1) \in \sigma(M)$. Then, given any $l > n$, $FP(\langle f(t)\rangle_{t=l}^{\infty}) \subseteq \sigma(F_l)$. □

Countability may not be necessary for the existence of many adequate sequences. For example, let X be any infinite set, let $S = \mathcal{P}_f(X)$, and for $F, G \in \mathcal{P}_f(X)$, if $F \cap G = \emptyset$, define $F \uplus G = F \cup G$, leaving $F \uplus G$ undefined otherwise. Then any sequence of pairwise disjoint members of S is adequate. To see this, let $\langle F_n\rangle_{n=1}^{\infty}$ be a sequence of pairwise disjoint members of S and let $\mathcal{H} \in \mathcal{P}_f(S)$. Let $H = \bigcup \mathcal{H}$. Then there is some $m \in \mathbb{N}$ such that $F_n \cap H = \emptyset$ whenever $n \geq m$, and so $FP(\langle F_n \rangle_{n=m}^{\infty}) \subseteq \sigma(\mathcal{H})$.

The above example is commutative. For an example of an uncountable noncommutative adequate partial semigroup with many adequate sequences, let ω_1 be the first uncountable ordinal and let S be the free semigroup on the alphabet $\{a_\eta : \eta < \omega_1\}$. For $W \in S$, let $\text{supp}(w) = \{\eta < \omega_1 : a_\eta \text{ occurs in } w\}$. For $w, v \in S$, define $w * v$ to be the usual concatenation of words, defined if and only if $\text{supp}(w) \cap \text{supp}(v) = \emptyset$. (Then $w * v$ is defined if and only if $v * w$ is defined, but $w * v \neq v * w$.) As above, any sequence in S with pairwise disjoint supports is adequate.

If S is uncountable, there may not be any adequate sequences. (In this case the Central Sets Theorem is vacuously true.) For example, let $S = \mathcal{P}_f(\omega_1)$, and for $F, G \in S$, if $\max F < \min G$, define $F * G = F \cup G$, leaving $F * G$ undefined otherwise. Then $(S, *)$ is an adequate partial semigroup. Suppose one has an adequate sequence $\langle F_n \rangle_{n=1}^{\infty}$ in S. Then for each n, $\max F_n < \min F_{n+1}$. (Recall that $\prod_{n \in H} f(n)$ is defined to be the product in increasing order of indices.) Let $\mu = \sup \bigcup_{n=1}^{\infty} F_n$. Then $FP(\langle F_n \rangle_{n=1}^{\infty}) \cap \varphi(\{\mu\}) = \emptyset$.

In [5, Definition 3.2], the author defines the notion of a J_δ-set for a commutative adequate partial semigroup. We repeat this definition in an equivalent form.

Definition 4.2. Let $(S, *)$ be a commutative adequate partial semigroup. A set $A \subseteq S$ is a J_δ-set if and only if for all $F \in \mathcal{P}_f(\mathcal{F})$ and all $L \in \mathcal{P}_f(S)$, there exist $b \in S$ and $H \in \mathcal{P}_f(\mathbb{N})$ such that for all $f \in F$, $b * \prod_{t \in H} f(t) \in A \cap \sigma(L)$.

Note that if S is commutative, then any J-set in S is a J_δ-set. (Let $m \in \mathbb{N}$, $a \in S^{m+1}$, and $t \in \mathcal{J}_m$ be as in the definition of a J-set. Let $b = \prod_{i=1}^{m+1} a(i)$ and let $H = \{t(1), t(2), \ldots, t(m)\}$.) We set out now to show that if S is countable and commutative, the converse holds. Again, we do not know whether countability is required.

Lemma 4.3. *Assume that $(S, *)$ is a countable adequate partial semigroup enumerated as $\langle s_t \rangle_{t=1}^{\infty}$. For each $n \in \mathbb{N}$, let $W_n = \{s_1, s_2, \ldots, s_n\}$ and let $F \in \mathcal{P}_f(\mathcal{F})$. There exist an increasing sequence $\langle k(t) \rangle_{t=1}^{\infty}$ in $\mathbb{N}$ and a sequence $\langle b(t) \rangle_{t=1}^{\infty}$ in S so that for each $f \in F$, $\langle f\big(k(t)\big) * b(t) \rangle_{t=1}^{\infty}$ is an adequate sequence with $FP(\langle f\big(k(t)\big) * b(t) \rangle_{t=n}^{\infty}) \subseteq \sigma(W_n)$ for each $n \in \mathbb{N}$.*

Proof. Let $M_1 = W_1$ and pick $k(1) \in \mathbb{N}$ such that $f\big(k(1)\big) \in \sigma(M_1)$ for each $f \in F$. Pick $b(1) \in \sigma\big(\bigcup_{f \in F} M_1 * f\big(k(1)\big)\big)$. Inductively, let $n \in \mathbb{N}$ and assume that we have chosen $M_i \in \mathcal{P}_f(S)$, $k(i) \in \mathbb{N}$, and $b(i) \in S$ for $i \in \{1, 2, \ldots, n\}$ such that $k(i) > k(i-1)$ if $i > 1$, and $f\big(k(i)\big) * b(i) \in \sigma(M_i)$ for each $f \in F$.

Let $M_{n+1} = M_n \cup W_{n+1} \cup \bigcup_{f\in F} \big(M_n * f\big(k(n)\big) * b(n)\big)$. Pick $k(n+1) > k(n)$ such that $f\big(k(n+1)\big) \in \sigma(M_{n+1})$ for each $f \in F$. Pick $b(n+1) \in \sigma\big(\bigcup_{f\in F} M_{n+1} * f\big(k(n+1)\big)\big)$. Then $f\big(k(n+1)\big) * b(n+1) \in \sigma(M_{n+1})$ for each $f \in F$.

The construction being complete, we claim that for each $f \in F$ and each $H \in \mathcal{P}_f(\mathbb{N})$, if $l = \min H$, then $\prod_{t\in H} f\big(k(t)\big) * b(t)$ is defined and $\prod_{t\in H} f\big(k(t)\big) * b(t) \in \sigma(M_l)$. So, in particular, $\prod_{t\in H} f\big(k(t)\big) * b(t) \in \sigma(W_l)$.

So let $f \in F$ be given. We proceed by induction on $|H|$. Assume first that $H = \{l\}$. Then $f\big(k(l)\big) \in \sigma(M_l)$ and $b(l) \in \sigma\big(M_l * f\big(k(l)\big)\big)$, so $f\big(k(l)\big) * b(l) \in \sigma(M_l)$.

Now assume that $|H| > 1$, let $l = \min H$, let $G = H \setminus \{l\}$, and let $p = \min G$. Then

$$\prod_{t\in G} f\big(k(t)\big) * b(t) \in \sigma(M_p) \subseteq \sigma(M_{l+1}) \subseteq \sigma\big(M_l * f\big(k(l)\big) * b(l)\big).$$

So, for each $w \in M_l$,

$$w * f\big(k(l)\big) * b(l) * \prod_{t\in G} f\big(k(t)\big) * b(t) = w * \prod_{t\in H} f\big(k(t)\big) * b(t)$$

is defined, so $\prod_{t\in H} f\big(k(t)\big) * b(t) \in \sigma(M_l)$. □

Theorem 4.4. *Let $(S, *)$ be a countable commutative adequate partial semigroup and let A be a J_δ-set in S. Then A is a J-set.*

Proof. Assume that S has been enumerated as $\langle s_t\rangle_{t=1}^\infty$. Let $F \in \mathcal{P}_f(\mathcal{F})$ and $L \in \mathcal{P}_f(S)$ be given. Pick an increasing sequence $\langle k(t)\rangle_{t=1}^\infty$ in $\mathbb{N}$ and a sequence $\langle b(t)\rangle_{t=1}^\infty$ in S as guaranteed by Lemma 4.3 for F. For $f \in F$, define $g_f(t) = f\big(k(t)\big) * b(t)$. Then $\{g_f : f \in F\} \in \mathcal{P}_f(\mathcal{F})$, so pick $d \in S$ and $H \in \mathcal{P}_f(\mathbb{N})$ such that, for each $f \in F$, $d * \prod_{t\in H} g_f(t) \in A \cap \sigma(L)$. Let $m = |H|$ and enumerate H in order as $\langle n(j)\rangle_{j=1}^m$. Define $a \in S^{m+1}$ by $a(1) = d$ and for $j \in \{2, 3, \ldots, m+1\}$, $a(j) = b\big(n(j-1)\big)$. Define $t \in \mathcal{J}_m$ by, for $j \in \{1, 2, \ldots, m\}$, $t(j) = k\big(n(j)\big)$. Then one has for each $f \in F$ that $\big(\prod_{j=1}^m a(j) * f\big(t(j)\big)\big) * a(m+1) = d * \prod_{j\in H} g_f(j) \in A \cap \sigma(L)$. □

We now set out to prove in Theorem 4.8 that if the union of two subsets of an adequate partial semigroup is a J-set, then one of them is. Our argument requires that S be countable. We do not know whether countability is required for the result.

Definition 4.5. A set $F \in \mathcal{P}_f(\mathcal{F})$ is a *strongly adequate set* if and only if whenever $\phi : \mathbb{N} \to F$ and $g : \mathbb{N} \to S$ is defined by, for $n \in \mathbb{N}$, $g(n) = \phi(n)(n)$, then g is an adequate sequence.

Given any finite set of adequate sequences, our next result establishes that there exists an increasing sequence in $\mathbb{N}$ such that the set of resulting sequences is strongly adequate. In other words, if a set F in $\mathcal{P}_f(\mathcal{F})$ is not strongly adequate, we can use it to build a strongly adequate set.

Lemma 4.6. *Assume that $(S,*)$ is a countable adequate partial semigroup and let $F \in \mathcal{P}_f(\mathcal{F})$. There is an increasing sequence $\langle \delta(t)\rangle_{t=1}^{\infty}$ in $\mathbb{N}$ such that if for each $f \in F$, $h_f : \mathbb{N} \to S$ is defined by, for $n \in \mathbb{N}$, $h_f(n) = f(\delta(n))$, then $\{h_f : f \in F\}$ is a strongly adequate set.*

Proof. Let $F \in \mathcal{P}_f(\mathcal{F})$ be given and let $p \in \mathbb{N}$ such that $|F| = p$ and write $F = \{f_1, f_2, \ldots, f_p\}$. Since S is countable, we can choose a sequence $\langle L_n\rangle_{n=1}^{\infty}$ such that for each n, $L_n \subseteq L_{n+1}$ and $S = \bigcup_{n=1}^{\infty} L_n$. We will inductively build $\langle \delta(n)\rangle_{n=1}^{\infty}$ in $\mathbb{N}$. Consider $L_1 \in \mathcal{P}_f(S)$. We can pick $k_1 \in \mathbb{N}$ such that $FP(\langle f(m)\rangle_{m=k_1}^{\infty}) \subseteq \sigma(L_1)$ for each $f \in F$. Let $\delta(1) = k_1$. Then $f_i(\delta(1)) \in \sigma(L_1)$ for each $i \in \{1, 2, \ldots, p\}$. Now let $n \geq 2$ and assume that for all $r \in \{1, 2, \ldots, n-1\}$, we have already chosen $\delta(r)$ so that

(1) if $r > 1$, then $\delta(r-1) < \delta(r)$, and
(2) whenever $\emptyset \neq H \subseteq \{1, 2, \ldots, n-1\}$ and $\varphi : H \to \{1, 2, \ldots, p\}$, if $r \leq \min H$, then $\prod_{t\in H} f_{\varphi(t)}(\delta(t)) \in \sigma(L_r)$.

Let

$$M_n = L_n \cup \{x * \textstyle\prod_{t\in H} f_{\ell_t}(\delta(t)) : x \in L_{n-1}, \emptyset \neq H \subseteq \{1, 2, \ldots, n-1\}, \text{ each } \ell_t \in \{1, 2, \ldots, p\} \text{ for } t \in H, \text{ and } \textstyle\prod_{t\in H} f_{\ell_t}(\delta(t)) \in \sigma(\{x\})\}.$$

Then $M_n \in \mathcal{P}_f(S)$. We can pick $k_n \in \mathbb{N}$ such that, for each $f \in F$, $FP(\langle f(m)\rangle_{m\geq k_n}) \subseteq \sigma(M_n)$. Let $\delta(n) = \max\{k_n, \delta(n-1)+1\}$. Let $1 \leq r \leq n$ and $\emptyset \neq G \subseteq \{r, r+1, \ldots, n\}$. For each $t \in G$, let $\ell_t \in \{1, 2, \ldots, p\}$. We claim $\prod_{t\in G} f_{\ell_t}(\delta(t)) \in \sigma(L_r)$.

If $n \notin G$, then our result is given by the induction hypothesis. Assume $n \in G$. If $G = \{n\}$, then $\prod_{t\in G} f_{\ell_t}(\delta(t)) = f_{\ell_n}(\delta(n))$. Recall $\delta(n) = \max\{k_n, \delta(n-1)+1\}$ and k_n was chosen so that $FP(\langle f(m)\rangle_{m\geq k_n}) \subseteq \sigma(M_n)$ for each $f \in F$. This implies $\prod_{t\in H} f_{\ell_t}(\delta(t)) \in \sigma(L_n) \subseteq \sigma(L_r)$. Now assume $G \setminus \{n\} \neq \emptyset$. Then $\prod_{t\in G\setminus\{n\}} f_{\ell_t}(\delta(t)) \in \sigma(L_r)$. So for $x \in L_r$, $x * \prod_{t\in G\setminus\{n\}} f_{\ell_t}(\delta(t)) \in M_n$, which implies $x * \prod_{t\in G\setminus\{n\}} f_{\ell_t}(\delta(t)) * f_{\ell_n}(\delta(n))$ is defined. As a result, $\prod_{t\in G} f_{\ell_t}(\delta(t)) \in \sigma(L_r)$. Thus, our induction hypothesis is satisfied.

We have just completed the construction of increasing $\langle \delta(t)\rangle_{t\geq 1}$ in $\mathbb{N}$. Note that for any $G \in \mathcal{P}_f(\mathbb{N})$, $r \leq \min G$ and $\varphi : G \to \{1, 2, \ldots, p\}$, $\prod_{t\in G} f_{\varphi(t)}(\delta(t)) \in \sigma(L_r)$. For each $i \in \{1, 2, \ldots, p\}$ and $n \in \mathbb{N}$, define $h_i(n) = f_i(\delta(n))$. We claim $\{h_i : i \in \{1, 2, \ldots, p\}\}$ is a strongly adequate set. Let $\varphi : \mathbb{N} \to \{1, 2, \ldots, p\}$ and $g : \mathbb{N} \to S$ so that $g(n) = h_{\varphi(n)}(n)$ for each $n \in \mathbb{N}$. We must show $g \in \mathcal{F}$.

(1) Let $H \in \mathcal{P}_f(\mathbb{N})$. Then

$$\prod_{t\in H} g(t) = \prod_{t\in H} h_{\varphi(t)}(t) = \prod_{t\in H} f_{\varphi(t)}(\delta(t))\,.$$

Since $\prod_{t\in H} f_{\varphi(t)}(\delta(t)) \in \sigma(L_r)$ where $r \leq \min H$, then the product must be defined. So the first condition is satisfied.

(2) Let $L \in \mathcal{P}_f(S)$. Then there exists $r \in \mathbb{N}$ such that $L \subseteq L_r$. We claim that $FP(\langle g(n)\rangle_{n=r}^{\infty}) \subseteq \sigma(L_r) \subseteq \sigma(L)$. To see this, let $K \in \mathcal{P}_f(\{r, r+1, \ldots,\})$. Then

$$\prod_{n\in K} g(n) = \prod_{n\in K} h_{\varphi(n)}(n) = \prod_{n\in K} f_{\varphi(n)}(\delta(n)) \in \sigma(L_r)$$

by induction hypothesis (2).

Hence, g is an adequate sequence. Therefore, $\{h_i : i \in \{1, 2, \ldots, p\}\}$ is a strongly adequate set. □

The proof of the following lemma is a routine exercise.

Lemma 4.7. *Let $(S, *)$ be an adequate partial semigroup. Let f be an adequate sequence in S, and let $\langle H_n\rangle_{n=1}^{\infty}$ be a sequence in $\mathcal{P}_f(\mathbb{N})$ such that* $\max H_n < \min H_{n+1}$ *for each $n \in \mathbb{N}$. Define $g : \mathbb{N} \to S$ such that for each $n \in \mathbb{N}$, $g(n) = \prod_{t\in H_n} f(t)$. Then g is an adequate sequence in S.*

Theorem 4.8. *Let $(S, *)$ be a countable adequate partial semigroup, let A be a J-set in S, and let $A = A_1 \cup A_2$. Then either A_1 is a J-set in S or A_2 is a J-set in S.*

Proof. Suppose not. Pick $F_1, F_2 \in \mathcal{P}_f(\mathcal{F})$ and $L_1, L_2 \in \mathcal{P}_f(S)$ such that for all $i \in \{1, 2\}$, all $m \in \mathbb{N}$, all $a \in S^{m+1}$, and all $t \in \mathcal{J}_m$, there exists $f \in F_i$ such that $\big(\prod_{j=1}^{m} a(j)f(t(j))\big)a(m+1) \notin A_i \cap \sigma(L_i)$. Let $F = F_1 \cup F_2$ and let $L = L_1 \cup L_2$. Let $p = |F|$ and write $F = \{f_1, f_2, \ldots, f_p\}$. Using Lemma 4.6, pick an increasing sequence $\langle\delta(n)\rangle_{n\geq 1}$ in $\mathbb{N}$, and for each $i \in \{1, 2, \ldots, p\}$, define $h_i : \mathbb{N} \to S$ by, for $n \in \mathbb{N}$, $h_i(n) = f_i(\delta(n))$. Then $H = \{h_i : i \in \{1, 2, \ldots, p\}\}$ is a strongly adequate set.

By [10, Lemma 14.8.1] (which is a version of the Hales–Jewett Theorem), we can pick $n \in \mathbb{N}$ such that whenever the set W of length n words over the alphabet $\{1, 2, \ldots, p\}$ is 2-colored, there is a variable word $w(v)$ beginning and ending with a constant and without successive occurrences of v such that $\{w(\ell) : \ell \in \{1, 2, \ldots, p\}\}$ is monochromatic. For each $w = b_1 b_2 \ldots b_n \in W$, define $g_w : \mathbb{N} \to S$ by, for $y \in \mathbb{N}$,

$$g_w(y) = \prod_{i=1}^{n} h_{b_i}(ny + i)\,.$$

Since H is a strongly adequate set, then the function r defined by $r(ny+i) = h_{b_i}(ny+i)$ for all $y \in \mathbb{N}$ and each $i \in \{1, 2, \ldots, n\}$ is adequate. By Lemma 4.7, g_w is adequate for each $w \in W$.

Since A is a J-set and $\{g_w : w \in W\} \in \mathcal{P}_f(\mathcal{F})$, we can pick $k \in \mathbb{N}$, $b \in S^{k+1}$, and $u \in \mathcal{J}_k$ such that, for each $w \in W$,

$$\big(\prod_{j=1}^{k} b(j) g_w(u(j))\big) b(k+1) \in A \cap \sigma(L)\,.$$

Define $\varphi : W \to \{1,2\}$ by $\varphi(w) = 1$ if $\big(\prod_{j=1}^{k} b(j) g_w(u(j))\big) b(k+1) \in A_1$ and $\varphi(w) = 2$ otherwise. Pick a variable word $w(v)$ beginning and ending with a constant without successive occurrences of v such that $\{w(\ell) : \ell \in \{1,2,\ldots,p\}\}$ is monochromatic. Without loss of generality, assume $\varphi(w(\ell)) = 1$ for each ℓ. In other words, $\big(\prod_{j=1}^{k} b(j) g_{w(\ell)}(u(j))\big) b(k+1) \in A_1$ for each ℓ.

Let $w(v) = b_1 b_2 \ldots b_n$ such that $b_i \in \{1,2,\ldots,p\} \cup \{v\}$ for each $i \in \{1,2,\ldots,n\}$. Then some $b_i = v$, $b_1 \neq v$, $b_n \neq v$, and if $b_i = v$, then $b_{i+1} \neq v$. Let r be the number of occurrences of v in $w(v)$. Pick $\langle G(x)\rangle_{x=1}^{r+1}$ in $\mathcal{P}_f(\mathbb{N})$ and $s \in \mathcal{J}_r$ such that $\max G(x) < s(x) < \min G(x+1)$ for each $x \in \{1,2,\ldots,r\}$, $G = \bigcup_{x=1}^{r+1} G(x) = \{i \in \{1,2,\ldots,n\} : b_i \in \{1,2,\ldots,p\}\}$, and $\{s(1), s(2), \ldots s(r)\} = \{i \in \{1,2,\ldots,n\} : b_i = v\}$. We claim that given any $y \in \mathbb{N}$, there exists $c_y \in S^{r+1}$ and $z_y \in \mathcal{J}_r$ such that for all $\ell \in \{1,2,\ldots,p\}$, $g_{w(\ell)}(y) = \big(\prod_{q=1}^{r} c_y(q) h_\ell(z_y(q))\big) c_y(r+1)$, and for each y, $z_y(r) < z_{y+1}(1)$.

Let y be given. For $q \in \{1,2,\ldots,r+1\}$, let $c_y(q) = \prod_{i \in G(q)} h_{b_i}(ny+i)$ and for $q \in \{1,2,\ldots,r\}$, let $z_y(q) = ny + s(q)$. Note $z_y(r) = ny + s(r) \leq ny + n < z_{y+1}(1)$. Now let $\ell \in \{1,2,\ldots,p\}$ be given. Then $w(\ell) = d_1 d_2 \ldots d_n$ where

$$d_i = \begin{cases} b_i & i \in G \\ \ell & i \in \{s(1), s(2, \ldots, s(r)\} \end{cases}$$

for each $i \in \{1,2,\ldots,n\}$.

Hence,

$$\begin{aligned}
& g_{w(\ell)}(y) \\
&= \prod_{i=1}^{n} h_{d_i}(ny+i) \\
&= \left(\prod_{q=1}^{r}\left(\prod_{i\in G(q)} h_{b_i}(ny+i)\right) h_\ell(ny+s(q))\right) \prod_{i\in G(r+1)} h_{b_i}(ny+i) \\
&= \left(\prod_{q=1}^{r}\left(\prod_{i\in G(q)} h_{b_i}(ny+i)\right) h_\ell(z_y(q))\right) \prod_{i\in G(r+1)} h_{b_i}(ny+i) \\
&= \big(\prod_{q=1}^{r} c_y(q) h_\ell(z_y(q))\big) c_y(r+1).
\end{aligned}$$

Thus, our claim is satisfied.

We claim that there exist $m \in \mathbb{N}$, $a \in S^{m+1}$, and $t \in \mathcal{J}_m$ such that, for each $\ell \in \{1, 2, \ldots, p\}$,

$$\big(\prod_{j=1}^{k} b(j) g_{w(\ell)}(u(j))\big) b(k+1) = \big(\prod_{x=1}^{m} a(x) h_\ell(t(x))\big) a(m+1)\,.$$

To see this, let $m = kr$. For $j \in \{1, 2, \ldots, k\}$ and $q \in \{1, 2, \ldots, r\}$, let $t\big((j-1)r + q\big) = z_{u(j)}(q)$. Let $a(1) = b(1) c_{u(1)}(1)$ and let $a(m+1) = c_{u(k)}(r+1) b(k+1)$. For $j \in \{1, 2, \ldots, k-1\}$, let $a(jr+1) = c_{u(j)}(r+1) b(j+1) c_{u(j+1)(1)}$. For $j \in \{1, 2, \ldots, k\}$ and $q \in \{2, 3, \ldots, r\}$, let $a\big((j-1)r + q\big) = c_{u(j)}(q)$. Then, for $\ell \in \{1, 2, \ldots, p\}$,

$$\begin{aligned}
&\big(\prod_{j=1}^{k} b(j) g_{w(\ell)}(u(j))\big) b(k+1) \\
&\quad = \bigg(\prod_{j=1}^{k} b(j) \big(\prod_{q=1}^{r} c_{u(j)}(q) h_\ell\big(z_{u(j)}(q)\big)\big) c_{u(j)}(r+1)\bigg) b(k+1) \\
&\quad = \big(\prod_{x=1}^{m} a(x) h_\ell(t(x))\big) a(m+1)\,.
\end{aligned}$$

Note that $\big\langle \delta\big(t(1)\big), \delta\big(t(2)\big), \ldots, \delta\big(t(m)\big)\big\rangle \in \mathcal{J}_m$. Pick $\ell \in \{1, 2, \ldots, p\}$ such that $f_\ell \in F_1$ and $\big(\prod_{x=1}^{m} a(x) f_\ell\big(\delta\big(t(x)\big)\big)\big) a(m+1) \notin A_1 \cap \sigma(L_1)$. That is to say,

$$\big(\prod_{x=1}^{m} a(x) h_\ell\big(t(x)\big)\big) a(m+1) \notin A_1 \cap \sigma(L_1)\,.$$

But

$$\begin{aligned}
\big(\prod_{x=1}^{m} a(x) h_\ell\big(t(x)\big)\big) a(m+1) &= \prod_{j=1}^{k} \big(b(j) g_{w(\ell)}(u(j))\big) b(k+1) \\
&\in A_1 \cap \sigma(L) \\
&\subseteq A_1 \cap \sigma(L_1)\,.
\end{aligned}$$

This is a contradiction. □

Corollary 4.9. *Let $(S, *)$ be a countable adequate partial semigroup and let $A \subseteq S$. Then $\overline{A} \cap J(S) \neq \emptyset$ if and only if A is a J-set.*

Proof. The necessity is trivial. By Theorem 4.8, if the union of finitely many subsets of S is a J-set, then one of them is. Thus. by [10, Theorem 3.11], if A is a J-set, then there exists $p \in J(S)$ such that $A \in p$. □

We now set out to show, as promised earlier, that if $(S, *)$ is the partial semigroup of located words introduced in Definition 3.8, there exists a set C which is a member of an idempotent in $J(S)$ but is not central.

If $(S, *)$ and $(T, \diamond)$ are partial semigroups, we say that $f : S \to T$ is a *partial semigroup homomorphism* if and only if, whenever $x, y \in S$ and $x * y$ is defined, then $f(x) \diamond f(y)$ is defined and $f(x * y) = f(x) \diamond f(y)$.

Lemma 4.10. *Let Σ be a finite nonempty set and let $(S, *)$ be the partial semigroup of located words over Σ. Define $\psi : S \to \mathbb{N}$ by, for $f \in S$, $\psi(f) = |\mathrm{dom}(f)|$. Then ψ is a surjective partial semigroup homomorphism. If A is a J-set in $\mathbb{N}$, then $\psi^{-1}[A]$ is a J-set in S.*

Proof. It is trivial that ψ is a surjective partial semigroup homomorphism to $(\mathbb{N},+)$. Assume that A is a J-set in $\mathbb{N}$. To see that $\psi^{-1}[A]$ is a J-set in S, let $F \in \mathcal{P}_f(\mathcal{F})$ and $W \in \mathcal{P}_f(S)$ be given. Pick $M \in \mathbb{N}$ such that for all $f \in W$, $\max\operatorname{supp}(f) < M$. By Lemma 4.3, pick an increasing sequence $\langle k(t)\rangle_{t=1}^\infty$ in $\mathbb{N}$ and a sequence $\langle b(t)\rangle_{t=1}^\infty$ in S such that for each $f \in F$, $\langle f\big(k(t)\big) * b(t)\rangle_{t=1}^\infty$ is an adequate sequence in S. We may presume that $\min\operatorname{supp}\big(f\big(k(1)\big)\big) > M$ and, therefore, for each $t \in \mathbb{N}$, $\min\operatorname{supp}\big(f\big(k(t)\big)\big) > M$.

For $f \in F$ and $t \in \mathbb{N}$, let $h_f(t) = \psi\big(f\big(k(t)\big) * b(t)\big)$. Since A is a J-set in $\mathbb{N}$ and $\mathbb{N}$ is commutative, we may pick $d \in \mathbb{N}$ and $H \in \mathcal{P}_f(\mathbb{N})$ such that for each $f \in F$, $d + \sum_{n\in H} h_f(n) \in A$. Enumerate H in order as $\langle v(1), v(2), \ldots, v(m)\rangle$ and pick $z \in \Sigma$. For $t \in \{1,2,\ldots,m\}$, let $j(t) = k\big(v(t)\big)$. Let $a(1) = \{(M,z)\}$ and for $t \in \{2,3,\ldots,m\}$, let $a(t) = b\big(v(t-1)\big)$. If $d = 1$, let $a(m+1) = b\big(v(m)\big)$. If $d > 1$, pick $w \in S$ such that $\psi(w) = d-1$ and $\min\operatorname{dom}(w) > \max\operatorname{dom}\big(b\big(v(m)\big)\big)$ and let $a(m+1) = b\big(v(m)\big) * w$.

We claim that for each $f \in F$, $\psi\big(\big(\prod_{t=1}^m a(t) * f\big(j(t)\big)\big) * a(m+1)\big) = d + \sum_{n\in H} h_f(n)$ so that $\big(\prod_{t=1}^m a(t) * f\big(j(t)\big)\big) * a(m+1) \in \psi^{-1}[A] \cap \sigma(W)$, as required. So let $f \in F$ be given. We will do the verification assuming that $m > 1$ and $d > 1$, the other cases being similar. Now

$$\begin{aligned}
&\big(\textstyle\prod_{t=1}^m a(t) * f\big(j(t)\big)\big) * a(m+1) = \\
&a(1) * \big(\textstyle\prod_{t=1}^{m-1} f\big(j(t)\big) * a(t+1)\big) * f\big(j(m)\big) * a(m+1) = \\
&a(1) * \big(\textstyle\prod_{t=1}^{m-1} f\big(k\big(v(t)\big)\big) * b\big(v(t)\big)\big) * f\big(k\big(v(m)\big)\big) * b\big(v(m)\big) * w\,,
\end{aligned}$$

so

$$\begin{aligned}
&\psi\Big(\big(\textstyle\prod_{t=1}^m a(t) * f\big(j(t)\big)\big) * a(m+1)\Big) = \\
&1 + \textstyle\sum_{t=1}^{m-1} h_f\big(v(t)\big) + h_f\big(v(m)\big) + d - 1 = d + \sum_{n\in H} h_f(n)\,.
\end{aligned}$$

□

Lemma 4.11. *Let Σ be a finite nonempty set and let $(S,*)$ be the partial semigroup of located words over Σ. Define $\psi : S \to \mathbb{N}$ by, for $f \in S$, $\psi(f) = |\operatorname{dom}(f)|$. Let p be an idempotent in $J(\mathbb{N},+)$. There is an idempotent $q \in J(S)$ such that $\widetilde{\psi}(q) = p$ where $\widetilde{\psi} : \beta S \to \beta\mathbb{N}$ is the continuous extension of ψ.*

Proof. By Lemma 4.10, if $A \in p$, then $\psi^{-1}[A]$ is a J-set in S, so by Corollary 4.9, $\overline{\psi^{-1}[A]} \cap J(S) \neq \emptyset$. Consequently, $\{\overline{\psi^{-1}[A]} \cap J(S) : A \in p\}$ is a set of closed subsets of βS with the finite intersection property and thus, $\bigcap_{A\in p}\big(\overline{\psi^{-1}[A]} \cap J(S)\big) \neq \emptyset$. Consequently, we have that $\{q \in J(S) : \widetilde{\psi}(q) = p\} \neq \emptyset$. By [10, Theorem 4.22.3], the restriction of $\widetilde{\psi}$ to δS is a homomorphism to $\beta\mathbb{N}$, so we have that $\{q \in J(S) : \widetilde{\psi}(q) = p\}$ is a compact subsemigroup of δS and thus has an idempotent. □

Theorem 4.12. *Let Σ be a finite nonempty set and let $(S,*)$ be the partial semigroup of located words over Σ. There exists a subset C of S which is a member of an idempotent in $J(S)$ and is not central. In fact, C is not piecewise syndetic. That is, $\overline{C} \cap c\ell K(\delta S) = \emptyset$.*

Proof. By [7, Theorem 5.5], pick an idempotent $p \in J(\mathbb{N}) \setminus c\ell K(\beta\mathbb{N})$. (It is an easy exercise to show that the set J defined in [7] is equal to $J(\mathbb{N})$.) We shall show that $\widetilde{\psi}[\delta S] = \beta\mathbb{N}$. Assume for now that we have done this. As we noted in the proof of Lemma 4.11, the restriction of $\widetilde{\psi}$ to δS is a homomorphism to $\beta\mathbb{N}$. Having shown that $\widetilde{\psi}[\delta S] = \beta\mathbb{N}$, we then have by [10, Exercise 1.7.3] that $\widetilde{\psi}[K(\delta S)] = K(\beta\mathbb{N})$ and, consequently, $\widetilde{\psi}[c\ell K(\delta S)] = c\ell K(\beta\mathbb{N})$. By Lemma 4.11, pick an idempotent $q \in J(S)$ such that $\widetilde{\psi}(q) = p$. Then $q \notin c\ell K(\delta S)$. Pick $C \in q$ such that $\overline{C} \cap c\ell K(\delta S) = \emptyset$.

To see that $\widetilde{\psi}[\delta S] = \beta\mathbb{N}$, let $r \in \beta\mathbb{N}$ be given. Let

$$\mathcal{A} = \{\psi^{-1}[A] \cap \sigma(W) : A \in r \text{ and } W \in \mathcal{P}_f(S)\}\,.$$

Note that if $s \in \beta S$ and $\mathcal{A} \subseteq s$, then $s \in \delta S$ and $\widetilde{\psi}(s) = r$. So it suffices to show that $\mathcal{A}$ has the finite intersection property. Given $n \in \mathbb{N}$, $A_1, A_2, \ldots, A_n \in r$, and $W_1, W_2, \ldots, W_n \in \mathcal{P}_f(S)$,

$$\psi^{-1}[\textstyle\bigcap_{i=1}^{n} A_i] \cap \sigma(\bigcup_{i=1}^{n} W_i) \subseteq \bigcap_{i=1}^{n} \big(\psi^{-1}[A_i] \cap \sigma(W_i)\big),$$

so it suffices to show that for $A \in r$ and $W \in \mathcal{P}_f(S)$, $\psi^{-1}[A] \cap \sigma(W) \neq \emptyset$. Pick $x \in A$. (Note that r could be principal, so $A = \{x\}$ is possible.) Let $M = \bigcup_{g \in W} \operatorname{dom}(g)$ and let $n = \max M$. Pick $a \in \Sigma$. Let $h = \{(n+1,a),(n+2,a),\ldots,(n+x,a)\}$. Then $h \in \sigma(W)$ and $\psi(h) = x$. □

The version of the Central Sets Theorem produced in [2] has an apparently weaker conclusion than Theorem 1.2 in which the increasing sequences $\langle t(j)\rangle_{j=1}^{m}$ in $\mathbb{N}$ are replaced by sequences $\langle H(j)\rangle_{j=1}^{m}$ in $\mathcal{P}_f(\mathbb{N})$ with the property that if $j < m$, then $\max H(j) < \min H(m)$. Since one could take $H(j) = \{t(j)\}$, that version obviously follows from Theorem 1.2. John H. Johnson, Jr. shows in [11] that the two versions are equivalent for semigroups.

To make the presentation simpler, we will deal with the corresponding issue for J-sets. The following is the definition of J-sets as given in [2] adjusted for partial semigroups. We will denote these as J'-sets solely to be able to talk about the two notions.

Definition 4.13. Let $(S,*)$ be an adequate partial semigroup.

(a) For $m \in \mathbb{N}$, $\mathcal{I}_m = \big\{\big(H(1),H(2),\ldots,H(m)\big) : \text{each } H(i) \in \mathcal{P}_f(\mathbb{N})$ and if $i < m$, $\max H(i) < \min H(i+1)\big\}$.

(b) A set $A \subseteq S$ is a J'-set if and only if for every $F \in \mathcal{P}_f(\mathcal{F})$ and every $L \in \mathcal{P}_f(S)$, there exist $m \in \mathbb{N}$, $a \in S^{m+1}$, and $H \in \mathcal{I}_m$ such that for each $f \in F$, $\big(\prod_{j=1}^m a(j) * \prod_{t \in H(j)} f(t)\big) * a(m+1) \in A \cap \sigma(L)$.

It is trivial that any J-set is a J'-set. We set out to show that the converse is true if S is countable. Again, we do not know whether countability is needed for the conclusion.

Theorem 4.14. *Let $(S,*)$ be a countable adequate partial semigroup and let $A \subseteq S$. Then A is a J-set if and only if A is a J'-set.*

Proof. For the necessity, given $m \in \mathbb{N}$ and $t \in \mathcal{J}_m$, let $H(j) = \{t(j)\}$.

Now assume that A is a J'-set and let $F \in \mathcal{P}_f(\mathcal{F})$ and $L \in \mathcal{P}_f(S)$ be given. Pick sequences $\langle k(t) \rangle_{t=1}^\infty$ and $\langle b(t) \rangle_{t=1}^\infty$ as guaranteed by Lemma 4.3. For $f \in F$ and $t \in \mathbb{N}$, let $g_f(t) = f\big(k(t)\big) * b(t)$. Then $\{g_f : f \in f\} \in \mathcal{P}_f(\mathcal{F})$, so pick $m \in \mathbb{N}$, $a \in S^{m+1}$, and $H \in \mathcal{I}_m$ such that, for each $f \in F$, $x_f = \big(\prod_{j=1}^m \big(a(j) * \prod_{t \in H(j)} g_f(t)\big)\big) * a(m+1) \in A \cap \sigma(L)$.

For each $j \in \{1, 2, \ldots, m\}$, let $l(j) = |H(j)|$ and enumerate $H(j)$ in order as $\langle s(j,i) \rangle_{i=1}^{l(j)}$. Then, for $f \in F$,

$$x_f = \prod_{j=1}^m \Big(a(j) * \Big(\prod_{i=1}^{l(j)} f\big(k(s(j,i))\big) * b\big(s(j,i)\big)\Big)\Big) * a(m+1)\,.$$

Let $p = \sum_{j=1}^m l(j) = |\bigcup_{j=1}^m H(j)|$. Let

$$\begin{aligned}
&\langle c(1), c(2), \ldots, c(p+1) \rangle = \\
&\langle a(1), b\big(s(1,1)\big), \ldots, b\big(s(1,l(1)-1)\big), b\big(s(1,l(1))\big) * a(2), \\
&\quad b\big(s(2,1)\big), \ldots, b\big(s(2,l(2)-1)\big), b\big(s(2,l(2))\big) * a(3), \\
&\qquad\qquad \vdots \\
&\quad b\big(s(m,1)\big), \ldots, b\big(s(m,l(m)-1)\big), b\big(s(m,l(m))\big) * a(m+1) \rangle
\end{aligned}$$

and let

$$\begin{aligned}
&\langle t(1), t(2), \ldots, t(p) \rangle = \\
&\langle k\big(s(1,1)\big), k\big(s(1,2)\big), \ldots, k\big(s(1,l(1))\big)\rangle, \\
&\langle k\big(s(2,1)\big), k\big(s(2,2)\big), \ldots, k\big(s(2,l(2))\big)\rangle, \\
&\qquad \vdots \\
&\langle k\big(s(m,1)\big), k\big(s(m,2)\big), \ldots, k\big(s(m,l(m))\big)\rangle\rangle\,.
\end{aligned}$$

Then, for $f \in F$, $x_f = \big(\prod_{i=1}^p c(i) * f\big(t(i)\big)\big) * c(p+1)$. □

5. An Alternate Proof That Piecewise Syndetic Sets Are J-Sets

We conclude by presenting an alternate proof that any piecewise syndetic set in an adequate partial semigroup is a J-set. The proof we give in section 3 is based on an idea of Furstenberg and Katznelson in [4]. The proof we are going to present is based on an idea of Andreas Blass, which is first used in [1]. The idea of this latter proof is simpler. It simply uses the fact that homomorphisms preserve the ordering of idempotents and that if p is a minimal idempotent and $q \leq p$, then $q = p$. The casual reader might even believe that the current proof is shorter than our previous proof. That is because we will not present the proof of Lemma 5.2. Its proof is straightforward, but by the time one fills in all of the details, that proof is longer than that of Lemma 3.2.

Definition 5.1. Let $(S, *)$ be an adequate partial semigroup and let $F \in \mathcal{P}_f(\mathcal{T})$. Let

$$\begin{aligned} T_F = \quad & \{\big(a(1), t(1), a(2), \ldots, a(m), t(m), a(m+1)\big) : m \in \mathbb{N} \\ & a \in S^{m+1}, t \in \mathcal{J}_m, \text{ and for all } f \in F, \\ & a(1) * f\big(t(1)\big) * * * f\big(t(m)\big) * a(m+1) \text{ is defined}\}. \end{aligned}$$

Assume that $S \cap T_F = \emptyset$ and let $R_F = S \cup T_F$. Define a partial operation $\diamond$ on R_F as follows.

(1) If $x, y \in S$, then $x \diamond y = x * y$, defined if and only if $x * y$ is defined.
(2) If $x \in S$ and $\vec{y} = \big(a(1), t(1), a(2), \ldots, a(m), t(m), a(m+1)\big) \in T_F$, then

$$x \diamond \vec{y} = \big(x * a(1), t(1), a(2), \ldots, a(m), t(m), a(m+1)\big)$$

defined if and only if for each $f \in F$, $x * a(1) * f\big(t(1)\big) * * * f\big(t(m)\big) * a(m+1)$ is defined and

$$\vec{y} \diamond x = \big(a(1), t(1), a(2), \ldots, a(m), t(m), a(m+1) * x\big)$$

defined if and only if for each $f \in F$, $a(1) * f\big(t(1)\big) * * * f\big(t(m)\big) * a(m+1) * x$ is defined.
(3) If $\vec{x} = \big(a(1), t(1), a(2), \ldots, a(m), t(m), a(m+1)\big) \in T_F$ and $\vec{y} = \big(b(1), s(1), b(2), \ldots, b(k), s(k), b(k+1)\big) \in T_F$, then

$$\vec{x} \diamond \vec{y} = \big(a(1), t(1), \ldots, t(m), a(m+1) * b(1), s(1), \ldots, s(k), b(k+1)\big)$$

defined if and only if $t(m) < s(1)$ and for all $f \in F$

$$a(1) * f\big(t(1)\big) * * * f\big(t(m)\big) * a(m+1) * b(1) * f\big(s(1)\big) * * * f\big(s(k)\big) * b(k+1)$$

is defined.

Note that the requirement that $S \cap T_F = \emptyset$ is not a serious restriction because one may always take an algebraic copy of S for which this is true.

Lemma 5.2. *Let $F \in \mathcal{P}_f(\mathcal{T})$. Then $(R_F, \diamond)$ is an adequate partial semigroup. Let $T' = \delta R_F \cap \delta T_F$. Then $T' = \delta R_F \cap \beta T_F$, $\delta R_F = \delta S \cup T'$, and T' is a two-sided ideal of δR_F.*

We do not know whether it must be true that $T' = \delta T$, which would be more elegant.

Theorem 5.3. *Let $(S,*)$ be an adequate partial semigroup and let A be a piecewise syndetic subset of S. Then A is a J-set.*

Proof. Let $F \in \mathcal{P}_f(\mathcal{T})$ and let $M \in \mathcal{P}_f(S)$. Let $T = T_F$, $R = R_F$, and $T' = \delta R \cap \delta T$. Since A is piecewise syndetic in S, pick $p \in \overline{A} \cap K(\delta S)$. Pick a minimal left ideal L of δS such that $p \in L$ and let r be an idempotent in L. Then $p = p * r$, so $\{x \in S : x^{-1}A \in r\} \in p$; therefore, pick $x \in \sigma(M)$ such that $x^{-1}A \in r$. By [10, Theorem 1.60], pick an idempotent $q \in K(\delta R)$ such that $q \leq r$. Then $q \in T' \subseteq \delta T$.

For $f \in F$, define $\nu_f : R \to S$ by $\nu_f(a) = a$ for $a \in S$ and for

$$\vec{y} = \big(a(1), t(1), a(2), \ldots, a(m), t(m), a(m+1)\big) \in T\,,$$

$\nu_f(\vec{y}) = a(1) * f\big(t(1)\big) * * * f\big(t(m)\big) * a(m+1)$. Note that each ν_f is surjective. Further, it is easy to verify that each ν_f is a partial semigroup homomorphism. For each $f \in F$, let $\widetilde{\nu_f} : \beta R \to \beta S$ be the continuous extension of ν_f. By [10, Theorem 4.22.3], the restriction of $\widetilde{\nu_f}$ to δR is a homomorphism into δS.

Thus, for each $f \in F$, $\widetilde{\nu_f}(q) \leq \widetilde{\nu_f}(r)$. Since ν_f is the identity on S, $\widetilde{\nu_f}(r) = r$. Since r is minimal in δS, we have $\widetilde{\nu_f}(q) = r$. Given $f \in F$, we have that $\sigma(M * x) \cap x^{-1}A \in r$, so we may pick $B_f \in q$ such that $\widetilde{\nu_f}[\,\overline{B_f}\,] \subseteq \overline{\sigma(M * x) \cap x^{-1}A}$.

Pick $\big(a(1), t(1), \ldots, t(m), a(m+1)\big) \in \bigcap_{f \in F} B_f$. Then, for $f \in F$,

$$a(1) * f\big(t(1)\big) * * * f\big(t(m)\big) * a(m+1) \in \sigma(M * x) \cap x^{-1}A,$$

so $x * a(1) * f\big(t(1)\big) * * * f\big(t(m)\big) * a(m+1) \in \sigma(M) \cap A$. □

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