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Dense Difference Sets and their Combinatorial Structure

Vitaly Bergelson[1]
Paul Erdős
Neil Hindman[1]
and
Tomasz Łuczak

Abstract. We show that if a set B of positive integers has positive upper density, then its difference set *D(B)* has extremally rich combinatorial structure, both additively and multiplicatively. If on the other hand only the density of *D(B)* rather than B is assumed to be positive one is not guaranteed any multiplicative structure at all and is guaranteed only a modest amount of additive structure.

1. **Introduction.** Given a subset B of the set $\mathbb{N}$ of positive integers, denote by $D(B)$ its "difference set". That is $D(B) = \{x - y : x, y \in B \text{ and } x > y\}$. We are concerned here with difference sets which are "large" in one of two senses. That is, we ask either that $\overline{d}(B) > 0$ or that $\overline{d}(D(B)) > 0$ where

$$\overline{d}(B) = \lim\sup_{n\to\infty} |A \cap \{1, 2, \ldots, n\}|/n.$$

We show in Section 2 that if $\overline{d}(B) > 0$, then $D(B)$ has an incredibly rich algebraic structure. We show for example that given any function $f : \mathbb{N} \longrightarrow \mathbb{N}$, there must exist a sequence $\langle x_n\rangle_{n=1}^{\infty}$ so that $\{\sum_{n\in F} a_n \cdot x_n : F$ is a finite nonempty subset of $\mathbb{N}$ and for each $n \in F$, $1 \le a_n \le f(n)\} \cup \{\prod_{n\in F} x_n^{a_n} : F$ is a finite nonempty subset of $\mathbb{N}$ and for each $n \in F$, $1 \le a_n \le f(n)\} \subseteq D(B)$.

With no sort of largeness assumptions at all (beyond the requirement that B should have at least three members) one must always be able to get some a and b with $\{a, b, a+b\} \subseteq D(B)$. (Given $x < y < z$ in B, let $a = y - x$ and $b = z - y$.) Infiniteness by itself doesn't help much. Indeed, it is easy to see that if $B = \{2^n : n \in \mathbb{N}\}$, then for no a, b, and c is $\{a, b, c, a+b, a+c, b+c, a+b+c\} \subseteq D(B)$. On the other

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hand, we show in Section 3 that if $\overline{d}(D(B)) > 0$, one can always find a, b, and c with $\{a, b, c, a+b, a+c, b+c, a+b+c\} \subseteq D(B)$.

We have not been able to determine whether $D(B)$ (where $\overline{d}(D(B)) > 0$) must contain some 4 elements with all of their sums. However, we do show in Section 3 that one can find sets B with $\overline{d}(D(B))$ arbitrarily close to $1/2$ such that $D(B)$ contains no five elements and all of their sums. We also show that we can find sets B with $\overline{d}(D(B))$ arbitrarily close to 1 such that $D(B)$ does not contain any $\{a, b, a \cdot b\}$.

2. **The difference set of a set of positive density**. We show here that if $\overline{d}(B) > 0$, then $D(B)$ has a rich additive and multiplicative structure. Many of the results in this section are from the dissertation of the first author [2]. We begin by stating a well known result about sets of positive upper density, whose proof we leave as an exercise.

2.1 **Lemma**. *Let $A \subseteq \mathbb{N}$ such that $\overline{d}(A) > 0$ and let $k \in \mathbb{N}$ such that $1/k < \overline{d}(A)$. Then given any $t_1, t_2, \ldots, t_k$ in $\mathbb{N}$ there exist some $i < j$ in $\{1, 2, \ldots, k\}$ with $\overline{d}((A - t_i) \cap (A - t_j)) > 0$.*

Note by way of contrast that it is easy to get two disjoint sets both with upper density equal to 1. It is an immediate consequence of Lemma 2.1 that if $\overline{d}(B) > 0$, then $D(B)$ is an IP*−set. That is, given any sequence $\langle x_n \rangle_{n=1}^{\infty}$ in $\mathbb{N}$ there is some finite nonempty subset F of $\mathbb{N}$ such that $\sum_{n \in F} x_n \in D(B)$. (To see this, for each i, let $a_i = \sum_{n=1}^{i} x_n$ and pick $i < j$ such that $\overline{d}((B - a_i) \cap (B - a_j)) > 0$. Then $\sum_{n=i+1}^{j} x_n \in D(B)$.) Therefore, by [4, Theorem 2.6] there is some sequence $\langle x_n \rangle_{n=1}^{\infty}$ with $\{\sum_{n \in F} x_n : F \text{ is a finite nonempty subset of } \mathbb{N}\} \cup \{\prod_{n \in F} x_n : F \text{ is a finite nonempty subset of } \mathbb{N}\} \subseteq D(B)$. We show in Theorem 2.6 below that a stronger conclusion holds, (without invoking any results from [4]).

We shall utilize in our proofs two results from ergodic theory. The first of these is Furstenberg's famous correspondence principle which was first used in his proof of Szemerédi's Theorem [6]. Recall that a *measure preserving system* is a quadruple $(X, \mathcal{B}, \mu, T)$ where X is a nonempty set, $\mathcal{B}$ is a σ-algebra of subsets of X, μ is a nonnegative σ-additive measure defined on $\mathcal{B}$ with $\mu(X) = 1$ (so that $(X, \mathcal{B}, \mu)$ is a probability measure space), and T is an invertible measure preserving transformation of X. (That is, T is continuous, and whenever $B \in \mathcal{B}$, $T^{-1}B \in \mathcal{B}$ and $\mu(T^{-1}B) = \mu(B)$.)

2.2 **Theorem** (Furstenberg). *Let $B \subseteq \mathbb{N}$ with $\overline{d}(B) > 0$. There exist a measure preserving system $(X, \mathcal{B}, \mu, T)$ and a set $A \in \mathcal{B}$ such that $\mu(A) = \overline{d}(B)$ and for all $n \in \mathbb{N}$, $\overline{d}(B \cap (B - n)) \geq \mu(A \cap T^n A)$.*

Proof. [6, Theorem 1.1].▯

Given measure preserving systems $(X_1, \mathcal{B}_1, \mu_1, T_1)$ and $(X_2, \mathcal{B}_2, \mu_2, T_2)$ we follow standard practice and denote by $(X_1 \times X_2, \mathcal{B}_1 \times \mathcal{B}_2, \mu_1 \times \mu_2, T_1 \times T_2)$ the system where $X_1 \times X_2$ is the cartesian product, $\mathcal{B}_1 \times \mathcal{B}_2$is the σ-algebra generated by sets of the form $A_1 \times A_2$ for $A_1 \in \mathcal{B}_1$ and $A_2 \in \mathcal{B}_2$, $\mu_1 \times \mu_2$ is the measure on $\mathcal{B}_1 \times scrb_2$ determined by $(\mu_1 \times \mu_2)(A_1 \times A_2) = \mu_1(A_1) \cdot \mu_2(A_2)$ and $T_1 \times T_2$ is the measure preserving transformation defined by $(T_1 \times T_2)((x_1, x_2)) = (T_1(x_1), T_2(x_2))$.

2.3 **Theorem**. *Let $T_1, T_2, \ldots, T_k$ be invertible commuting transformations of a probability measure space $(X, \mathcal{B}, \mu)$. Assume that $p_1(n), p_2(n), \ldots, p_k(n)$ are polynomials with integer coefficients such that $p_i(0) = 0$ for $i \in \{1, 2, \ldots, k\}$. Let $A \in \mathcal{B}$ with $\mu(A) > 0$. Then there exists $n \in \mathbb{N}$ such that $\mu(A \cap T_1^{p_1(n)} T_2^{p_2(n)} \ldots T_k^{p_k(n)} A) > 0$.*

Proof. This is exactly [3, Theorem 4.2] except that the conclusion there has $n \in \mathbb{Z} \backslash \{0\}$. To derive this version we utilize the product space $(X \times X, \mathcal{B} \times \mathcal{B}, \mu \times \mu)$. For $i \in \{1, 2, \ldots, k\}$, let $S_i = T_i \times \iota$, where ι is the identity. For $i \in \{k+1, k+2, \ldots, 2k\}$, let $S_i = \iota \times T_{i-k}$ and let $p_i(n) = p_{i-k}(-n)$. Then $S_1, S_2, \ldots, S_{2k}$ are invertible commuting transformations of $(X \times X, \mathcal{B} \times \mathcal{B}, \mu \times \mu)$ and $(\mu \times \mu)(A \times A) > 0$ so pick (using [3, Theorem 4.2]) $n \in \mathbb{Z} \backslash \{0\}$ such that $(\mu \times \mu)((A \times A) \cap S_1^{p_1(n)} S_2^{p_2(n)} \ldots S_{2k}^{p_{2k}(n)}(A \times A)) > 0$. If $n > 0$ we see from the first coordinate that $\mu(A \cap T_1^{p_1(n)} T_2^{p_2(n)} \ldots T_k^{p_k(n)} A) > 0$. If $n < 0$ we see from the second coordinate that $\mu(A \cap T_1^{p_1(-n)} T_2^{p_2(-n)}$ ... $T_k^{p_k(-n)} A) > 0$. ▯

We shall see in Theorem 2.6 that whenever $\overline{d}(B) > 0$, $\overline{d}(B)$ contains sums and products from a sequence where terms are allowed to repeat a restricted number of times. We present first a special case so we may introduce the proof in a relatively uncomplicated setting.

2.4 **Theorem**. *Let $B \subseteq \mathbb{N}$ with $\overline{d}(B) > 0$. Then there is some sequence $\langle x_n \rangle_{n=1}^{\infty}$ such that $\{\sum_{n \in F} a_n x_n$: F is a finite nonempty subset of $\mathbb{N}$ and for each $n \in F$, $a_n \in \{1, 2\}\} \cup \{\prod_{n \in F} x_n^{a_n}$: F is a finite nonempty subset of $\mathbb{N}$ and for each $n \in F$, $a_n \in \{1, 2\}\} \subseteq D(B)$.*

Proof. Pick by Theorem 2.2 a measure preserving system $(X, \mathcal{B}, \mu, T)$ and some $A \in \mathcal{B}$ such that $\mu(A) = \overline{d}(B)$ and for each $n \in \mathbb{N}$, $\overline{d}(B \cap (B - n)) \geq \mu(A \cap T^n A)$. Observe that $\{n \in \mathbb{N} : \mu(A \cap T^n A) > 0\} \subseteq D(B)$. For $m \in \mathbb{N}$ and a sequence $\langle x_n \rangle_{n=1}^{m}$ in $\mathbb{N}$ let $E(\langle x_n \rangle_{n=1}^{m}) = \{\sum_{n \in F} a_n x_n$: F is a nonempty subset of $\{1, 2, \ldots, m\}$ and for each $n \in F$, $a_n \in \{1, 2\}\}$ and let $C(\langle x_n \rangle_{n=1}^{m}) = \{\prod_{n \in F} x_n^{a_n}$: F is a nonempty subset of $\{1, 2, \ldots, m\}$ and for each $n \in F, a_n \in \{1, 2\}\}$. We construct a sequence $\langle x_n \rangle_{n=1}^{\infty}$ by induction so that for each m, $E(\langle x_n\rangle_{n=1}^m)\cup C(\langle x_n\rangle_{n=1}^m)\subseteq\{n\in\mathbb{N}:\mu(A\cap T^nA)>0\}$ which will suffice by our observation.

To ground the induction consider the measure space $(X\times X\times X,\mathcal{B}\times\mathcal{B}\times\mathcal{B},\mu\times\mu\times\mu)$, let $S_1=(T\times\iota\times\iota)$, $S_2=(\iota\times T\times\iota)$, $S_3=(\iota\times\iota\times T)$, $p_1(n)=n$, $p_2(n)=2n$, and $p_3(n)=n^2$. (Recall that ι is the identity.) Pick by Theorem 2.3 some $x_1\in\mathbb{N}$ such that $(\mu\times\mu\times\mu)((A\times A\times A)\cap S_1^{p_1(x_1)}S_2^{p_2(x_1)}S_3^{p_3(x_1)}(A\times A\times A))>0$. From the first coordinate we see that $\mu(A\cap T_1^{x_1}A)>0$, from the second coordinate we see that $\mu(A\cap T_1^{2x_1}A)>0$, and from the third coordinate we see that $\mu(A\cap T_1^{x_1^2}A)>0$. Since $E(\langle x_n\rangle_{n=1}^1)=\{x_1,2x_1\}$ and $C(\langle x_n\rangle_{n=1}^1)=\{x_1,x_1^2\}$, the grounding is complete.

Now let $m\in\mathbb{N}$ be given and assume we have chosen $\langle x_n\rangle_{n=1}^{m-1}$ with $E(\langle x_n\rangle_{n=1}^{m-1})\cup C(\langle x_n\rangle_{n=1}^{m-1})\subseteq\{n\in\mathbb{N}:\mu(A\cap T^nA)>0\}$. Let $b=3^{m-1}$ and enumerate (with repetitions if need be) $\{0\}\cup E(\langle x_n\rangle_{n=1}^{m-1})$ as $\langle y_j\rangle_{j=1}^b$ and enumerate $\{1\}\cup C(\langle x_n\rangle_{n=1}^{m-1})$ as $\langle z_j\rangle_{j=1}^b$. Now consider the measure space $(\times_{j=1}^{4b}X,\times_{j=1}^{4b}\mathcal{B},\times_{j=1}^{4b}\mu)$. Let $H=\times_{j=1}^{b}((A\cap T^{y_j}A)\times(A\cap T^{y_j}A)\times A\times A)$, let $\overline{\mu}=\times_{j=1}^{4b}\mu$, and note that $\overline{\mu}(H)>0$. (Our induction hypothesis tells us that each $\mu(A\cap T^{y_j}A)>0$.) Let $S_1=\times_{j=1}^{b}(T\times\iota\times\iota\times\iota)$, $S_2=\times_{j=1}^{b}(\iota\times T\times\iota\times\iota)$, $S_3=\times_{j=1}^{b}(\iota\times\iota\times T^{z_j}\times\iota)$, and $S_4=\times_{j=1}^{b}(\iota\times\iota\times\iota\times T^{z_j})$. Let $p_1(n)=n,p_2(n)=2n,p_3(n)=n$, and $p_4(n)=n^2$. Pick by Theorem 2.2, some $x_m\in\mathbb{N}$ such that $\overline{\mu}(H\cap S_1^{p_1(x_m)}S_2^{p_2(x_m)}S_3^{p_3(x_m)}S_4^{p_4(x_m)}H)>0$.

To see that $E(\langle x_n\rangle_{n=1}^m)\subseteq\{n\in\mathbb{N}:\mu(A\cap T^nA)>0\}$, let $\emptyset\neq F\subseteq\{1,2,\ldots,m\}$ and for each $n\in F$, let $a_n\in\{1,2\}$. If $m\notin F$, then $\sum_{n\in F}a_nx_n\in E(\langle x_n\rangle_{n=1}^{m-1})$, so we assume $m\in F$. Pick $j\in\{1,2,\ldots,b\}$ such that $\sum_{n\in F}a_nx_n=y_j+a_mx_m$. If $a_m=1$, we see by looking at coordinate $4j-3$ that $\mu(A\cap T^{y_j}A\cap T^{x_m}(A\cap T^{y_j}A))>0$; in particular $\mu(A\cap T^{y_j+x_m})>0$. If $a_m=2$, we see by looking at coordinate $4j-2$ that $\mu(A\cap T^{y_j}A\cap T^{2x_m}(A\cap T^{y_j}A))>0$; in particular $\mu(A\cap T^{y_j+2x_m}A)>0$.

To see that $C(\langle x_n\rangle_{n=1}^m)\subseteq\{n\in\mathbb{N}:\mu(A\cap T^nA)>0\}$, let $\emptyset\neq F\subseteq\{1,2,\ldots,m\}$ and for each $n\in F$, let $a_n\in\{1,2\}$. If $m\notin F$, then $\prod_{n\in F}x_n^{a_n}\in C(\langle x_n\rangle_{n=1}^{m-1})$, so we assume $m\in F$. Pick $j\in\{1,2,\ldots,b\}$ such that $\prod_{n\in F}x_n^{a_n}=z_j\cdot x_x^{a_m}$. If $a_m=1$, we see by looking at coordinate $4b-1$ that $\mu(A\cap(T^{z_j})^{x_m}A)>0$ so that $\mu(A\cap T^{z_jx_m}A)>0$. If $a_m=2$, we see by looking at coordinate $4b$ that $\mu(A\cap(T^{z_j})^{x_m^2}A)>0$ so that $\mu(A\cap T^{z_jx_m^2}A)>0$. []

We observe in fact that if one has sets $B_1,B_2,\ldots,B_n$ with each $\overline{d}(B_i)>0$, then the conclusion of Theorem 2.4 applies to $\bigcap_{i=1}^n D(B_1)$. To see this one simply starts with the product system $(\times_{i=1}^nX_i,\times_{i=1}^n\mathcal{B}_i,\times_{i=1}^n\mu_i,\times_{i=1}^nT_i)$ where $(X_i,\mathcal{B}_i,\mu_i,T_i)$ is the system given by Theorem 2.2 for B_i.

Recall that a set $B\subseteq\mathbb{N}$ is an IP* set if and only if whenever $\langle x_n\rangle_{n=1}^\infty$ is a sequence in $\mathbb{N}$, one has $FS(\langle x_n\rangle_{n=1}^\infty) \cap B \neq \emptyset$. We pause now to observe that neither of the conclusions of Theorem 2.4 follow from the fact that $D(B)$ is an IP* set.

2.5 **Theorem**. *There is an* IP* *set* A *such that for no sequence* $\langle x_n\rangle_{n=1}^\infty$ *is* $\{\sum_{n\in F} a_n x_n : F$ *is a finite nonempty subset of* $\mathbb{N}$ *and for each* $n \in \mathbb{N}, a_n \in \{1,2\}\} \subseteq A$ *and for no sequence* $\langle y_n\rangle_{n=1}^\infty$ *is* $\{\prod_{n\in F} y_n^{a_n} : F$ *is a finite nonempty subset of* $\mathbb{N}$ *and for each* $n \in \mathbb{N}$, $a_n \in \{1,2\}\} \subseteq A$.

Proof. Let $B = \mathbb{N}\backslash\{x^2 : x \in \mathbb{N}\}$. Since one clearly cannot get any sequence $\langle x_n\rangle_{n=1}^\infty$ with $\{\sum_{n\in F} x_n : F$ is a finite nonempty subset of $\mathbb{N}\} \subseteq \{x^2 : x \in \mathbb{N}\}$, one has that B is an IP* set. And no sequence $\langle y_n\rangle_{n=1}^\infty$ has any $y_n^2 \in B$.

Now by [5, Theorem 3.14], there is a partition $\mathbb{N} = C_1 \cup C_2$ such that for no sequence $\langle x_n\rangle_{n=1}^\infty$ is $\{\sum_{n\in F} x_n : F$ is a finite nonempty subset of $\mathbb{N}\} \subseteq C_1$ and for no sequence $\langle y_n\rangle_{n=1}^\infty$ is $\{\sum_{n\in F_1} y_n + \sum_{n\in F_2} 2y_n : F_1$ and F_2 are finite nonempty subsets of $\mathbb{N}$ and $\max F_1 < \min F_2\} \subseteq C_2$. Then C_2 is an IP* set. Let $A = B \cap C_2$. Since the intersection of two IP* sets is again an IP* set (see [4]), we have that A is as required. ▯

The next theorem is our major result of this section. Considerably stronger statements are in fact available with the same proof. However, we are trying to keep the results easily comprehensible.

2.6 **Theorem**. *Let* $B \subseteq \mathbb{N}$ *with* $\overline{d}(B) > 0$ *and let* $f : \mathbb{N} \to \mathbb{N}$. *Then there is some sequence* $\langle x_n\rangle_{n=1}^\infty$ *such that* $\{\sum_{n\in F} a_n x_n : F$ *is a finite nonempty subset of* $\mathbb{N}$ *and for each* $n \in F$, $a_n \in \{1,2,\ldots,f(n)\}\} \cup \{\prod_{n\in F} x_n^{a_n} : F$ *is a finite nonempty subset of* $\mathbb{N}$ *and for each* $n \in F$, $a_n \in \{1,2,\ldots,f(n))\} \subseteq D(B)$.

Proof. We describe how to modify the proof of Theorem 2.3. First define $E(\langle x_n\rangle_{n=1}^m)$ and $C(\langle x_n\rangle_{n=1}^m)$ analogously. At the grounding level one takes the measure space $(\times_{i=1}^{2f(1)-1} X, \times_{i=1}^{2f(1)-1} \mathcal{B}, \times_{i=1}^{2f(1)-1} \mu)$. One lets $p_i(n) = i \cdot n$ for $i \in \{1,2,\ldots,f(1)\}$ and lets $p_i(n) = n^{i-f(1)-1}$ for $i \in \{f(1)+1, f(1)+2, \ldots, 2f(1)-1\}$.

At the induction stage, one lets $b = \prod_{i=1}^{m-1}(f(i)+1)$ and enumerates $E(\langle x_n\rangle_{n=1}^{m-1}) \cup \{0\}$ as $\langle y_j\rangle_{j=1}^b$ and enumerates $\{1\} \cup C(\langle x_n\rangle_{n=1}^{m-1})$ as $\langle z_j\rangle_{j=1}^b$. Then one uses the measure space $(\times_{j=1}^{b\cdot 2\cdot f(m)} X, \times_{j=1}^{b\cdot 2\cdot f(m)} \mathcal{B}, \times_{j=1}^{b\cdot 2\cdot f(m)} \mu)$, and lets $H = \times_{j=1}^b (\times_{i=1}^{f(m)} (A \cap T^{y_j} A) \times \times_{i=1}^{f(m)} A)$. Using the obvious definitions of $S_1, S_2, \ldots, S_{2\cdot f(m)}$ and $p_1, p_2, \ldots, p_{2\cdot f(m)}$ one completes the proof. ▯

3. **Additive structure in dense difference sets**. For the remainder of the paper we look at difference sets $D(B)$ where we no longer require that $\overline{d}(B) > 0$, but only that $\overline{d}(D(B)) > 0$.

Because difference sets are defined additively one would not necessarily expect them to have any multiplicative structure. On the other hand, Theorem 2.6 might make one suspect that they would have some multiplicative structure. We begin this section by showing that they need not.

3.1 **Theorem**. *Let $\epsilon > 0$. There is a set B such that $\overline{d}(D(B)) > 1-\epsilon$ and there do not exist a and b in $\mathbb{N}$ with $\{a,b,a\cdot b\} \subseteq D(B)$.*

Proof. Pick $\alpha \in \mathbb{N}$ such that $1/2^\alpha < \epsilon$. Define a sequence $\langle f(r)\rangle_{r=0}^{\infty}$ by $f(0) = 2+\alpha$ and $f(r+1) = 2(f(r)+\alpha)+1$. Let $\langle x_n\rangle_{n=1}^{\infty}$ enumerate $\bigcup_{r=0}^{\infty}\{2^{f(r)}, 2^{f(r)}+1, \ldots, 2^{f(r)+\alpha}-1\}$ in increasing order and note that for all n in $\mathbb{N}$, $x_n < 2^{f(n)}$. Let $B = \{2^{f(n)} : n \in \mathbb{N}\} \cup \{2^{f(n)} + x_n : n \in \mathbb{N}\}$. Then $D(B) = \{x_n : n \in \mathbb{N}\} \cup \{2^{f(n)} + x_n - 2^{f(m)} : m, n \in \mathbb{N}$ and $m < n\} \cup \{2^{f(n)} - 2^{f(m)} : m, n \in \mathbb{N}$ and $m < n\} \cup \{2^{f(n)} + x_n - 2^{f(m)} - x_m : m, n \in \mathbb{N}$ and $m < n\} \cup \{2^{f(n)} - 2^{f(m)} - x_m : m, n \in \mathbb{N}$ and $m < n\}$. Now given any $r \in \mathbb{N}$ we have $|\{x_n : n \in \mathbb{N}\} \cap \{1, 2, \ldots, 2^{f(r)+\alpha}\}| > 2^{f(r)+\alpha} - 2^{f(r)}$ so $\overline{d}(D(B)) \geq 1 - 1/2^\alpha > 1 - \epsilon$.

If $a = x_n$, then for some $r \in \mathbb{N} \cup \{0\}$, we have $2^{f(r)} \leq a < 2^{f(r)+\alpha}$. If $a \in D(B)\backslash\{x_n : n \in \mathbb{N}\}$ then there exist m and n in $\mathbb{N}$ with $m < n$ such that $2^{f(n)} - 2^{f(m)} - x_m \leq a \leq 2^{f(n)} + x_n - 2^{f(n)}$. Since $2^{f(n)} - 2^{f(m)} - x_m > 2^{f(n)-1}$ we conclude that for any $a \in D(B)$ there is some $n \in \mathbb{N} \cup \{0\}$ with $2^{f(n)-1} < a < 2^{f(n)+\alpha}$. Now suppose we have $a \leq b$ in $D(B)$ such that $a \cdot b \in D(B)$. Pick $m \leq n \leq r$ in $\mathbb{N} \cup \{0\}$ such that $2^{f(m)-1} < a < 2^{f(m)+\alpha}$, $2^{f(n)-1} < b < 2^{f(n)+\alpha}$, and $a^{f(r)-1} < a \cdot b < 2^{f(r)+\alpha}$. If $n < r$ we have $2^{f(n)-1} < a \cdot b < 2^{f(m)+f(n)+2\alpha}$ so $f(r) \leq f(m)+f(n)+2\alpha \leq 2f(n)+2\alpha < f(r)$, a contradiction. Thus $n = r$ so that $2^{f(m)+f(n)-2} < a \cdot b < 2^{f(r)+\alpha} = 2^{f(n)+\alpha}$. Then $f(m) < \alpha + 2 = f(0) \leq f(m)$, a contradiction. $\Box$

3.2 **Theorem**. *Let $B \subseteq \mathbb{N}$ and assume $\overline{d}(D(B)) > 0$. There exist a, b, c in $\mathbb{N}$ such that $\{a, b, c, a+b, a+c, b+c, a+b+c\} \subseteq D(B)$.*

Proof. If $\overline{d}(B) > 0$ we are done by Theorem 2.6 so we assume $\overline{d}(B) = 0$. Enumerate B in order as $\langle x_n\rangle_{n=1}^{\infty}$. The result of this theorem is almost free. That is given any $r > s > n > k$, if we let $a = x_r - x_s, b = x_s - x_n$, and $c = x_n - x_k$, then $a + b = x_r - x_n, b + c = x_s - x_k$, and $a + b + c = x_r - x_k$. The only problem then is to find $r > s > n > k$ such that $x_r - x_s + x_n - x_k \in D(B)$.

Let $\alpha = \overline{d}(D(B))$ and pick $\ell \in \mathbb{N}$ such that $1/\ell < \alpha$. For each t we have $\overline{d}(B - t) = \overline{d}(B) = 0$. Let $E = D(B)\backslash \bigcup_{k=1}^{\ell}(B - x_k)$. Then $E = \{x_r - x_s : r, s \in \mathbb{N}$ and $r > s > \ell\}$ and $\overline{d}(E) = \alpha$. Pick by Lemma 2.1 some $k < n \leq \ell$ such that $\overline{d}((E - x_k) \cap (E - x_n)) > 0$. In particular $(E - x_k) \cap (E - x_n) \neq \emptyset$ so pick $r > s > \ell$ and $t > m > \ell$ such that

$x_r - x_s - x_k = x_t - x_m - x_n$. Then $r > s > \ell \geq n > k$ and $x_r - x_s + x_n - x_k = x_t - x_m$ as required. $\Box$

We now set out to show that we can produce sets B with $\overline{d}(D(B))$ arbitrarily close to 1/2 such that $D(B)$ does not contain $FS(\langle a_n \rangle_{n=1}^5)$ for any a_1, a_2, a_3, a_4, a_5. (Here $FS(\langle a_n \rangle_{n=1}^m) = \{\sum_{n \in F} a_n : \emptyset \neq F \subseteq \{1, 2, \ldots, m\}\}$.) We first introduce the sets B (whose dependence on α is suppressed).

3.3 **Definition**. Fix $\alpha \in \mathbb{N}$ with $\alpha > 4$. Let $\langle x_n \rangle_{n=1}^\infty$ enumerate in increasing order $(\mathbb{N}2+1) \cap (\bigcup_{t=0}^\infty \{2^{\alpha t+2}, 2^{\alpha t+2}+1, \ldots, 2^{\alpha t+\alpha-2}-1\})$. Let $B = \{2^{\alpha n} : n \in \mathbb{N}\} \cup \{2^{\alpha n} + x_n : n \in \mathbb{N}\}$.

One sees immediately that one can get a_1, a_2, a_3, and a_4 with $FS(\langle a_n \rangle_{n=1}^4) \subseteq D(B)$. Indeed let $s < m$ be given, pick ℓ and r such that $2^{\alpha r+2} < x_\ell < x_\ell + 2^{\alpha m} - 2^{\alpha s} < 2^{\alpha r+\alpha-2}$, let $x_k = x_\ell + 2^{\alpha m} - 2^{\alpha s}$, and pick v and t such that $2^{\alpha t+2} < x_v < x_v + 2^{\alpha k} - 2^{\alpha s} < 2^{\alpha t+\alpha-2}$. Then let $a_1 = 2^{\alpha m} - 2^{\alpha s} = x_k - x_\ell$, $a_2 = 2^{\alpha \ell} - 2^{\alpha m}, a_3 = 2^{\alpha k} - 2^{\alpha \ell}$, and $a_4 = x_v$. Then $FS(\langle a_n \rangle_{n=1}^4) \subseteq D(B)$. In fact, one can show that any sequence of length 4 with its sums contained in $D(B)$ must fit this description. The computations are longer and more painful than those on which we are embarking, so we omit them.

3.4 **Definition**. Let α and $\langle x_n \rangle_{n=1}^\infty$ be as in Definition 3.3. Then $A_1 = \{x_n : n \in \mathbb{N}\}$, $A_2 = \{2^{\alpha n} + x_n - 2^{\alpha m} : n, m \in \mathbb{N} \text{ and } m < n\}$, $A_3 = \{2^{\alpha n} - 2^{\alpha m} - x_m : n, m \in \mathbb{N} \text{ and } m < n\}$, $A_4 = \{2^{\alpha n} - 2^{\alpha m} : n, m \in \mathbb{N} \text{ and } m < n\}$, and $A_5 = \{2^{\alpha n} + x_n - 2^{\alpha m} - x_m : n, m \in \mathbb{N} \text{ and } m < n\}$.

Observe that $D(B) = \bigcup_{i=1}^5 A_i$.

We next prove two lemmas to aid in our computations.

3.5 **Lemma**. *Let $n_1, n_2, m_1, m_2 \in \mathbb{N}$ and let $\gamma_1, \gamma_2, \delta_1, \delta_2 \in \{0, 1\}$ with $n_2 \geq n_1$ and $m_2 \geq m_1$. If $2^{\alpha n_2} + 2^{\alpha n_1} + \gamma_2 x_{n_2} + \gamma_1 x_{n_1} = 2^{\alpha m_2} + 2^{\alpha m_1} + \delta_2 x_{m_2} + \delta_1 x_{m_1}$, then*

(1) $(n_2, n_1, \gamma_2, \gamma_1) = (m_2, m_1, \delta_2, \delta_1)$ or

(2) $n_2 = n_1$ and $(n_2, n_1, \gamma_2, \gamma_1) = (m_2, m_1, \delta_1, \delta_2)$.

Proof. We assume without loss of generality that $n_2 \geq m_2$. If we had $n_2 > m_2$ we would have $2^{\alpha m_2} + 2^{\alpha m_1} + \delta_2 x_{m_2} + \delta_1 x_{m_1} < 4 \cdot 2^{\alpha m_2} = 2^{\alpha m_2+2} < 2^{\alpha n_2} < 2^{\alpha n_2} + 2^{\alpha n_1} + \gamma_2 x_{n_2} + \gamma_1 x_{n_1}$, a contradiction. Thus $n_2 = m_2$. Assume first that $\gamma_2 \neq \delta_2$ and assume without loss of generality that $\gamma_2 = 1$ and $\delta_2 = 0$. Then $x_{n_2} = 2^{\alpha m_1} - 2^{\alpha n_1} + \delta_1 x_{m_1} - \gamma_1 x_{n_1}$. We claim $m_1 = n_1$. If we had $m_1 < n_1$ we would have $x_{n_2} \leq 2^{\alpha m_1} - 2^{\alpha n_1} + \delta_1 x_{m_1} < 2 \cdot 2^{\alpha m_1} - 2^{\alpha n_1} < 0$. Suppose now $m_1 > n_1$. Then $x_{n_2} < 2^{\alpha m_1} + \delta_1 x_{m_1} < 2^{\alpha m_1+1}$ and $x_{n_2} \geq 2^{\alpha m_1} - 2^{\alpha n_1} - \gamma_1 x_{n_1} > 2^{\alpha m_1} - 2 \cdot 2^{\alpha n_1} > 2^{\alpha m_1-1}$.

But for some r we have $2^{\alpha r+2} < x_{n_2} < 2^{\alpha r+\alpha-2}$, a contradiction. Thus $m_1 = n_1$ so $x_{n_2} = (\delta_1 - \gamma_1) \cdot x_{n_1}$ and hence $\delta_1 = 1, \gamma_1 = 0$ and $n_1 = n_2$ so that conclusion (2) holds.

Now assume $\gamma_2 = \delta_2$. Then we have $2^{\alpha m_1} + \gamma_1 n_1 = 2^{\alpha m_1} + \delta_1 m_1$. As in the first paragraph we see $n_1 = m_1$ so $\gamma_1 n_1 = \delta_1 n_1$ so $\gamma_1 = \delta_1$. ▯

3.6 **Lemma**. *Let $n_1, n_2, n_3, m_1, m_2, m_3 \in \mathbb{N}$ and let $\gamma_1, \gamma_2, \gamma_3, \delta_1, \delta_2, \delta_3 \in \{0, 1\}$ with $n_3 \geq n_2 \geq n_1$ and $m_3 \geq m_2 \geq m_1$. Assume $2^{\alpha n_3} + 2^{\alpha n_2} + 2^{\alpha n_1} + \gamma_3 x_{n_3} + \gamma_2 x_{n_2} + \gamma_1 x_{n_1} = 2^{\alpha m_3} + 2^{\alpha m_2} + 2^{\alpha m_1} + \delta_3 x_{m_3} + \delta_2 x_{m_2} + \delta_1 x_{m_1}$. Then some one of the following conclusions holds. In any event we have $\gamma_1 + \gamma_2 + \gamma_3 = \delta_1 + \delta_2 + \delta_3$ and* $\max\{n_1, n_2, n_3\} = \max\{m_1, m_2, m_3\}$.

(1) $(n_3, n_2, n_1, \gamma_3, \gamma_2, \gamma_1) = (m_3, m_2, m_1, \delta_3, \delta_2, \delta_1)$

(2) $n_2 = n_1$ *and* $(n_3, n_2, n_1, \gamma_3, \gamma_2, \gamma_1) = (m_3, m_2, m_1, \delta_3, \delta_1, \delta_2)$

(3) $n_3 = n_2$ *and* $(n_3, n_2, n_1, \gamma_3, \gamma_2, \gamma_1) = (m_3, m_2, m_1, \delta_2, \delta_3, \delta_1)$

(4) $n_3 = n_2 = n_1$ *and* $(n_3, n_2, n_1, \gamma_3, \gamma_2, \gamma_1) = (m_3, m_2, m_1, \delta_1, \delta_2, \delta_3)$

(5) $(n_3, n_2, \gamma_3, \gamma_2, \gamma_1) = (m_3, m_2, \delta_2, \delta_3, \delta_1)$ *and* $\gamma_3 \neq \gamma_2$ *and* $n_1 \neq m_1$.

Proof. We assume without loss of generality that $n_3 \geq m_3$. If we had $n_3 > m_3$ we would have $2^{\alpha m_3} + 2^{\alpha m_2} + 2^{\alpha m_1} + \delta_3 x_{m_3} + \delta_2 x_{m_2} + \delta_1 x_{m_1} < 6 \cdot 2^{\alpha m_3} < 2^{\alpha n_3} < 2^{\alpha n_3} + 2^{\alpha n_2} + 2^{\alpha n_1} + \gamma_3 x_{n_3} + \gamma_2 x_{n_2} + \gamma_1 x_{n_1}$, a contradiction. Thus we must have $n_3 = m_3$. If also $\gamma_3 = \delta_3$ we have $2^{\alpha n_2} + 2^{\alpha n_1} + \gamma_2 x_{n_2} + \gamma_1 x_{n_1} = 2^{\alpha m_2} + 2^{\alpha m_1} \delta_2 x_{m_2} + \delta_1 x_{m_1}$ so Lemma 3.5 applies and yields conclusion (1) or conclusion (2).

Thus we assume $\gamma_3 \neq \delta_3$ and assume without loss of generality that $\gamma_3 = 1$ and $\delta_3 = 0$. Then $x_{n_3} = 2^{\alpha m_2} - 2^{\alpha n_2} + 2^{\alpha m_1} - 2^{\alpha n_1} + \delta_2 x_{m_2} - \gamma_2 x_{n_2} + \delta_1 x_{m_1} - \gamma_1 x_{n_1}$. We observe that if we had $m_2 < n_2$ we would have $x_{n_3} < 4 \cdot 2^{\alpha m_2} - 2^{\alpha n_2} < 0$. Consequently $m_2 \geq n_2$. We claim in fact $m_2 = n_2$ so suppose instead that $m_2 > n_2$. Then $x_{n_3} < 4 \cdot 2^{\alpha m_2} = 2^{\alpha m_2 + 2}$ and $x_{n_3} > 2^{\alpha m_2} - 4 \cdot 2^{\alpha n_2} > 2^{\alpha m_2 - 1}$. But there is some $r \in \mathbb{N}$ such that $2^{\alpha r+2} < x_{n_3} < 2^{\alpha r+\alpha-2}$, a contradiction. Thus $m_2 = n_2$ as claimed. Consequently we have $x_{n_3} = 2^{\alpha m_1} - 2^{\alpha n_1} + (\delta_2 - \gamma_2) x_{n_2} + \delta_1 x_{m_1} - \gamma_1 x_{n_1}$.

Case 1. $\delta_2 = \gamma_2$. Then we have $x_{n_3} = 2^{\alpha m_1} - 2^{\alpha n_1} + \delta_2 x_{m_1} - \gamma_1 x_{n_1}$. Reasoning as above we conclude $m_1 = n_1$. Then $x_{n_3} = (\delta_1 - \gamma_1) \cdot x_{n_1}$ so $\delta_1 = 1, \gamma_1 = 0$, and $n_3 = n_1$. Then conclusion (4) holds.

Case 2. $\delta_2 \neq \gamma_2$. We claim that we must have $\delta_2 = 1$ and $\gamma_2 = 0$. To see this suppose instead $\delta_2 = 0$ and $\gamma_2 = 1$. Then $x_{n_3} = 2^{\alpha m_1} - 2^{\alpha n_1} - x_{n_2} + \delta_1 x_{m_1} - \gamma_1 x_{n_1}$. One cannot have $n_1 > m_1$ for then one would have $x_{n_3} < 0$. If we had $n_1 = m_1$ we would have $x_{n_3} = -x_{n_2} + (\delta_1 - \gamma_1) x_{n_1}$. Since $x_{n_3} > 0$ one would have to have $\delta_1 = 1$ and $\gamma_1 = 0$. But then one would have $x_{n_3} + x_{n_2} = x_{n_1}$ forcing x_{n_1} to be even. Thus one must have $n_1 < m_1$.

Now we claim that $x_{n_2} > 2^{\alpha m_1 - 1}$. Suppose instead that $x_{n_2} < 2^{\alpha m_1 - 1}$. Now $x_{n_3} < 2 \cdot 2^{\alpha m_1}$ and for some r $2^{\alpha r + 2} < x_{n_3} < 2^{\alpha r + \alpha - 2}$ so $x_{n_3} < 2^{\alpha m_1 - 2}$. That is $2^{\alpha m_1} - 2^{\alpha n_1} - x_{n_2} + \delta_1 x_{m_1} - \gamma_1 x_{n_1} < 2^{\alpha m_1 - 2}$ so $2^{\alpha m_1} + \delta_1 x_{m_1} < 2^{\alpha m_1 - 2} + 2^{\alpha n_1} + x_{n_2} + \gamma_1 x_{n_1} < 2^{\alpha m_1 - 2} + 2^{\alpha m_1 - 2} + 2^{\alpha m_1 - 1} = 2^{\alpha m_1}$, a contradiction. Thus we have $x_{n_2} > 2^{\alpha m_1 - 1}$.

But now for some s we have $2^{\alpha s + 2} < x_{n_2} < 2^{\alpha s + \alpha - 2}$ so $x_{n_2} > 2^{\alpha m_1 + 2}$. But then we have $x_{n_3} = 2^{\alpha m_1} - 2^{\alpha n_1} - x_{n_2} + \delta_1 x_{m_1} - \gamma_1 x_{n_1} < 2^{\alpha m_1} + \delta_1 x_{m_1} - 2^{\alpha m_1 + 2} < 0$, a contradiction. Thus we have established that $\delta_2 = 1$ and $\gamma_2 = 0$.

Then we have that $x_{n_3} = 2^{\alpha m_1} - 2^{\alpha n_1} + x_{m_2} + \delta_1 x_{m_1} - \gamma_1 x_{n_1}$. Since $x_{n_3}, x_{m_2}, x_{m_1}$, and x_{n_1} are all odd we conclude $\delta_1 = \gamma_1$. If also $m_1 = n_1$ we conclude that $x_{n_3} = x_{m_2}$ so $n_3 = m_2 = n_2$ and conclusion (3) holds. Thus we assume $m_1 \neq n_1$. In this case conclusion (5) holds. $\square$

We now begin an embarrassingly long sequence of computational lemmas.

3.7 **Lemma**. *If $a, b \in A_1 \cup A_2$ then $a + b \notin D(B)$.*

Proof. Suppose $a, b \in A_1 \cup A_2$ and $a + b \in D(B)$. Then $a + b$ is even so $a + b \in A_4 \cup A_5$. Pick $s < r$ and $\delta \in \{0, 1\}$ such that $a + b = 2^{\alpha r} - 2^{\alpha s} + \delta(x_r - x_s)$. We consider 3 cases.

Case 1. $a, b \in A_1$. Pick $n, m \in \mathbb{N}$ such that $a = x_n$ and $b = x_m$. Then $x_n + x_m + 2^{\alpha s} + \delta x_s = 2^{\alpha r} + \delta x_r$ so adding $2^{\alpha n} + 2^{\alpha m}$ to both sides we get by Lemma 3.6 that $1 + 1 + \delta = \delta$, a contradiction.

Case 2. $a, b \in A_2$. Pick $m < n$ and $\ell < k$ such that $a = 2^{\alpha n} + x_n - 2^{\alpha m}$ and $b = 2^{\alpha k} + x_k - 2^{\alpha \ell}$. Then $2^{\alpha n} + x_n - 2^{\alpha m} + 2^{\alpha k} + x_k - 2^{\alpha \ell} = 2^{\alpha r} - 2^{\alpha s} + \delta(x_r - x_s)$ so $2^{\alpha n} + 2^{\alpha k} + 2^{\alpha s} + x_n + x_k + \delta x_s = 2^{\alpha r} + 2^{\alpha m} + 2^{\alpha \ell} + \delta x_r$ so by Lemma 3.6, $1 + 1 + \delta = \delta$, a contradiction.

Case 3. Not case 1 or case 2. Without loss of generality $a \in A_1$ and $b \in A_2$. Pick n such that $a = x_n$ and pick $\ell < k$ such that $b = 2^{\alpha k} + x_k - 2^{\alpha \ell}$. Then $x_n + 2^{\alpha k} + x_k - 2^{\alpha \ell} = 2^{\alpha r} - 2^{\alpha s} + \delta(x_r - x_s)$ so we again get a contradiction using Lemma 3.6. $\square$

3.8 **Lemma**. *If $a, b \in A_3$, then $a + b \notin D(B)$.*

Proof. Pick $n > m$ and $k > \ell$ such that $a = 2^{\alpha n} - 2^{\alpha m} - x_m$ and $b = 2^{\alpha k} - 2^{\alpha \ell} - x_\ell$. Suppose $a + b \in D(B)$, in which case since it is even, $a + b \in A_4 \cup A_5$. Pick $\delta \in \{0, 1\}$ and $s < r$ such that $a + b = 2^{\alpha r} - 2^{\alpha s} + \delta(x_r - x_s)$. Then $2^{\alpha n} + 2^{\alpha k} + 2^{\alpha s} + \delta x_s = 2^{\alpha r} + 2^{\alpha m} + 2^{\alpha \ell} + x_\ell + x_m + \delta x_r$ so that by Lemma 3.6, $\delta = 1 + 1 + \delta$, a contradiction. $\square$

3.9 **Lemma**. *Let $m < n$ and $\ell < k$ be given and let $a = 2^{\alpha n} - 2^{\alpha m} - x_m$ and $b = 2^{\alpha k} - 2^{\alpha \ell}$. If $a + b \in D(B)$, then $\ell = n$.*

Proof. Since $a+b$ is odd we have $a+b \in A_1$ or $a+b \in A_2$ or $a+b \in A_3$. We show first that the first two possibilities cannot hold. Indeed if we had $a+b \in A_1$, then for some r, $2^{\alpha n} - 2^{\alpha m} - x_m + 2^{\alpha k} - 2^{\alpha \ell} = x_r$ so that $2^{\alpha n} + 2^{\alpha k} + 2^{\alpha r} = 2^{\alpha m} + 2^{\alpha \ell} + 2^{\alpha r} + x_m + x_r$ so that by Lemma 3.6, $1+1=0$. A similar contradiction is obtained from the assumption that $a+b \in A_2$. Thus we may pick $s < r$ such that $a+b = 2^{\alpha r} - 2^{\alpha s} - x_s$. Then $2^{\alpha n} + 2^{\alpha k} + 2^{\alpha s} + x_s = 2^{\alpha r} + 2^{\alpha m} + 2^{\alpha \ell} + x_m$. By Lemma 3.6 we have that $\max\{n,k,s\} = \max\{r,m,\ell\}$. Since $\ell < k \leq \max\{n,k,s\}$ we have $\ell \neq \max\{r,m,\ell\}$. Similarly $m \neq \max\{r,m,\ell\}$ and $s \neq \max\{n,k,s\}$. Thus $\max\{r,m,\ell\} = r$. Assume first, $k \leq n$. Then $n = \max\{n,k,s\}$ so $n = r$ so $2^{\alpha k} + 2^{\alpha s} + x_s = 2^{\alpha m} + 2^{\alpha \ell} + x_m$. By Lemma 3.5 we have $\max\{k,s\} = \max\{m,\ell\}$. Since $\ell < k$ we have $\ell \neq \max\{m,\ell\}$ so $\ell < m$ so conclusion (2) of Lemma 3.5 cannot hold. If we had $k \leq s$ we would have $(m,\ell) = (s,k)$, while $\ell < k$. Thus $s < k$ so $(k,s,0,1) = (m,\ell,1,0)$, a contradiction. Thus we have $n < k$ so that $k = \max\{n,k,s\}$ and hence $k = r$. Then $2^{\alpha n} + 2^{\alpha s} + x_s = 2^{\alpha m} + 2^{\alpha \ell} + x_m$. By Lemma 3.5 $\max\{n,s\} = \max\{m,\ell\}$. Since $m < n$ we have $m \neq \max\{m,\ell\}$ so $(\ell,m) = (n,s)$ or $(\ell,m) = (s,n)$. The latter is impossible since $m < n$ so in particular $n = \ell$. $\square$

3.10 **Lemma**. *Let $\ell < k$ and $m < n$ in $\mathbb{N}$ be given with $k \geq n$ and let $\mu, \tau \in \{0,1\}$. Let $a = 2^{\alpha k} - 2^{\alpha \ell} + \tau(x_k - x_\ell)$ and let $b = 2^{\alpha n} - 2^{\alpha m} + \mu(x_n - x_m)$ and assume that $a+b \in D(B)$. Then some one of the following holds:*

(1) $n = \ell$ and $\mu = \tau$;

(2) $n = \ell$ and $\mu = 0$ and $\tau = 1$ and there is some $v < m$ such that $x_k - x_\ell = 2^{\alpha m} - 2^{\alpha v}$;

(3) $n \leq \ell$ and $\mu = 1$ and $\tau = 0$ and there is some $v > m$ such that $x_k - x_\ell = 2^{\alpha v} - 2^{\alpha m} + x_v - x_m$; if $n < \ell$, then $v = n$; or

(4) $n \leq \ell$ and $\mu = \tau = 0$ and $x_k - x_\ell = 2^{\alpha n} - 2^{\alpha m}$.

Proof. Since $a+b$ is even we must have $a+b \in A_4 \cup A_5$. So pick $r > s$ in $\mathbb{N}$ and $\nu \in \{0,1\}$ such that $a+b = 2^{\alpha r} - 2^{\alpha s} + \nu(x_r - x_s)$. Then $2^{\alpha k} + 2^{\alpha n} + 2^{\alpha s} + \tau x_k + \mu x_n + \nu x_s = 2^{\alpha r} + 2^{\alpha \ell} + 2^{\alpha m} + \nu x_r + \tau x_\ell + \mu x_m$. By Lemma 3.6, $\max\{k,n,s\} = \max\{r,\ell,m\}$. Since $\ell < k, m < n$, and $s < r$ we have $\max\{r,\ell,m\} = r$ and $s \neq \max\{k,n,s\}$. Since $k \geq n$, $k = \max\{k,n,s\}$.

Case 1. $n \geq s$. If we had $m \geq \ell$ we would then have $k \geq n \geq s$ and $r > m \geq \ell$ so that by Lemma 3.6 we would have $(k,n) = (r,m)$ while $m < n$. Thus $\ell > m$. We then have $k \geq n \geq s$ and $r > \ell > m$ so by Lemma 3.6 some one of the following holds:

(a) $(k,n,s,\tau,\mu,\nu) = (r,\ell,m,\nu,\tau,\mu)$,

(b) $n = s$ and $(k,n,s,\tau,\mu,\nu) = (r,\ell,m,\nu,\mu,\tau)$,

(c) $k = n$ and $(k, n, s, \tau, \mu, \nu) = (r, \ell, m, \tau, \nu, \mu)$,

(d) $k = n = s$ and $(k, n, s, \tau, \mu, \nu) = (r, \ell, m, \mu, \tau, \nu)$, or

(e) $(k, n, \tau, \mu, \nu) = (r, \ell, \tau, \nu, \mu)$ and $\tau \neq \mu$ and $s \neq m$.

If $\mu = \tau$ we have that conclusion (1) of the current lemma holds. So assume $\mu \neq \tau$. This eliminates (a) and (d) above. The fact that $m < \ell$ eliminates (b) above. The fact that $\ell < k$ eliminates (c) above. Thus we have (e) must hold. Observe also that $\tau \neq \nu$. (If so one would have $2^{\alpha n} + 2^{\alpha s} + \mu x_n + \nu x_s = 2^{\alpha \ell} + 2^{\alpha m} + \tau x_\ell + \mu x_m$ so that by Lemma 3.5 one would have $m = s$, which is forbidden by (e).)

There are thus two possibilities. First one could have $\mu = \nu = 0$ and $\tau = 1$. In this case $2^{\alpha s} + x_k = 2^{\alpha m} + x_\ell$ so $2^{\alpha m} - 2^{\alpha s} = x_k - x_\ell > 0$ so $s < m$ and conclusion (2) of the current lemma holds. Second one could have $\mu = \nu = 1$ and $\tau = 0$. In this case $2^{\alpha s} + x_\ell + x_s = 2^{\alpha m} + x_k + x_m$ so that $x_k - x_\ell = 2^{\alpha s} - 2^{\alpha m} + x_s - x_m$ and conclusion (3) of the current lemma holds.

Case 2. $n < s$. Since $s < r = k$ we have then $k > s > n$. By Lemma 3.6 we then have that $(k, s) = (r, \ell)$ or $(k, s) = (r, m)$. Since $m < n$, the latter alternative is impossible and hence $m < \ell$. Also $\ell < k = r$ so we have $r > \ell > m$. Since $n \neq m$ we have only one possibility from Lemma 3.6, namely that $(k, s, \tau, \nu, \mu) = (r, \ell, \tau, \nu, \mu)$ and $\tau \neq \nu$. Since $k = r$ and $s = \ell$ we then have $2^{\alpha n} + \tau x_k + \mu x_n + \nu x_\ell = 2^{\alpha m} + \nu x_k + \tau x_\ell + \mu x_m$. Suppose $\tau = 1$. Then we have $\nu = 0$ so $x_k - x_\ell = 2^{\alpha m} - 2^{\alpha n} + \mu(x_m - x_n) < 0$, which is impossible. Thus $\tau = 0$ and $\nu = 1$ and hence $x_k - x_\ell = 2^{\alpha n} - 2^{\alpha m} + \mu(x_n - x_m)$. If $\mu = 1$ this gives conclusion (3) of the current lemma while if $\mu = 0$ it gives conclusion (4). $\square$

3.11 **Lemma**. *Assume $a \geq b \geq c$ and $\{a, b, c\} \subseteq A_4 \cup A_5$ and $\{a + b, a + c, b + c, a + b + c\} \subseteq D(B)$. Then there exist $k > \ell > m > s$ in $\mathbb{N}$ such that $a = 2^{\alpha k} - 2^{\alpha \ell}$, $b = 2^{\alpha \ell} - 2^{\alpha m}$, and $c = 2^{\alpha m} - 2^{\alpha s} = x_k - x_\ell$.*

Proof. Since a, b, and c are in $A_4 \cup A_5$ we have $k > \ell$,$n > m$, and $r > s$ in $\mathbb{N}$ and τ, μ, ν in $\{0, 1\}$ such that $a = 2^{\alpha k} - 2^{\alpha \ell} + \tau \cdot (x_k - x_\ell), b = 2^{\alpha n} - 2^{\alpha m} + \mu \cdot (x_n - x_m)$ and $c = 2^{\alpha r} - 2^{\alpha s} + \nu \cdot (x_r - x_s)$. Since $a \geq b \geq c$ we have $k \geq n \geq r$. Applying Lemma 3.10 to $a + b$ we have one of:

(1) $n = \ell$ and $\mu = \tau$;

(2) $n = \ell$ and $\mu = 0$ and $\tau = 1$ and there is some $v < m$ such that $x_k - x_\ell = 2^{\alpha m} - 2^{\alpha v}$;

(3) $n \leq \ell$ and $\mu = 1$ and $\tau = 0$ and there is some $v > m$ such that $x_k - x_\ell = 2^{\alpha v} - 2^{\alpha m} + x_v - x_m$; if $n < \ell$, then $v = n$; or

(4) $n < \ell$ and $\mu = \tau = 0$ and $x_k - x_\ell = 2^{\alpha n} - 2^{\alpha m}$.

Applying Lemma 3.10 to $b+c$ we have one of:

$(1)'$ $r=m$ and $\nu=\mu$;

$(2)'$ $r=m$ and $\nu=0$ and $\mu=1$ and there is some $t<s$ such that $x_n-x_m=2^{\alpha s}-2^{\alpha t}$;

$(3)'$ $r\le m$ and $\nu=1$ and $\mu=0$ and there is some $t>s$ such that $x_n-x_m=2^{\alpha t}-2^{\alpha s}+x_t-x_s$; if $r<m$, then $t=r$; or

$(4)'$ $r<m$ and $\nu=\mu=0$ and $x_n-x_m=2^{\alpha r}-2^{\alpha s}$.

Now from $(1)'$, $(2)'$,$(3)'$ and $(4)'$ we see that in any event $r\le m$ and from (1), (2), (3) and (4) we see that $n\le \ell$. Thus $r\le m<n\le\ell$. Thus applying Lemma 3.10 to $a+c$ we have one of:

$(3)^*$ $r<\ell$ and $\nu=1$ and $\tau=0$ and $x_k-x_\ell=2^{\alpha r}-2^{\alpha s}+x_r-x_s$; or

$(4)^*$ $r<\ell$ and $\nu=\tau=0$ and $x_k-x_\ell=2^{\alpha r}-2^{\alpha s}$.

We show first that (1) must hold. From $(3)^*$ or $(4)^*$ we conclude $\tau=0$ so (2) cannot hold.

Now suppose that (3) or (4) holds and pick $v>m$ and $\gamma\in\{0,1\}$ such that $x_k-x_\ell=2^{\alpha v}-2^{\alpha m}+\gamma\cdot(x_v-x_m)$. Since $(3)^*$ or $(4)^*$ holds pick $\lambda\in\{0,1\}$ such that $x_k-x_\ell=2^{\alpha r}-2^{\alpha s}+\lambda\cdot(x_r-x_s)$. Then $2^{\alpha v}+2^{\alpha s}+\gamma x_v+\lambda x_s=2^{\alpha r}+2^{\alpha m}+\lambda x_r+\gamma x_s$. Since $s<r$ and $m<v$ we conclude from Lemma 3.5 that $(v,s)=(r,m)$. But we have already observed that $r\le m$ so $r\le m=s<r$, a contradiction.

We have thus established that (1) holds. In particular we know $\mu=\tau$ from (1) and $\tau=0$ from $(3)^*$ or $(4)^*$ so $\mu=\tau=0$. We now show that $(1)'$ holds. Since $\mu=0$ we know $(2)'$ cannot hold.

Since (1) holds we know that $a=2^{\alpha k}-2^{\alpha\ell}$ and $b=2^{\alpha\ell}-2^{\alpha m}$ so that $a+b+c=2^{\alpha k}-2^{\alpha m}+2^{\alpha r}-2^{\alpha s}+\nu\cdot(x_r-x_s)$. Also $a+b+c\in A_4\cup A_5$ so pick $w>u$ in $\mathbb{N}$ and $\rho\in\{0,1\}$ such that $a+b+c=2^{\alpha w}-2^{\alpha u}+\rho\cdot(x_w-x_u)$. Then $2^{\alpha k}+2^{\alpha r}+2^{\alpha u}+\nu x_r+\rho x_u=2^{\alpha w}+2^{\alpha m}+2^{\alpha s}+\rho x_w+\nu x_s$. Now $\max\{k,r,u\}=\max\{w,m,s\}$ and $m<k$ and $s<r$ so $w=\max\{w,m,s\}$. Also $m\ge r>s$ so we have $w>m>s$. Since $r\le m<k$ and $u<w$ we have $k=\max\{k,r,u\}$. Thus $k=w$. We suppose $(3)'$ or $(4)'$ holds and consider two cases.

Case 1. $m=r$. Then $(4)'$ cannot hold so $(3)'$ holds and hence $\nu=1$. We also conclude that $r\ge u$. (For if $r<u$ then by Lemma 3.6 we have $(k,u)=(w,m)$ so $m=u>r=m$.) Now since $w>m>s$ the only possibilities in Lemma 3.6 are for conclusion (1) or (5) to hold. If conclusion (1) held we would have $(k,r,u,0,1,\rho)=(w,m,s,\rho,0,1)$ which is impossible. Thus $(k,r,0,1,\rho)=(w,m,0,\rho,1)$ so $\rho=1$. Thus we have $2^{\alpha u}+x_m+x_u=2^{\alpha s}+x_k+x_s$ so $x_k-x_m=2^{\alpha u}-2^{\alpha s}+x_u-x_s$ and hence

$u > s$. Also by $(3)'$ pick $t > s$ such that $x_n - x_m = 2^{\alpha t} - 2^{\alpha s} + x_t - x_s$. Since $\nu = 1$, $(3)^*$ holds so we have $x_k - x_\ell = 2^{\alpha r} - 2^{\alpha s} + x_r - x_s$. Since $\ell = n$ we then have $x_k - x_m = 2^{\alpha t} + 2^{\alpha r} - 2 \cdot 2^{\alpha s} + x_t + x_r - 2 \cdot x_s$. Thus $2^{\alpha u} - 2^{\alpha s} + x_u - x_s = 2^{\alpha t} + 2^{\alpha r} - 2 \cdot 2^{\alpha s} + x_t + x_r - 2 \cdot x_s$ so that $2^{\alpha u} + 2^{\alpha s} + x_u + x_s = 2^{\alpha t} + 2^{\alpha r} + x_t + x_r$. Thus by Lemma 3.5 we have $(u, s) = (t, r)$ or $(u, s) = (r, t)$. But $r > s$ and $t > s$, a contradiction.

Case 2. $m > r$. Then from $(3)'$ or $(4)'$ we have that $x_n - x_m = 2^{\alpha r} - 2^{\alpha s} + \nu \cdot (x_r - x_s)$. Now $w > m > s$ and $(w, m) \neq (k, r)$ so by Lemma 3.6 we must have $k > u > r$. Since $s < r$ we must then have conclusion (5) of Lemma 3.6 must hold and consequently $\rho \neq 0$, i.e. $\rho = 1$. Thus $2^{\alpha r} + \nu x_r + x_m = 2^{\alpha s} + \nu x_s + x_k$ so that $x_k - x_m = 2^{\alpha r} - 2^{\alpha s} + \nu \cdot (x_r - x_s)$ so $x_k - x_m = x_n - x_m$ and hence $k = n$. Since $n = \ell < k$, this is a contradiction.

Thus we have established that $(1)'$ holds. Thus $\mu = \tau = \nu$ so $(3)^*$ does not hold so $(4)^*$ holds. The conjunction of (1), $(1)'$, and $(4)^*$ is precisely the conclusion of this lemma. []

3.12 **Lemma**. *Let a_1, a_2, a_3, and a_4 in $\mathbb{N}$ be given such that $FS(\langle a_n \rangle_{n=1}^4) \subseteq D(B)$. Then there is some $i \in \{1, 2, 3, 4\}$ such that $a_i \in A_1 \cup A_2$ and $\{a_j : j \in \{1, 2, 3, 4\}$ and $j \neq i\} \subseteq A_4 \cup A_5$.*

Proof. Suppose first that $\{a_1, a_2, a_3, a_4\} \subseteq A_4 \cup A_5$ and assume without loss of generality that $a_1 \geq a_2 \geq a_3 \geq a_4$. Applying Lemma 3.11 to a_1, a_2, and a_3 we pick $k > \ell > m > s$ in $\mathbb{N}$ such that $a_1 = 2^{\alpha k} - 2^{\alpha \ell}, a_2 = 2^{\alpha \ell} - 2^{\alpha m}$, and $a_3 = 2^{\alpha m} - 2^{\alpha s}$. Applying Lemma 3.11 to a_1, a_3, and a_4 we conclude that $m = \ell$, a contradiction.

Now by Lemma 3.7 at most one i has $a_i \in A_1 \cup A_2$ and by Lemma 3.8 at most one i has $a_i \in A_3$ so to complete the proof it suffices to show that no $a_i \in A_3$. Suppose we have some $a_i \in A_3$ and assume without loss of generality that $a_1 \in A_3$.

Case 1. Some j has $a_j \in A_1 \cup A_2$. Without loss of generality $a_2 \in A_1 \cup A_2$. We may further assume without loss of generality that $a_3 \geq a_4$. Since $FS(< a_1 + a_2, a_3, a_4 >) \subseteq A_4 \cup A_5$ we have by Lemma 3.11 some $k > \ell \geq m > s$ such that $a_3 = 2^{\alpha k} - 2^{\alpha \ell}$ and $a_4 = 2^{\alpha m} - 2^{\alpha s}$. (If $a_1 + a_2$ is between a_3 and a_4 we have $\ell > m$. Otherwise equality holds). Pick $u > v$ in $\mathbb{N}$ such that $a_1 = 2^{\alpha u} - 2^{\alpha v} - x_v$. Since $a_1 + a_3 \in D(B)$ we have by Lemma 3.9 that $\ell = u$. Since $a_1 + a_4 \in D(B)$ we have by Lemma 3.9 that $s = u$. But $s < \ell$, a contradiction.

Case 2. $\{a_2, a_3, a_4\} \subseteq A_4 \cup A_5$. Without loss of generality $a_2 \geq a_3 \geq a_4$. Then by Lemma 3.11 we have some $k \geq \ell > m > s$ such that $a_2 = 2^{\alpha k} - 2^{\alpha \ell}$, $a_3 = 2^{\alpha \ell} - 2^{\alpha m}$, and $a_4 = 2^{\alpha m} - 2^{\alpha s}$. Applying Lemma 3.9 to (a_1, a_2) and (a_1, a_4) we again get $\ell = u = s$, a contradiction. []

We temporarily abandon our assumption that α has a fixed value in order to state the next theorem.

3.13 **Theorem**. *Let $\epsilon > 0$ be given. There is a set $B \subseteq \mathbb{N}$ with $\overline{d}(D(B)) > 1/2 - \epsilon$ such that no a_1, a_2, a_3, a_4, and a_5 have $FS(\langle a_n \rangle_{n=1}^5) \subseteq D(X)$.*

Proof. Pick $\alpha \in \mathbb{N}$ such that $1/2^{\alpha-5} < \epsilon$. Define B as in Definition 3.3. Observe that $A_1 \subseteq D(B)$ and $\overline{d}(A_1) \geq 1/2 - 1/2^{\alpha-5}$ since $|A_1 \cap \{1, 2, \ldots, 2^{\alpha t+\alpha-2}\}| \geq \frac{1}{2}((2^{\alpha t+\alpha-2} - 2^{\alpha t+2}))$.

Suppose now one has a_1, a_2 , a_3, a_4, and a_5 with $FS(\langle a_n \rangle_{n=1}^5) \subseteq D(B)$. Applying Lemma 3.12 first to a_1, a_2, a_3, and a_4 one has without loss of generality that $a_1 \in A_1 \cup A_2$ and $\{a_2, a_3, a_4\} \subseteq A_4 \cup A_5$. Applying Lemma 3.12 to a_2, a_3, a_4, and a_5 one sees that $a_5 \in A_1 \cup A_2$. Then applying Lemma 3.7 to a_1 and a_5 one obtains a contradiction. $\Box$

We close with two questions which are raised by Theorems 3.2 and 3.13.

3.14 **Question**. *If $B \subseteq \mathbb{N}$ and $\overline{d}(D(B)) > 0$, must there exist a_1, a_2, a_3, and a_4 in $\mathbb{N}$ with $FS(\langle a_n \rangle_{n=1}^4) \subseteq D(B)$?*

Since always $\overline{d}(B \cap (B - t)) \geq 2 \cdot \overline{d}(B) - 1$, one easily sees that if $\overline{d}(D(X)) > 1 - 1/2^{m-1}$, there will exist $a_1, a_2, \ldots, a_m$ with $FS(\langle a_i \rangle_{i=1}^m) \subseteq D(X)$. (See [8, Theorem 4.5].) To utilize this to obtain $FS(\langle a_n \rangle_{n=1}^5)$ one needs $\overline{d}(D(X)) > 1 - 1/16$.

3.15 **Question**. *If $\overline{d}(D(X)) = 1/2$ or even if $\overline{d}(D(X)) > 1/2$ must there exist a_1, a_2, a_3, a_4, and a_5 in $\mathbb{N}$ with $FS(\langle a_n \rangle_{n=1}^5) \subseteq D(X)$?*

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Addresses:

Vitaly Bergelson
Department of Mathematics
Ohio State University
Columbus, OH 43210
USA

Paul Erdős
Mathematical Institute of the Hungarian Academy of Sciences
P.O. Box 127
Realtanoda u. 13-15
H-1364 Budapest
Hungary

Neil Hindman
Department of Mathematics
Howard University
Washington, D.C. 20059
USA

Tomasz Łuczak
Mathematics Institute of The Polish Academy of Sciences
Poznań
Poland
(On leave from:
Department of Discrete Mathematics
Adam Mickiewicz University
Poznań
Poland)