This article was published in *Semigroup Forum* **73** (2006), 273-300. To the best of my knowledge, this is the final version as it was submitted to the publisher. – NH

Density in Arbitrary Semigroups

Neil Hindman[1]

and

Dona Strauss

Abstract. We introduce some notions of density in an arbitrary semigroup S which extend the usual notions in countable left amenable semigroups in which density is based on Følner sequences. The new notions are based on nets of finite sets. We show that under certain conditions on the nets and on S these notions relate nicely to some established notions of size in S such as *central*, *syndetic*, and *piecewise syndetic*. And we investigate the conditions under which these notions have other desirable properties such as translation invariance. We obtain new information about the algebraic structure of the Stone-Čech compactification βS of S and derive generalizations of some known Ramsey Theoretic results, including Bergelson's density version of Schur's Theorem.

1. Introduction

Our starting point is various notions of size in the set $\mathbb{N}$ of positive integers. One of the earliest of these (whose origin is lost in antiquity) is the notion of *upper asymptotic density* which is defined by

$$\overline{d}(A) = \limsup_{n\to\infty} \frac{|A \cap \{1, 2, \ldots, n\}|}{n}\,.$$

This notion has some nice properties. It is partition regular in the sense that if $\overline{d}(A \cup B) > 0$, then either $\overline{d}(A) > 0$ or $\overline{d}(B) > 0$. (More generally, a family of subsets of a set S is *partition regular* for S if and only if whenever S is partitioned into finitely many cells, one of these cells must contain a member of the given family.) And it is translation invariant in the sense that $\overline{d}(x + A) = \overline{d}(A) = \overline{d}(-x + A)$ for all $x \in \mathbb{N}$ (where $-x + A = \{y \in \mathbb{N} : x + y \in A\}$). And, while $\overline{d}$ is not additive, it is for translations of a given set. That is if $x, y \in \mathbb{N}$ and $\overline{d}\big((x + A) \cap (y + A)\big) = 0$, then $\overline{d}\big((x + A) \cup (y + A)\big) = \overline{d}(x + A) + \overline{d}(y + A)$.

The related notion of *lower asymptotic density*, defined by

$$\underline{d}(A) = \liminf_{n\to\infty} \frac{|A \cap \{1, 2, \ldots, n\}|}{n}\,,$$

[1]This author acknowledges support received from the National Science Foundation (USA) via grants DMS-0243586 and DMS-0554803.

is not as nicely behaved, although sets for which $\underline{d}(A) = \overline{d}(A)$ are of considerable interest.

Another notion of density for subsets of $\mathbb{N}$ was introduced by Polya [20] in 1929:

$$d^*(A) = \sup\left\{\alpha : (\forall m \in \mathbb{N})(\exists n \geq m)(\exists x \in \mathbb{N})(\frac{|A \cap \{x+1, x+2, \ldots, x+n\}|}{n} \geq \alpha\right\}.$$

This notion is also partition regular, translation invariant, and additive for translations.

Now let us consider several other notions of size, whose origins come from topological dynamics. (Given a set X we write $\mathcal{P}_f(X)$ for the set of finite nonempty subsets of X.)

1.1 Definition. Let $A \subseteq \mathbb{N}$.

(a) A is *thick* if and only if for every $n \in \mathbb{N}$ there exists $x \in \mathbb{N}$ such that $\{x+1, x+2, \ldots, x+n\} \subseteq A$.

(b) A is *syndetic* if and only if there exists $n \in \mathbb{N}$ such that for every $x \in \mathbb{N}$, $\{x+1, x+2, \ldots, x+n\} \cap A \neq \emptyset$.

(c) A is *piecewise syndetic* if and only if there exists $b \in \mathbb{N}$ such that $\bigcup_{t=1}^{b}(-t+A)$ is thick.

(d) A is a Δ-*set* if and only if there exists a sequence $\langle x_n \rangle_{n=1}^{\infty}$ in $\mathbb{N}$ such that $\{x_m - x_n : n < m\} \subseteq A$.

(e) A is an *IP-set* if and only if there exists a sequence $\langle x_n \rangle_{n=1}^{\infty}$ in $\mathbb{N}$ such that $FS(\langle x_n \rangle_{n=1}^{\infty}) = \{\sum_{n \in F} x_n : F \in \mathcal{P}_f(\mathbb{N})\} \subseteq A$.

Another important notion of size was introduced by Furstenberg in [13], namely that of *central* sets. These sets are most easily defined in terms of the algebraic structure of the semigroup $(\beta\mathbb{N}, +)$ so we will postpone the definition until we have discussed that structure.

Several of these notions have important applications in Ramsey Theory. For example, any piecewise syndetic set contains arbitrarily long arithmetic progressions and any central set in $\mathbb{N}$ contains a solution set for any partition regular system of homogeneous linear equations. See [16, Part III] for details and many more examples.

Finally, given any notion of size, say P, there is the corresponding notion P^*. A set A is a P^*-set if and only if it has nonempty intersection with every P-set. (Notice that a thick set is the same as a syndetic* set.) We abbreviate (piecewise syndetic)* by PS*.

One has then that the following relationships among these notions hold for subsets of $\mathbb{N}$ and none of the missing implications is valid. Once we have introduced algebraic characterizations of these notions, most of the implications become trivial. Examples establishing that none of the missing implications hold are provided in [6, p. 24] and [4, Theorem 2.20], the latter being an old result of Ernst Straus.

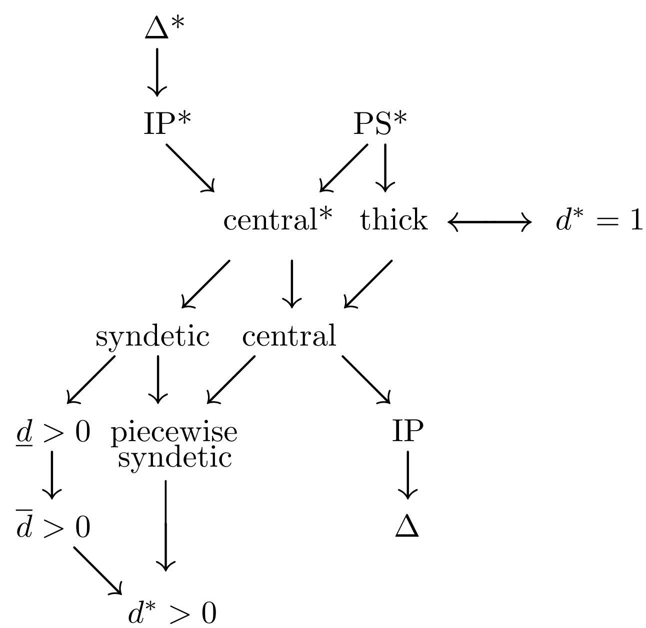

Figure 1

All of the notions above except the ones involving density make sense in any semigroup. Here are the extensions of the notions defined in Definition 1.1. In a semigroup $(S,\cdot)$, if $A \subseteq S$ and $x \in S$, then $x^{-1}A = \{y \in S : xy \in A\}$.

1.2 Definition. Let $(S,\cdot)$ be a semigroup and let $A \subseteq S$.

(a) A is *thick* if and only if for every $F \in \mathcal{P}_f(S)$ there exists $x \in S$ such that $Fx \subseteq A$.

(b) A is *syndetic* if and only if there exists $H \in \mathcal{P}_f(S)$ such that $S = \bigcup_{t\in H} t^{-1}A$.

(c) A is *piecewise syndetic* if and only if there exists $H \in \mathcal{P}_f(S)$ such that $\bigcup_{t\in H} t^{-1}A$ is thick.

(d) A is a Δ-*set* if and only if there exists a sequence $\langle x_n\rangle_{n=1}^\infty$ in S such that for all $n < m$, $x_m \in x_n \cdot A$.

(e) A is an *IP-set* if and only if there exists a sequence $\langle x_n\rangle_{n=1}^\infty$ in S such that $FP(\langle x_n\rangle_{n=1}^\infty) = \{\prod_{n\in F} x_n : F \in \mathcal{P}_f(\mathbb{N})\} \subseteq A$, where the products are taken in increasing order of indices.

Since we are not assuming that S is commutative, the above notions all have left-right switches.

Notice that "thick" is equivalent to an apparently stronger notion. That is, a subset A of S is thick if and only if for each $F \in \mathcal{P}_f(S)$ there exists $x \in A$ such that $Fx \subseteq A$. To see this, pick $y \in S$ such that $Fy \subseteq A$ and pick $z \in S$ such that $(Fy \cup \{y\})z \subseteq A$.

Given an infinite semigroup $(S, \cdot)$ there is an extension of the operation to the Stone-Čech compactification βS of S such that $(\beta S, \cdot)$ is a right topological semigroup (meaning that for all $p \in \beta S$, the function $\rho_p : \beta S \to \beta S$ defined by $\rho_p(q) = q \cdot p$ is continuous) with S contained in its topological center (meaning that for all $x \in S$, the function $\lambda_x : \beta S \to \beta S$ defined by $\lambda_x(p) = x \cdot p$ is continuous). We take the points of βS to be the ultrafilters on S, identifying the points of S with the principal ultrafilters. Given $A \subseteq S$ and $p \in \beta S$, $p \in c\ell A$ if and only if $A \in p$. Given $p, q \in \beta S$ and $A \subseteq S$, $A \in p \cdot q$ if and only if $\{x \in S : x^{-1}A \in q\} \in p$.

Any compact Hausdorff right topological semigroup T has a smallest two sided ideal $K(T)$ which is the union of all minimal right ideals and is the union of all minimal left ideals. The intersection of any minimal right ideal with any minimal left ideal is a group. In particular, there are idempotents in $K(T)$. Such an idempotent is said to be *minimal*. (An idempotent p in T is in $K(T)$ if and only if it is minimal with respect to the ordering of idempotents which has $p \leq q$ if and only if $p = p \cdot q = q \cdot p$.) See [16] for proofs of the above assertions as well as any other unfamiliar algebraic facts mentioned in this paper.

We can now provide the simple definition of *central* sets and simple algebraic characterizations of most of the other notions defined above. (There are algebraic characterizations of Δ-sets and Δ^*-sets, but they are not particularly simple. See [6, Lemma 1.9].)

1.3 Definition. Let $(S, \cdot)$ be a semigroup and let $A \subseteq S$. Then A is *central* if and only if there is a minimal idempotent p in βS such that $p \in c\ell A$.

1.4 Lemma. *Let $(S, \cdot)$ be a semigroup and let $A \subseteq S$.*

(a) A is an IP-set if and only if there is some idempotent of βS in $c\ell A$.

(b) A is piecewise syndetic if and only if $c\ell A \cap K(\beta S) \neq \emptyset$.

(c) A is syndetic if and only if for every left ideal L of βS, $L \cap c\ell A \neq \emptyset$.

(d) A is thick if and only if there is some left ideal of βS contained in $c\ell A$.

(e) A is a central-set if and only if every minimal idempotent of βS is in $c\ell A$.*

(f) A is a PS-set if and only if $K(\beta S) \subseteq c\ell A$.*

(g) A is an IP-set if and only if every idempotent of βS is in $c\ell A$.*

Proof. Statement (a) is [16, Theorem 5.12]. Statement (b) is [16, Theorem 4.40]. Statement (c) is [7, Theorem 2.9(d)]. Statements (d), (f), and (g) follow from statements (c), (b), and (a) respectively (using the fact that "thick" is syndetic*) and statement (e) follows from the definition of central. □

All of the implications in Figure 1 among the notions of largeness that do not involve density follow immediately from Lemma 1.4 and the fact that if $FP(\langle x_n\rangle_{n=1}^{\infty}) \subseteq A$ and $y_n = \prod_{t=1}^{n} x_t$, then whenever $n < m$, $y_m \in y_n \cdot A$, so any IP-set is a Δ-set and any Δ^*-set is an IP*-set.

In $(\mathbb{N}, +)$ if $n \in \mathbb{N}$ and for every $x \in \mathbb{N}$, $\{x+1, x+2, \ldots, x+n\} \cap A \neq \emptyset$, then $\underline{d}(A) \geq \frac{1}{n}$. That A is thick if and only if $d^*(A) = 1$ is routine. It is also relatively easy to establish that if A is piecewise syndetic, then $d^*(A) > 0$. However that statement also follows from the fact that the set $\mathbf{D}^*$ defined below is a two sided ideal of $(\beta\mathbb{N}, +)$ [16, Theorems 20.5 and 20.6]. By contrast, the sets $\mathbf{D}$ and $\beta\mathbb{N} \setminus (\mathbb{N} \cup \mathbf{D})$ are both left ideals of $(\beta\mathbb{N}, +)$ [16, Theorems 6.79 and 6.80], so that $\mathbf{D}$ is far from being a right ideal. (We customarily use the notations Δ^* and Δ for $\mathbf{D}^*$ and $\mathbf{D}$, but do not wish to cause confusion here with the notions of Δ^*-set and Δ-set.)

1.5 Definition.

(a) $\mathbf{D}^* = \{p \in \beta\mathbb{N} : (\forall A \in p)(d^*(A) > 0)\}$.

(b) $\mathbf{D} = \{p \in \beta\mathbb{N} : (\forall A \in p)(\overline{d}(A) > 0)\}$.

By way of contrast with the notions of thick, central, syndetic, piecewise syndetic, IP, and Δ and their duals, the various notions of density appear to only have been studied in a limited class of semigroups, namely those which contain a *Følner sequence*, a proper subclass of the left amenable semigroups.

A semigroup S is *left amenable* if and only if there exists a left invariant mean μ on the space of bounded real (or complex) valued functions on S under the supremum norm. That is μ is a positive linear functional, the norm of μ is 1, and for every bounded $f : S \to \mathbb{R}$ and every $x \in S$, $\mu(f \circ \lambda_s) = \mu(f)$. Right amenable semigroups are defined similarly. For groups left and right amenability are equivalent and such groups are called simply amenable.

In [11] Følner showed that any amenable group S satisfies a condition which we denote (FC) (for Følner Condition) and Frey [12] showed that any left amenable semigroup satisfies (FC). (For a simplified proof see [18].)

(FC) $\big(\forall H \in \mathcal{P}_f(S)\big)\big(\forall \epsilon > 0\big)\big(\exists K \in \mathcal{P}_f(S)\big)\big(\forall s \in H\big)\big(|sK \setminus K| < \epsilon \cdot |K|\big)$.

In [1] Argabright and Wilde introduced a Strong Følner Condition and showed that any semigroup satisfying (SFC) is left amenable. In particular, if S is left cancellative, then left amenability, (FC), and (SFC) are equivalent. They also showed that all commutative semigroups satisfy (SFC). In [17] Klawe showed that certain semidirect products of semigroups satisfying (SFC) also satisfy (SFC). We shall have more to say about these products in Section 4, including some details.

(SFC) $\big(\forall H \in \mathcal{P}_f(S)\big)\big(\forall \epsilon > 0\big)\big(\exists K \in \mathcal{P}_f(S)\big)\big(\forall s \in H\big)\big(|K \setminus sK| < \epsilon \cdot |K|\big)$.

Notice that (using an argument borrowed from [19]) for any $K \in \mathcal{P}_f(S)$ and any $s \in S$, $|K \setminus sK| = |K| - |K \cap sK|$ and $|sK \setminus K| = |sK| - |K \cap sK|$ so $|K \setminus sK| - |sK \setminus K| = |K| - |sK| \geq 0$ so $|K \setminus sK| \geq |sK \setminus K|$. Thus one has directly that (SFC) implies (FC) and that (SFC) may be restated in the following apparently stronger form.

(SFC) $\big(\forall H \in \mathcal{P}_f(S)\big)\big(\forall \epsilon > 0\big)\big(\exists K \in \mathcal{P}_f(S)\big)\big(\forall s \in H\big)\big(|K \triangle sK| < \epsilon \cdot |K|\big)$.

If S is countable, it is easy to see that (SFC) is equivalent to the existence of a *left Følner sequence* in $\mathcal{P}_f(S)$.

1.6 Definition. Let S be a semigroup. A *left Følner sequence* in $\mathcal{P}_f(S)$ is a sequence $\langle F_n \rangle_{n=1}^{\infty}$ in $\mathcal{P}_f(S)$ such that for each $s \in S$, $\displaystyle\lim_{n\to\infty} \frac{|sF_n \triangle F_n|}{|F_n|} = 0$.

Given a left Følner sequence $\mathcal{F} = \langle F_n \rangle_{n=1}^{\infty}$ in S, there is a natural notion of upper density associated with $\mathcal{F}$, namely

$$\overline{d}_{\mathcal{F}}(A) = \limsup_{n\to\infty} \frac{|A \cap F_n|}{|F_n|}\,.$$

The first specific mention of this density that we can find in the literature is in [10] by P. Feit. (Følner sequences in $\mathbb{Z}^n$ are defined by Furstenberg in [13].) This density shares with the densities $\overline{d}$ and d^* in $(\mathbb{N}, +)$ the property of partition regularity and, if S is left cancellative, it is also left translation invariant and left inverse translation invariant and additive for translates.

Of course, if S does not satisfy (SFC), in particular if S is not left amenable, then one can not define density in this fashion. Also, if S is uncountable and satisfies a very weak form of right cancellation, there do not exist any Følner sequences in S. (Specifically, assume that $|S| = \kappa > \omega$ and either (i) The cofinality of κ is uncountable and for each $a, b \in S$, $|\{s \in S : sa = b\}| < \kappa$ or (ii) there exists $\delta < \kappa$ such that for each $a, b \in S$, $|\{s \in S : sa = b\}| \leq \delta$. Then given any sequence $\langle F_n \rangle_{n=1}^{\infty}$ in $\mathcal{P}_f(S)$ one can choose $x \in S$ such that $xF_n \cap F_n = \emptyset$ for all $n \in \mathbb{N}$.)

In Section 2 of this paper we define a notion of density determined by a net in $\mathcal{P}_f(S)$ and derive conditions which guarantee that this density has some or all of the desirable properties that we have described above. In Section 3 we consider density determined by a net of finite products of a set of generators of S, showing that it often provides a reasonable notion of density even in some nonamenable semigroups such as free semigroups. In Section 4 we introduce Følner nets, the natural analogue for uncountable semigroups of Følner sequences. Such nets exist precisely in semigroups satisfying (SFC). We show that such semigroups have a single naturally defined notion of density which is very well behaved. In Section 5 we present some applications to Ramsey Theory and the structure of βS.

Several of our results require a weak form of right or left cancellation, so we introduce terminology for these.

1.7 Definition. Let S be a semigroup and let $b \in \mathbb{N}$. Then S is *b-weakly left cancellative* (respectively *b-weakly right cancellative*) if and only if for all $x, y \in S$, $|\{s \in S : xs = y\}| \leq b$ (respectively $|\{s \in S : sx = y\}| \leq b$).

Recall that weakly left cancellative means that for all $x, y \in S$, $\{s \in S : xs = y\}$ is finite, so the above is a stronger assumption.

2. Density determined by nets in $\mathcal{P}_f(S)$

Given any net in $\mathcal{P}_f(S)$, there are corresponding natural notions of density for subsets of S.

2.1 Definition. Let S be a semigroup, let $\mathcal{F} = \langle F_n \rangle_{n \in D}$ be a net in $\mathcal{P}_f(S)$, and let $A \subseteq S$.

(a) $\underline{d}_{\mathcal{F}}(A) = \sup\{\alpha : (\exists m \in D)(\forall n \geq m)(|A \cap F_n| \geq \alpha \cdot |F_n|)\}$.

(b) $\overline{d}_{\mathcal{F}}(A) = \sup\{\alpha : (\forall m \in D)(\exists n \geq m)(|A \cap F_n| \geq \alpha \cdot |F_n|)\}$.

(c) $d^*_{\mathcal{F}}(A) = \sup\{\alpha : (\forall m \in D)(\exists n \geq m)(\exists x \in S \cup \{1\})(|A \cap F_n x| \geq \alpha \cdot |F_n|)\}$.

We are not assuming that S has an identity. When we write above that $(\exists x \in S \cup \{1\})(|A \cap (F_n \cdot x)| \geq \alpha \cdot |F_n|)$ this is simply an abbreviation for "either $|A \cap F_n| \geq \alpha \cdot |F_n|$ or $(\exists x \in S)(|A \cap (F_n \cdot x)| \geq \alpha \cdot |F_n|)$."

Notice that if S is $(\mathbb{N}, +)$ and $\mathcal{F} = \langle \{1, 2, \ldots, n\} \rangle_{n \in \mathbb{N}}$ then $\underline{d}_{\mathcal{F}}$, $\overline{d}_{\mathcal{F}}$, and $d^*_{\mathcal{F}}$ are respectively $\underline{d}$, $\overline{d}$, and d^* as defined in the introduction.

We shall be concerned with two subsets of βS determined by these upper densities. Recall that in $(\beta\mathbb{N}, +)$, $\mathbf{D}^*$ is a two sided ideal and $\mathbf{D}$ is a left ideal.

2.2 Definition. Let S be a semigroup and let $\mathcal{F}$ be a net in $\mathcal{P}_f(S)$.

(a) $\mathbf{D}_{\mathcal{F}} = \{p \in \beta S : (\forall A \in p)(\overline{d}_{\mathcal{F}}(A) > 0)\}$.

(b) $\mathbf{D}^*_{\mathcal{F}} = \{p \in \beta S : (\forall A \in p)(d^*_{\mathcal{F}}(A) > 0)\}$.

We easily see that no extra conditions on $\mathcal{F}$ are required for partition regularity of $\overline{d}_{\mathcal{F}}$ and $d^*_{\mathcal{F}}$.

2.3 Lemma. *Let S be a semigroup and let $\mathcal{F}$ be a net in $\mathcal{P}_f(S)$. If A and B are subsets of S, then $\overline{d}_{\mathcal{F}}(A \cup B) \leq \overline{d}_{\mathcal{F}}(A) + \overline{d}_{\mathcal{F}}(B)$ and $d^*_{\mathcal{F}}(A \cup B) \leq d^*_{\mathcal{F}}(A) + d^*_{\mathcal{F}}(B)$. Consequently if $A \subseteq S$ and $\overline{d}_{\mathcal{F}}(A) > 0$, then $c\ell A \cap \mathbf{D}_{\mathcal{F}} \neq \emptyset$ and if $A \subseteq S$ and $d^*_{\mathcal{F}}(A) > 0$, then $c\ell A \cap \mathbf{D}^*_{\mathcal{F}} \neq \emptyset$.*

Proof. The proofs of the first two assertions are routine exercises. For the third assertion, let $\mathcal{R} = \{A \subseteq S : \overline{d}_{\mathcal{F}}(A) > 0\}$. Then $\mathcal{R}$ is partition regular so by [16, Theorem 3.11] there is some ultrafilter p on S with $A \in p \subseteq \mathcal{R}$. Then $p \in c\ell A \cap \mathbf{D}_{\mathcal{F}}$. The proof of the fourth assertion is identical. $\square$

A weak right cancellation assumption guarantees that $\mathbf{D}^*_{\mathcal{F}}$ is a right ideal of βS. (We do not expect $\mathbf{D}_{\mathcal{F}}$ to be a right ideal of βS because we know $\mathbf{D}$ is not a right ideal of $\beta\mathbb{N}$.)

2.4 Theorem. *Let S be a semigroup and let $\mathcal{F} = \langle F_n \rangle_{n \in D}$ be a net in $\mathcal{P}_f(S)$. Assume that there is some $b \in \mathbb{N}$ such that S is b-weakly right cancellative. Then $\mathbf{D}^*_{\mathcal{F}}$ is a right ideal of βS.*

Proof. By Lemma 2.3, $\mathbf{D}^*_{\mathcal{F}} \neq \emptyset$. Let $p \in \mathbf{D}^*_{\mathcal{F}}$, let $q \in \beta S$, and let $A \in p \cdot q$. Let $B = \{t \in S : t^{-1}A \in q\}$. Then $B \in p$ so pick α such that $d^*_{\mathcal{F}}(B) > \alpha > 0$. We claim that $d^*_{\mathcal{F}}(A) \geq \frac{\alpha}{b}$. So let $m \in D$ be given. Pick $n \geq m$ and $x \in S \cup \{1\}$ such that $|B \cap F_n x| \geq \alpha \cdot |F_n|$. Pick $y \in \bigcap\{t^{-1}A : t \in B \cap F_n x\}$. (This set is in q and is therefore nonempty.) Then $(B \cap F_n x)y \subseteq A$ so $|A \cap F_n xy| \geq |(B \cap F_n x)y| \geq \frac{1}{b} \cdot |B \cap F_n x| \geq \frac{\alpha}{b} \cdot |F_n|$. $\square$

Other desirable properties for density are the implications of Figure 1.

2.5 Remark. *Let S be a right cancellative semigroup and let $\mathcal{F} = \langle F_n \rangle_{n \in D}$ be a net in $\mathcal{P}_f(S)$. If A is a thick subset of S, then $d^*_{\mathcal{F}}(A) = 1$.*

We now introduce three requirements on the net $\mathcal{F}$ that will guarantee certain desirable properties of the densities determined by $\mathcal{F}$.

2.6 Definition. Let S be a semigroup and let $\mathcal{F} = \langle F_n \rangle_{n \in D}$ be a net in $\mathcal{P}_f(S)$. Following are three properties that $\mathcal{F}$ might satisfy.

($*$) $(\forall \epsilon > 0)(\forall t \in S)(\exists c \in \mathbb{N})(\exists m \in D)(\forall n \geq m)(\exists k \geq n)(\exists z \in S \cup \{1\})$ $(|tF_n \setminus F_k z| < \epsilon \cdot |F_n|$ and $|F_k| \leq c \cdot |F_n|)$.

($*'$) $(\forall \epsilon > 0)(\forall t \in S)(\exists c \in \mathbb{N})(\exists m \in D)(\forall n \geq m)(\exists k \geq n)(|tF_n \setminus F_k| < \epsilon \cdot |F_n|$ and $|F_k| \leq c \cdot |F_n|)$.

($**$) $\big(\forall H \in \mathcal{P}_f(S)\big)(\exists c \in \mathbb{N})(\exists m \in D)(\forall n \geq m)(|F_n| \leq c \cdot |\bigcap_{a \in H} a^{-1}F_n|)$.

Note that ($*$) is automatically satisfied for any $\mathcal{F}$ if S is commutative. We see that ($*$) and a weak form of left cancellation guarantee that $\mathbf{D}^*_{\mathcal{F}}$ is a left ideal of βS.

2.7 Theorem. *Let S be a semigroup and let $\mathcal{F} = \langle F_n \rangle_{n \in D}$ be a net in $\mathcal{P}_f(S)$. Assume that there is some $b \in \mathbb{N}$ such that S is b-weakly left cancellative. If $\mathcal{F}$ satisfies ($*$), $B \subseteq S$, $t \in S$, and $d^*_{\mathcal{F}}(t^{-1}B) > 0$, then $d^*_{\mathcal{F}}(B) > 0$. In particular $\mathbf{D}^*_{\mathcal{F}}$ is a left ideal of βS.*

Proof. Pick $\alpha > 0$ such that $d^*_{\mathcal{F}}(t^{-1}B) > \alpha$ and let $\epsilon = d^*_{\mathcal{F}}(t^{-1}B) - \alpha$. Pick $c \in \mathbb{N}$ and $m \in D$ as guaranteed by ($*$) for $\frac{\epsilon}{2b}$ and t. We claim that $d^*_{\mathcal{F}}(B) \geq \frac{\alpha}{bc}$. To this end, let $r \in D$. Pick $n \in D$ and $x \in S \cup \{1\}$ such that $n \geq r$, $n \geq m$, and $|t^{-1}B \cap F_n x| \geq (\alpha + \frac{\epsilon}{2}) \cdot |F_n|$. Pick $k \geq n$ and $z \in S \cup \{1\}$ such that $|tF_n \setminus F_k z| < \frac{\epsilon}{2b} \cdot |F_n|$ and $|F_k| \leq c \cdot |F_n|$. Now $\lambda_t : t^{-1}B \cap F_n x \to B \cap tF_n x$ so $|B \cap tF_n x| \geq \frac{1}{b} \cdot |t^{-1}B \cap F_n x|$. Also $(tF_n x \setminus F_k z x) \subseteq (tF_n \setminus F_k z)x$ so $|tF_n x \setminus F_k z x| \leq |(tF_n \setminus F_k z)x| \leq |tF_n \setminus F_k z|$. Now $B \cap tF_n x \subseteq (B \cap F_k z x) \cup (tF_n x \setminus F_k z x)$ so

$$\begin{aligned}
|B \cap tF_n x| &\leq |B \cap F_k z x| + |tF_n x \setminus F_k z x| \\
&\leq |B \cap F_k z x| + |tF_n \setminus F_k z| \text{ so} \\
|B \cap F_k z x| &\geq |B \cap tF_n x| - |tF_n \setminus F_k z| \\
&> \tfrac{1}{b} \cdot |t^{-1}B \cap F_n x| - \tfrac{\epsilon}{2b} \cdot |F_n| \\
&\geq \tfrac{\alpha}{b} \cdot |F_n| \\
&\geq \tfrac{\alpha}{bc} \cdot |F_k| \,.
\end{aligned}$$

To see that $\mathbf{D}^*_{\mathcal{F}}$ is a left ideal of βS, let $p \in \mathbf{D}^*_{\mathcal{F}}$, let $q \in \beta S$, and let $B \in q \cdot p$. Then $\{t \in S : t^{-1}B \in p\} \in q$ so pick t such that $t^{-1}B \in p$. Then $d^*_{\mathcal{F}}(t^{-1}B) > 0$ so $d^*_{\mathcal{F}}(B) > 0$. □

The proof of the following theorem is nearly identical, so we omit it.

2.8 Theorem. *Let S be a semigroup and let $\mathcal{F} = \langle F_n \rangle_{n \in D}$ be a net in $\mathcal{P}_f(S)$. Assume that there is some $b \in \mathbb{N}$ such that S is b-weakly left cancellative. If $\mathcal{F}$ satisfies ($*'$),*

$B \subseteq S$, $t \in S$, and $\overline{d}_{\mathcal{F}}(t^{-1}B) > 0$, then $\overline{d}_{\mathcal{F}}(B) > 0$. In particular $\mathbf{D}_{\mathcal{F}}$ is a left ideal of βS.

The condition $(*)$ along with weak right and left cancellation assumptions guarantees another of the implications from Figure 1.

2.9 Theorem. *Let S be a semigroup and let $\mathcal{F} = \langle F_n \rangle_{n \in D}$ be a net in $\mathcal{P}_f(S)$. Assume that there is some $b \in \mathbb{N}$ such that S is b-weakly left cancellative and b-weakly right cancellative. If $\mathcal{F}$ satisfies $(*)$ and B is a piecewise syndetic subset of S, then $d^*_{\mathcal{F}}(B) > 0$.*

Proof. Pick $H \in \mathcal{P}_f(S)$ such that $\bigcup_{t \in H} t^{-1}B$ is thick. Given $m \in D$, pick $x \in S$ such that $F_m x \subseteq \bigcup_{t \in H} t^{-1}B$. Since $|F_m x| \geq \frac{1}{b} \cdot |F_m|$ we conclude that $d^*_{\mathcal{F}}(\bigcup_{t \in H} t^{-1}B) \geq \frac{1}{b}$. Thus by Lemma 2.3 there is some $t \in H$ such that $d^*_{\mathcal{F}}(t^{-1}B) > 0$ so by Theorem 2.7 $d^*_{\mathcal{F}}(B) > 0$. $\square$

Now we turn our attention to consequences of $(**)$, beginning with another of the implications in Figure 1.

2.10 Theorem. *Let S be a semigroup and let $\mathcal{F} = \langle F_n \rangle_{n \in D}$ be a net in $\mathcal{P}_f(S)$. Assume that there is some $b \in \mathbb{N}$ such that S is b-weakly left cancellative. If $\mathcal{F}$ satisfies $(**)$ and B is syndetic, then $\underline{d}_{\mathcal{F}}(B) > 0$.*

Proof. Pick $H \in \mathcal{P}_f(S)$ such that $S = \bigcup_{t \in H} t^{-1}B$. Pick $c \in \mathbb{N}$ and $m \in D$ as guaranteed by $(**)$ for H. Let $k = |H|$ and let $n \geq m$. We shall show that $|B \cap F_n| \geq \frac{1}{bck} \cdot |F_n|$ and thus $\underline{d}_{\mathcal{F}}(B) \geq \frac{1}{bck}$. Let $G = \bigcap_{a \in H} a^{-1}F_n$, so that $|F_n| \leq c \cdot |G|$. Define $\tau : G \to (B \cap F_n) \times H$ as follows. For $s \in G$ pick $t \in H$ such that $ts \in B$. Since $s \in G$, $ts \in F_n$. Let $\tau(s) = (ts, t)$. Since S is b-weakly left cancellative we have that for any $(x, y) \in (B \cap F_n) \times H$, $|\tau^{-1}[\{(x, y)\}]| \leq b$ so $|G| \leq b \cdot |B \cap F_n| \cdot |H|$ as required. $\square$

2.11 Theorem. *Let S be a semigroup and let $\mathcal{F} = \langle F_n \rangle_{n \in D}$ be a net in $\mathcal{P}_f(S)$. Assume that there is some $b \in \mathbb{N}$ such that S is b-weakly left cancellative and b-weakly right cancellative. If $\mathcal{F}$ satisfies $(**)$ and $d^*_{\mathcal{F}}(B) = 1$, then B is thick.*

Proof. Suppose that B is not thick, so that $S \setminus B$ is syndetic. Pick $H \in \mathcal{P}_f(S)$ such that $S = \bigcup_{t \in H} t^{-1}(S \setminus B)$ and pick $c \in \mathbb{N}$ and $m \in D$ as guaranteed by $(**)$ for H. Let $k = |H|$. Since $d^*_{\mathcal{F}}(B) > 1 - \frac{1}{b^2ck}$ pick $x \in S \cup \{1\}$ and $n \geq m$ such that $|B \cap F_n x| > (1 - \frac{1}{b^2ck}) \cdot |F_n|$. Let $G = \bigcap_{a \in H} a^{-1}F_n$ and for each $t \in H$, let $E_t = \{s \in G : tsx \in S \setminus B\}$. Then $G = \bigcup_{t \in H} E_t$ and $|G| \geq \frac{1}{c} \cdot |F_n|$ so pick $t \in H$ such that $|E_t| \geq \frac{1}{ck} \cdot |F_n|$. Now $tE_t x \subseteq (S \setminus B) \cap F_n x$ so $|B \cap F_n x| \leq |F_n x| - |tE_t x| \leq |F_n| - \frac{1}{b^2}|E_t| \leq (1 - \frac{1}{b^2ck}) \cdot |F_n|$, a contradiction. $\square$

2.12 Corollary. *Let S be a semigroup and let $\mathcal{F} = \langle F_n \rangle_{n \in D}$ be a net in $\mathcal{P}_f(S)$. Assume that S is right cancellative and there is some $b \in \mathbb{N}$ such that S is b-weakly left cancellative. If $\mathcal{F}$ satisfies $(**)$, then B is thick if and only if $d^*_{\mathcal{F}}(B) = 1$.*

Proof. Remark 2.5 and Theorem 2.11. □

If the directed set D is countable and if $\mathcal{F}$ satisfies some very natural conditions, then $(**)$ is exactly what is required for the conclusion of Theorem 2.11. (We do not know whether the countability assumption is required.) In this proof we will use the fact, noted earlier, that if a subset B of S is thick, then for any $H \in \mathcal{P}_f(S)$ there is some $x \in B$ such that $Hx \subseteq B$.

2.13 Theorem. *Let S be a semigroup and let $\mathcal{F} = \langle F_n \rangle_{n \in D}$ be a net in $\mathcal{P}_f(S)$ such that $\lim_{n \in D} |F_n| = \infty$ and $F_m \subseteq F_n$ whenever $m \leq n$ in D. Assume that D is countable and that whenever $B \subseteq S$ and $d^*_{\mathcal{F}}(B) = 1$, one must have that B is thick. Then $\mathcal{F}$ satisfies $(**)$.*

Proof. Enumerate D as $\langle y_n \rangle_{n=1}^{\infty}$. Suppose that $(**)$ fails and pick $H \in \mathcal{P}_f(S)$ such that for all $c \in \mathbb{N}$ and all $m \in D$, there is some $x \geq m$ such that $|F_x| > c \cdot |E_x|$, where $E_x = \bigcap_{a \in H} a^{-1} F_x$. We inductively choose a sequence $\langle x_n \rangle_{n=1}^{\infty}$ in D. Choose $x_1 \in D$ such that $x_1 \geq y_1$ and $|E_{x_1}| < |F_{x_1}|$. Given $n \in \mathbb{N}$ and $x_1, x_2, \ldots, x_n$, choose $x_{n+1} \in D$ such that $x_{n+1} \geq x_n$, $x_{n+1} \geq y_{n+1}$, $|F_{x_{n+1}}| \geq (n+1) \cdot |\bigcup_{t=1}^{n} HF_{x_t}|$, and $|F_{x_{n+1}}| > (n+1) \cdot |E_{x_{n+1}}|$. (Choose $z \in D$ satisfying the first three of these requirements, and they apply the choice of H with $c = n+1$ and $m = z$ to choose x_{n+1}.)

Inductively define B_n for $n \in \mathbb{N}$ as follows. Let $B_1 = F_{x_1} \setminus E_{x_1}$. Having defined B_n, let $B_{n+1} = F_{x_{n+1}} \setminus (E_{x_{n+1}} \cup \bigcup_{t=1}^{n} HB_t)$. Let $B = \bigcup_{n=1}^{\infty} B_n$.

We show that $\overline{d}_{\mathcal{F}}(B) = 1$. Let $m \in \mathbb{N}$ be given and let $z \in D$. Pick $r \in \mathbb{N}$ such that $z = y_r$ and let $n \geq \max\{m, r\}$. Then $x_n \geq z$ and $|B \cap F_{x_n}| \geq |B_n \cap F_{x_n}| \geq (1 - \frac{2}{n}) \cdot |F_{x_n}|$.

Therefore B is thick so pick $s \in B$ such that $Hs \subseteq B$. Pick the least $m \in \mathbb{N}$ such that $Hs \cup \{s\} \subseteq \bigcup_{k=1}^{m} B_k$. Since $\bigcup_{k=1}^{m} B_k \subseteq F_{x_m}$, we have that $s \in E_{x_m}$, so $s \notin B_m$ and thus $s \in B_k$ for some $k < m$. Since $HB_k \cap B_m = \emptyset$, we have that $Hs \cup \{s\} \subseteq \bigcup_{k=1}^{m-1} B_t$, a contradiction. □

We see that the conclusion of Theorem 2.9 follows also from $(**)$. (We shall see at the end of this section that neither of $(*)$ nor $(**)$ implies the other.)

2.14 Theorem. *Let S be a semigroup and let $\mathcal{F} = \langle F_n \rangle_{n \in D}$ be a net in $\mathcal{P}_f(S)$. Assume that there is some $b \in \mathbb{N}$ such that S is b-weakly left cancellative and b-weakly right* *cancellative. If $\mathcal{F}$ satisfies (∗∗) and B is a piecewise syndetic subset of S, then $d^*_{\mathcal{F}}(B) > 0$.*

Proof. Pick $H \in \mathcal{P}_f(S)$ such that $\bigcup_{t\in H} t^{-1}B$ is thick. Pick $c \in \mathbb{N}$ and $m \in D$ as guaranteed for H by (∗∗). Let $k = |H|$. Let $n \geq m$ and let $G = \bigcap_{a\in H} a^{-1}F_n$. Pick $x \in S$ such that $Gx \subseteq \bigcup_{t\in H} t^{-1}B$. Define $\tau : G \to (B \cap F_n x) \times H$ as follows. Given $y \in G$, pick $t \in H$ such that $tyx \in B$. Let $\tau(y) = (tyx, t)$. Then $|G| \leq b^2 \cdot |B\cap F_n x| \cdot |H|$ so $|B \cap F_n x| \geq \frac{1}{b^2 k} \cdot |G| \geq \frac{1}{b^2 ck} \cdot |F_n|$. □

2.15 Theorem. *Let S be a semigroup and let $\mathcal{F} = \langle F_n\rangle_{n\in D}$ be a net in $\mathcal{P}_f(S)$. Assume that there is some $b \in \mathbb{N}$ such that S is b-weakly left cancellative. If B is a syndetic subset of S and $\mathcal{F}$ satisfies (∗′) or (∗∗), then $\overline{d}_{\mathcal{F}}(B) > 0$.*

Proof. If $\mathcal{F}$ satisfies (∗∗), the conclusion follows immediately from Theorem 2.10 so assume that $\mathcal{F}$ satisfies (∗′). Pick $H \in \mathcal{P}_f(S)$ such that $S = \bigcup_{t\in H} t^{-1}B$. Then $\overline{d}_{\mathcal{F}}(\bigcup_{t\in H} t^{-1}B) = 1$ so by Lemma 2.3 there is some $t \in H$ such that $\overline{d}_{\mathcal{F}}(t^{-1}B) > 0$ and thus by Theorem 2.8 $\overline{d}_{\mathcal{F}}(B) > 0$. □

We close this section with a discussion of the relationships among the conditions of Definition 2.6. Trivially (∗′) implies (∗). In the semigroup $(\mathbb{N}, +)$, if for each $n \in \mathbb{N}$, $F_n = \{1, 2, \ldots, n\} \cup \big\{2t : t \in \{n+1, n+2, \ldots, n^n\}\big\}$ and $\mathcal{F} = \langle F_n\rangle_{n\in\mathbb{N}}$, then $\mathcal{F}$ is a sequence in $(\mathbb{N}, \cdot)$ which satisfies (∗), since $(\mathbb{N}, \cdot)$ is commutative, but not (∗′) with $\epsilon = \frac{1}{2}$ and $t = 2$. If S is the free semigroup over the alphabet $\{a, b\}$, for each $n \in \mathbb{N}$, $F_n = \{w \in S : \ell(w) \leq n^2\}$ (where $\ell(w)$ is the length of w), and $\mathcal{F} = \langle F_n\rangle_{n\in\mathbb{N}}$, then $\mathcal{F}$ satisfies (∗∗) but not (∗). (To see that $\mathcal{F}$ satisfies (∗∗), let $H \in \mathcal{P}_f(S)$ be given and let $k = \max\{\ell(w) : w \in H\}$. If $n^2 \geq k + 1$, then $|F_n| \leq 2^{k+1} \cdot |\bigcap_{w\in H} w^{-1}F_n|$. To see that $\mathcal{F}$ does not satisfy (∗) let $\epsilon = \frac{1}{2}$ and $t = a$. Given c, if $2^n > c$, then $|F_{n+1}| > c \cdot |F_n|$ so if $k \geq n$ is chosen to satisfy (∗), then $k = n$. But then for any $z \in S \cup \{1\}$, $|aF_n \setminus F_n z| \geq \frac{1}{2} \cdot |F_n|$.) We shall see in Theorems 3.2 and 3.3 that there are natural examples satifying (∗′) but not (∗∗).

3. Density determined by the FP-net

Throughout this section we shall be concerned with density defined in terms of a particular net in $\mathcal{P}_f(S)$ determined by a set of generators of S. In many of the semigroups with which we deal there is a natural choice for such a set of generators.

3.1 Definition. Let S be a semigroup and let Γ be a set of generators for S. Let D_Γ be the set of finite sequences in Γ and order D_Γ by agreeing that $\vec{x} \leq \vec{y}$ if and only if

$\vec{x}$ is a subsequence of $\vec{y}$. For $\vec{x} = \langle x_1, x_2, \ldots, x_n \rangle \in D_\Gamma$, let $F_{\vec{x}} = FP(\vec{x}) = \{\prod_{t \in F} x_t : \emptyset \neq F \subseteq \{1, 2, \ldots, n\}\}$. Then $\langle F_{\vec{x}} \rangle_{\vec{x} \in D_\Gamma}$ is the *FP-net determined by* Γ.

Note that the FP-net determined by Γ has the property that $F_{\vec{x}} \subsetneq F_{\vec{y}}$ whenever $\vec{x} \leq \vec{y}$ in D and if S is infinite, then $\lim_{n \in D} |F_{\vec{x}} = \infty$. Notice also that in the semigroup $(\mathbb{N}, +)$ with $\Gamma = \{1\}$, the FP-net determined by Γ is the same as $\langle \{1, 2, \ldots, n\} \rangle_{n \in \mathbb{N}}$ and thus the densities determined by this net are just the ordinary ones.

3.2 Theorem. *Let S be any infinite semigroup and let Γ be a set of generators of S. Let $\mathcal{F} = \langle F_{\vec{x}} \rangle_{\vec{x} \in D_\Gamma}$ be the FP-net determined by Γ. Then $\mathcal{F}$ satisfies ($*'$), so also ($*$).*

Proof. Let $\epsilon > 0$ and let $t \in S$. Pick $\vec{x} = \langle x_1, x_2, \ldots, x_n \rangle \in D_\Gamma$ such that $t = \prod_{i=1}^n x_i$, and let $c = |F_{\vec{x}}| + 1$. Let $\vec{y} \geq \vec{x}$ and let $\vec{w} = \vec{x}^\frown \vec{y}$. (That is, $\vec{w}$ is the sequence $\vec{x}$ followed by the sequence $\vec{y}$.) Then $tF_{\vec{y}} \subseteq F_{\vec{w}}$ (so $|tF_{\vec{y}} \setminus F_{\vec{w}}| < \epsilon \cdot |F_{\vec{x}}|$) and $F_{\vec{w}} = F_{\vec{x}} \cup F_{\vec{y}} \cup (F_{\vec{x}} \cdot F_{\vec{y}}) = F_{\vec{y}} \cup (F_{\vec{x}} \cdot F_{\vec{y}})$. Also $|F_{\vec{x}} \cdot F_{\vec{y}}| \leq |F_{\vec{x}}| \cdot |F_{\vec{y}}|$ so $|F_{\vec{w}}| \leq (1 + |F_{\vec{x}}|) \cdot |F_{\vec{y}}|$. $\square$

On the other hand, even in a countable cancellative semigroup ($**$) need not hold.

3.3 Theorem. *Let S be the free semigroup on the alphabet $\Gamma = \{a, b\}$ and let $\mathcal{F} = \langle F_{\vec{x}} \rangle_{\vec{x} \in D_\Gamma}$ be the FP-net determined by Γ. Then $\mathcal{F}$ does not satisfy ($**$).*

Proof. Let $H = \{a\}$. Suppose that we have $c \in \mathbb{N}$ and $\vec{x} \in D_\Gamma$ such that whenever $\vec{y} \geq \vec{x}$, $|F_{\vec{y}}| \leq c \cdot |a^{-1} F_{\vec{y}}|$. Let $n = (c+1) \cdot |F_{\vec{x}}|$ and let $\vec{y}$ be the sequence consisting of n occurrences of b followed by $\vec{x}$, then $|F_{\vec{y}}| > c \cdot |F_{\vec{x}}|$ and $a^{-1} F_{\vec{y}} \subseteq F_{\vec{x}}$. $\square$

3.4 Theorem. *Let S be an infinite commutative semigroup and let Γ be a set of generators of S. Let $\mathcal{F} = \langle F_{\vec{x}} \rangle_{\vec{x} \in D_\Gamma}$ be the FP-net determined by Γ. Then $\mathcal{F}$ satisfies ($**$).*

Proof. Let $H \in \mathcal{P}_f(S)$ and pick $\vec{w} \in D_\Gamma$ such that $H \subseteq F_{\vec{w}}$. Let $\vec{x} = \vec{w}^\frown \vec{w}$ and let $c = 1 + |F_{\vec{w}}|$. Let $\vec{y} > \vec{x}$. We claim that $|F_{\vec{y}}| \leq c \cdot |\bigcap_{a \in H} a^{-1} F_{\vec{y}}|$. Since S is commutative we may presume that $\vec{y} = \vec{x}^\frown \vec{z}$ for some $\vec{z} \in D_\Gamma$. Let $\vec{v} = \vec{w}^\frown \vec{z}$. Then $\vec{y} = \vec{w}^\frown \vec{v}$ and $F_{\vec{v}} \subseteq \bigcap_{a \in H} a^{-1} F_{\vec{y}}$. Also $F_{\vec{y}} \subseteq F_{\vec{v}} \cup F_{\vec{w}} \cdot F_{\vec{v}}$ so $|F_{\vec{y}}| \leq (1 + |F_{\vec{w}}|) \cdot |F_{\vec{v}}| \leq c \cdot |\bigcap_{a \in H} a^{-1} F_{\vec{y}}|$. $\square$

As a consequence of Theorems 2.9, 2.10, 3.2, and 3.4 and Corollary 2.12 we know that if S is commutative and cancellative, then all of the implications in Figure 1 are valid for the densities determined by the FP-net.

We see now that weak cancellation assumptions guarantee that the density determined by the FP-net has several desirable properties.

3.5 Theorem. *Let S be an infinite semigroup and let Γ be a set of generators of S. Assume that there is some $b \in \mathbb{N}$ such that S is b-weakly left cancellative and b-weakly right cancellative and let $\mathcal{F} = \langle F_{\vec{x}} \rangle_{\vec{x} \in D_\Gamma}$ be the FP-net determined by Γ. Then $\mathbf{D}^*_{\mathcal{F}}$ is a two sided ideal of βS, $\mathbf{D}_{\mathcal{F}}$ is a left ideal of βS, and all of the implications in Figure 2 below hold, except the fact that if A is thick then $d^*_{\mathcal{F}}(A) = 1$. That fact requires the additional assumption that S is right cancellative.*

Proof. The conclusions about $\mathbf{D}^*_{\mathcal{F}}$ and $\mathbf{D}_{\mathcal{F}}$ follow from Theorems 2.4, 2.7, 2.8, and 3.2. The implications in the diagram not involving density have already been established. The implications involving density follow from Remark 2.5 and Theorems 2.9, 2.15, and 3.2. □

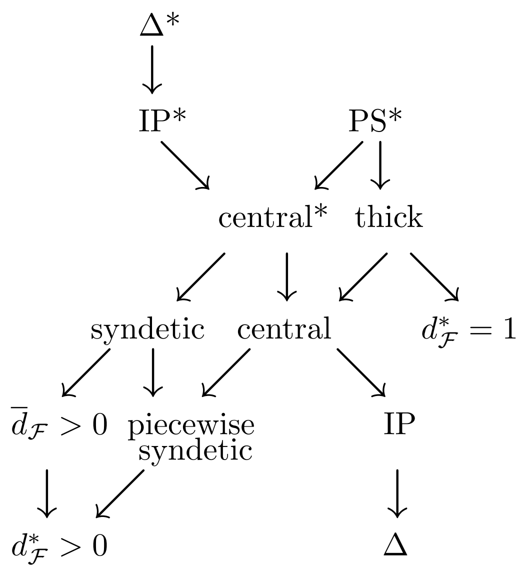

Figure 2

We have not indicated the trivial fact that if $d^*_{\mathcal{F}}(A) = 1$, then $d^*_{\mathcal{F}}(A) > 0$. We do not know whether the fact that $d^*_{\mathcal{F}}(A) = 1$ implies that A has any or all of the properties of being central, piecewise syndetic, an IP-set, or a Δ-set.

4. Følner nets and density in uncountable semigroups

Recall that Følner sequences provide quite satisfactory notions of density. We will primarily be interested in this section in uncountable semigroups where we know Følner sequences do not exist.

The following is the obvious extension of the notion of a Følner sequence. Følner nets have been used by various authors without being called such. (See for example [14, p. 65].) They do not appear to have been used to define densities before.

4.1 Definition. Let S be a semigroup and let $\mathcal{F} = \langle F_n \rangle_{n \in D}$ be a net in $\mathcal{P}_f(S)$. Then $\mathcal{F}$ is a *left Følner net* if and only if for each $s \in S$, the net

$$\left\langle \frac{|sF_n \triangle F_n|}{|F_n|} \right\rangle_{n \in D}$$

converges to 0. Also $\mathcal{F}$ is a *right Følner net* if and only if for each $s \in S$, the net

$$\left\langle \frac{|F_n s \triangle F_n|}{|F_n|} \right\rangle_{n \in D}$$

converges to 0.

We shall be almost exclusively concerned with the left version and will write "Følner net" rather than "left Følner net".

4.2 Theorem. *Let S be a semigroup. There exists a Følner net in $\mathcal{P}_f(S)$ if and only if S satisfies (SFC).*

Proof. The necessity is trivial. For the sufficiency, let $D = \mathbb{N} \times \mathcal{P}_f(S)$ and direct D by agreeing that $(n, H) \leq (n', H')$ if and only if $n \leq n'$ and $H \subseteq H'$. Given $(n, H) \in D$, pick $F_{n,H} \in \mathcal{P}_f(S)$ such that for all $s \in H$, $|sF_{n,H} \triangle F_{n,H}| < \frac{1}{n} \cdot |F_{n,H}|$. Then $\langle F_{n,H} \rangle_{(n,H) \in D}$ is a Følner net in $\mathcal{P}_f(S)$. □

4.3 Lemma. *Let S be a left cancellative semigroup and let $\mathcal{F} = \langle F_n \rangle_{n \in D}$ be a Følner net in $\mathcal{P}_f(S)$. Then for each $s \in S$, the net*

$$\left\langle \frac{|s^{-1}F_n \triangle F_n|}{|F_n|} \right\rangle_{n \in D}$$

converges to 0.

Proof. Given $s \in S$ and $n \in D$, $|s^{-1}F_n \triangle F_n| \leq |sF_n \triangle F_n|$. □

4.4 Lemma. *Let S be a left cancellative semigroup, let $F \in \mathcal{P}_f(S)$, and let $t, x \in S$. Then $\big||t^{-1}A \cap Fx| - |A \cap Fx|\big| \leq |tF \triangle F|$.*

Proof.

$$\begin{aligned} |t^{-1}A \cap Fx| &= |A \cap tFx| \\ &\leq |A \cap Fx| + |tFx \setminus Fx| \\ &\leq |A \cap Fx| + |tF \setminus F| \text{ and} \\ |A \cap Fx| &\leq |A \cap tFx| + |Fx \setminus tFx| \\ &\leq |A \cap tFx| + |F \setminus tF| \\ &= |t^{-1}A \cap Fx| + |F \setminus tF| \,. \end{aligned}$$

□

We see that density induced by Følner nets behaves as nicely as that induced by Følner sequences.

4.5 Theorem. *Let S be an infinite left cancellative semigroup and let $\mathcal{F} = \langle F_n \rangle_{n \in D}$ be a Følner net in $\mathcal{P}_f(S)$. Then for all $A \subseteq S$ and all $s \in S$,*
(a) $\underline{d}_{\mathcal{F}}(A) = \underline{d}_{\mathcal{F}}(sA) = \underline{d}_{\mathcal{F}}(s^{-1}A)$,
(b) $\overline{d}_{\mathcal{F}}(A) = \overline{d}_{\mathcal{F}}(sA) = \overline{d}_{\mathcal{F}}(s^{-1}A)$, *and*
(c) $d^*_{\mathcal{F}}(A) = d^*_{\mathcal{F}}(sA) = d^*_{\mathcal{F}}(s^{-1}A)$.

Proof. That the densities of A and $s^{-1}A$ are equal follows from Lemma 4.4. Then use the fact that $A = s^{-1}(sA)$. □

The next two theorems are well known in the context of countable semigroups when applied to Følner sequences.

4.6 Theorem. *Let S be an infinite left cancellative semigroup, let $\mathcal{F} = \langle F_n \rangle_{n \in D}$ be a Følner net in $\mathcal{P}_f(S)$, and let $A \subseteq S$. There is a countably additive regular measure μ on the set $\mathcal{B}$ of Borel subsets of βS such that*
(1) $\mu(\overline{A}) = d^*_{\mathcal{F}}(A)$,
(2) for all $B \subseteq S$, $\mu(\overline{B}) \leq d^*_{\mathcal{F}}(B)$,
(3) for all $B \in \mathcal{B}$ *and all* $t \in S$, $\mu(t^{-1}B) = \mu(B) = \mu(tB)$, *and*
(4) $\mu(\beta S) \leq 1$ *and if* $b \in \mathbb{N}$ *and* S *is* b*-weakly right cancellative, then* $\mu(\beta S) \geq \frac{1}{b}$.

Proof. Let $\alpha = d^*_{\mathcal{F}}(A)$. Let $E = D \times \mathbb{N}$ and direct E by agreeing that $(m,k) \leq (m',k')$ if and only if $m \leq m'$ and $k \leq k'$. For $(m,k) \in E$ pick $n(m,k) \in D$ and $x_{m,k} \in S \cup \{1\}$ such that $n(m,k) \geq m$ and $|A \cap F_{n(m,k)} x_{m,k}| > (\alpha - \frac{1}{k}) \cdot |F_{n(m,k)}|$.

Let $\mathfrak{C}$ denote the Banach space of continuous complex-valued functions defined on βS with the uniform norm. For each $(m,k) \in E$ we define $T_{m,k} : \mathfrak{C} \to \mathbb{C}$ by $T_{m,k}(f) = \frac{1}{|F_{n(m,k)}|} \cdot \sum_{t \in F_{n(m,k)}} f(t x_{m,k})$. We observe that, for each $A \subseteq S$, $T_{m,k}(\chi_{\overline{A}}) = \frac{1}{|F_{n(m,k)}|} \cdot |A \cap F_{n(m,k)} x_{m,k}|$. For each $f \in C$, let $C_f = \{z \in \mathbb{C} : |z| \leq \|f\|\}$. Let $T : \mathfrak{C} \to \mathbb{C}$ be a limit point of the net $\langle T_{m,k} \rangle_{(m,k) \in E}$ in the compact space $\prod_{f \in C} C_f$.

It is easy to establish each of the following statements:

(i) T is a continuous linear functional on $\mathfrak{C}$;
(ii) $T(\chi_{\overline{A}}) = d^*(A)$;
(iii) $T(\chi_{\overline{B}}) \leq d^*(B)$ for every $B \subseteq S$.
(iv) $T(\chi_{s^{-1}\overline{B}}) = T(\chi_{\overline{B}})$ for every $B \subseteq S$ and every $s \in S$.

By the Riesz Representation Theorem, T corresponds to a regular Borel measure μ on βS. By (iv), μ is left invariant on the algebra of functions in $\mathfrak{C}$ which take only a finite number of values. It follows from the Stone-Weierstrass Theorem that μ is left invariant. $\square$

By an identical proof, choosing in each case $x_{m,k} = 1$, one has also the following theorem.

4.7 Theorem. *Let S be an infinite left cancellative semigroup, let $\mathcal{F} = \langle F_n \rangle_{n \in D}$ be a Følner net in $\mathcal{P}_f(S)$, and let $A \subseteq S$. There is a countably additive measure μ on the set $\mathcal{B}$ of Borel subsets of βS such that*
(1) $\mu(\overline{A}) = \overline{d}_{\mathcal{F}}(A)$,
(2) for all $B \subseteq S$, $\mu(\overline{B}) \leq \overline{d}_{\mathcal{F}}(B)$,
(3) for all $B \in \mathcal{B}$ and all $t \in S$, $\mu(t^{-1}B) = \mu(B) = \mu(tB)$, and
(4) $\mu(\beta S) = 1$.

4.8 Corollary. *Let S be an infinite left cancellative semigroup and let $\mathcal{F} = \langle F_n \rangle_{n \in D}$ be a Følner net in $\mathcal{P}_f(S)$. Let $A \subseteq S$ and let $H \in \mathcal{P}_f(S)$.*
(a) If for all $a \neq b$ in H, $\overline{d}_{\mathcal{F}}(a^{-1}A \cap b^{-1}A) = 0$, then $\overline{d}_{\mathcal{F}}(\bigcup_{a \in H} a^{-1}A) = |H| \cdot \overline{d}_{\mathcal{F}}(A)$.
(b) If for all $a \neq b$ in H, $\overline{d}_{\mathcal{F}}(aA \cap bA) = 0$, then $\overline{d}_{\mathcal{F}}(\bigcup_{a \in H} aA) = |H| \cdot \overline{d}_{\mathcal{F}}(A)$.
(c) If for all $a \neq b$ in H, $d^*_{\mathcal{F}}(a^{-1}A \cap b^{-1}A) = 0$, then $d^*_{\mathcal{F}}(\bigcup_{a \in H} a^{-1}A) = |H| \cdot d^*_{\mathcal{F}}(A)$.
(d) If for all $a \neq b$ in H, $d^*_{\mathcal{F}}(aA \cap bA) = 0$, then $d^*_{\mathcal{F}}(\bigcup_{a \in H} aA) = |H| \cdot d^*_{\mathcal{F}}(A)$.

Proof. We do part (a) only. Pick μ as guaranteed by Theorem 4.7 for A. Then for $a \neq b$ in H, $\mu(\overline{a^{-1}A \cap b^{-1}A}) = 0$ so $\mu(\overline{\bigcup_{a \in H} a^{-1}H}) = \sum_{a \in H} \mu(\overline{a^{-1}A})$ so

$$\begin{aligned} |H| \cdot \overline{d}_{\mathcal{F}}(A) &= \textstyle\sum_{a \in H} \overline{d}_{\mathcal{F}}(a^{-1}A) \\ &\geq \overline{d}_{\mathcal{F}}(\textstyle\bigcup_{a \in H} a^{-1}A) \\ &\geq \mu(\overline{\textstyle\bigcup_{a \in H} a^{-1}A}) \\ &= \textstyle\sum_{a \in H} \mu(\overline{a^{-1}A}) \\ &= \textstyle\sum_{a \in H} \mu(\overline{A}) \\ &= |H| \cdot \overline{d}_{\mathcal{F}}(A)\,. \end{aligned}$$ $\square$

We observe that if S is cancellative, $\mathcal{F}$ is a Følner net in $\mathcal{P}_f(S)$, and $A \subseteq S$, one can obtain a Følner net with respect to which A has actual density equal to $d^*_{\mathcal{F}}(A)$.

4.9 Theorem. *Let S be an infinite cancellative semigroup, let $\mathcal{F} = \langle F_n \rangle_{n \in D}$ be a Følner net in $\mathcal{P}_f(S)$, and let $A \subseteq S$. There is a Følner net $\mathcal{G}$ in $\mathcal{P}_f(S)$ such that $\underline{d}_{\mathcal{G}}(A) = \overline{d}_{\mathcal{G}}(A) = d^*_{\mathcal{F}}(A)$ and for all $B \subseteq S$, $\overline{d}_{\mathcal{G}}(B) \leq d^*_{\mathcal{F}}(B)$.*

Proof. Let $\alpha = d^*_{\mathcal{F}}(A)$. Let $E = D \times \mathbb{N}$ and direct E by agreeing that $(m,k) \le (m',k')$ if and only if $m \le m'$ and $k \le k'$. For $(m,k) \in E$ pick $n(m,k) \in D$ and $x_{m,k} \in S \cup \{1\}$ such that $n(m,k) \ge m$ and $|A \cap F_{n(m,k)} x_{m,k}| > (\alpha - \frac{1}{k}) \cdot |F_{n(m,k)}|$. For $(m,k) \in E$ let $G_{m,k} = F_{n(m,k)} x_{m,k}$ and let $\mathcal{G} = \langle G_{m,k} \rangle_{(m,k) \in E}$. It is routine to verify the conclusions. □

4.10 Remark. *If S is a right cancellative semigroup, $\mathcal{F} = \langle F_n \rangle_{n \in D}$ is a Følner net in $\mathcal{P}_f(S)$, and for each $n \in D$, $x_n \in S \cup \{1\}$, then $\langle F_n x_n \rangle_{n \in D}$ is also a Følner net in $\mathcal{P}_f(S)$. In particular, if there exists a Følner net in $\mathcal{P}_f(S)$ and A is a thick subset of S, then there is a Følner net $\mathcal{G}$ in $\mathcal{P}_f(S)$ such that $\underline{d}_{\mathcal{G}}(A) = \overline{d}_{\mathcal{G}}(A) = 1$.*

We see now that Følner nets must satisfy the properties of Section 2.

4.11 Theorem. *Let $\mathcal{F} = \langle F_n \rangle_{n \in D}$ be a Følner net in a semigroup S. Then $\mathcal{F}$ satisfies (∗∗) and (∗′), and therefore (∗).*

Proof. It is trivial that $\mathcal{F}$ satisfies (∗′). Notice that, given any $F \subseteq S$ and any $a \in S$, we have that $\lambda_a : a^{-1}F \cap F \xrightarrow{\text{onto}} F \cap aF$ so

$$\begin{aligned} |F| &= |F \cap aF| + |F \setminus aF| \\ &\le |a^{-1}F \cap F| + |F \setminus aF| \\ &= |F| - |F \setminus a^{-1}F| + |F \setminus aF| \end{aligned}$$

and consequently $|F \setminus a^{-1}F| \le |F \setminus aF|$.

Now let $H \in \mathcal{P}_f(S)$ be given. We shall show that there exists $m \in D$ such that for all $n \ge m$, $|F_n| \le 2 \cdot |\bigcap_{a \in H} a^{-1}F_n|$. Let $k = |H|$ and for $a \in H$ pick $t_a \in D$ such that for all $n \ge t_a$, $|F_n \setminus a^{-1}F_n| \le \frac{1}{2k} \cdot |F_n|$. Pick $m \in D$ such that for all $a \in H$, $t_a \le m$ and let $n \ge m$. Then

$$\begin{aligned} |F_n| &= |F_n \cap \textstyle\bigcap_{a \in H} a^{-1}F_n| + |F_n \setminus \bigcap_{a \in H} a^{-1}F_n| \\ &= |F_n \cap \textstyle\bigcap_{a \in H} a^{-1}F_n| + |\bigcup_{a \in H} (F_n \setminus a^{-1}F_n)| \\ &\le |F_n \cap \textstyle\bigcap_{a \in H} a^{-1}F_n| + \sum_{a \in H} |F_n \setminus a^{-1}F_n| \\ &< |F_n \cap \textstyle\bigcap_{a \in H} a^{-1}F_n| + \frac{1}{2} \cdot |F_n| \end{aligned}$$

so $\frac{1}{2} \cdot |F_n| < |F_n \cap \bigcap_{a \in H} a^{-1}F_n| \le |\bigcap_{a \in H} a^{-1}F_n|$ as required. □

As a consequence we see that any cancellative semigroup satisfying (SFC) has notions of density preserving all of the relationships of Figure 1.

4.12 Corollary. *Let $\mathcal{F}$ be a Følner net in a semigroup S and assume that S is right cancellative and there is some $b \in \mathbb{N}$ such that S is b-weakly left cancellative. Then with $d_{\mathcal{F}}$ replacing d, all of the implications of Figure 1 are valid.*

Proof. Remark 2.5 and Theorems 2.10, 2.11, 2.14, and 4.11. □

It is interesting to note that no cancellation assumptions are needed in the following lemma.

4.13 Lemma. *Let S be a semigroup and let $\mathcal{F} = \langle F_n \rangle_{n\in D}$ be a net in $\mathcal{P}_f(S)$. Let $s, t \in S$. If*

$$\left\langle \frac{|sF_n \triangle F_n|}{|F_n|} \right\rangle_{n\in D} \text{ and } \left\langle \frac{|tF_n \triangle F_n|}{|F_n|} \right\rangle_{n\in D} \text{ converge to } 0\,,$$

$$\text{then } \left\langle \frac{|tsF_n \triangle F_n|}{|F_n|} \right\rangle_{n\in D} \text{ converges to } 0\,.$$

Proof. We have that $tsF_n \triangle tF_n \subseteq t(sF_n \triangle F_n)$ so $|tsF_n \triangle tF_n| \le |sF_n \triangle F_n|$. Also $tsF_n \triangle F_n \subseteq (tsF_n \triangle tF_n) \cup (tF_n \triangle F_n)$. □

We see in particular that FP-nets on commutative and cancellative semigroups induce these nice density notions.

4.14 Theorem. *Let S be an infinite commutative and cancellative semigroup and let Γ be a set of generators of S. Let $\mathcal{F} = \langle F_{\vec{x}} \rangle_{\vec{x}\in D_\Gamma}$ be the FP-net determined by Γ. Then $\mathcal{F}$ is a Følner net.*

Proof. By Lemma 4.13, it suffices to let $s \in \Gamma$ and prove that $\left\langle \frac{|sF_{\vec{x}} \triangle F_{\vec{x}}|}{|F_{\vec{x}}|} \right\rangle_{\vec{x}\in D_\Gamma}$ converges to 0. And, as we have already observed, for any $K \in \mathcal{P}_f(S)$, $|sK \setminus K| \le |K \setminus sK|$, so it suffices to show that $\left\langle \frac{|F_{\vec{x}} \setminus sF_{\vec{x}}|}{|F_{\vec{x}}|} \right\rangle_{\vec{x}\in D_\Gamma}$ converges to 0. To this end let $n \in \mathbb{N}$ and let $\vec{y}$ be the sequence consisting of n occurrences of s. Let $\vec{x} > \vec{y}$ be given. We shall show that $|F_{\vec{x}} \setminus sF_{\vec{x}}| < \frac{1}{n+1} \cdot |F_{\vec{x}}| + 1$. Since S is commutative, we may presume that we have some $\vec{z} \in D_\Gamma$ such that $\vec{x} = \vec{y}^\frown \vec{z}$. Let k be the length of $\vec{x}$.

Let $E = F_{\vec{z}} \setminus \bigcup_{t=1}^{n} s^t F_{\vec{z}}$. We claim that $F_{\vec{x}} \setminus sF_{\vec{x}} \subseteq E \cup \{s\}$. So assume that $a \in F_{\vec{x}} \setminus sF_{\vec{x}}$. Pick nonempty $H \subseteq \{1, 2, \ldots, k\}$ such that $a = \prod_{i\in H} x_i$. If $H \subseteq \{1, 2, \ldots, n\}$, then $a \in \{s, s^2, \ldots, s^n\}$. Since $\{s^2, s^3, \ldots, s^n\} \subseteq sF_{\vec{x}}$, we have then that $a = s$. So assume that $H \setminus \{1, 2, \ldots, n\} \ne \emptyset$. If $H \cap \{1, 2, \ldots, n\} \ne \emptyset$ pick $j \in H \cap \{1, 2, \ldots, n\}$. Then $a = s \cdot \prod_{i\in H\setminus\{j\}} x_i \in sF_{\vec{x}}$. So we have $H \cap \{1, 2, \ldots, n\} = \emptyset$ and thus $a \in F_{\vec{z}}$. Since $\bigcup_{t=1}^{n} s^t F_{\vec{z}} \subseteq sF_{\vec{x}}$ we have that $a \in E$.

We have established that $|F_{\vec{x}} \setminus sF_{\vec{x}}| \le |E| + 1$. Notice that if $0 \le t < j \le n$, then $s^t E \cap s^j E = \emptyset$. (We are not assuming that S has an identity. By $s^0 E$ we mean simply E.) Since $\bigcup_{j=0}^{n} s^j E \subseteq F_{\vec{x}}$, we have that $(n+1) \cdot |E| \le |F_{\vec{x}}|$. Thus $|F_{\vec{x}} \setminus sF_{\vec{x}}| \le \frac{1}{n+1} \cdot |F_{\vec{x}}| + 1$, as required. □

Recall from the introduction the statement of the *Strong Følner Condition*:

(SFC) $\big(\forall H \in \mathcal{P}_f(S)\big)\big(\forall \epsilon > 0\big)\big(\exists K \in \mathcal{P}_f(S)\big)\big(\forall s \in H\big)\big(|K \triangle sK| < \epsilon \cdot |K|\big)$.

This suggests a natural definition of density in any semigroup satisfying (SFC):

4.15 Definition. If S satisifies (SFC) and $A \subseteq S$, then

$$d_{\text{Fø}}(A) = \sup\Big\{\alpha : \big(\forall H \in \mathcal{P}_f(S)\big)\big(\forall \epsilon > 0\big)\big(\exists K \in \mathcal{P}_f(S)\big)\Big(|A \cap K| \geq \alpha \cdot |K| \text{ and } \big(\forall s \in H\big)\big(|K \triangle sK| < \epsilon \cdot |K|\big)\Big)\Big\}\,.$$

4.16 Theorem. *Let S be a semigroup which satsifies (SFC) and let $A \subseteq S$. Then $d_{\text{Fø}}(A) = \sup\{\overline{d}_{\mathcal{F}}(A) : \mathcal{F}$ is a Følner net in $\mathcal{P}_f(S)\}$ and there is a Følner net $\mathcal{F}$ such that $d_{\text{Fø}}(A) = \overline{d}_{\mathcal{F}}(A)$. If S is countable, then $d_{\text{Fø}}(A) = \sup\{\overline{d}_{\mathcal{F}}(A) : \mathcal{F}$ is a Følner sequence in $\mathcal{P}_f(S)\}$ and there is a Følner sequence $\mathcal{F}$ such that $d_{\text{Fø}}(A) = \overline{d}_{\mathcal{F}}(A)$.*

Proof. Let $\gamma = d_{\text{Fø}}(A)$. Let $D = \mathcal{P}_f(S) \times \mathbb{N}$ and for (H, n) and (K, m) in D, agree that $(H, n) \leq (K, m)$ if and only if $H \subseteq K$ and $n \leq m$. For $d = (H, n) \in D$, pick $K_d \in \mathcal{P}_f(S)$ such that $|A \cap K_d| \geq (\gamma - \frac{1}{n}) \cdot |K_d|$ and $(\forall s \in H)(|K_d \triangle sK_d| < \frac{1}{n} \cdot |K_d|)$. Then $\mathcal{K} = \langle K_d \rangle_{d \in D}$ is a Følner net in S and $\gamma \leq \overline{d}_{\mathcal{K}}(A)$. To complete the proof of the first assertion, it suffices to show that $\sup\{\overline{d}_{\mathcal{F}}(A) : \mathcal{F}$ is a Følner net in $S\} \leq \gamma$, so suppose instead that we have α with $\gamma < \alpha < \sup\{\overline{d}_{\mathcal{F}}(A) : \mathcal{F}$ is a Følner net in $S\}$. Pick $F \in \mathcal{P}_f(S)$ and $\epsilon > 0$ such that $\big(\forall K \in \mathcal{P}_f(S)\big)\big(|A \cap K| \geq \alpha \cdot |K| \Rightarrow (\exists s \in F)(|K \triangle sK| \geq \epsilon \cdot |K|)\big)$. Pick a Følner net $\mathcal{F} = \langle F_n \rangle_{n \in D}$ in $\mathcal{P}_f(S)$ such that $\overline{d}_{\mathcal{F}}(A) > \alpha$. Since $\mathcal{F}$ is a Følner net, pick $b \in D$ such that for all $a \geq b$ and all $s \in F$, $|F_a \triangle sF_a| < \epsilon \cdot |F_a|$. Since $\overline{d}_{\mathcal{F}}(A) > \alpha$ pick $a \geq b$ such that $|A \cap F_a| \geq \alpha \cdot |F_a|$. Then pick $s \in F$ such that $|F_a \triangle sF_a| \geq \epsilon \cdot |F_a|$. This contradiction completes the proof.

Now assume that S is countable and let $\langle s_n \rangle_{n=1}^{\infty}$ enumerate S. Since each Følner sequence is a Følner net, it suffices to show that there is a Følner sequence $\mathcal{K} = \langle K_n \rangle_{n=1}^{\infty}$ such that $\overline{d}_{\mathcal{K}}(A) \geq \gamma$. For each $n \in \mathbb{N}$ pick $K_n \in \mathcal{P}_f(S)$ such that $|A \cap K_n| \geq (\gamma - \frac{1}{n}) \cdot |K_n|$ and for all $t \in \{1, 2, \ldots, n\}$, $|K_n \triangle s_t K_n| < \frac{1}{n} \cdot |K_n|$. □

In [3] Bergelson established that *multiplicatively large* subsets of $\mathbb{N}$ have substantial additive and multiplicative structure. He defined a multiplicatively large set A as one for which there is some Følner sequence $\mathcal{F}$ in $(\mathbb{N}, \cdot)$ such that $\overline{d}_{\mathcal{F}}(A) > 0$. By virtue of the above theorem we see that multiplicatively large subsets of $\mathbb{N}$ are exactly those for which $d_{\text{Fø}}(A) > 0$.

In left cancellative semigroups Følner density is left translation invariant and left inverse translation invariant.

4.17 Theorem. *If S satisfies (SFC) and is left cancellative, then for all $A \subseteq S$ and all $a \in S$, $d_{\mathrm{F}\emptyset}(A) = d_{\mathrm{F}\emptyset}(aA) = d_{\mathrm{F}\emptyset}(a^{-1}A)$.*

Proof. Pick by Theorem 4.16 Følner nets $\mathcal{F}$, $\mathcal{G}$, and $\mathcal{H}$ such that $d_{\mathrm{F}\emptyset}(A) = \overline{d}_{\mathcal{F}}(A)$, $d_{\mathrm{F}\emptyset}(aA) = \overline{d}_{\mathcal{G}}(aA)$, and $d_{\mathrm{F}\emptyset}(a^{-1}A) = \overline{d}_{\mathcal{H}}(a^{-1}A)$. By Theorem 4.5, $\overline{d}_{\mathcal{F}}(A) = \overline{d}_{\mathcal{F}}(aA) = \overline{d}_{\mathcal{F}}(a^{-1}A)$, $\overline{d}_{\mathcal{G}}(A) = \overline{d}_{\mathcal{G}}(aA) = \overline{d}_{\mathcal{G}}(a^{-1}A)$, and $\overline{d}_{\mathcal{H}}(A) = \overline{d}_{\mathcal{H}}(aA) = \overline{d}_{\mathcal{H}}(a^{-1}A)$. Thus, invoking Theorem 4.16 again we have

$$d_{\mathrm{F}\emptyset}(A) = \overline{d}_{\mathcal{F}}(aA) \leq d_{\mathrm{F}\emptyset}(aA) = \overline{d}_{\mathcal{G}}(a^{-1}A) \leq d_{\mathrm{F}\emptyset}(a^{-1}A) = \overline{d}_{\mathcal{H}}(A) \leq d_{\mathrm{F}\emptyset}(A)\,.$$

We see that in an infinite left cancellative semigroup S with minimal right cancellation assumptions, Følner density understands that sets with lower cardinality than S are small. Notice that they hypotheses of the following theorem hold if S is not equal to the union of λ sets of the form $\{s \in S : b = sc\}$ with $b, c \in S$, and in particular if S is weakly right cancellative.

4.18 Theorem. *Let S be an infinite left cancellative semigroup of cardinality κ which satisfies (SFC). Let A be an infinite subset of S with $|A| = \lambda < \kappa$. Assume that for all $m \in \mathbb{N}$ there exists $H \subseteq S$ with $|H| = m$ such that $sA \cap tA = \emptyset$ whenever s and t are distinct members of H. Then $d_{\mathrm{F}\emptyset}(A) = 0$.*

Proof. Suppose that $d_{\mathrm{F}\emptyset}(A) > 0$ and pick $m \in \mathbb{N}$ such that $d_{\mathrm{F}\emptyset}(A) > \frac{2}{m}$. Pick $H \subseteq S$ with $|H| = m$ such that $sA \cap tA = \emptyset$ whenever s and t are distinct members of H and let $\epsilon = \frac{1}{m}$. Pick $K \in \mathcal{P}_f(S)$ such that $|A \cap K| \geq \frac{2}{m} \cdot |K|$ and for all $s \in H$, $|K \triangle sK| < \epsilon \cdot |K|$. Given $s \in H$,

$$\begin{aligned} |A \cap K| &\leq |A \cap sK| + |K \setminus sK| \\ &< |A \cap sK| + \tfrac{1}{m} \cdot |K| \end{aligned}$$

so that $|A \cap sK| > |A \cap K| - \frac{1}{m} \cdot |K|$. Now $\bigcup_{s \in H} (sA \cap K) \subseteq K$ so

$$\begin{aligned} |K| &\geq |\textstyle\bigcup_{s \in H} (sA \cap K)| \\ &= \textstyle\sum_{s \in H} |sA \cap K| \\ &> \textstyle\sum_{s \in H} (|A \cap K| - \frac{1}{m} \cdot |K|) \\ &= m \cdot |A \cap K| - |K| \end{aligned}$$

so $\frac{2}{m} \cdot |K| > |A \cap K|$, a contradiction. □

We conclude this section by showing that there are plentiful examples of uncountable, noncommutative, and cancellative semigroups satisfying (SFC). By Corollary 4.12 and Theorems 4.16, 4.17, and 4.18, $d_{\mathrm{F}\emptyset}$ is a very satisfactory notion of density for such semigroups.

4.19 Definition. Let U and T be semigroups, let $\mathrm{End}(U)$ be the group of endomorphisms of U and let $\sigma : T \to \mathrm{End}(U)$ be a homomorphism. The *semidirect product of U by T with respect to σ*, denoted by $U \times_\sigma T$, is the set $U \times T$ with operation $(a, b)(c, d) = (a \cdot \sigma(b)(c), bd)$.

See [8] for a considerable amount of information about semidirect products. It is routine to verify that semidirect products are semigroups and that, if U and T are right cancellative, then $U \times_\sigma T$ is right cancellative. Further, if U and T are left cancellative and $\sigma(b)$ is injective for each $b \in T$, then $U \times_\sigma T$ is left cancellative.

4.20 Theorem (Klawe). *Let U and T be semigroups, let $\mathrm{Aut}(U)$ be the group of automorphisms of U and let $\sigma : T \to \mathrm{Aut}(U)$ be a homomorphism. If U and T both satisfy (SFC), then so does $U \times_\sigma T$.*

Proof. [17, Proposition 4.6]. □

Now consider the semigroups $(\mathbb{R}, +)$ and $(\mathbb{R}^+, \cdot)$. Define $\sigma : \mathbb{R}^+ \to \mathrm{Aut}(\mathbb{R})$ as follows. Given $b \in \mathbb{R}^+$ and $c \in \mathbb{R}$, $\sigma(b)(c) = b \cdot c$. The operation in $\mathbb{R} \times_\sigma \mathbb{R}^+$ is then given by $(a, b)(c, d) = (a + bc, bd)$. Since $(\mathbb{R}, +)$ and $(\mathbb{R}^+, \cdot)$ are both commutative, they satisfy (SFC) and thus by Theorem 4.20, so does $\mathbb{R} \times_\sigma \mathbb{R}^+$. Thus $\mathbb{R} \times_\sigma \mathbb{R}^+$ is an uncountable, noncommutative, and cancellative semigroup satisfying (SFC).

The semidirect product is a one-sided notion. We shall see in Theorem 4.22 that a stronger version of the left-right switch of Theorem 4.20 holds. For this we need to state the left-right switch of (SFC):

(RSFC) $\big(\forall H \in \mathcal{P}_f(S)\big)\big(\forall \epsilon > 0\big)\big(\exists K \in \mathcal{P}_f(S)\big)\big(\forall s \in H\big)\big(|K \setminus Ks| < \epsilon \cdot |K|\big)$.

4.21 Lemma. *Let U and T be semigroups and let $\sigma : T \to \mathrm{End}(U)$ be a homomorphism. Let $(c, d) \in U \times_\sigma T$, let $F \in \mathcal{P}_f(U)$, and let $G \in \mathcal{P}_f(T)$. Then*

$$(F \times G) \setminus (F \times G) \cdot (c, d) \subseteq \big(F \times (G \setminus Gd)\big) \cup \{(x, bd) : b \in G \text{ and } x \in F \setminus F \cdot \sigma(b)(c)\}\,.$$

Proof. Let $(x, y) \in (F \times G) \setminus (F \times G) \cdot (c, d)$. If $y \notin Gd$, then we are done, so assume we have $b \in G$ such that $y = bd$. We claim that $x \notin F \cdot \sigma(b)(c)$ so suppose instead we have some $a \in F$ such that $x = a \cdot \sigma(b)(c)$. Then $(x, y) = (a, b) \cdot (c, d)$, a contradiction.□

4.22 Theorem. *Let U and T be semigroups and let $\sigma : T \to \mathrm{End}(U)$ be a homomorphism. If U and T both satisfy (RSFC), then so does $U \times_\sigma T$.*

Proof. By the left-right switch of Theorem 4.2 pick right Følner nets $\langle F_n \rangle_{n \in D}$ in $\mathcal{P}_f(U)$ and $\langle G_m \rangle_{m \in E}$ in $\mathcal{P}_f(T)$. To see that $U \times_\sigma T$ satisfies (RSFC), let $H \in \mathcal{P}_f(U \times_\sigma T)$ and

$\epsilon > 0$ be given. Pick $m \in E$ such that $|G_m \backslash G_m \cdot d| < \frac{\epsilon}{2} \cdot |G_m|$. For $b \in G_m$ and $(c,d) \in H$, pick $t_{b,c,d} \in D$ such that for all $n \in D$, if $n \geq t_{b,c,d}$, then $|F_n \setminus F_n \cdot \sigma(b)(c)| < \frac{\epsilon}{2} \cdot |F_n|$. Pick $n \in D$ such that $n \geq t_{b,c,d}$ for each $b \in G_m$ and each $(c,d) \in H$.

We claim that $F_n \times G_m$ is as required for (RSFC). So let $(c,d) \in H$ be given. By Lemma 4.21 it suffices to observe that $|F_n \times (G_m \setminus G_m \cdot d)| < \frac{\epsilon}{2} \cdot |F_n| \cdot |G_m|$ by the choice of m and that $|\{(x, bd) : b \in G_m \text{ and } x \in F_n \setminus F_n \cdot \sigma(b)(c)\}| < \frac{\epsilon}{2} \cdot |F_n| \cdot |G_m|$ by the choice of n. □

5. Applications

We begin by recording in Theorem 5.3 an extension of a result about density in $\mathbb{N}$. In [2] Bergelson showed that whenever $\mathbb{N}$ is partitioned into finitely many cells, one cell C satisfies $\overline{d}(\{n \in C : \overline{d}\big(C \cap (-n + C)\big) > 0\}) > 0$. This was a generalization of Schur's Theorem (which says that whenever $\mathbb{N}$ is partitioned into finitely many cells one of them contains n, m, and $n + m$ for some n and m). This result says that there are many $n \in C$ with the property that for many m, both m and $n + m$ are in C. We shall show that a corresponding statement is valid in any left cancellative semigroup which has a Følner net.

5.1 Lemma. *Let S be an infinite left cancellative semigroup and let $\mathcal{F} = \langle F_n \rangle_{n \in D}$ be a Følner net in $\mathcal{P}_f(S)$. Let $\langle x_n \rangle_{n=1}^{\infty}$ be a sequence in S and let $B \subseteq S$. If $\overline{d}_{\mathcal{F}}(B) > 0$, then there exist $k < m$ in $\mathbb{N}$ such that $\overline{d}_{\mathcal{F}}(B \cap (\prod_{t=k+1}^{m} x_t)^{-1} B) > 0$.*

Proof. For each $m \in \mathbb{N}$, let $s_m = \prod_{t=1}^{m} x_t$. Pick $n \in \mathbb{N}$ such that $\overline{d}_{\mathcal{F}}(B) > \frac{1}{n}$. We claim that there exist $k < m \in \mathbb{N}$ such that $\overline{d}_{\mathcal{F}}(s_k B \cap s_m B) > 0$. By Theorem 4.5, for each $t \in \{1, 2, \ldots, n\}$, $\overline{d}_{\mathcal{F}}(s_t B) = \overline{d}_{\mathcal{F}}(B)$. If some $k < m$ in $\{1, 2, \ldots, n\}$ have $s_k = s_m$, then our claim is satisfied, so we may assume that $s_k \neq s_m$ whenever $k < m$ in $\{1, 2, \ldots, n\}$. By Corollary 4.8(b), if each $\overline{d}_{\mathcal{F}}(s_k B \cap s_m B) = 0$, then $\overline{d}_{\mathcal{F}}(\bigcup_{t=1}^{n} s_t B) \geq n \cdot \overline{d}_{\mathcal{F}}(B) > 1$, so there exist $k < m \in \mathbb{N}$ such that $\overline{d}_{\mathcal{F}}(s_k B \cap s_m B) > 0$ as claimed.

Now by Theorem 4.5 again, $\overline{d}_{\mathcal{F}}\big((s_m)^{-1}(s_k B \cap s_m B)\big) = \overline{d}_{\mathcal{F}}(s_k B \cap s_m B)$. Since $(s_m)^{-1}(s_k B \cap s_m B) = B \cap (\prod_{t=k+1}^{m} x_t)^{-1} B$, we are done. □

The proof of the following lemma is nearly identical, so we omit it.

5.2 Lemma. *Let S be an infinite left cancellative semigroup and let $\mathcal{F} = \langle F_n \rangle_{n \in D}$ be a Følner net in $\mathcal{P}_f(S)$. Let $\langle x_n \rangle_{n=1}^{\infty}$ be a sequence in S and let $B \subseteq S$. If $d^*_{\mathcal{F}}(B) > 0$, then there exist $k < m$ in $\mathbb{N}$ such that $d^*_{\mathcal{F}}(B \cap (\prod_{t=k+1}^{m} x_t)^{-1} B) > 0$.*

The following theorem extends Bergelson's result cited above to uncountable semigroups, using essentially the original proof.

5.3 Theorem. *Let S be an infinite left cancellative semigroup and let $\mathcal{F} = \langle F_n \rangle_{n \in D}$ be a Følner net in $\mathcal{P}_f(S)$. If $r \in \mathbb{N}$ and $S = \bigcup_{i=1}^{r} C_i$, then there exists $i \in \{1, 2, \ldots, r\}$ such that $\overline{d}_{\mathcal{F}}(\{s \in C_i : \overline{d}_{\mathcal{F}}(C_i \cap s^{-1}C_i) > 0\}) > 0$.*

Proof. We may assume that we have $k \in \{1, 2, \ldots, r\}$ such that for each $i \in \{1, 2, \ldots, k\}$, $\overline{d}_{\mathcal{F}}(C_i) > 0$ and for each $i \in \{k+1, k+2, \ldots, r\}$, if any, $\overline{d}_{\mathcal{F}}(C_i) = 0$. For each $i \in \{1, 2, \ldots, k\}$, let $R_i = \{s \in C_i : \overline{d}_{\mathcal{F}}(C_i \cap s^{-1}C_i) > 0\}$ and suppose that for each $i \in \{1, 2, \ldots, k\}$, $\overline{d}_{\mathcal{F}}(R_i) = 0$. Then by Lemma 2.3 $\overline{d}_{\mathcal{F}}(\bigcup_{i=1}^{k} R_i \cup \bigcup_{i=k+1}^{r} C_i) = 0$ so $\mathbf{D}_{\mathcal{F}} \subseteq \overline{\bigcup_{i=1}^{k}(C_i \setminus R_i)}$. By Theorem 4.11 $\mathcal{F}$ satisfies $(*')$ so by Theorem 2.8, $\mathbf{D}_{\mathcal{F}}$ is a left ideal of βS so by [9, Corollary 2.10] pick an idempotent $p \in \mathbf{D}_{\mathcal{F}}$. Pick $i \in \{1, 2, \ldots, k\}$ such that $C_i \setminus R_i \in p$ and pick by [16, Theorem 5.8] a sequence $\langle x_n \rangle_{n=1}^{\infty}$ such that $FP(\langle x_n \rangle_{n=1}^{\infty}) \subseteq C_i \setminus R_i$. Pick by Lemma 5.1 $k < m$ in $\mathbb{N}$ such that $\overline{d}_{\mathcal{F}}(C_i \cap (\prod_{t=k+1}^{m} x_t)^{-1} C_i) > 0$. Then $\prod_{t=k+1}^{m} x_t \in R_i$, a contradiction. $\square$

We remark that hypotheses that guarantee that $\mathbf{D}_{\mathcal{F}}$ or $\mathbf{D}^*_{\mathcal{F}}$ are left ideals of βS (or even subsemigroups) guarantee that if $S = \bigcup_{i=1}^{r} C_i$ there will exist $i \in \{1, 2, \ldots, r\}$ and the ability to chose a sequence $\langle x_n \rangle_{n=1}^{\infty}$ with $FP(\langle x_n \rangle_{n=1}^{\infty}) \subseteq C_i$, and having a set of positive density ($\overline{d}_{\mathcal{F}}$ or $d^*_{\mathcal{F}}$) from which to make the choice of each x_n. For a precise description of this phenomenon, see [16, p. 292].

We shall have need of the following deep result of Furstenberg. Recall that a *measure space* is a triple $(X, \mathcal{B}, \mu)$, where X is a set, $\mathcal{B}$ is a σ-algebra of subsets of X, and μ is a countably additive measure on $\mathcal{B}$ such that $0 < \mu(X) < \infty$. Recall also that $T : X \to X$ is *measure preserving* provided that for each $B \in \mathcal{B}$, $T^{-1}[B] \in \mathcal{B}$ and $\mu(T^{-1}[B]) = \mu(B)$.

5.4 Theorem. *Let $(X, \mathcal{B}, \mu)$ be a measure space, let $k \in \mathbb{N}$, let T_1, T_2, $\ldots$, T_k be commuting measure preserving transformations of $(X, \mathcal{B}, \mu)$, and let $B \in \mathcal{B}$ such that $\mu(B) > 0$. Then there exists $d \in \mathbb{N}$ such that*

$$\mu\big((T_1)^{-d}[B] \cap (T_2)^{-d}[B] \cap \ldots \cap (T_k)^{-d}[B]\big) > 0\,.$$

Proof. [13, Theorem 7.15]. $\square$

5.5 Theorem. *Let $b \in \mathbb{N}$ and let S be an infinite left cancellative and b-weakly right cancellative semigroup. Let $\mathcal{F} = \langle F_n \rangle_{n \in D}$ be a Følner net in $\mathcal{P}_f(S)$, let $s \in S$, and let $A \subseteq S$ such that $d^*_{\mathcal{F}}(A) > 0$. Then for each $k \in \mathbb{N}$ there is some $d \in \mathbb{N}$ such that $d^*_{\mathcal{F}}\big(\{b \in S : \{s^d b, s^{2d} b, \ldots, s^{kd} b\} \subseteq A\}\big) > 0$.*

Proof. Let $\mathcal{B}$ be the set of Borel subsets of βS. Pick by Theorem 4.6 a countably additive measure on $\mathcal{B}$ such that

(1) $\mu(\,\overline{A}\,) = d^*_{\mathcal{F}}(A)$,

(2) for all $B \subseteq S$, $\mu(\,\overline{B}\,) \leq d^*_{\mathcal{F}}(B)$,

(3) for all $B \in \mathcal{B}$ and all $t \in S$, $\mu(t^{-1}B) = \mu(B)$, and

(4) $\frac{1}{b} \leq \mu(\beta S) \leq 1$.

Let $k \in \mathbb{N}$ be given. For $t \in \{1, 2, \ldots, k\}$, let $T_t = \lambda_{s^t}$. Since each λ_{s^t} is continuous one has that $(T_t)^{-1}[B] \in \mathcal{B}$ whenever $B \in \mathcal{B}$ and thus $T_1, T_2, \ldots, T_k$ are commuting measure preserving transformations of the measure space $(\beta S, \mathcal{B}, \mu)$. Since $\mu(\,\overline{A}\,) = d^*_{\mathcal{F}}(A) > 0$, pick by Theorem 5.4 some $d \in \mathbb{N}$ such that

$$\mu\big((T_1)^{-d}[\,\overline{A}\,] \cap (T_2)^{-d}[\,\overline{A}\,] \cap \ldots \cap (T_k)^{-d}[\,\overline{A}\,]\big) > 0\,.$$

Now $(T_1)^{-d}[\,\overline{A}\,] \cap (T_2)^{-d}[\,\overline{A}\,] \cap \ldots \cap (T_k)^{-d}[\,\overline{A}\,] = \overline{(s^d)^{-1}A \cap (s^{2d})^{-1}A \cap \ldots \cap (s^{kd})^{-1}A}$ so by (2) $d^*_{\mathcal{F}}\big((s^d)^{-1}A \cap (s^{2d})^{-1}A \cap \ldots \cap (s^{kd})^{-1}A\big) > 0$. □

The same proof, using Theorem 4.7 instead of Theorem 4.6 yields the following theorem. (We no longer need the b-weakly right cancellative assumption which was only used to guarantee that $\mu(\beta S) > 0$.)

5.6 Theorem. *Let S be an infinite left cancellative semigroup. Let $\mathcal{F} = \langle F_n \rangle_{n \in D}$ be a Følner net in $\mathcal{P}_f(S)$, let $s \in S$, and let $A \subseteq S$ such that $\overline{d}_{\mathcal{F}}(A) > 0$. Then for each $k \in \mathbb{N}$ there is some $d \in \mathbb{N}$ such that $\overline{d}_{\mathcal{F}}\big(\{b \in S : \{s^d b, s^{2d} b, \ldots, s^{kd} b\} \subseteq A\}\big) > 0$.*

We obtain as a consequence of Theorem 5.5 a new result about piecewise syndetic sets in left amenable semigroups.

5.7 Theorem. *Let S be an infinite left amenable left cancellative semigroup and assume that there is some $b \in \mathbb{N}$ such that S is b-weakly right cancellative. Let A be a piecewise syndetic subset of S and let $s \in S$. For each $k \in \mathbb{N}$ there exist $b \in S$ and $d \in \mathbb{N}$ such that $\{b, sb^d, s^{2d}b, \ldots, s^{kd}b\} \subseteq A$.*

Proof. Pick a Følner net $\mathcal{F}$ in $\mathcal{P}_f(S)$. By Theorems 2.4, 2.7, and 4.11, $\mathbf{D}^*_{\mathcal{F}}$ is a two sided ideal of βS so $K(\beta S) \subseteq \mathbf{D}^*_{\mathcal{F}}$. Since $\overline{A} \cap K(\beta S) \neq \emptyset$ we have that $\overline{A} \cap \mathbf{D}^*_{\mathcal{F}} \neq \emptyset$ and so $d^*_{\mathcal{F}}(A) > 0$. Pick by Theorem 5.5 some $d \in \mathbb{N}$ such that $d^*_{\mathcal{F}}\big(\{b \in S : \{s^d b, s^{2d} b, \ldots, s^{(k+1)d} b\} \subseteq A\}\big) > 0$ and pick $b' \in S$ such that $\{s^d b', s^{2d} b', \ldots, s^{(k+1)d} b'\} \subseteq A$. Let $b = s^d b'$. □

Geometric progressions are commonly written in the form $bs^d, bs^{2d}, \ldots, bs^{kd}$ and one naturally wonders whether any or all of Theorems 5.5, 5.6, or 5.7 hold in that form.

We see now that they do not. (By Remark 4.10, if A and S are as in the following theorem, there is a Følner net $\mathcal{G}$ in $\mathcal{P}_f(S)$ with respect to which $\underline{d}_{\mathcal{G}}(A) = 1$.)

5.8 Theorem. *There exist a countable cancellative semigroup S with identity, a Følner sequence $\mathcal{F} = \langle F_n\rangle_{n=1}^\infty$ in $\mathcal{P}_f(S)$, and a subset A of S such that A is thick (and therefore central and piecewise syndetic) and there is some $s \in S$ such that $\{bs^n : n \in \mathbb{N}$ and $b \in S\} \cap A = \emptyset$.*

Proof. Let $S = \omega \times \mathbb{N}$ with the operation defined by $(a, b) \cdot (c, d) = (c + ad, bd)$. (This is the left-right switch of the semidirect product of the semigroup $(\omega, +)$ with the semigroup $(\mathbb{N}, \cdot)$ determined by σ where $\sigma(d)(a) = da$.)

Let $\langle G_n\rangle_{n=1}^\infty$ be any Følner sequence in $(\mathbb{N}, \cdot)$. Given $n \in \mathbb{N}$, let $k(n) = n^2 \cdot \max G_n$ and let $F_n = \{0, 1, \ldots, k(n)\} \times G_n$. Then $\langle F_n\rangle_{n=1}^\infty$ is a Følner sequence in S. (Given $(a, b) \in S$ and $\epsilon > 0$, if $n > a$, $|G_n \setminus bG_n| < \frac{\epsilon}{2}$, and $\frac{1}{n} < \frac{\epsilon}{2}$, then one sees as in the proof of Theorem 4.22 that $|F_n \setminus (a, b) \cdot F_n| < \epsilon \cdot |F_n|$.)

Let $A = (2 \cdot \omega) \times \mathbb{N}$. Then trivially $\underline{d}_{\mathcal{F}}(A) = \overline{d}_{\mathcal{F}}(A) = \frac{1}{2}$. Also $S \cdot (2, 2) \subseteq A$ so A is thick. Let $s = (1, 2)$. Then for each $n \in \mathbb{N}$, $s^n = (2^n - 1, 2^n)$ and so for any $(a, b) \in S$, $(a, b) \cdot s^n = (2^n - 1 + 2^n a, 2^n b) \notin A$. □

It was proved in [5, Lemma 2.3] that if A is a piecewise syndetic subset of an arbitrary semigroup S and $\mathcal{H}$ is a partition regular family of finite subsets of S, then there must exist $t, x \in S$ and $H \in \mathcal{H}$ such that $tHx \subseteq A$. It was also shown by example that both translates may be needed. (Although, of course, if S is left amenable and left cancellative, then by Theorem 5.7 one only needs the right translate, and it may be chosen in A.) We see now in Theorem 5.9 that in general one may require one or the other of the translates to be in A, and in Theorem 5.10 that one cannot require both of the translates to be in A.

5.9 Theorem. *Let $(S, \cdot)$ be a semigroup, let $\mathcal{H}$ be a partition regular family of finite subsets of S, and let A be a piecewise syndetic subset of S. Let $\mathcal{K}$ be a family of finite sets with the property that any piecewise syndetic set contains a member of $\mathcal{K}$ and let $\mathcal{L}$ be a family of sets (finite or infinite) with the property that any piecewise syndetic set contains a member of $\mathcal{L}$. Then there exist $K \in \mathcal{K}$, $L \in \mathcal{L}$, $H \in \mathcal{H}$, and $x, t \in S$ such that $K \subseteq A$, $KHx \subseteq A$, $L \subseteq A$, and $tHL \subseteq A$. Also there exist $K \in \mathcal{K}$, $H, H' \in \mathcal{H}$, and $x, t \in S$ such that $K \subseteq A$, $tHK \subseteq A$, and $KH'x \subseteq A$.*

Proof. Pick $q \in \beta S$ such that for each $B \in q$ there is some $H \in \mathcal{H}$ such that $H \subseteq B$. (Such q exists because $\mathcal{H}$ is partition regular. See for example [16, Theorem 3.11].) Pick by Lemma 1.4 some $p \in c\ell A \cap K(\beta S)$. Then there exist a minimal right ideal R and a minimal left ideal L such that $p \in R \cap L = RL$ and RL is a group by [16, Theorems 1.51]. Also by [16, Theorem 1.31] $R = p\beta S$ and $L = \beta S p$ so $RL = p\beta S\beta S p \subseteq p\beta S p \subseteq R \cap L = RL$ so $p\beta S p$ is a group to which p belongs. Now pqp is a member of this group, so there exist $u, v \in p\beta S p$ such that $p = pqpu = vpqp$. Let $u' = pu$ and $v' = vp$.

Now $A \in pqu' = v'qp$ so $\{s \in S : s^{-1}A \in qu'\} \in p$ and $\{t \in S : t^{-1}A \in qp\} \in v'$. Pick $t \in S$ such that $t^{-1}A \in qp$. Then $\{s \in S : s^{-1}(t^{-1}A) \in p\} \in q$. Pick $K \in \mathcal{K}$ such that $K \subseteq \{s \in A : s^{-1}A \in qu'\}$ and let $B = \bigcap_{s \in K} s^{-1}A$. Then $B \in qu'$ so $\{s \in S : s^{-1}B \in u'\} \in q$. Pick $H \in \mathcal{H}$ such that

$$H \subseteq \{s \in S : s^{-1}(t^{-1}A) \in p\} \cap \{s \in S : s^{-1}B \in u'\}.$$

Pick $L \in \mathcal{L}$ such that $L \subseteq A \cap \bigcap_{s \in H} s^{-1}(t^{-1}A)$. Pick $x \in \bigcap_{s \in H} s^{-1}B$. Then $KHx \subseteq A$ and $tHL \subseteq A$.

Now, as above, pick $t \in S$ such that $t^{-1}A \in qp$. Then $\{s \in S : s^{-1}(t^{-1}A) \in p\} \in q$ so pick $H \in \mathcal{H}$ such that $H \subseteq \{s \in S : s^{-1}(t^{-1}A) \in p\}$. Pick $K \in \mathcal{K}$ such that $K \subseteq \{s \in A : s^{-1}A \in qu'\} \cap \bigcap_{s \in H} s^{-1}(t^{-1}A)$. Let $C = \bigcap_{s \in K} s^{-1}A$. Then

$$\{y \in S : y^{-1}C \in u'\} \in q$$

so pick $H' \in \mathcal{H}$ such that $H' \subseteq \{y \in S : y^{-1}C \in u'\}$ and pick $x \in \bigcap_{y \in H'} y^{-1}C$. Then $KH'x \subseteq A$ and $tHK \subseteq A$. □

Notice that if S is any semigroup, $s \in S$, and $n \in \mathbb{N}$, and $\mathcal{H} = \big\{\{s^n\}\big\}$ then $\mathcal{H}$ is a partition regular family of finite subsets of S, so the following result shows that one cannot extend Theorem 5.9 by requiring in any of the conclusions that either $x \in A$ or $t \in A$.

5.10 Theorem. *Let S be the free semigroup on the set of generators $\{a, b\}$. There exist a central subset A of S and $s \in S$ such that there do not exist u and v in A and $n \in \mathbb{N}$ with $us^nv \in A$.*

Proof. Let $A = \{bwb : w \in S$ and w has an even number of blocks of a's$\}$. We have that $c\ell(bS)$ is a right ideal of βS and $c\ell(Sb)$ is a left ideal of βS by [16, Theorem 4.17] so there is a minimal idempotent $p \in c\ell(bS) \cap c\ell(Sb)$. Since $p = pp$, $\{bwb : w \in S$ and w has an odd number of blocks of a's$\} \notin p$ so $A \in p$. That is A is central. Let $s = a$. Given any $u, v \in A$ and any $n \in \mathbb{N}$, $us^nv \notin A$. □

We know from Theorem 3.5 that it is common for $\mathbf{D}^*_{\mathcal{F}}$ to be a two sided ideal of βS. However, this fact may well be trivial. For example, if $D = \mathcal{P}_f(S)$ and for $H \in D$,

$F_H = H$, then $\mathcal{F} = \langle F_H \rangle_{H \in D}$ is a net in $\mathcal{P}_f(S)$. However for $B \subseteq S$, $d^*_{\mathcal{F}}(B) > 0$ if and only if B is infinite (in which case $\overline{d}_{\mathcal{F}}(B) = 1$) and so $\mathbf{D}^*_{\mathcal{F}} = \beta S \setminus S$, which is an ideal of βS if and only if S is both weakly left cancellative and weakly right cancellative by [16, Theorem 4.36]. The following theorem, which has not been noted before even for $\mathbb{N}$, shows that $\mathbf{D}^*_{\mathcal{F}}$ can indeed be quite small.

5.11 Theorem. *Let S be an infinite cancellative semigroup, let $\mathcal{F} = \langle F_n \rangle_{n \in D}$ be a Følner net in $\mathcal{P}_f(S)$, and let $E = \{p \in \beta S : p \cdot p = p\}$. Then $\mathbf{D}^*_{\mathcal{F}} \subseteq c\ell(E \cdot \mathbf{D}^*_{\mathcal{F}})$. In fact, given any $B \subseteq S$ with $d^*_{\mathcal{F}}(B) > 0$, $\{p \in E : \overline{B} \cap p \cdot \mathbf{D}^*_{\mathcal{F}} \neq \emptyset\}$ is dense in E. Consequently $\mathbf{D}^*_{\mathcal{F}} = c\ell(E \cdot \mathbf{D}^*_{\mathcal{F}})$.*

Proof. Let $B \subseteq S$ with $d^*_{\mathcal{F}}(B) > 0$, let $r \in E$, and let $C \in r$. We need to show that $\overline{C} \cap \{p \in E : \overline{B} \cap p \cdot \mathbf{D}^*_{\mathcal{F}} \neq \emptyset\} \neq \emptyset$. Pick by [16, Theorem 5.8] a sequence $\langle x_n \rangle_{n=1}^{\infty}$ with $FP(\langle x_n \rangle_{n=1}^{\infty}) \subseteq C$. Let $B_0 = B$. Pick by Lemma 5.2 $k_1 \leq m_1$ in $\mathbb{N}$ such that $d^*_{\mathcal{F}}(B_0 \cap (\prod_{t=k_1}^{m_1} x_t)^{-1} B_0) > 0$. Let $y_1 = \prod_{t=k_1}^{m_1} x_t$ and let $B_1 = B_0 \cap {y_1}^{-1} B_0$. Inductively, given B_{n-1} and $k_{n-1} \leq m_{n-1}$ pick $k_n \leq m_n$ in $\mathbb{N}$ with $k_n > m_{n-1}$ such that $d^*_{\mathcal{F}}(B_{n-1} \cap (\prod_{t=k_n}^{m_n} x_t)^{-1} B_{n-1}) > 0$. Let $y_n = \prod_{t=k_n}^{m_n} x_t$ and let $B_n = B_{n-1} \cap {y_n}^{-1} B_{n-1}$.

Note that $FP(\langle y_n \rangle_{n=1}^{\infty}) \subseteq FP(\langle x_n \rangle_{n=1}^{\infty})$. We claim that for each $H \in \mathcal{P}_f(S)$, if $n = \max H$, then $B_n \subseteq (\prod_{l \in H} y_l)^{-1} B$. We establish this by induction on $|H|$. If $H = \{n\}$ we have $B_n \subseteq {y_n}^{-1} B_{n-1} \subseteq {y_n}^{-1} B$. So assume $|H| > 1$, let $K = H \setminus \{n\}$ and let $v = \max K$. Then $B_v \subseteq (\prod_{l \in K} y_l)^{-1} B$ so $B_n \subseteq {y_n}^{-1} B_v \subseteq {y_n}^{-1} (\prod_{l \in K} y_l)^{-1} B = (\prod_{l \in H} y_l)^{-1} B$.

Pick by [16, Lemma 5.11] some $p \in E$ such that $FP(\langle y_n \rangle_{n=1}^{\infty}) \in p$ and pick by Lemma 2.3 and [16, Theorem 3.11] some $q \in \mathbf{D}^*_{\mathcal{F}}$ such that $\{B_n : n \in \mathbb{N}\} \subseteq q$. Then as we have shown in the paragraph above, $FP(\langle y_n \rangle_{n=1}^{\infty}) \subseteq \{z \in S : z^{-1} B \in q\}$ so $B \in p \cdot q$ as required.

For the final conclusion note that by Theorem 4.11 $\mathcal{F}$ satisfies $(*)$ so by Theorem 2.7 $\mathbf{D}^*_{\mathcal{F}}$ is a left ideal of βS and so $E \cdot \mathbf{D}^*_{\mathcal{F}} \subseteq \mathbf{D}^*_{\mathcal{F}}$ and the latter set is closed. □

References

[1] L. Argabright and C. Wilde, *Semigroups satisfying a strong Følner condition*, Proc. Amer. Math. Soc. **18** (1967), 587-591.

[2] V. Bergelson, *A density statement, generalizing Schur's Theorem*, J. Comb. Theory (Series A) **43** (1986), 338-343.

[3] V. Bergelson, *Multiplicatively large sets and ergodic Ramsey Theory*, Israel J. Math., to appear.

[4] V. Bergelson, M. Beiglböck, N. Hindman, and D. Strauss, *Multiplicative structures in additively large sets*, J. Comb. Theory (Series A), to appear*.

[5] V. Bergelson, M. Beiglböck, N. Hindman, and D. Strauss, *Some new results in multiplicative and additive Ramsey Theory*, manuscript*.

[6] V. Bergelson and N. Hindman, *Partition regular structures contained in large sets are abundant*, J. Comb. Theory (Series A) **93** (2001), 18-36.

[7] V. Bergelson, N. Hindman and R. McCutcheon, *Notions of size and combinatorial properties of quotient sets in semigroups*, Topology Proceedings **23** (1998), 23-60.

[8] J. Berglund, H. Junghenn, and P. Milnes, *Analysis on semigroups*, Wiley, N.Y., 1989.

[9] R. Ellis, *Lectures on topological dynamics*, Benjamin, New York, 1969.

[10] P. Feit, *Følner sequences and Hilbert's irreducibility theorem over* **Q**, Israel J. Math. **69** (1990), 289-320.

[11] E. Følner, *On groups with full Banach mean values*, Math. Scand. **3** (1955), 243-254.

[12] A. Frey, Jr., *Studies on amenable semigroups*, Thesis, University of Washington, 1960.

[13] H. Furstenberg, *Recurrence in ergodic theory and combinatorical number theory*, Princeton University Press, Princeton, 1981.

[14] F. Greenleaf, *Invariant means on topological groups and their applications*, Van Nostrand, New York, 1969.

[15] E. Hewitt and K. Stromberg, *Real and Abstract Analysis*, Springer-Verlag, Berlin, 1965.

[16] N. Hindman and D. Strauss, *Algebra in the Stone-Čech compactification: theory and applications*, de Gruyter, Berlin, 1998.

[17] M. Klawe, *Semidirect product of semigroups in relation to amenability, cancellation properties, and strong Følner conditions*, Pacific J. Math. **73** (1977), 91-106.

[18] I. Namioka, *Følner's conditions for amenable semi-groups*, Math. Scand. **15** (1964), 18-28.

[19] A. Patterson, *Amenability*, American Mathematical Society, Providence, 1988.

[20] G. Polya, *Untersuchungen über Lücken und Singularitaten von Potenzreihen*, Math. Zeit. **29** (1929), 549-640.

* (Currently available at `http://members.aol.com/nhindman/`.)

Neil Hindman
Department of Mathematics
Howard University
Washington, DC 20059
USA
nhindman@aol.com

Dona Strauss
Department of Pure Mathematics
University of Hull
Hull HU6 7RX
UK
d.strauss@hull.ac.uk