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Density and Invariant Means in Left Amenable Semigroups

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Abstract. A left cancellative and left amenable semigroup S satisfies the *Strong Følner Condition*. That is, given any finite subset H of S and any $\epsilon > 0$, there is a finite nonempty subset F of S such that for each $s \in H$, $|sF \triangle F| < \epsilon \cdot |F|$. This condition is useful in defining a very well behaved notion of density, which we call Følner density, via the notion of a *left Følner net*, that is a net $\langle F_\alpha \rangle_{\alpha \in D}$ of finite nonempty subsets of S such that for each $s \in S$, $(|sF_\alpha \triangle F_\alpha|)/|F_\alpha|$ converges to 0. Motivated by a desire to show that this density behaves as it should on cartesian products, we were led to consider the set $LIM_0(S)$ which is the set of left invariant means which are weak* limits in $l_\infty(S)^*$ of left Følner nets. We show that the set of all left invariant means is the weak* closure of the convex hull of $LIM_0(S)$. (If S is a left amenable group, this is a relatively old result of C. Chou.) We obtain our desired density result as a corollary. We also show that the set of left invariant means on $(\mathbb{N}, +)$ is actually equal to $LIM_0(\mathbb{N})$.We also derive some properties of the extreme points of the set of left invariant means on S, regarded as measures on βS, and investigate the algebraic implications of the assumption that there is a left invariant mean on S which is non-zero on some singleton subset of βS.

1. Introduction

If E is a Banach space, its dual, the space of continuous linear functionals defined on E, will be denoted by E^*. We recall that E^* is a Banach space with norm defined by $\|f\| = \sup(\{|f(x)| : x \in E, \|x\| \leq 1\})$ for $f \in E^*$.

Throughout this paper, S will denote a discrete semigroup. We shall use $l_\infty(S)$ to denote the real Banach space of bounded real valued functions on S with the supremum norm, denoted by $\| \ \|_\infty$. A *mean* on S is a member μ of $l_\infty(S)^*$ such that $||\mu||_\infty = 1$ and $\mu(g) \geq 0$ whenever $g \in l_\infty(S)$ and for all $s \in S$ $g(s) \geq 0$. A *left invariant mean* on S is a mean μ such that for all $s \in S$ and all $g \in l_\infty(S)$, $\mu(s \cdot g) = \mu(g)$, where $s \cdot g = g \circ \lambda_s$ and $\lambda_s : S \to S$ is defined by $\lambda_s(t) = st$.

1.1 Definition. Let S be a discrete semigroup. Then $MN(S)$ is the set of means on S and $LIM(S)$ is the set of left invariant means on S. A semigroup S is *left amenable* if and only if $LIM(S) \neq \emptyset$.

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For any set X, let $\mathcal{P}_f(X)$ be the set of finite nonempty subsets of X. In [1] Argabright and Wilde showed that a left cancellative semigroup S is left amenable if and only if it satisfies the *strong Følner condition*:

$$\text{(SFC)} \qquad \big(\forall H \in \mathcal{P}_f(S)\big)(\forall \epsilon > 0)\big(\exists F \in \mathcal{P}_f(S)\big)(\forall s \in H)(|F \setminus sF| < \epsilon \cdot |F|)$$

(See [17, page 145] for the short proof that this version is equivalent to the version stated in the abstract.) They also showed that any commutative semigroup satisfies SFC. In particular, any commutative semigroup is left amenable.

If $F \setminus sF$ is replaced by $sF \setminus F$ in the statement of SFC, one obtains the original Følner condition (FC). In [5] Følner showed that any amenable group S satisfies (FC) and Frey [6] showed that any left amenable semigroup satisfies (FC). (For a simplified proof see [16, Theorem 3.5].)

The strong Følner condition corresponds naturally to the notion of a left Følner net.

1.2 Definition. Let S be a semigroup and let $\langle F_\alpha \rangle_{\alpha \in D}$ be a net in $\mathcal{P}_f(S)$. Then $\langle F_\alpha \rangle_{\alpha \in D}$ is a *left Følner net* if and only if for each $s \in S$, the net

$$\left\langle \frac{|sF_\alpha \triangle F_\alpha|}{|F_\alpha|} \right\rangle_{\alpha \in D}$$

converges to 0.

Equivalently, $\langle F_\alpha \rangle_{\alpha \in D}$ is a *left Følner net* if and only if for each $s \in S$, the net $\left\langle \frac{|F_\alpha \setminus sF_\alpha|}{|F_\alpha|} \right\rangle_{\alpha \in D}$ converges to 0.

The notion of a left Følner net in turn leads naturally to a very well behaved notion of density in any semigroup which satisfies SFC. We remark that the concept of density is significant in Ramsey Theory. For example Szemerédi's Theorem ([19], [7]) says that any subset A of $\mathbb{N}$ with $d(A) > 0$ contains arbitrarily long arithmetic progressions.

1.3 Definition. Let S be a semigroup which satisifies SFC and let $A \subseteq S$. Then

$$d(A) = \sup\Big\{\alpha : \big(\forall H \in \mathcal{P}_f(S)\big)\big(\forall \epsilon > 0\big)\big(\exists K \in \mathcal{P}_f(S)\big)\Big(|A \cap K| \geq \alpha \cdot |K| \text{ and } \big(\forall s \in H\big)\big(|K \triangle sK| < \epsilon \cdot |K|\big)\Big)\Big\}\,.$$

See [11, Section 4] for verification of the niceness of this notion of density. In particular, we have the following.

1.4 Theorem. *Let S be a semigroup which satisifies SFC and let $A \subseteq S$. There exists a left Følner net $\langle F_\alpha\rangle_{\alpha\in D}$ in $\mathcal{P}_f(S)$ such that*

$$d(A) = \lim_{\alpha\in D} \frac{|A \cap F_\alpha|}{|F_\alpha|}\,.$$

Proof. [11, Theorem 4.16]. □

In recent research involving small sets satisfying a version of the Central Sets Theorem, we wanted to know that if S and T are left amenable left cancellative semigroups, $A \subseteq S$, $B \subseteq T$, and either $d(A) = 0$ or $d(B) = 0$, then $d(A \times B) = 0$. In fact we suspected that $d(A \times B) = d(A) \cdot d(B)$. The desire to prove that fact provided the initial motivation for this paper. It is easy to prove in an entirely elementary fashion that $d(A \times B) \geq d(A) \cdot d(B)$. (See Lemma 3.1.) The proof of the reverse inequality leads us back into the world of invariant means.

1.5 Definition. Let S be a discrete semigroup and let $F \in \mathcal{P}_f(S)$. Then $\mu_F \in l_\infty(S)^*$ is defined by $\mu_F(g) = \frac{1}{|F|} \cdot \sum_{t\in F} g(t)$ for every $g \in l_\infty(S)$.

Notice that each μ_F is a mean on S.

We remind the reader that, for any Banach space E with dual E^*, the weak topology on E is the topology for which a net $\langle x_\alpha\rangle_{\alpha\in D}$ in E converges to $x \in E$ if and only if $\langle f(x_\alpha)\rangle_{\alpha\in D}$ converges to $f(x)$ for every $f \in E^*$. The weak* topology on E^* is the topology for which a net $\langle f_\alpha\rangle_{\alpha\in D}$ in E^* converges to $f \in E^*$ if and only if $\langle f(x_\alpha)\rangle_{\alpha\in D}$ converges to $f(x)$ for every $x \in E$. Equivalently, the weak* topology is the restriction to E^* of the product topology on $\times_{g\in E} \mathbb{R}$.

We shall need to use the following well-known theorem.

1.6 Theorem. *If E is any Banach space, the unit ball of E^* is compact in the weak* topology.*

Proof. See [13, 17.4]. □

1.7 Definition. Let S be a discrete semigroup. Then $LIM_0(S) = \{\eta \in l_\infty(S)^* :$ there exists a left Følner net $\langle F_\alpha\rangle_{\alpha\in D}$ in $\mathcal{P}_f(S)$ such that $\langle \mu_{F_\alpha}\rangle_{\alpha\in D}$ weak* converges to $\eta\}$.

It is routine to show that $LIM_0(S) \subseteq LIM(S)$ and that $LIM(S)$ is convex and weak* closed. C. Chou showed in [3, Theorem 3.2(a)] that if G is a countable left amenable discrete group, then $LIM(G)$ is the weak* closed convex hull of $LIM_0(G)$. (He actually dealt with σ-compact locally compact topological groups. Also, in groups left amenability implies the existence of a two sided invariant mean. See [17, Section

0.17].) This result was extended to an arbitrary left amenable group by A. Patterson in [17, Theorem 4.17]. (Again, he actually dealt with locally compact topological groups.) The proofs of these results strongly use algebraic properties of groups, and do not apply directly to semigroups. In Section 2, we show that for any left cancellative left amenable semigroup S, $LIM(S)$ is the weak* closed convex hull of $LIM_0(S)$.

In Section 3 we use the above result to show that if S and T are left cancellative left amenable semigroups, $A \subseteq S$, and $B \subseteq T$, then $d(A \times B) = d(A) \cdot d(B)$.

The question naturally arises whether $LIM(S)$ properly contains $LIM_0(S)$. We cannot answer this question in general, but we do show in Section 4 that $LIM(\mathbb{N},+) = LIM_0(\mathbb{N},+)$.

In Section 5 we derive some properties of the extreme points of $LIM(S)$ where S is a left cancellative and left amenable semigroup.

In Section 6 we consider the implications for the algebra of βS of the assumption that there exists $\mu \in LIM(S)$, regarded as a measure on βS, and an element $x \in \beta S$ for which $\mu(\{x\}) > 0$. A. T.-M. Lau proved in [15] that this assumption holds if and only if every minimal left ideal of βS is a finite group. We include a proof of this fact because our terminology and methods of proof differ significantly from those used in [15]. The papers [14] and [8] also contain results relevant to Section 6.

2. The weak* closed convex hull of $\mathbf{LIM_0(S)}$

We use $l_1(S)$ to denote the Banach space of mappings $f : S \to \mathbb{R}$ such that $\sum_{s\in S} |f(s)| < \infty$, with norm $||f||_1 = \sum_{s\in S} |f(s)|$ for $f \in l_1(S)$. Of course, $l_\infty(S)$ is the dual of $l_1(S)$, the duality being defined by $\langle g, f\rangle = \sum_{t\in S} g(t)f(t)$ for $g \in l_\infty$ and $f \in l_1(S)$.

Given $A \subseteq S$ and $s \in S$, $s^{-1}A = \{u \in S : su \in A\}$.

2.1 Definition. Let S be a semigroup.

(a) Define $\tau : l_1(S) \to l_\infty(S)^*$ by $\tau(f)(g) = \langle g, f\rangle$ for $f \in l_1(S)$ and $g \in l_\infty(S)$.

(b) For $s \in S$ and $f \in l_1(S)$, $s \cdot f \in l_1(S)$ is defined by $(s \cdot f)(t) = \sum_{u\in s^{-1}\{t\}} f(u)$.

Thus $\tau : l_1(S) \to l_\infty(S)^*$ is the natural embedding of $l_1(S)$ in its second dual. It is an injective linear map and an isometry.

Note that if s is left cancelable, then $(s \cdot f)(t) = \begin{cases} f(u) & \text{if } su = t \\ 0 & \text{if } t \notin sS\,. \end{cases}$

2.2 Lemma. *Let S be a semigroup, let $s \in S$, and let $f \in l_1(S)$. Then $\tau(s \cdot f) = s \cdot \tau(f)$.*

Proof. Let $A = \{(t,u) : t \in S \text{ and } su = t\}$ and let $g \in l_\infty(S)$. Then

$$\begin{aligned}\big(s\cdot\tau(f)\big)(g) &= \textstyle\sum_{u\in S}\, g(su)f(u)\\ &= \textstyle\sum_{(t,u)\in A}\, g(t)f(u)\\ &= \tau(s\cdot f)(g)\,.\end{aligned}$$

□

We shall need the following two well-known results.

2.3 Lemma. *Let E be a real locally convex topological vector space, let C be a closed convex subset of E and let $x \in E \setminus C$. Then there is a continuous linear functional T on E for which $f(x) < \inf T[C]$.*

Proof. [9, B26]. □

2.4 Lemma. *Let E be a real normed linear space and let C be a convex subset of E. Then the weak closure of C and the norm closure of C are equal.*

Proof. [13, 17.1]. □

2.5 Definition. Let S be a semigroup. Then $\Phi = \Phi(S)$ is the set of all $f \in l_1(S)$ such that

(1) $(\forall s \in S)(f(s) \geq 0)$,

(2) $\{s \in S : f(s) > 0\}$ is finite, and

(3) $||f||_1 = 1$.

Note that if $f \in \Phi$, then $\tau(f)$ is a mean, called a *finite mean*.

The following lemma is also well known. We give a short proof because we do not have an explicit reference.

2.6 Lemma. *Let $\mu \in MN(S)$. Then μ is in the weak* closure of $\tau[\Phi]$.*

Proof. If $\mu \in MN(S)$ were not in the weak* closed convex hull of $\tau[\Phi]$, there would be a weak* continuous linear functional T on $l_\infty(S)^*$ and a real number a for which $T(\mu) > a$ and $T\big(\tau(\phi)\big) < a$ for every $\phi \in \Phi$, by Lemma 2.3. Now there exists $g \in l_\infty(S)$ such that $\langle T, \nu\rangle = \langle \nu, g\rangle$ for every $\nu \in l_\infty(S)^*$, by [13, 17.6]. Since $\langle \tau(\phi), g\rangle < a$ for every $\phi \in \Phi$, $g(s) < a$ for every $s \in S$ and so $g < a \cdot 1$, where 1 denotes the function constantly equal to 1 on S. This implies that $\langle \mu, g\rangle \leq \langle \mu, a\cdot 1\rangle = a$ – a contradiction. □

The following lemma was proved in [16] with a left-right switch.

2.7 Lemma. *Let S be an arbitrary semigroup. Let $\phi \in \Phi$. Then ϕ can be written in the form $\phi = \sum_{i=1}^{n} c_i \mu_{A_i}$ where, for each $i \in \{1, 2, \ldots, n\}$, $c_i \in [0, 1]$, $A_i \in \mathcal{P}_f(S)$ and $\sum_{i=1}^{n} c_i = 1$. Furthermore, for every $s \in S$, $||s \cdot \phi - \phi||_1 \geq \sum_{i=1}^{n} c_i \cdot \frac{|sA_i \setminus A_i|}{|A_i|}$.*

Proof. [16, Lemma 3.3]. □

2.8 Definition. Let S be a semigroup and let $A \in \mathcal{P}_f(S)$. We define $\mu_A \in \Phi$ by $\mu_A = \frac{1}{|F|} \chi_F$.

We omit the routine proof of the following lemma.

2.9 Lemma. *Let S be a left cancellative semigroup, let $F \in \mathcal{P}_f(S)$, and let $s \in S$. Then*

$$||s \cdot \mu_F - \mu_F||_1 = \frac{|sF \bigtriangleup F|}{|F|} .$$

2.10 Lemma. *Let S be a left cancellative semigroup, let $A \in \mathcal{P}_f(S)$ and let $s \in S$. Then $|sA \setminus A| = |A \setminus sA|$.*

Proof. We have $|F \setminus sF| = |F| - |F \cap sF|$, $|sF \setminus F| = |sF| - |F \cap sF|$ and $|F| = |sF|$. □

The proof of the following lemma is essentially the same as the elegant proof of Theorem 2.2 in [16].

2.11 Lemma. *Let S be a left cancellative and left amenable semigroup. Let $\mu \in LIM(S)$, let W be a convex weak* neighborhood of μ, and let $U = \Phi \cap \tau^{-1}[W]$, let $F \in \mathcal{P}_f(S)$ and let $\delta > 0$. Then there exists $f \in U$ such that for each $s \in F$, $||s \cdot f - f||_1 < \delta$.*

Proof. Let $F = \{s_1, s_2, \ldots, s_n\}$, let $E = \times_{i=1}^{n} l_1(S)$, and for $\vec{x} \in E$, let $||\vec{x}|| = \max \big\{ ||x_i||_1 : i \in \{1, 2, \ldots, n\} \big\}$. Then the weak topology on E is the product of the weak topologies on the coordinate spaces [13, 17.13]. By Lemma 2.6, there is a net $\langle f_\alpha \rangle_{\alpha \in D}$ in U for which $\langle \tau(f_\alpha) \rangle_{\alpha \in D}$ converges to μ in the weak* topology of $l_\infty(S)^*$. For each $s \in S$, the mapping $\nu \mapsto s \cdot \nu$ is a weak* continuous mapping from $l_\infty(S)^*$ to itself. So $\langle s \cdot \tau(f_\alpha) \rangle_{\alpha \in D}$ converges to $s \cdot \mu = \mu$ and $\langle s \cdot \tau(f_\alpha) - \tau(f_\alpha) \rangle_{\alpha \in D}$ converges to 0 in the weak* topology. Since the weak* topology on $\tau[l_1(S)]$ corresponds to the weak topology on $l_1(S)$, it follows that $\langle s \cdot f_\alpha - f_\alpha \rangle_{\alpha \in D}$ converges to 0 in the weak topology of $l_1(S)$. Thus the net $\langle s_1 \cdot f_\alpha - f_\alpha, s_2 \cdot f_\alpha - f_\alpha, \ldots, s_n \cdot f_\alpha - f_\alpha) \rangle_{\alpha \in D}$ converges to 0 in the weak topology of E. So 0 is in the weak closure of the subset $\{(s_1 \cdot f - f, s_2 \cdot f - f, \ldots, s_n \cdot f - f) : f \in U\}$ of E. By Lemma 2.4, 0 is in the norm closure of this set. So there exists $f \in U$ such that $\|s \cdot f - f\|_1 < \delta$ for every $s \in F$. □

2.12 Theorem. *Let S be left cancellative and left amenable. Then $LIM(S)$ is the weak* closed convex hull of $LIM_0(S)$.*

Proof. Let C denote the weak* closed convex hull of $LIM_0(S)$ and assume that there exists $\mu \in LIM(S) \setminus C$. By Lemma 2.3, there is a weak* continuous linear functional T on $l_\infty(S)^*$ and $b \in \mathbb{R}$ such that $T(x) < b < \inf T[C]$.

For each $F \in \mathcal{P}_f(S)$ and each $\delta > 0$, put

$$\Phi(F, \delta) = \{f \in \Phi : \|s \cdot f - f\|_1 < \delta \text{ for all } s \in F\}.$$

We claim that there exists $F_0 \in \mathcal{P}_f(S)$ and $\delta_0 > 0$ such that $T\big(\tau(f)\big) \geq b$ for every $f \in \Phi(F_0, \delta_0)$. To see this assume that, on the contrary, there exists $f(F,\delta) \in \Phi(F,\delta)$ for every $F \in \mathcal{P}_f(S)$ and every $\delta > 0$ such that $T\Big(\tau\big(f(F,\delta)\big)\Big) < b$. We give $\mathcal{P}_f(S) \times \mathbb{R}^+$ a directed set ordering by putting $(F_1, \delta_1) < (F_2, \delta_2)$ if $F_1 \subseteq F_2$ and $\delta_1 > \delta_2$. Then every limit point ν of the net $\langle \tau\big(f(F,\delta)\big)\rangle_{(F,\delta) \in \mathcal{P}_f(S) \times \mathbb{R}}$ is in $LIM_0(S)$ and satisfies $T(\nu) \leq b$, a contradiction.

Let $U = \{\nu \in l_\infty(S)^* : T(\nu) < b\}$. Since U is a weak* convex neighbourhood of μ, it follows from Lemmas 2.10 and 2.11 that there exists f in the convex hull of $\Phi(F_0, \delta_0)$ for which $\tau(f) \in U$. This is a contradiction because $T\big(\tau(f)\big) \geq b$ for every f in the convex hull of $\Phi(F_0, \delta_0)$. □

We remark that Theorem 2.12 does not hold if the assumption of left cancellativity is deleted. Every right cancellative left amenable semigroup which is not left cancellative does not satisfy SFC and so has no Følner nets [17, p.145].

2.13 Corollary. *If S is left cancellative and left amenable, $LIM_0(S)$ contains all the extreme points of $LIM(S)$.*

Proof. [13, Theorem 15.2]. □

2.14 Theorem. *Suppose that S is a semigroup which satisfies SFC. Then, for every $A \subseteq S$, $d(A) \leq \sup\{\mu(A) : \mu \in LIM(S)\}$. If $LIM(S)$ is the weak* closed convex hull of $LIM_0(S)$ then, for every subset A of S, $d(A) = \sup(\{\mu(\chi_A) : \mu \in LIM(S)\})$.*

Proof. By Theorem 1.4, there is a Følner net $\langle F_\alpha \rangle_{\alpha \in D}$ such that $d(A) = \lim \mu_{F_\alpha}(\chi_A)$. If μ is any weak* limit point of the net $\langle \mu_{F_\alpha} \rangle_{\alpha \in D}$, then $\mu \in LIM_0(S)$ and $\mu(\chi_A) = d(A)$. So $d(A) \leq \sup\{\mu(\chi_A) : \mu \in LIM(S)\}$.

Now $\{\mu \in LIM(S) : \mu(\chi_A) \leq d(A)\}$ is a weak* closed convex subset of $LIM(S)$ which contains $LIM_0(S)$. It therefore contains $LIM(S)$ if $LIM(S)$ is the weak* closed convex hull of $LIM_0(S)$. □

3. Density of products

We show in this section that for semigroups S satisfying SFC such that $LIM(S)$ is the weak* closed convex hull of $LIM_0(S)$, density in cartesian products behaves as it should. We first record the very simple elementary proof of one desired inequality. For this one does not need any special assumptions (beyond, of course, the assumption of SFC, which is needed for density to be defined).

3.1 Lemma. *Let S and T be semigroups which satisfy SFC, let $A \subseteq S$ and $B \subseteq T$. Then $S \times T$ satisfies SFC and $d(A \times B) \geq d(A) \cdot d(B)$.*

Proof. Pick by Lemma 1.4 a left Følner net $\langle F_\alpha \rangle_{\alpha \in D}$ in $\mathcal{P}_f(S)$ and a left Følner net $\langle G_\delta \rangle_{\delta \in E}$ in $\mathcal{P}_f(T)$ such that $d(A) = \lim_{\alpha \in D} \frac{|A \cap F_\alpha|}{|F_\alpha|}$ and $d(B) = \lim_{\delta \in E} \frac{|B \cap G_\delta|}{|G_\delta|}$. Direct $D \times E$ by $(\alpha, \delta) \leq (\alpha', \delta')$ if and only if $\alpha \leq \alpha'$ and $\delta \leq \delta'$. Then $\langle F_\alpha \times G_\delta \rangle_{(\alpha,\delta) \in D \times E}$ is a left Følner net in $\mathcal{P}_f(S \times T)$ and $d(A) \cdot d(B) = \lim_{(\alpha,\delta) \in D \times E} \frac{|(A \times B) \cap (F_\alpha \times G_\delta)|}{|F_\alpha \times G_\delta|}$.□

3.2 Definition. Let S be a semigroup. $C(S) = \{\chi_A : A \subseteq S\}$ and $\mathfrak{S}(S)$, the set of simple functions on S, is the linear span of $C(S)$.

3.3 Lemma. *Let S be an arbitrary semigroup. Assume $\nu : C(S) \to [0,1]$ such that $\nu(\chi_S) = 1$, $\nu(\chi_{A \cup B}) = \nu(\chi_A) + \nu(\chi_B)$ whenever A and B are disjoint subsets of S, and $\nu(\chi_{s^{-1}A}) = \nu(\chi_A)$ whenever $A \subseteq S$ and $s \in S$. Then ν extends to a member of $LIM(S)$.*

Proof. Let $f \in \mathfrak{S}(S)$ be written as $f = \sum_{i=1}^m a_i \chi_{A_i}$ where $a_1, a_2, \ldots, a_m \in \mathbb{R}$ and $A_1, A_2, \ldots, A_m \subseteq S$. We claim that the number $\sum_{i=1}^m a_i \nu(\chi_{A_i})$ is uniquely determined by f.

We first consider the case in which we also have $f = \sum_{j=1}^n b_j \chi_{B_j}$, where $(B_1, B_2, \ldots, B_n)$ is a disjoint partition of S and, for every i and j, $B_j \subseteq A_i$ or $B_j \cap A_i = \emptyset$. We observe that, for every j and every $s \in B_j$, $f(s) = b_j = \sum\{a_i : B_j \cap A_i \neq \emptyset\}$. So $b_j \nu(\chi_{B_j}) = \sum\{a_i \nu(\chi_{B_j}) : B_j \cap A_i \neq \emptyset\} = \sum_{i=1}^m a_i \nu(\chi_{A_i \cap B_j})$. So $\sum_{j=1}^n b_j \nu(\chi_{B_j}) = \sum_{j=1}^n \sum_{i=1}^m a_i \nu(\chi_{A_i \cap B_j}) = \sum_{i=1}^m \sum_{j=1}^n a_i \nu(A_i \cap B_j) = \sum_{i=1}^m a_i \nu(A_i)$.

Now suppose that f can also be written as $f = \sum_{i=1}^r a'_i \chi_{A'_i}$. We then we have $\sum_{i=1}^n a_i \nu(\chi_{A_i}) = \sum_{i=1}^r a'_i \nu(\chi_{A'_i})$ because we can choose a disjoint partition $(B_1, B_2, \ldots, B_n)$ of S such that, for every $j \in \{1, 2, \ldots, n\}$, every $i \in \{1, 2, \ldots, m\}$ and every $k \in \{1, 2, \ldots, r\}$, $B_j \subseteq A_i$ or $B_j \cap A_i = \emptyset$ and $B_j \subseteq A'_k$ or $B_j \cap A'_k = \emptyset$.

Thus we can extend ν to $\mathfrak{S}(S)$ by putting $\nu(f) = \sum_{i=1}^m a_i \nu(\chi_{A_i})$.

It is obvious that ν is linear on $\mathfrak{S}(S)$ and that $|\nu(f)| \le \|f\|_\infty$ for every $f \in \mathfrak{S}(S)$. So ν can be extended to a continuous linear functional defined on $l_\infty(S)$ because $\mathfrak{S}(S)$ is uniformly dense in $l_\infty(S)$. It is clear that ν is a mean. Given $s \in S$ and $A \subseteq S$, we have $\chi_A \circ \lambda_s = \chi_{s^{-1}A}$ and so $\nu(s \cdot f) = \nu(f)$ for every $f \in \mathfrak{S}(S)$. Consequently ν is left invariant on $l_\infty(S)$. □

3.4 Theorem. *Let S and T be semigroups satisfying SFC and assume that $LIM(S)$ is the weak* closed convex hull of $LIM_0(S)$. For every $A \subseteq S$ and $B \subseteq T$, $d(A \times B) = d(A)d(B)$.*

Proof. By Lemma 3.1, $S \times T$ satisfies SFC and $d(A \times B) \ge d(A)d(B)$.

By Theorem 2.14, $d(A \times B) \le \sup\{\rho(\chi_{A\times B}) : \rho \in LIM(S \times T)\}$. So the reverse inequality will follow from the claim that $\rho(\chi_{A\times B}) \le d(A)d(B)$ for every $\rho \in LIM(S \times T)$. To prove this, we may clearly suppose that $\rho(\chi_{A\times B}) > 0$.

We define functions μ and ν on $C(S)$ and $C(T)$ respectively by putting $\mu(\chi_X) = \rho(X \times T)$ for every $X \subseteq S$ and $\nu(Y) = \frac{\rho(A\times Y)}{\rho(A\times T)}$ for every $Y \subseteq T$. We clearly have $\mu(\chi_S) = \nu(\chi_T) = 1$, $\mu(\chi_{X_1\cup X_2}) = \mu(\chi_{X_1}) + \mu(\chi_{X_2})$ whenever X_1 and X_2 are disjoint subsets of S and $\nu(\chi_{Y_1\cup Y_2}) = \nu(\chi_{Y_1}) + \nu(\chi_{Y_2})$ whenever Y_1 and Y_2 are disjoint subsets of T. Furthermore, we claim that μ and ν are left invariant. To see this, observe that for every $a \in S$ and every $b \in T$, we have $\mu(\chi_X) = \rho(\chi_{(a,b)^{-1}(X\times T)}) = \rho(\chi_{a^{-1}X\times T})$ for every $X \subseteq S$ and $\nu(\chi_Y) = \frac{\rho(\chi_{(a,b)^{-1}(A\times Y)})}{\rho(\chi_{A\times T})}$ for every $Y \subseteq T$. So $\mu(\chi_{a^{-1}X}) = \mu(\chi_X)$ and, for every $t \in T$, $\nu(\chi_{t^{-1}Y}) = \frac{\rho(\chi_{(a,b)^{-1}(A\times t^{-1}Y)})}{\rho(\chi_{A\times T})} = \frac{\rho(\chi_{(a,tb)^{-1}(A\times Y)})}{\rho(\chi_{A\times T})} = \nu(\chi_Y)$.

It now follows from Theorem 2.14 and Lemma 3.3 that $\mu(\chi_A) \le d(A)$ and $\nu(\chi_B) \le d(B)$. Since $\mu(\chi_A)\nu(\chi_B) = \rho(\chi_{A\times B})$, we have that $\rho(\chi_{A\times B}) \le d(A)d(B)$. □

4. $\mathbf{LIM_0}(\mathbb{N})$ is convex

We let $\mathbb{N}$ be the semigroup of positive integers under addition. It is easy to see that for any left amenable left cancellative semigroup S, $LIM_0(S)$ is weak* closed. Therefore, by Theorem 2.12, to see that $LIM(\mathbb{N}) = LIM_0(\mathbb{N})$, it suffices to show that $LIM_0(\mathbb{N})$ is convex.

We write ${}^{\mathbb{N}}\{0,1\}$ for the set of sequences in $\{0,1\}$.

4.1 Lemma. *Let $\eta \in LIM_0(\mathbb{N})$, let $l, m \in \mathbb{N}$, let $K \in \mathcal{P}_f\left({}^{\mathbb{N}}\{0,1\}\right)$, and let $\epsilon > 0$. There exists $F \in \mathcal{P}_f(\mathbb{N})$ such that*

(a) $\min F > l$,

(b) F is the union of blocks each of length an integer multiple of m, and

(c) for all $g \in K$, $|\mu_F(g) - \eta(g)| < \epsilon$.

Proof. Pick a left Følner net $\langle H_\alpha \rangle_{\alpha \in D}$ in $\mathcal{P}_f(\mathbb{N})$ such that $\langle \mu_{H_\alpha} \rangle_{\alpha \in D}$ converges to η in the weak* topology on $l_\infty B(\mathbb{N})^*$. Pick $w \in \mathbb{N}$ such that $w > \frac{6}{\epsilon}$ and let

$$L = \{x \in \mathbb{N} : (\exists a > l)(x \in \{a, a+1, a+2, \ldots, a+wm\} \subseteq H)\}\,.$$

Note that for any $\alpha \in D$, $(1 + H_\alpha) \setminus H_\alpha \neq \emptyset$ and so $\lim_{\alpha \in D} |H_\alpha| = \infty$. Pick $\gamma \in D$ such that for all $\alpha \in D$, if $\alpha \geq \gamma$, then $|H_\alpha| > \frac{12l}{\epsilon}$ and for all $i \in \{1, 2, \ldots, wm\}$, $|(i + H_\alpha) \setminus H_\alpha| < \frac{\epsilon}{12wm} \cdot |H_\alpha|$ and note that $|H_\alpha \setminus (-i + H_\alpha)| = |(i + H_\alpha) \setminus H_\alpha|$. If $\alpha \geq \gamma$, then $H_\alpha \setminus L \subseteq \{1, 2, \ldots, l\} \cup \bigcup_{i=1}^{wm} (H_\alpha \setminus (-i + H_\alpha)$ so $|H_\alpha \setminus L| < l + \frac{\epsilon}{12} \cdot |H_\alpha| < \frac{\epsilon}{6} \cdot |H_\alpha|$.

Pick $\delta \in D$ such that whenever $\alpha \in D$ and $\alpha \geq \delta$, one has that for each $g \in K$, $|\mu_{H_\alpha}(g) - \eta(g)| < \frac{\epsilon}{3}$. Pick $\alpha \in D$ such that $\alpha \geq \gamma$ and $\alpha \geq \delta$.

Pick $r \in \mathbb{N}$, $\langle a_j \rangle_{j=1}^r$, and $\langle s_j \rangle_{j=1}^r$ such that $L = \bigcup_{j=1}^r \{a_j + 1, a_j + 2, \ldots, a_j + s_j\}$ where for $j \in \{1, 2, \ldots, r\}$, $s_j \geq wm$, and if $j < r$, $a_j + s_j \leq a_{j+1}$. For $j \in \{1, 2, \ldots, r\}$, let $t_j = \lfloor \frac{s_j}{m} \rfloor$. Let $F = \bigcup_{j=1}^r \{a_j + 1, a_j + 2, \ldots, a_j + mt_j\}$. Then $|L| \geq rwm$ and so $|L \setminus F| < rm \leq \frac{1}{w} \cdot |L| < \frac{\epsilon}{6} \cdot |H_\alpha|$. Therefore, $|H_\alpha \setminus F| < \frac{\epsilon}{3} \cdot |H_\alpha|$. Let $g \in K$. We have that $|\mu_{H_\alpha}(g) - \eta(g)| < \frac{\epsilon}{3}$ so it suffices to show that $|\mu_F(g) - \mu_{H_\alpha}(g)| < \frac{2\epsilon}{3}$. We have that

$$\begin{aligned}
|\mu_{H_\alpha}(g) - \mu_F(g)| &= \left| \frac{\sum_{t \in H_\alpha} g(t) - \sum_{t \in F} g(t)}{|H_\alpha|} + \frac{\sum_{t \in F} g(t)}{|H_\alpha|} - \frac{\sum_{t \in F} g(t)}{|F|} \right| \\
&\leq \frac{\sum_{t \in H_\alpha \setminus F} |g(t)|}{|H_\alpha|} + \frac{\sum_{t \in F} |g(t)|}{|F|} \cdot \frac{|H_\alpha| - |F|}{|H_\alpha|} \\
&\leq \frac{|H_\alpha \setminus F|}{|H_\alpha|} + \frac{|H_\alpha \setminus F|}{|H_\alpha|} \\
&< \frac{\epsilon}{3} + \frac{\epsilon}{3} = \frac{2\epsilon}{3}\,.
\end{aligned}$$

□

4.2 Lemma. *Let* $\eta \in LIM_0(\mathbb{N})$, *let* $\epsilon > 0$, *let* $K \in \mathcal{P}_f\left({}^{\mathbb{N}}\{0,1\}\right)$, *and let* $n \in \mathbb{N}$. *There exists* $B_0 \in \mathbb{N}$ *such that for all* $l \in \mathbb{N}$ *and all* $B \in \mathbb{N}$ *with* $B \geq B_0$, *there exists* $G \in \mathcal{P}_f(\mathbb{N})$ *such that*

(a) $\min G > l$,

(b) G *is the union of blocks each of length at least* n,

(c) $|G| = Bn$, *and*

(d) for all $g \in K$, $|\mu_G(g) - \eta(g)| < \epsilon$.

Proof. Let $X = {}^K\{0, \frac{1}{n}, \frac{2}{n}, \ldots, 1\}$ and pick $B_0 \in \mathbb{N}$ such that $B_0 > \dfrac{2 \cdot |X|}{\epsilon}$. Let $l \in \mathbb{N}$ and $B \geq B_0$ be given. Let $m = 2Bn$ and pick by Lemma 4.1, $F \in \mathcal{P}_f(\mathbb{N})$ such that

$\min F > l$, F is the union of blocks each of length an integer multiple of m, and for all $g \in K$, $|\mu_F(g) - \mu(g)| < \frac{\epsilon}{2}$. Let $v = \frac{|F|}{n}$ (so $v \geq 2B$) and choose an increasing sequence $\langle x_t \rangle_{t=1}^{v}$ such that $F = \bigcup_{t=1}^{v} \{x_t + 1, x_t + 2, \ldots, x_t + n\}$.

For $g \in K$ and $t \in \{1, 2, \ldots, v\}$, let $f_t(g) = \frac{1}{n} \cdot \sum_{j=1}^{n} g(x_t + j) \in \{0, \frac{1}{n}, \frac{2}{n}, \ldots, 1\}$. For $\psi \in X$, let $H_\psi = \{t \in \{1, 2, \ldots, v\} : f_t = \psi\}$. Note that $\sum_{\psi \in X} |H_\psi| = v$.

We now claim that we can choose $\langle a_\psi \rangle_{\psi \in X}$ in ω such that

(1) for all $\psi \in X$, $a_\psi \leq |H_\psi|$,

(2) for all $\psi \in X$, $\frac{B \cdot |H_\psi|}{v} - 1 < a_\psi < \frac{B \cdot |H_\psi|}{v} + 1$, and

(3) $\sum_{\psi \in X} a_\psi = B$.

Indeed, we shall always have either $a_\psi = \left\lceil \frac{B \cdot |H_\psi|}{v} \right\rceil$ or $a_\psi = \left\lfloor \frac{B \cdot |H_\psi|}{v} \right\rfloor$, so (2) will hold. If one always chooses $a_\psi = \left\lceil \frac{B \cdot |H_\psi|}{v} \right\rceil$, then $\sum_{\psi \in X} a_\psi \geq B \cdot \sum_{\psi \in X} \frac{|H_\psi|}{v} = B$ and if one always chooses $a_\psi = \left\lfloor \frac{B \cdot |H_\psi|}{v} \right\rfloor$, then $\sum_{\psi \in X} a_\psi \leq B \cdot \sum_{\psi \in X} \frac{|H_\psi|}{v} = B$ so one may make such choices so that $\sum_{\psi \in X} a_\psi = B$. To see that (1) holds, let $\psi \in X$. If $H_\psi = \emptyset$, then $a_\psi = 0$. Otherwise, let $k = |H_\psi|$. Then $\frac{B \cdot |H_\psi|}{v} \leq \frac{k}{2} < k$ so $a_\psi \leq \left\lceil \frac{B \cdot |H_\psi|}{v} \right\rceil \leq k$.

For each $\psi \in X$, pick $G_\psi \in [H_\psi]^{a_\psi}$ and let

$$G = \bigcup_{\psi \in X} \bigcup_{t \in G_\psi} \{x_t + 1, x_t + 2, \ldots, x_t + n\}.$$

Then $|G| = \sum_{\psi \in X} n a_\psi = Bn$.

To complete the proof, let $g \in K$. We shall show that $|\mu_G(g) - \mu_F(g)| < \frac{\epsilon}{2}$ so that $|\mu_G(g) - \mu(g)| < \epsilon$ as required. Now $F = \bigcup_{\psi \in X} \bigcup_{t \in H_\psi} \{x_t + 1, x_t + 2, \ldots, x_t + n\}$ so

$$\mu_F(g) = \frac{\sum_{\psi \in X} \sum_{t \in H_\psi} n f_t(g)}{vn} = \frac{\sum_{\psi \in X} \sum_{t \in H_\psi} f_t(g)}{v} = \frac{\sum_{\psi \in X} |H_\psi| \psi(g)}{v}.$$

Also $|G| = Bn$ and

$$\mu_G(g) = \frac{\sum_{\psi \in X} \sum_{t \in G_\psi} n f_t(g)}{Bn} = \frac{\sum_{\psi \in X} \sum_{t \in G_\psi} f_t(g)}{B} = \frac{\sum_{\psi \in X} a_\psi \psi(g)}{B}.$$

Now $\frac{B \cdot |H_\psi|}{v} - 1 < a_\psi < \frac{B \cdot |H_\psi|}{v} + 1$. So $\left| \frac{a_\psi}{B} - \frac{|H_\psi|}{v} \right| < \frac{1}{B}$. Hence

$$|\mu_F(g) - \mu_G(g)| \leq \sum_{\psi \in X} \psi(g) \cdot \left| \frac{a_\psi}{B} - \frac{|H_\psi|}{v} \right| < \sum_{\psi \in X} \psi(g) \cdot \frac{1}{B} \leq \frac{|X|}{B} < \frac{\epsilon}{2}. \qquad \square$$

Recall that $MN(\mathbb{N})$ is the space of all means on $\mathbb{N}$. The next lemma says that the topology on $MN(\mathbb{N})$ is determined by sequences in $\{0, 1\}$.

4.3 Lemma. *Let $\eta \in MN(\mathbb{N})$ and let U be a weak* neighborhood of η in $MN(\mathbb{N})$. There exist $K \in \mathcal{P}_f\left({}^{\mathbb{N}}\{0, 1\}\right)$ and $\epsilon > 0$ such that $\bigcap_{g \in K} \{\nu \in MN(\mathbb{N}) : |\eta(g) - \nu(g)| < \epsilon\} \subseteq U$.*

Proof. It suffices to assume that U is a subbasic neighborhood of η in $MN(\mathbb{N})$ so pick $f \in l_\infty(\mathbb{N})$ and $\delta > 0$ such that $U = \{\nu \in MN(\mathbb{N}) : |\eta(f) - \nu(f)| < \delta\}$. Let $b > ||f||$ and choose $n \in \mathbb{N}$ such that $n > \frac{3b}{\delta}$. Let $\epsilon = \frac{\delta}{6nb}$. For $i \in \{0, 1, \ldots, 2n\}$ let $c_i = -b + \frac{bi}{n}$ and for $i \in \{0, 1, \ldots, 2n-1\}$ let g_i be the characteristic function of $f^{-1}\big[[c_i, c_{i+1})\big]$. Then $||f - \sum_{i=0}^{2n-1} c_i g_i|| < \frac{b}{n} < \frac{\delta}{3}$. Consequently, given any $\nu \in MN(\mathbb{N})$, since $||\nu|| = 1$, $|\nu(f - \sum_{i=0}^{2n-1} c_i g_i)| < \frac{\delta}{3}$.

Assume now that $\nu \in MN(\mathbb{N})$ and for each $i \in \{0, 1, \ldots, 2n-1\}$, $|\eta(g_i) - \nu(g_i)| < \epsilon$. Then $|\nu(f) - \nu(\sum_{i=0}^{2n-1} c_i g_i)| < \frac{\delta}{3}$ and $|\eta(f) - \eta(\sum_{i=0}^{2n-1} c_i g_i)| < \frac{\delta}{3}$. Finally

$$\begin{aligned} |\nu(\textstyle\sum_{i=0}^{2n-1} c_i g_i) - \eta(\sum_{i=0}^{2n-1} c_i g_i)| &= \big|\textstyle\sum_{i=0}^{2n-1} c_i\big(\nu(g_i) - \eta(g_i)\big)\big| \\ &\leq \textstyle\sum_{i=0}^{2n-1} |c_i| \cdot |\nu(g_i) - \eta(g_i)| < 2nb\epsilon = \frac{\delta}{3}\,. \end{aligned}$$

Thus $|\nu(f) - \eta(f)| < \delta$. □

4.4 Lemma. *Let $\eta \in MN(\mathbb{N})$. If for each $K \in \mathcal{P}_f\left({}^{\mathbb{N}}\{0,1\}\right)$, each $n \in \mathbb{N}$, and each $\epsilon > 0$, there exists $F \in \mathcal{P}_f(\mathbb{N})$ such that F is the union of blocks, each of length at least n, and for each $g \in K$, $|\mu_F(g) - \eta(g)| < \epsilon$, then $\eta \in LIM_0(\mathbb{N})$.*

Proof. Let $D = \mathcal{P}_f\left({}^{\mathbb{N}}\{0,1\}\right) \times \mathbb{R}^+ \times \mathbb{N}$ and order D by $(K, \epsilon, n) \leq (K', \epsilon', n')$ if and only if $K \subseteq K'$, $\epsilon \geq \epsilon'$, and $n \leq n'$. For each $(K, \epsilon, n) \in D$, pick $F(K, \epsilon, n) \in \mathcal{P}_f(\mathbb{N})$ such that $F(K, \epsilon, n)$ is the union of blocks, each of length at least n, and for each $g \in K$, $|\mu_{F(K,\epsilon,n)}(g) - \eta(g)| < \epsilon$. Then by Lemma 4.3, $\langle \mu_{F(K,\epsilon,n)}\rangle_{(K,\epsilon,n)\in D}$ converges to η in the weak* topology on $MN(\mathbb{N})$ so it suffices to show that $\langle F(K, \epsilon, n)\rangle_{(K,\epsilon,n)\in D}$ is a left Følner net in $\mathcal{P}_f(\mathbb{N})$. To see this note that, given any $k \in \mathbb{N}$ and any $(K, \epsilon, n) \in D$, if $F = F(K, \epsilon, n)$ and $n > k$, then $|F \setminus (k + F)| \leq \frac{k}{n} \cdot |F|$. □

4.5 Theorem. *$LIM_0(\mathbb{N})$ is convex.*

Proof. Since $LIM_0(\mathbb{N})$ is weak* closed and the dyadic rationals are dense in $[0, 1]$, it suffices to let $\mu, \nu \in LIM_0(\mathbb{N})$ and show that $\eta = \frac{1}{2}\mu + \frac{1}{2}\nu \in LIM_0(\mathbb{N})$. Let $\epsilon > 0$, let $K \in \mathcal{P}_f\left({}^{\mathbb{N}}\{0,1\}\right)$, and let $n \in \mathbb{N}$. By Lemma 4.4 it suffices to produce $H \in \mathcal{P}_f(\mathbb{N})$ such that H is the union of blocks each of length at least n and for each $g \in K$, $|\mu_H(g) - \eta(g)| < \epsilon$.

Pick B_0 in $\mathbb{N}$ as guaranteed by Lemma 4.2 for μ, ϵ, K and n and pick B_1 in $\mathbb{N}$ as guaranteed by Lemma 4.2 for ν, ϵ, K and n. Let $B = \max\{B_0, B_1\}$. Pick $F \in \mathcal{P}_f(\mathbb{N})$ such that F is the union of blocks each of length at least n, $|F| = Bn$, and for all $g \in K$, $|\mu_F(g) - \mu(g)| < \epsilon$. Let $l = \max F$ and pick $G \in \mathcal{P}_f(\mathbb{N})$ such that $\min G > l$, G is the union of blocks each of length at least n, $|G| = Bn$, and for all $g \in K$,

$|\mu_G(g) - \nu(g)| < \epsilon$. Let $H = F \cup G$. Then $\mu_H = \frac{1}{2}\mu_F + \frac{1}{2}\mu_G$ and so for any $g \in K$, $|\mu_H(g) - \eta(g)| \leq \frac{1}{2}|\mu_F(g) - \mu(g)| + \frac{1}{2}|\mu_G(g) - \nu(g)| < \epsilon$. □

4.6 Corollary. $LIM(\mathbb{N}) = LIM_0(\mathbb{N})$.

Proof. By Theorem 2.12, $LIM(\mathbb{N})$ is the weak* closed convex hull of $LIM_0(\mathbb{N})$. Also $LIM_0(\mathbb{N})$ is weak* closed and by Theorem 4.5 $LIM_0(\mathbb{N})$ is convex. □

5. Further properties of left invariant means

We shall explore some of the connections between invariant means and the algebra of the Stone-Čech compactification βS of a discrete left amenable semigroup S. Since $l_\infty(S)$ can be identified with $C(\beta S)$, the Banach space of continuous real-valued functions defined on βS, it follows from the Riesz Representation Theorem that a mean on S corresponds to a regular Borel probability measure on βS. More precisely, for every $\mu \in LIM(S)$, there exists a unique regular Borel probability measure $\widetilde{\mu}$ on βS for which $\mu(f) = \int \widetilde{f} d\widetilde{\mu}$ for every $f \in l_\infty(S)$, where $\widetilde{f} : \beta S \to \mathbb{R}$ denotes the continuous extension of f. In particular, $\widetilde{\mu}(\overline{A}) = \mu(\chi_A)$ for every $A \subseteq S$. It follows easily from the regularity of μ that $\widetilde{\mu}(s^{-1}B) = \widetilde{\mu}(B)$ for every $s \in S$ and every Borel subset B of βS.

In the remainder of this paper, we shall regard $LIM(S)$ as the space of left invariant regular Borel probability measures defined on βS. We shall use a symbol such as μ to denote an element of $LIM(S)$, in preference to the more cumbersome $\widetilde{\mu}$ used in the preceding paragraph.

We regard βS as the space of ultrafilters on S, with the points of S identified with the principle ultrafilters. The topology of βS is defined by choosing the sets of the form $\overline{A} = \{p \in \beta S : A \in p\}$, where A denotes a subset of S, as a base for the open sets. $\overline{A}$ is then a clopen subset of βS, with $\overline{A} = cl_{\beta S}(A)$, and all the clopen subsets of βS are of this form.

We shall need to use the well-known fact that the semigroup operation of S can be extended to βS and that βS is then a right topological semigroup with S contained in its topological center. This means that, for every $x \in \beta S$, the map $\rho_x : \beta S \to \beta S$ defined by $\rho_x(y) = yx$ is continuous and, for every $s \in S$, the map $\lambda_s : \beta S \to \beta S$ defined by $\lambda_s(y) = sy$ is continuous. Given $p, q \in \beta S$ and $A \subseteq S$, one has that $A \in pq$ if and only if $\{s \in S : s^{-1}A \in q\} \in p$.

The fact that βS is a compact right topological semigroup has important algebraic consequences. Among these is the fact that βS contains a smallest ideal $K(\beta S)$ which is the union of all the minimal left ideals of βS, as well as the union of all the minimal right ideals of βS. Any two minimal left ideals of βS are isomorphic, as are any two minimal right ideals. The intersection of any minimal left ideal and any minimal right ideal of βS is a group, and every minimal left ideal of βS is closed. See [10] for derivations of these facts and further information.

Most of the statements in the following theorem are well-known. We give proofs rather than references because the proofs are so simple.

5.1 Theorem. *Let S be a discrete left amenable semigroup and let $\mu \in LIM(S)$. Then the following statements hold:*

(a) For every Borel subset B of βS, every $s \in S$ and every $\mu \in LIM(S)$, if sB is a Borel subset of βS, $\mu(sB) \geq \mu(B)$. In the case in which S is left cancellative, $\mu(sB) = \mu(B)$.

(b) The support of μ is a left ideal of βS;

(c) Every minimal left ideal of βS is the support of a measure in $LIM(S)$;

(d) If L is a minimal left ideal of βS, then $sL = L$ for every $s \in S$;

(e) If R is a right ideal of S, $\mu(\overline{R}) = 1$;

(f) $\mu(\bigcap\{\overline{R} : R$ is a right ideal of $S\}) = 1$;

(g) $K(\beta S) \subseteq \bigcap\{\overline{R} : R$ is a right ideal of $S\}$.

Proof. (a) In any semigroup S, $B \subseteq s^{-1}sB$. In the case in which S is left cancellative, $s^{-1}sB = B$.

(b) Let C denote the support of μ. For every $s \in S$, $\mu(s^{-1}C) = 1$ and so $C \subseteq s^{-1}C$. Hence, for every $x \in C$, $Sx \subseteq C$ and therefore $(\beta S)x = \overline{Sx} \subseteq C$.

(c) Let L be a minimal left ideal in βS and let $p \in L$. We can define a a left invariant Borel measure ν on βS by putting $\nu(B) = \mu(\rho_p^{-1}[B])$ for every Borel subset B of βS. To see that ν is regular, let B be a Borel subset of βS and let $\varepsilon > 0$. We can choose a compact subset C of $\rho_p^{-1}[B]$ for which $\mu(\rho_p^{-1}[B] \setminus C) < \varepsilon$. Since $\rho_p^{-1}\big[B \setminus \rho_p[C]\big] \subseteq \rho_p^{-1}[B] \setminus C$, it follows that $\nu(B \setminus \rho_p[C]) < \varepsilon$. So $\nu \in LIM(S)$. Now ${\rho_p}^{-1}[\beta S \setminus L] = \emptyset$, and thus the support of ν is contained in L. By (b), the support of ν is a left ideal and is therefore equal to L. (d) By (c), L is the support of a measure $\mu \in \beta S$. By (a), $\mu(sL) = 1$. So $L \subseteq sL$ and hence $sL = L$.

(e) Choose $s \in R$. Since $\overline{R}$ is a right ideal of βS by [10, Corollary 4.18], $\beta S \subseteq s^{-1}\overline{R}$.

(f) Since μ is regular, for any downward directed family $\mathcal{C}$ of compact subsets of βS, $\mu(\bigcap \mathcal{C}) = \inf(\{\mu(C) : C \in \mathcal{C}\})$. So (f) follows from (e).

(g) This follows from (d), the fact that $K(\beta S)$ is the union of the minimal left ideals of βS, and the fact already mentioned that if R is a right ideal of S, then $\overline{R}$ is a right ideal of βS. $\square$

A semigroup S is said to be *weakly left cancellative* provided that for all $u, v \in S$, $\{x \in S : ux = v\}$ is finite.

5.2 Corollary. *Let S be an infinite discrete left amenable semigroup. If S is weakly left cancellative, then $|LIM(S)| = 2^{2^{|S|}}$.*

Proof. This follows from Theorem 5.1(c) and the fact that βS has $2^{2^{|S|}}$ left ideals ([10, Theorem 6.42]). $\square$

We shall now derive some properties of the extreme points of $LIM(S)$. An interesting characterisation of these points is given in [4].

5.3 Lemma. *Let S be a discrete left amenable semigroup and let $\mu, \nu \in LIM(S)$. If $\nu = \nu_1 + \nu_2$ where ν_1 is absolutely continuous with respect to μ and ν_2 is singular with respect to μ, then ν_1 and ν_2 are left invariant.*

Proof. Let $s \in S$. We shall show that $s \cdot \nu_1$ is absolutely continuous with respect to μ and that $s \cdot \nu_2$ is singular with respect to μ. It will follow from the uniqueness of ν_1 and ν_2 that $\nu_1 = s \cdot \nu_1$ and that $\nu_2 = s \cdot \nu_2$.

If B is a Borel subset of βS for which $\mu(B) = 0$, then $\mu(s^{-1}B) = 0$. So $\nu_1(s^{-1}B) = 0$. Thus $s \cdot \nu_1$ is absolutely continuous with respect to μ.

Now we can write βS as the union of two disjoint Borel subsets, B_1 and B_2, for which $\mu(B_1) = \nu_2(B_2) = 0$. Then $\nu_1(s^{-1}B_1) = \nu_1(B) = 0$. So $\nu_2(s^{-1}B_1) = \nu(s^{-1}B_1) = \nu(B_1) = \nu_2(B_1)$. Since $0 = \nu_2(B_2) = 1 - \nu_2(B_1) = 1 - \nu_2(s^{-1}B_1) = \nu_2(s^{-1}B_2)$, it follows that $s \cdot \nu_2$ is singular with respect to μ. $\square$

5.4 Lemma. *Let S be a discrete left amenable semigroup and let ν be an extreme point of $LIM(S)$. For every $\mu \in LIM(S)$, either ν is singular with respect to μ or absolutely continuous with respect to μ.*

Proof. We write $\nu = \nu_1 + \nu_2$, where ν_1 is absolutely continuous with respect to μ and ν_2 is singular with respect to μ. If ν_1 and ν_2 are both non-zero, we obtain a contradicition by writing $\nu = \|\nu_1\| \frac{\nu_1}{\|\nu_1\|} + \|\nu_2\| \frac{\nu_2}{\|\nu_2\|}$ and noting that, by Lemma 5.3, $\frac{\nu_1}{\|\nu_1\|}$ and $\frac{\nu_2}{\|\nu_2\|}$ are in $LIM(S)$. $\square$

5.5 Definition. Let S be a discrete left amenable semigroup and let μ be a non-negative Borel measure on βS. We shall say that a Borel subset B of βS is μ *left invariant* if $\mu(s^{-1}B \triangle B) = 0$ for every $s \in S$.

5.6 Lemma. *Let S be a discrete left amenable semigroup and let $\mu \in LIM(S)$. Suppose that S is left cancellative. Let ν be an extreme point of $LIM(S)$. If ν is absolutely continuous with respect to μ, then there is a μ-left invariant Borel subset E of βS such that $\nu(B) = \frac{\mu(B\cap E)}{\mu(E)}$ for every Borel subset B of βS.*

Proof. Let $f : \beta S \to \mathbb{R}$ be a Borel measurable function with the property that $\nu(B) = \int_B f d\mu$ for every Borel subset B of βS. We claim that, for every given $s \in S$, $f(st) = f(t)$ for every t in the complement of some μ-null subset N_s of βS.

To see this, note that, for every $x \in s\beta S$, the mapping $x \mapsto s^{-1}x$ is well defined on $s\beta S$ (by [10, Lemma 8.1]) and is easily seen to be continuous. We note that $\mu(s\beta S) = 1$ by Theorem 5.1(e). Let B be a Borel subset of $s\beta S$. We have $\int f(s^{-1}t)\chi_B(t)d\mu(t) = \int f(t)\chi_B(st)d\mu(t) = \int f(t)\chi_{s^{-1}B}(t)d\mu(t) = \nu(s^{-1}B) = \nu(B) = \int f(t)\chi_B(t)d\mu(t)$. It follows that there exists a μ-null subset N_s of βS such that $f(s^{-1}t) = f(t)$ for every $t \in \beta S \setminus N_s$. We then have $f(t) = f(s^{-1}st) = f(st)$ for every $t \in s^{-1}N_s$.

Let U_1 and U_2 be disjoint subsets of $\mathbb{R}$ which partition $\mathbb{R}$ and let $B_1 = f^{-1}[U_1]$ and $B_2 = f^{-1}[U_2]$. Then B_1 and B_2 are disjoint Borel subsets of βS which partition βS. They are μ-left invariant because, for every $s \in S$, $s^{-1}B_1 \triangle B_1$ and $s^{-1}B_2 \triangle B_2$ are contained in N_s. For $i \in \{1, 2\}$, define a measure ν_i by putting $\nu_i(B) = \int_{B_i \cap B} f d\mu$. We claim that ν_i is left invariant. To see this, observe that, for every Borel subset B of βS and every $s \in S$,

$$\begin{aligned}\nu_i(B) &= \int f\chi_B\chi_{B_1}d\mu \\ &= \int f(st)\chi_B(st)\chi_{B_i}(st)d\mu(t) \\ &= \int f(t)\chi_{s^{-1}B}(t)\chi_{B_i}(t)d\mu(t) \\ &= \nu_i(s^{-1}B).\end{aligned}$$

We have $\nu = \nu_1 + \nu_2$. If ν_1 and ν_2 are both non-zero, $\nu = \|\nu_1\|\frac{\nu_1}{\|\nu_1\|} + \|\nu_2\|\frac{\nu_2}{\|\nu_2\|}$, contradicting the assumption that ν is an extreme point of $LIM(S)$. So $\nu_1 = 0$ or $\nu_2 = 0$. It follows that $\mu(B_1) = 0$ or $\mu(B_2) = 0$. Thus there exists a unique number $c \in \mathbb{R}$ such that $\mu(f^{-1}[U]) > 0$ for every open neighbourhood U of c, So $\mu(f^{-1}[\mathbb{R} \setminus \{c\}]) = 0$ and $f = c$ except on a μ-null set.

Our claim is now established with $E = f^{-1}[\{c\}]$. □

5.7 Theorem. *Let S be a discrete left amenable semigroup. If S is left cancellative, any two distinct extreme points of $LIM(S)$ are mutually singular.*

Proof. Let μ and ν be extreme points of $LIM(S)$. If they are not mutually singular, then each is absolutely continuous with respect to the other by Lemma 5.4. By Lemma 5.6, there exist Borel subsets B and C of βS such that $\mu(E) = \frac{\nu(E\cap B)}{\nu(B)}$ and $\nu(E) = \frac{\mu(E\cap C)}{\mu(C)}$ for every Borel subset E of βS. This implies that $\mu(\beta S \setminus B) = 0 = \nu(\beta S \setminus B)$. So $\mu(B) = \nu(B) = 1$. Similarly, $\mu(C) = \nu(C) = 1$. So $\mu(E) = \nu(E)$ if E is a Borel subset of $B \cap C$ or if E is a Borel subset of $\beta S \setminus (B \cap C)$. It follows that $\mu = \nu$. $\square$

We conclude this section with some results about the algebra of βS.

5.8 Definition. Let S be a semigroup which satisfies SFC.

$$\Delta^*(S) = \{p \in \beta S : (\forall A \in p)(d(A) > 0)\}.$$

Recall that by Theorems 2.12 and 2.14 the hypotheses of the following theorem are satisfied by any left cancellative left amenable semigroup.

5.9 Theorem. *Let S be a semigroup which satisfies SFC. If for every $A \subseteq S$, $d(A) = \sup\{\mu(\overline{A}) : \mu \in LIM(S)\}$, then $\Delta^*(S)$ is a closed two sided ideal of βS.*

Proof. It is immediate that $\Delta^*(S)$ is closed. To see that it is a left ideal, let $p \in \Delta^*(S)$. We show that that $Sp \subseteq \Delta^*(S)$ and hence that $c\ell_{\beta S} Sp = \beta Sp \subseteq \Delta^*(S)$. To this end, let $s \in S$ and let $A \in sp$. Then $s^{-1}A \in p$ and so $d(s^{-1}A) > 0$. Therefore there is some $\mu \in LIM(S)$ such that $\mu(\overline{s^{-1}A}) > 0$. Since $\mu(\overline{A}) = \mu(\overline{s^{-1}A})$, one has $d(A) > 0$ as required.

Now to see that $\Delta^*(S)$ is a right ideal, let $p \in \Delta^*(S)$, let $q \in \beta S$, and let $A \in pq$. Let $B = \{s \in S : s^{-1}A \in q\}$. Then $B \in p$ so $d(B) > 0$ and therefore there is some $\mu \in LIM(S)$ such that $\mu(\overline{B}) > 0$. Note that $\overline{B} = \rho_q^{-1}[\overline{A}]$. Define $\nu \in LIM(S)$ by $\nu(X) = \mu(\rho_q^{-1}[X])$ for every Borel subset X of βS. Then $\nu(\overline{A}) > 0$ so $d(A) > 0$. $\square$

Recall that a set $A \subseteq S$ is *thick* provided that for each $F \in \mathcal{P}_f(S)$ there exists $t \in S$ such that $Ft \subseteq A$. The final result of this section is about the concept of weakly thick sets, introduced in [12], where it was used in determining which products of central subsets of semigroups are central.

5.10 Definition. Let A be a subset of a semigroup S. A is said to be *weakly thick* if there exists $s \in S$ such that $s^{-1}A$ is thick.

We remark that being weakly thick is trivially equivalent to being thick in a commutative semigroup, but that these two concepts are not equivalent in general. For example, if S is the free semigroup on two generators a and b, aS is weakly thick but not thick.

5.11 Theorem. *Let S be a discrete left amenable semigroup. If $A \subseteq S$ is weakly thick, then A is thick.*

Proof. Suppose that A is weakly thick. Then $s^{-1}A$ is thick for some $s \in S$. By [2, Theorem 2.9(c)], a subset T of S is thick if and only if $\overline{T}$ contains a left ideal. So there is a minimal left ideal L of βS for which $L \subseteq \overline{s^{-1}A}$. Then $sL \subseteq \overline{A}$. By Theorem 5.1(d), $sL = L$ and so A is thick. □

6. Semigroups S for which βS has finite minimal left ideals

In the context of the current paper, we were interested in the possibility that one might have some $\mu \in LIM(S)$ and some $x \in \beta S$ such that $\mu(\{x\}) > 0$. It was shown by A. T.-M. Lau in [15] that this is equivalent to the statement that the minimal left ideals of βS are finite groups. In this section, we derive characterizations of semigroups with some $\mu \in LIM(S)$ and some $x \in \beta S$ such that $\mu(\{x\}) > 0$ and of those with some $\mu \in LIM(S)$ and some $x \in S$ such that $\mu(\{x\}) > 0$. We shall see that, if any such point x exists, $\{x \in \beta S : \mu(\{x\}) > 0$ for some $\mu \in LIM(S)\} = K(\beta S)$. We shall then turn our attention to showing that there is a rich class of semigroups, occurring naturally in mathematics, which have this property.

6.1 Lemma. *Let S be a semigroup, assume that some minimal left ideal L of βS is finite and that $R = K(\beta S)$ is a minimal right ideal of βS. Any idempotent in R is a left identity for R. Let p be an idempotent in L and let $E = \{s \in S : sp = p\}$. Then for every idempotent q in R, $E = \{s \in S : sq = q\}$, $E \in q$, and for each $s \in E$, $\{u \in S : su = u\} \in q$.*

Proof. Let q be an idempotent in R. Then q is a left identity for R. (We have $qR = R$ so given any $x \in R$, $x = qy$ for some $y \in R$ so $qx = qqy = qy = x$.) Thus if $s \in S$ we have that $sp = p$ if and only if $sq = q$ so $E = \{x \in S : sq = q\}$. To see that $E \in q$, pick $A \subseteq S$ such that $\overline{A} \cap L = \{p\}$. Then $A \in p = qp$ so $\{s \in S : s^{-1}A \in p\} \in q$. It suffices to show that $\{s \in S : s^{-1}A \in p\} \subseteq E$ so let $s \in S$ such that $s^{-1}A \in p$. Then $sp \in \overline{A} \cap L$ so $sp = p$. Now, given $s \in E$ one has that $\lambda_s(q) = q$ and so by [10, Theorem 3.35], $\{u \in S : su = u\} \in q$. □

The following theorem is purely topological-algebraic; it does not involve means, invariant or otherwise. Recall that a subset E of S is a central* set if and only if E is a member of every idempotent in $K(\beta S)$. When we say that objects A and B are *topologically isomorphic* we mean that there is a function from A to B which is both an isomorphism and a homeomorphism. A semigroup S is a *right zero* semigroup if and only if $ab = b$ for all a and b in S.

6.2 Theorem. *Let S be a semigroup and assume that some minimal left ideal of βS is finite (and thus every minimal left ideal of βS is finite). Statements (a) through (i) are equivalent and imply statement (j).*

(a) βS has a unique minimal right ideal $R = K(\beta S)$.

(b) $K(\beta S)$ is a compact minimal right ideal of βS.

(c) Each minimal left ideal of βS is a group.

(d) For each $s \in S$ and each minimal left ideal L of βS, $\lambda_{s|L}$ is injective.

(e) For each $s \in S$ and some minimal left ideal L of βS, $\lambda_{s|L}$ is injective.

(f) If L is a minimal left ideal of $K(\beta S)$ and $T = \{z \in K(\beta S) : zz = z\}$, then L is a finite group, T is a compact right zero semigroup, and $K(\beta S)$ is topologically isomorphic to $L \times T$.

(g) There exist a finite group G and a compact right zero semigroup T such that $K(\beta S)$ is topologically isomorphic to $G \times T$.

(h) There is a central subset E of S such that for each $s \in E$, $U_s = \{u \in S : su = u\}$ is non-empty and therefore, being a right ideal in S, satisfies $K(\beta S) \subseteq \overline{U_s}$.*

(i) There exist a central subset E of S and an idempotent $p \in K(\beta S)$ such that for each $s \in E$, $\{u \in S : su = u\} \in p$.*

(j) If L is a minimal left ideal of βS and p is an idempotent in L, then the function $s \mapsto sp$ is a homomorphism from S onto L. If $G \in \mathcal{P}_f(S)$ and if V is a subset of S for which $\overline{V} \cap K(\beta S) \neq \emptyset$, then there exists $v \in V$ such that $sv = tv$ whenever $s, t \in G$ and $sp = tp$.

Proof. That (a) and (c) are equivalent follows immediately from the fact that the intersection of any minimal left ideal and any minimal right ideal is a group (and distinct minimal right ideals are disjoint).

Trivially (b) implies (a). To see that (a) implies (b), let p be an idempotent in $R = K(\beta S)$. Let $L = \beta S p$ and let $E = \{s \in S : sp = p\}$. By Lemma 6.1 $E \in p$. Note that for any $s \in S$, $\{x \in \beta S : sx = x\}$ is closed. It thus suffices to show that $R = \bigcap_{s \in E} \{x \in \beta S : sx = x\}$. Indeed, if $x \in R$ and $s \in E$, then $x = px = spx = sx$. Conversely, if $x \in \beta S$ and for all $s \in E$, $sx = x$, then ρ_x is constantly equal to x on E and therefore $px = x$ so $x \in R$.

To see that (a) implies (d), let L be a minimal left ideal of βS, let $x, y \in L$, and assume that $sx = sy$. Let p be the identity of L. Let $A = \{a \in S : ax = ay\}$. Then A is a left ideal of S so $\overline{A}$ is a left ideal of βS. Pick a minimal left ideal L' of βS such that $L' \subseteq \overline{A}$ and let p' be the identity of L'. Then for all $a \in A$, $axp' = ayp'$ and so $\rho_{xp'}$ and $\rho_{yp'}$ agree on a member of p' and therefore $p'xp' = p'yp'$. Since p' is a left identity for $R = K(\beta S)$, we have that $xp' = p'xp' = p'yp' = yp'$ and so $x = xp = xp'p = yp'p = yp = y$.

Trivially (d) implies (e). To see that (e) implies (a), pick a minimal left ideal L of βS such that for each $s \in S$, $\lambda_{s|L}$ is injective. Suppose that βS has distinct minimal left ideals R and R'. Let p be the identity of $L \cap R$ and let p' be the identity of $L \cap R'$. Pick $A \subseteq S$ such that $\overline{A} \cap L = \{p\}$. Since $pp = p$, $\{s \in S : s^{-1}A \in p\} \in p$. Since p' is a right identity for L, $pp' = p$ and so $\{s \in S : s^{-1}A \in p'\} \in p$. Pick $s \in S$ such that $s^{-1}A \in p$ and $s^{-1}A \in p'$. Then $sp \in \overline{A} \cap L$ and $sp' \in \overline{A} \cap L$ so $sp = sp' = p$. Since λ_s is injective on L, $p = p'$, a contradiction.

Trivially (f) implies (g). To see that (g) implies (c), note that the minimal left ideals of $G \times T$ are the sets of the form $G \times \{a\}$ for $a \in T$. It follows quickly from [10, Lemma 1.43(c)] that L is a minimal left ideal of βS if and only if L is a minimal left ideal of $K(\beta S)$.

To see that (b) implies (f) note that each member of T is a left identity for $R = K(\beta S)$, and in particular T is a right zero semigroup. Also, $T = \{z \in R : zx = x$ for all $x \in R\}$ so $T = \bigcap_{x \in R} (R \cap \rho_x^{-1}[\{x\}])$ and thus T is compact. Since (b) implies (c), we know that L is a group. Let p be the identity of L. Define $\varphi : L \times T \to R$ by $\varphi(x, z) = xz$. To see that φ is a homomorphism, let $(x_1, z_1), (x_2, z_2) \in L \times T$. Then $\varphi(x_1, z_1)\varphi(x_2, z_2) = x_1z_1x_2z_2 = x_1x_2z_2 = x_1x_2z_1z_2 = \varphi(x_1x_2, z_1z_2)$.

To see that φ is surjective, let $x \in R$ and pick the minimal left ideal L' of βS such that $x \in L'$. If z is the identity of L', then $xz = x$ and so $\varphi(xp, z) = xpz = xz = x$.

Now assume that $(x_1, z_1), (x_2, z_2) \in L \times T$ and $\varphi(x_1, z_1) = \varphi(x_2, z_2)$. Then z_1 and z_2 are idempotents in the same minimal left ideal of βS, which is a group, and so $z_1 = z_2$. Also $x_1 = x_1p = x_1z_1p = x_2z_2p = x_2p = x_2$.

Finally, to see that φ is continuous, let $(x, z) \in L \times T$ and let U be a neighborhood of $\varphi(x, z)$. Pick $A \in xz$ such that $\overline{A} \subseteq U$. Pick $B \subseteq S$ such that $\overline{B} \cap L = \{x\}$. Since $xp = x$, $\{s \in S : s^{-1}B \in p\} \in x$ so pick $s \in S$ such that $s^{-1}B \in p$. Then $sp \in \overline{B} \cap L$ so $sp = x$. Since p is a left identity for R, we have that for all $y \in R$, $sy = spy = xy$. We have that $\{x\}$ is open in L. Also, since $sz = xz \in \overline{A}$, we have that $\overline{s^{-1}A} \cap R$ is a neighborhood of z in R. We claim that $\varphi[\{x\} \times (\overline{s^{-1}A} \cap R)] \subseteq \overline{A}$. Let $y \in \overline{s^{-1}A} \cap R$.

Then $\varphi(x,y) = xy = sy \in \overline{A}$.

To see that (a) implies (h), note that by Lemma 6.1, $U_s \neq \emptyset$ and is therefore a right ideal of S. It then follows from Lemma 5.1(g) that $K(\beta S) \subseteq \overline{U_s}$.

Trivially (h) implies (i). To see that (i) implies (a), pick an idempotent p in $K(\beta S)$ such that for each $s \in E$, $\{u \in S : su = u\} \in p$. Pick a minimal right ideal R of βS such that $p \in R$. Let q be an arbitrary idempotent in $K(\beta S)$ and pick a minimal right ideal R' of βS such that $q \in R'$. We shall show that $p \in R'$, and so $R' = R$. Given $s \in E$, λ_s agrees with the identity on a member of p and thus $sp = p$. Since $E \in q$, one has that $\{s \in S : sp = p\} \in q$. Since ρ_p is constant on a member of q, we have that $qp = p$ and so $p \in R'$ as claimed.

Now assume that statements (a) through (i) hold. Let L be a minimal left ideal of βS and let p be an idempotent in L. Define $h : S \to L$ by $h(s) = sp$. By Lemma 6.1, p is a left identity for $K(\beta S)$ so if $s, t \in S$ then $h(s)h(t) = sptp = stp = h(st)$. Since $L = \beta Sp = c\ell(Sp) = Sp$, h is surjective.

For $s, t \in S$ such that $sp = tp$, let $R_{s,t} = \{v \in \beta S : sv = tv\}$. We claim first that $R_{s,t} \neq \emptyset$. To this end, pick $u \in S$ such that up is the inverse of sp in the group L. Then $p = spup = sup$ and $p = tpup = tup$ so by [10, Theorem 3.35], $\{w \in S : w = suw\} \in p$ and $\{w \in S : w = tuw\} \in p$. Choosing w in the intersection of these two sets, we have that $uw \in R_{s,t}$. Next we note that if $s, t \in S$ such that $sp = tp$, then $R_{s,t} \in p$. Indeed, since $R_{s,t} \neq \emptyset$, it is a right ideal of S and consequently $\overline{R_{s,t}}$ is a right ideal of βS which therefore contains $K(\beta S)$, which is the unique minimal right ideal of βS. If $V \subseteq S$ satisfies $\overline{V} \cap K(\beta S) \neq \emptyset$ and if $G \in \mathcal{P}_f(S)$, we can choose $v \in V \cap \bigcap\{R_{s,t} : s, t \in G$ and $sp = tp\}$. □

We remark that statement (j) of Theorem 6.2 does not imply the other statements. To see this, D be a two element left zero semigroup and let T be an infinite right zero semigroup. Let $S = D \times T$. Then the minimal left ideals of βS are copies of D, so statement (c) fails. It is routine to verify that S satisfies statement (j).

In the following theorem we shall show that, if S is any semigroup for which βS contains a point x such that $\mu(\{x\}) > 0$ for some $\mu \in LIM(S)$, then S has properties reminiscent of a right zero semigroup.

6.3 Theorem. *Let S be a discrete semigroup. The following statements are equivalent.*

(a) There exist $\mu \in LIM(S)$ and $x \in \beta S$ such that $\mu(\{x\}) > 0$.

(b) Every minimal left ideal L of βS is finite and has the property that the mapping λ_s is injective on L for every $s \in S$.

(c) Every minimal left ideal L of βS is a finite group, and there exists a Følner net $\langle F_\alpha\rangle_{\alpha\in D}$ in $\mathcal{P}_f(S)$ such that $\{|F_\alpha| : \alpha\in D\}$ is bounded and $\langle \mu_{F_\alpha}\rangle_{\alpha\in D}$ converges in the weak topology to the unique measure in $LIM(S)$ with support L.*

(d) There exists a Følner net $\langle F_\alpha\rangle_{\alpha\in D}$ for which $\langle |F_\alpha|\rangle_{\alpha\in D}$ is bounded.

If S satisfies these equivalent statements, then it satisfies each of the statements of Theorem 6.2. In addition for every $\nu\in LIM(S)$, $\nu\big(K(\beta S)\big)=1$.

Proof. To see that (a) implies (b), assume that $x\in\beta S$ and $\mu\in LIM(S)$ satisfy $\mu(\{x\})>0$. Let L be a minimal left ideal of βS. We may suppose that $x\in L$, because we could choose any $y\in L$ and replace x by xy and μ by the measure which maps each Borel subset B of βS to $\mu(\rho_y^{-1}[B])$.

For each $s\in S$, $\mu(\{sx\})\geq\mu(\{x\})$ by Theorem 5.1(a) and so Sx is finite. Since $L=\overline{Sx}$, $L=Sx$. Furthermore, $sL=L$ for every $s\in S$ by Theorem 5.1(d), and so λ_s is injective on L.

To see that (b) implies (c), Let L be a minimal left ideal of βS and let p be an idempotent in L. We have by Theorem 6.2 that L is a finite group, λ_s is injective on L for each $s\in S$, the mapping $s\mapsto sp$ is a homomorphism from S onto L, and if $V\subseteq S$ satisfies $\overline{V}\cap K(\beta S)\neq\emptyset$ and $G\in\mathcal{P}_f(S)$, then there exists $v\in V$ such that $sv=tv$ whenever $s,t\in G$ and $sp=tp$.

Since L is finite and $Sp\subseteq L$, $L=\beta Sp=c\ell(Sp)=Sp$. Pick $K\in\mathcal{P}_f(S)$ such that $|K|=|L|$ and $Kp=L$. Given $s\in S$, $sKp=sL=L$ because L is finite and λ_s is injective on L. For $F\in\mathcal{P}_f(S)$, let $G_F=FK\cup K$. Let $\mathcal{A}=\{A\subseteq S : L\subseteq\overline{A}\}$ and let $A\in\mathcal{A}$. For each $s\in S$ we have $L\subseteq\overline{s^{-1}A}$. Let $V_{F,A}=\bigcap_{s\in G_F}s^{-1}A$ so that $L\subseteq\overline{V}_{F,A}$ and $sv\in A$ for every $s\in G_F$ and every $v\in V_{F,A}$. Pick $v_{F,A}\in V_{F,A}$ such that $sv_{F,A}=tv_{F,A}$ whenever $s,t\in G_F$ and $sp=tp$, and let $H_{F,A}=Kv_{F,A}$. Then $|H_{F,A}|\leq|K|=|L|$. Direct $\mathcal{P}_f(S)\times\mathcal{A}$ by putting $(F',A')\leq(F,A)$ if and only if $F'\subseteq F$ and $A\subseteq A'$. We claim that $\langle H_{F,A}\rangle_{(F,A)\in\mathcal{P}_f(S)\times\mathcal{A}}$ is a Følner net in $\mathcal{P}_f(S)$. To see this, it suffices to let $F\in\mathcal{P}_f(S)$, let $A\in\mathcal{A}$, let $s\in F$, and show that $H_{F,A}\subseteq sH_F$. To this end, let $t\in H_{F,A}$ and pick $u\in K$ such that $t=uv_{F,A}$. Now $up\in L=sKp$ so pick $w\in K$ such that $up=swp$. Then $u,sw\in G_F$ so $t=uv_{F,A}=swv_{F,A}\in sH_{F,A}$. Since $H_{F,A}\subseteq A$, it follows that, for every $B\in\mathcal{A}$, $\mu_{H_{F,A}}(\overline{B})=1$ whenever $(F,A)\in\mathcal{P}_f(S)\times\mathcal{A}$ satisfies $A\subseteq B$. So $\langle\mu_{H_{F,A}}\rangle_{(F,A)\in\mathcal{P}_f(S)\times\mathcal{A}}$ converges in the weak* topology to the measure in $LIM(S)$ with support L.

It is obvious that (c) implies (d). To see that (d) implies (a), assume that $\langle F_\alpha\rangle_{\alpha\in D}$ is a Følner net in $\mathcal{P}_f(S)$ such that $\{|F_\alpha| : \alpha\in D\}$ is bounded and let $n=\max\{|F_\alpha| :$

$\alpha \in D\}$.

Let μ be a weak* limit point of the net $\langle \mu_{F_\alpha} \rangle_{\alpha \in D}$. (In our current context, given a Borel subset B of βS and $F \in \mathcal{P}_f(S)$, $\mu_F(B) = \frac{|B \cap F|}{|F|}$.) Pick a subnet $\langle \mu_{G_\delta} \rangle_{\delta \in E}$ of $\langle \mu_{F_\alpha} \rangle_{\alpha \in D}$ which converges to μ. Let $x \in \bigcap_{\delta \in E} c\ell(\bigcup_{\beta > \delta} G_\beta)$ and suppose that $\mu(\{x\}) = 0$. Pick $A \in x$ such that $\mu(\overline{A}\,) < \frac{1}{2n}$ and let $U = \{\nu \in MN(S) : \nu(\overline{A}\,) < \frac{1}{2n}\}$. Pick $\delta \in E$ such that for every $\beta > \delta$, $\mu_{G_\beta} \in U$. Now also $\bigcup_{\beta > \delta} G_\beta \in x$ so pick $\beta > \delta$ such that $A \cap G_\beta \neq \emptyset$. Then $\mu_{G_\beta}(\overline{A}\,) > \frac{1}{n}$ so $\mu_{G_\beta} \notin U$, a contradiction.

Now assume that these equivalent statements hold and let $R = K(\beta S)$. Let $\nu \in LIM(S)$. We shall show that $\nu(R) = 1$. Let p be an idempotent in R. By Lemma 6.1 for every $s \in E$, if $U_s = \{u \in S : su = u\}$, then $U_s \in p$. Let $X = \bigcap_{s \in E} \bigcap_{u \in U_s} u\beta S$. By Theorem 5.1(g), $\nu(X) = 1$. It suffices to show that $X \subseteq R$, so let $x \in X$. We claim that for each $s \in E$, $sx = x$, so let $s \in E$ be given and pick $u \in U_s$. Then $x \in u\beta S$ so pick $y \in \beta S$ such that $x = uy$. Then $sx = suy = uy = x$. Since for each $s \in E$, $sx = x$ we have that $px = x$ and therefore $x \in R$ as required. □

The preceding results give us a great deal of information about the structure of $LIM(S)$ when S denotes a discrete semigroup with the property that βS contains a singleton subset which has positive measure for some mean in $LIM(S)$.

6.4 Definition. Let S denote a discrete semigroup with the property that there is a mean in $LIM(S)$ which assumes a positive value on a singleton subset of βS. Let $Z(S) = \{p \in K(\beta S) : pp = p\}$ and let $G(S)$ denote the finite group isomorphic to each minimal left ideal of βS.

We have seen that $Z(S)$ is a compact right zero semigroup which is equal to the set of left identities of $K(\beta S)$.

The fact that the extreme points of $LIM(S)$ are the measuresin $LIM(S)$ whose supports are minimal left ideals of βS under the hypotheses of the following Corollary, was proved in [18] by different methods.

6.5 Corollary. *Let S denote a discrete semigroup with the property that there is a mean in $LIM(S)$ which assumes a positive value on a singleton subset of βS. We can then identify $K(\beta S)$ with $G(S) \times Z(S)$ by Theorem 6.2(f). Let μ denote the unique left invariant mean on $G(S)$. Then for each $\nu \in LIM(S)$ there is a regular Borel probability measure ρ on $Z(S)$ such that ν is the product measure $\mu \otimes \rho$. The mapping $\nu \mapsto \rho$ is a bijection from $LIM(S)$ onto the set of all regular Borel probability measures on $Z(S)$. The extreme points of $LIM(S)$ are the means in $LIM(S)$ whose supports are minimal left ideals of βS, and $LIM(S)$ is the weak* closed convex hull of $LIM_0(S)$.*

Proof. We claim that a regular Borel measure ν on βS whose support is contained in $K(\beta S)$, is left invariant if and only if $\nu(x^{-1}B) = \nu(B)$ for every Borel subset B of βS and every $x \in K(\beta S)$. To see this, let p be the identity of the minimal left ideal of βS which contains x. Then $x = xp = sp$ for some $s \in S$. So $xy = sy$ for every $y \in K(\beta S)$. Conversely, for each $s \in S$, we can put $x = sp$ and deduce that $xy = sy$ for every $y \in K(\beta S)$. Thus $\lambda_{s|K(\beta S)} = \lambda_{x|K(\beta S)}$.

Now let ρ be any regular Borel probability measure on $Z(S)$. The product measure $\mu \otimes \rho$ on $G(S) \times Z(S)$ has the property that $\mu \otimes \rho\big((g, z)^{-1}B\big) = \mu \otimes \rho(B)$ for every Borel subset B of $G(S) \times Z(S)$, every $g \in G(S)$ and every $z \in Z(S)$. Thus $\mu \otimes \rho$ corresponds to a measure in $LIM(S)$. Conversely, if $\nu \in LIM(S)$, we can define a regular Borel probability measure ρ on $Z(S)$ by putting $\rho(B) = \nu(\pi_2^{-1}[B])$ for every Borel subset B of $Z(S)$. It is then easy to verify that $\nu = \mu \otimes \rho$. We observe that the mapping $\nu \mapsto \rho$ preserves convex combinations.

Now the extreme points of the set of regular Borel probability measures on $Z(S)$ are the point measures δ_z where $z \in Z(S)$ by [13, 15.J]. Under the correspondence described in the preceding paragraph, δ_z corresponds to the unique measure in $LIM(S)$ whose support is the minimal left ideal $G(S) \times \{z\}$ of $G(S) \times Z(S) \sim K(\beta S)$ which contains z. So the extreme points of $LIM(S)$ are the measures in $LIM(S)$ whose supports are minimal left ideals of $K(\beta S)$. By Theorem 6.3(c), each such measure is in $LIM_0(S)$. Thus it follows from the Krein Milman Theorem ([13, 15.1]) that $LIM(S)$ is the weak* closed convex hull of $LIM_0(S)$. □

Recall that a left amenable semigroup need not satisfy SFC, and so we might not be able to apply our definition of density. As a consequence of Theorem 6.3 if there exist $\mu \in LIM(S)$ and $x \in \beta S$ such that $\mu(\{x\}) > 0$, then S does satisfy SFC and we have the following corollary.

6.6 Corollary. *Let S be a discrete left amenable semigroup and assume that $LIM(S)$ contains a measure which is non-zero on a singleton subset of βS. Suppose that every minimal left ideal of βS has n elements. Then there is a central* subset E of S such $\mu(\overline{E}) = \frac{1}{n}$ for every $\mu \in LIM(S)$ and, for every $s \in E$, $\mu(\overline{\{u \in S : su = u\}}) = 1$ for every $\mu \in LIM(S)$. In particular, $d(E) = \frac{1}{n}$ and, for every $s \in E$, $d(\{u \in S : su = u\}) = 1$.*

Proof. Let p be an idempotent in $K(\beta S)$ and let L be the minimal left ideal in βS which contains p. Let $E = \{s \in S : sp = p\}$. By Theorem 6.3, S satisfies the statements of Theorem 6.3. Therefore, by Lemma 6.1 E is central*.

We claim that the sets of the form $t^{-1}\overline{E} \cap K(\beta S)$, where $t \in S$, partition $K(\beta S)$ into n disjoint subsets. Let $y \in K(\beta S)$ and suppose that y belongs to a minimal left ideal L' with identity p'. Since $L' = Sy$, there exists $t \in S$ such that $ty = p' \in \overline{E}$. This shows that $K(\beta S) \subseteq \bigcup_{t \in S} t^{-1}\overline{E}$. Now let $s, t \in S$. Suppose first that $sp = tp$. Let $y \in K(\beta S)$. Then $sy = spy = tpy = ty$. So $K(\beta S) \cap s^{-1}\overline{E} = K(\beta S) \cap t^{-1}\overline{E}$. Now suppose that $sp \neq tp$ and that there exists $y \in K(\beta S) \cap s^{-1}\overline{E} \cap t^{-1}\overline{E}$. Then $syp = typ = p$. Multiplying on the right by the inverse of yp in L we have that $sp = tp$, a contradiction. So $K(\beta S) \cap s^{-1}\overline{E}$ and $K(\beta S) \cap t^{-1}\overline{E}$ are disjoint. This shows that the sets of the form $t^{-1}\overline{E}$ partition $K(\beta S)$ into n disjoint sets, as claimed. If $\mu \in LIM(S)$, then $\mu(K(\beta S)) = 1$ by Theorem 6.3 and so $\mu(\overline{E}) = \frac{1}{n}$. If $s \in E$, $\{u \in S : us = u\}$ is non-empty by Lemma 6.1 and is therefore a right ideal in S. So $\mu(\overline{\{u \in S : su = u\}}) = 1$ by Theorem 5.1(f).

As shown in the proof Theorem 2.14, for any subset A of S there exists $\mu \in LIM(S)$ such that $d(A) = \mu(\overline{A})$. It follows that $d(A) = \frac{1}{n}$ and that $d(\{u \in S : su = u\}) = 1$ for every $s \in E$. □

We now investigate the possibility that $\mu(S) > 0$ for some $\mu \in LIM(S)$.

6.7 Theorem. *Let S be a discrete semigroup. The following statements are equivalent.*

(a) There exists $\mu \in LIM(S)$ for which $\mu(S) = 1$.

(b) There exists $\mu \in LIM(S)$ for which $\mu(S) > 0$.

(c) There exist $\mu \in LIM(S)$ and $x \in S$ for which $\mu(\{x\}) > 0$.

(d) S contains a finite left ideal L which is a group and therefore has the property that λ_s is injective on L for every $s \in S$.

Proof. Trivially (a) implies (b). To see that (b) implies (c), assume that $\mu(S) > 0$. Since μ is regular, $\mu(F) > 0$ for some finite subset F of S, because the compact subsets of S are finite. So $\mu(\{x\}) > 0$ for some $x \in F$.

To see that (c) implies (d) assume that $\mu(\{x\}) > 0$ for some $\mu \in LIM(S)$ and some $x \in S$. By Theorem 5.1(a), $\mu(\{sx\}) \geq \mu(\{x\})$ for each $s \in S$ and thus Sx is finite. Thus $Sx = c\ell(Sx) = \beta Sx$ so Sx is a left ideal of βS and thus contains a minimal left ideal L. By Theorem 6.3 the statements of Theorem 6.2 apply to L. In particular L is a group and λ_s is injective on L for each $s \in S$.

To see that (d) implies (a), pick a finite left ideal L of S which is a group and has the property that λ_s is injective on L for every $s \in S$. Since λ_s is injective, and thus bijective, on L for each $s \in S$, we have that for any $B \subseteq \beta S$ and any $s \in S$,

$|s^{-1}B \cap L| = |B \cap L|$ and thus $\mu_L(s^{-1}B) = \mu_L(B)$. That is $\mu_L \in LIM(S)$. Since $L \subseteq S$, $\mu_L(S) = 1$. $\square$

We remark that the statements of Theorem 6.7 are strictly stronger than the statements of Theorem 6.3. Trivially Theorem 6.7(c) implies Theorem 6.3(a). Consider now the semigroup $S = (\mathbb{N}, \max)$. We have that $K(S) = \beta\mathbb{N} \setminus \mathbb{N}$ and $K(S)$ is a right zero semigroup, so Theorem 6.3(b) is trivially satisfied. But $(\mathbb{N}, \max)$ has no finite left ideals, so Theorem 6.7(d) is not satisfied.

We are naturally interested in knowing about the class of semigroups that have the property of Theorem 6.3(b). We shall investigate in particular those for which minimal left ideals of βS are singletons. That is the same as saying that $K(\beta S)$ is a right zero semigroup. (Since any right ideal meets any left ideal, $K(\beta S)$ must be a minimal right ideal, and then any member of $K(\beta S)$, being an idempotent, must be a left identity for $K(\beta S)$.)

6.8 Definition. **RZ** is the class of semigroups S such that $K(\beta S)$ is a right zero semigroup.

Given any member T of **RZ** and any finite group G, we have that $S = G \times T$ has the property that any minimal left ideal L of βS is a copy of G. It is not true that every semigroup S which satisfies Theorem 6.3(b) is of the form $G \times T$ for some finite group G and some $T \in$ **RZ** because if S is any semigroup which satisfies Theorem 6.3(b) and S' is S with an identity adjoined, then also S' satisfies Theorem 6.3(b). The following theorem tells us however that if S satisfies Theorem 6.3(b), then $K(\beta S)$ is topologically isomorphic to $K\big(\beta(G \times T)\big)$ for some finite group G and some $T \in$ **RZ**.

6.9 Theorem. *Let S be a semigroup with the property that βS has a finite minimal left ideal L which is a group. There exists $T \in$ **RZ** such that $K(\beta S)$ is topologically isomorphic to $L \times K(\beta T)$.*

Proof. Let p be the identity of L and let $T = \{s \in S : sp = p\}$. By Lemma 6.1 if $qq = q \in K(\beta S)$, then $T \in q$. In particular, $T \neq \emptyset$ so T is a semigroup and $\beta T = c\ell T$. Also $\beta T \cap K(\beta S) \neq \emptyset$ so by [10, Theorem 1.65], $K(\beta T) = \beta T \cap K(\beta S)$. We claim that $\beta T \cap K(\beta S) = \{q \in K(\beta S) : qq = q\}$ so that $K(\beta T) = \{q \in K(\beta S) : qq = q\}$. We have already seen that $\{q \in K(\beta S) : qq = q\} \subseteq \beta T \cap K(\beta S)$. Let $q \in \beta T \cap K(\beta S)$ and let r be the identity of $\beta S q$. Then $T = \{s \in S : sr = r\}$ by Lemma 6.1 so $T \subseteq \{s \in S : sq = q\}$ and thus $qq = q$. By Theorem 6.2(f) we have that $K(\beta S)$ is topologically isomorphic to $L \times K(\beta T)$. $\square$

One trivial class of semigroups $S \in \mathbf{RZ}$ is those which have a right zero element. (If $b \in S$ and $ab = b$ for all $a \in S$, then $xb = b$ for all $x \in \beta S$ so $\{b\}$ is a minimal left ideal of βS.) We set out to show that $\mathbf{RZ}$ includes all left amenable bands. (Recall that a *band* is a semigroup all of whose members are idempotents.)

6.10 Lemma. *If S is a left amenable semigroup, then $\bigcap_{s\in S} s\beta S \neq \emptyset$ and if $x \in \bigcap_{s\in S} s\beta S$, and $ss = s \in S$, then $sx = x$.*

Proof. By Theorem 5.1(g), $K(\beta S) \subseteq \bigcap_{s\in S} s\beta S \neq \emptyset$. If $ss = s$, then s is a left identity for $s\beta S$. □

We do not have an example of a semigroup satisfying the hypotheses of the following theorem which is not already a band.

6.11 Theorem. *Let S be a left amenable semigroup such that every element of S is a product of idempotents. Then $S \in \mathbf{RZ}$ and $K(\beta S) = \bigcap_{s\in S} s\beta S$.*

Proof. By Lemma 6.10 we have that for each $x \in \bigcap_{s\in S} s\beta S$, $Sx = \{x\}$ and therefore $\beta Sx = \{x\}$, so that $\{x\}$ is a minimal left ideal. Therefore $S \in \mathbf{RZ}$ and $\bigcap_{s\in S} s\beta S \subseteq K(\beta S)$. By Theorem 5.1(g), $K(\beta S) \subseteq \bigcap_{s\in S} s\beta S$. □

We thus have several common examples of members of $\mathbf{RZ}$, such as $(\mathbb{N}, \max)$, $(\mathbb{N}, \text{lcm})$ and $(\mathcal{P}_f(X), \cup)$ where X is an arbitrary set. If $S = (\mathbb{N}, \max)$, then $K(\beta S) = \beta\mathbb{N} \setminus \mathbb{N}$. If $S = (\mathbb{N}, \text{lcm})$, then $K(\beta S) = \bigcap_{n\in\mathbb{N}} \overline{n\mathbb{N}}$. If $S = (\mathcal{P}_f(X), \cup)$, then $K(\beta S) = \bigcap_{F\in\mathcal{P}_f(X)} cl_{\beta S}(\{G \in \mathcal{P}_f(X) : F \subseteq G\})$.

We now show that $\mathbf{RZ}$ is closed under arbitrary cartesian products and direct sums.

6.12 Theorem. *Let A be a set and for each $i \in A$, let $S_i \in \mathbf{RZ}$.*

(a) If $S = \times_{i\in A} S_i$, then $S \in \mathbf{RZ}$.

(b) If for each $i \in A$, S_i has an identity e_i and $S = \bigoplus_{i\in A} S_i$, then $S \in \mathbf{RZ}$.

Proof. The first part of the proofs of (a) and (b) are identical. For each $F \in \mathcal{P}_f(S)$ and each $i \in A$, let $T_{i,F} = \{t \in S_i : (\forall s \in F)(\pi_i(s) \cdot t = t)\}$. Let $F \in \mathcal{P}_f(S)$ and let $i \in A$. We claim that $K(\beta S_i) \subseteq c\ell T_{i,F}$. To see this, let $p \in K(\beta S_i)$. Then $\beta S_i p = \{p\}$ so for all $s \in S_i$, $sp = p$. Therefore, for each $s \in F$ we have by [10, Theorem 3.35] that $\{v \in S_i : \pi_i(s) \cdot v = v\} \in p$ and so $T_{i,F} = \bigcap_{s\in F} \{v \in S_i : \pi_i(s) \cdot v = v\} \in p$.

Now assume that $S = \times_{i\in A} S_i$. For $F \in \mathcal{P}_f(S)$, let $T_F = \times_{i\in A} T_{i,F}$. If $F \in \mathcal{P}_f(S)$, $s \in F$, and $v \in T_F$, then $sv = v$. Therefore, if $p \in c\ell T_F$ and $s \in F$, we have that $sp = p$. Since $\{c\ell T_F : F \in \mathcal{P}_f(S)\}$ has the finite intersection property, one may pick $p \in \bigcap_{F\in\mathcal{P}_f(S)} c\ell T_F$. Then $Sp = \{p\}$ and so $\beta Sp = \{p\}$.

Finally assume that for each $i \in A$, S_i has an identity e_i and $S = \bigoplus_{i \in A} S_i$. For $F \in \mathcal{P}_f(S)$ and $G \in \mathcal{P}_f(A)$, let $T_{F,G} = \{x \in S : (\forall i \in G)(\pi_i(x) \in T_{i,F})\}$. Let $F \in \mathcal{P}_f(S)$. Then $\{c\ell T_{F,G} : G \in \mathcal{P}_f(A)\}$ has the finite intersection property so $\bigcap_{G \in \mathcal{P}_f(A)} c\ell T_{F,G} \neq \emptyset$. Let $p \in \bigcap_{G \in \mathcal{P}_f(A)} c\ell T_{F,G}$ and let $s \in F$. We claim that $sp = p$, for which it suffices that $\{v \in S : sv = v\} \in p$. Let $G = \{i \in A : \pi_i(s) \neq e_i\}$. If $G = \emptyset$, then s is the identity of S so $sp = p$, so we may assume $G \in \mathcal{P}_f(A)$. Then $T_{F,G} \in p$ and $T_{F,G} \subseteq \{v \in S : sv = v\}$.

Now $\{c\ell T_{F,G} : F \in \mathcal{P}_f(S) \text{ and } G \in \mathcal{P}_f(A)\}$ has the finite intersection property. Pick $p \in \bigcap_{F \in \mathcal{P}_f(S)} \bigcap_{G \in \mathcal{P}_f(A)} c\ell T_{F,G}$. Then for all $s \in S$, $sp = p$ and so $Sp = \{p\}$ and thus $\beta Sp = \{p\}$. □

As a consequence of Theorem 6.12 we can give an example of a member of **RZ** which is not a band and does not have a right zero. Let $S = (\mathbb{N}, \max)$ and $T = (\omega, \cdot)$. Then S and T are commutative and thus left amenable. Also, S is a band and T has a right zero. Therefore $S \times T \in \mathbf{RZ}$, and $S \times T$ is not a band and does not have a right zero.

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