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An Algebraic Proof of Deuber's Theorem

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Abstract. Deuber's Theorem says that, given any m, p, c, r in $\mathbb{N}$, there exist n, q, μ in $\mathbb{N}$ such that whenever an (n, q, c^μ)-set is r-coloured, there is a monochrome (m, p, c)-set. This theorem has been used in conjunction with the algebraic structure of the Stone-Čech compactification $\beta\mathbb{N}$ of $\mathbb{N}$ to derive several strengthenings of itself. We present here an algebraic proof of the main results in $\beta\mathbb{N}$ and derive Deuber's Theorem as a consequence.

1. Introduction.

In [4], Deuber introduced the notion of (m, p, c)-sets and used them to prove a conjecture of Rado, namely that the property of containing solutions to any partitition regular system of homogeneous linear equations is itself a partition regular property. (See [9, p. 80] for a description of how this partition regularity follows.)

1.1 Definition. Let $m, p, c \in \mathbb{N}$. A set $A \subseteq \mathbb{N}$ is an (m, p, c)*-set* if and only if there exists $\vec{x} \in \mathbb{N}^m$ such that $A = \{\Sigma_{i=1}^m \lambda_i \cdot x_i : \{\lambda_1, \lambda_2, \ldots, \lambda_m\} \subseteq \{0, 1, \ldots, p\}$ and there is some $j \in \{1, 2, \ldots, m\}$ such that $\lambda_j = c$ and $\lambda_i = 0$ for $i < j\}$.

Often, in the definition of (m, p, c)-set, one allows the coefficients λ_i to come from $\{-p, -p+1, \ldots, p-1, p\}$ and adds the requirement that the resulting sum be positive. It is well known that there is no substantive difference between these definitions. (See for example [2, p. 309].)

Note also, that if $c > p$ then any (m, p, c)-set is empty.

1.2 Theorem (Deuber). *Let* $m, p, c, r \in \mathbb{N}$. *There exist* $n, q, \mu \in \mathbb{N}$ *such that whenever* A *is an* (n, q, c^μ)*-set and* $A = \bigcup_{i=1}^r B_i$, *there exist* $i \in \{1, 2, \ldots, r\}$ *and an* (m, p, c)*-set* C *such that* $C \subseteq B_i$.

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Proof. This is [4, Satz 3.1]. See also [13] or [14]. □

Deuber's Theorem is often stated with an unspecified d where we have written c^μ. But the fact that d can be chosen to be a power of c (and, in particular, that d can be chosen equal to 1 if $c = 1$) is included in Deuber's original paper [4].

Two strong extensions of Deuber's Theorem were obtained using the algebraic structure of the compact right topological semigroup $(\beta\mathbb{N}, +)$. These extensions involve the notion of finite sums from a sequence of sets.

1.3 Definition. (a) Let $\langle Y_t\rangle_{t=1}^\infty$ be a sequence of subsets of $\mathbb{N}$. For any $D \subseteq \mathbb{N}$,

$$FS(\langle Y_t\rangle_{t\in D}) = \big\{\Sigma_{t\in F}\, x_t : F \text{ is a finite nonempty subset of } D \text{ and for each } t \in F,\ x_t \in Y_t\big\}\ .$$

In particular, given $m, n \in \mathbb{N}$,

$$FS(\langle Y_t\rangle_{t=m}^\infty) = \big\{\Sigma_{t\in F}\, x_t : F \text{ is a finite nonempty subset of } \{m, m+1, m+2, \ldots\} \text{ and for each } t \in F,\ x_t \in Y_t\big\}$$

and

$$FS(\langle Y_t\rangle_{t=m}^n) = \big\{\Sigma_{t\in F}\, x_t : F \text{ is a finite nonempty subset of } \{m, m+1, \ldots, n\} \text{ and for each } t \in F,\ x_t \in Y_t\big\}\ .$$

(b) A subset A of $\mathbb{N}$ is an $(\mathcal{M}, \mathcal{P}, \mathcal{C})$*-system if and only if* there is a sequence $\langle Y_t\rangle_{t=1}^\infty$ such that $A = FS(\langle Y_t\rangle_{t=1}^\infty)$ and for any $m, p, c \in \mathbb{N}$ there is some $t \in \mathbb{N}$ such that Y_t is an (m, p, c)-set.

(c) An (m, p, c)-matrix is a matrix A with m columns which satisfies the following conditions:

(i) No row of A is identically 0;
(ii) The first (leftmost) nonzero entry of every row is equal to c;
(iii) All the entries of A are in $\{0, 1, 2, \ldots, p\}$; and
(iv) All possible rows which satisfy these conditions occur in A.

We observe that there is a close connection between (m, p, c)-sets and (m, p, c)-matrices. If A is an (m, p, c)-matrix and $\vec{x} \in \mathbb{N}^m$, then the set of entries of $A\vec{x}$ is an (m, p, c)-set and all (m, p, c)-sets arise in this way.

It was shown in [5] that whenever $\mathbb{N}$ is finitely coloured there is a monochrome $(\mathcal{M}, \mathcal{P}, \mathcal{C})$-system. This result was extended in [12], where it was shown that the $(\mathcal{M}, \mathcal{P}, \mathcal{C})$-systems are themselves partition regular. That is, given any finite colouring of any $(\mathcal{M}, \mathcal{P}, \mathcal{C})$-system, there must be some monochrome $(\mathcal{M}, \mathcal{P}, \mathcal{C})$-system.

The stronger of these results was based on the fact that a certain subset of $(\beta\mathbb{N}, +)$ is a compact subsemigroup, and hence contains an idempotent. But the proof that this subset is nonempty required the use of Deuber's Theorem (Theorem 1.2). (And the weaker of the results depended on a corollary of Deuber's Theorem.) We found this fact to be aesthetically unpleasing. (A similar situation had held for a number of years with regard to van der Waerden's Theorem. One was able to prove many strong extensions of van der Waerden's Theorem using the algebraic structure of $(\beta\mathbb{N}, +)$, but these all depended on van der Waerden's Theorem itself. This situation changed when an algebraic proof of van der Waerden's Theorem was found [1].)

In Section 2 of this paper we present an algebraic derivation of a result which implies that the subsemigroup used in [12] is nonempty. In Section 3 we use this result to derive Deuber's Theorem.

As we have previously indicated, we use the semigroup $(\beta\mathbb{N}, +)$ where $\beta\mathbb{N}$ is the Stone-Čech compactification of the set $\mathbb{N}$ of positive integers and $+$ denotes the extension of ordinary addition to $\beta\mathbb{N}$ which makes $(\beta\mathbb{N}, +)$ a right topological semigroup with $\mathbb{N}$ contained in its topological centre.

We take the points of $\beta\mathbb{N}$ to be the ultrafilters on $\mathbb{N}$, the principal ultrafilters being identified with the points of $\mathbb{N}$. Given $A \subseteq \mathbb{N}$, $\overline{A} = c\ell A = \{p \in \beta\mathbb{N} : A \in p\}$. The set $\{\overline{A} : A \subseteq \mathbb{N}\}$ is a basis for the open sets (as well as a basis for the closed sets) of $\beta\mathbb{N}$. A fundamental topological property of $\beta\mathbb{N}$ which we shall need, is that every neighbourhood $U \subseteq \beta\mathbb{N}$ of an ultrafilter $p \in \beta\mathbb{N}$ satisfies $U \cap \mathbb{N} \in p$.

When we say that $(\beta\mathbb{N}, +)$ is a right topological semigroup we mean that for each $p \in \beta\mathbb{N}$ the function $\rho_p : \beta\mathbb{N} \longrightarrow \beta\mathbb{N}$, defined by $\rho_p(q) = q + p$, is continuous. When we say that $\mathbb{N}$ is contained in the topological centre of $(\beta\mathbb{N}, +)$ we mean that for each $x \in \mathbb{N}$, the function $\lambda_x : \beta\mathbb{N} \longrightarrow \beta\mathbb{N}$ defined by $\lambda_x(q) = x + q$ is continuous. The operation $+$ on $\beta\mathbb{N}$ is characterized as follows: Given $A \subseteq \mathbb{N}$, $A \in p + q$ if and only if $\{x \in \mathbb{N} : -x + A \in q\} \in p$ where $-x + A = \{y \in \mathbb{N} : x + y \in A\}$. This operation could also be defined toplogically by stating that $x + y = \lim_{m_\alpha \to x} \lim_{n_\beta \to y}(m_\alpha + n_\beta)$, where $\langle m_\alpha \rangle_{\alpha \in D}$ and $\langle n_\beta \rangle_{\beta \in E}$ denote nets in $\mathbb{N}$ converging to x, y respectively in $\beta\mathbb{N}$. See [10] for a detailed construction of $\beta\mathbb{N}$ and derivations of some of the basic algebraic facts, with the caution that there $(\beta\mathbb{N}, +)$ is taken to be left rather than right topological.

When we say that $p \in \beta\mathbb{N}$ is idempotent, we mean that $p + p = p$. We shall need to use the fact that, for every idempotent $p \in \beta\mathbb{N}$ and every $c \in \mathbb{N}$, $c\mathbb{N} \in p$. (This can easily be verified by noting that the natural map $h : \mathbb{N} \to \mathbb{Z}_c$ has a continuous extension $\tilde{h} : \beta\mathbb{N} \to \mathbb{Z}_c$ which is a homomorphism and therefore satisfies $\tilde{h}(p) = 0$.)

Given any compact right topological semigroup S, we denote by $K(S)$ the smallest two sided ideal of S. An idempotent in S is *minimal* if and only if it is a member of $K(S)$. See [3] for any unfamiliar facts about compact right topological semigroups.

Given a set X we take the members of X^m to be column vectors, and given a vector $\vec{x}$ we take as usual x_i to be the i^{th} entry of $\vec{x}$.

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2. Idempotents and Image Partition Regular Matrices.

Let $\ell, m \in \mathbb{N}$ and let A be an $\ell \times m$ matrix with entries from $\mathbb{Q}$. In terminology suggested by Deuber, the matrix A is said to be *image partition regular* provided whenever $r \in \mathbb{N}$ and $\mathbb{N} = \bigcup_{i=1}^{r} B_i$, there exist $i \in \{1, 2, \ldots, r\}$ and $\vec{x} \in \mathbb{N}^m$ such that $A\vec{x} \in B_i^\ell$. In [11] several characterizations of image partition regular matrices were found, some of which were given in terms of the notion of a "first entries" matrix. (As the reader can observe, the notion of a first entries matrix is in turn based on the notion of an (m, p, c)-set.) We modify the usual definition slightly by requiring that all "first entries" be 1, but as can easily be seen, this is not a substantive modification.

In this section, we prove our main Theorem by using basic results about the algebraic structure of compact right topological semigroups. The purpose of the preliminary lemmas is to construct a compact subsemigroup of a product of copies of $\beta\mathbb{N}$ and an ideal within this subsemigroup. The reader who is familiar with the algebraic proof of van der Waerden's Theorem will recognise that our methods are inspired by this proof.

2.1 Definition. Let and let A be a matrix with entries from $\mathbb{Q}$. Then A satisfies the *first entries condition* if and only if no row of A is $\vec{0}$ and the first (leftmost) nonzero entry in each row is 1.

We shall restrict our attention in this section to matrices that have nonnegative rational entries.

2.2 Definition. Let F be a finite subset of $\mathbb{Q} \cap [0, \infty)$ with $\{0, 1\} \subseteq F$.

(a) For each $k \in \mathbb{N}$ let $P(k) = \{\Pi_{i=1}^{k} a_i : \{a_1, a_2, \ldots, a_k\} \subseteq F\}$.

(b) $\mathcal{M} = \{A : A$ is a finite dimensional matrix which satisfies the first entries condition and has all of its entries from $\bigcup_{k=1}^{\infty} P(k)\}$.

(c) Fix an enumeration of $\mathcal{M}$ as $\langle A_t \rangle_{t=1}^{\infty}$, and let $m(t)$ denote the number of columns of A_t.

(d) Fix $\vec{\alpha} \in \times_{t=1}^{\infty} \mathbb{N}^{m(t)}$ such that for each $t \in \mathbb{N}$, all the entries of $A_t\vec{\alpha}_t$ are in $\mathbb{N}$. For each $t \in \mathbb{N}$ let Y_t be the set of entries of $A_t\vec{\alpha}_t$.

(e) For each $n \in \mathbb{N}$, let $S_n = FS(\langle Y_t \rangle_{t=n}^{\infty})$ and let $T_n = FS(\langle Y_t \rangle_{t=1}^{n})$.

(f) For each $n, k \in \mathbb{N}$ define $V(n, k) \subseteq S_n$ by stating that $v \in V(n, k)$ if and only if $v \in S_n$ and there exist $x_1, x_2, \ldots, x_k \in \mathbb{N}$ such that

(*) $$x_{r+1} + \Sigma_{i=r+2}^{k} u_i x_i + u_{k+1} v \in S_n$$

for every $r \in \{0, 1, \ldots, k-1\}$ and every choice of $u_i \in P(k)$, where $i \in \{r+2, r+3, \ldots, k+1\}$.

All of the notions defined in Definition 2.2 depend on the choice of the set F and the sets $Y_t \subseteq \mathbb{N}$ in part (d) (and, consequently, the sets S_n and $V(n,k)$) depend on the choice of $\vec{\alpha}$, but we supress that dependence in the notation.

In the expression (*) of part (f), of course, if $r = k - 1$, we take $x_{r+1} + \Sigma_{i=r+2}^{k} u_i x_i + u_{k+1} v = x_{r+1} + u_{k+1} v$.

Observe that $P(k) \subseteq P(k+1)$ for every $k \in \mathbb{N}$, because $1 \in F$.

We first prove two simple algebraic lemmas.

2.3 Lemma. *Let $p \in \beta\mathbb{N}$ be an idempotent and let $B \in p$. Define $B^* = \{b \in B : -b + B \in p\}$, then $B^* \in p$. Furthermore, for every $b \in B^*$, $-b + B^* \in p$.*

Proof. Since $B \in p = p + p$, $\{b \in \mathbb{N} : -b + B \in p\} \in p$ so $B^* = B \cap \{b \in \mathbb{N} : -b + B \in p\} \in p$.

Now let $b \in B^*$. Then $-b + B \in p$ so $(-b + B)^* \in p$, and $(-b + B)^* = -b + B^*$. □

In the following Lemma, we denote the semigroup operation additively, because we shall want to apply it to a product of copies of $(\beta\mathbb{N}, +)$. However, we do not assume that the operation is commutative.

2.4 Lemma. *Let $(S, +)$ be a compact right topological semigroup with dense topological centre Λ. Suppose that $\langle E_n \rangle_{n=1}^{\infty}$ and $\langle I_n \rangle_{n=1}^{\infty}$ are decreasing sequences of non-empty subsets of Λ, with $I_n \subseteq E_n$ for each $n \in \mathbb{N}$. Suppose that, for each $k \in \mathbb{N}$ and each $a \in E_k$, there exists $n \in \mathbb{N}$ such that $a + E_n \subseteq E_k$ and $a + I_n \subseteq I_k$. Suppose, in addition, that $a + E_n \subseteq I_k$ if we also have $a \in I_k$. Then $\bigcap_{n=1}^{\infty} c\ell_S E_n$ is a subsemigroup of S and $\bigcap_{n=1}^{\infty} c\ell_S I_n$ is an ideal in this subsemigroup.*

Proof. Let $E = \bigcap_{n=1}^{\infty} c\ell_S E_n$ and $I = \bigcap_{n=1}^{\infty} c\ell_S I_n$. Let $x, y \in E$. Choose any $k \in \mathbb{N}$. We show that $x + y \in c\ell_S E_k$. First we show that $E_k + y \subseteq c\ell_S E_k$, so let $a \in E_k$ and let $n \in \mathbb{N}$ such that $a + E_n \subseteq E_k$. Since $a + y = \lambda_a(y) \in a + c\ell_S E_n = c\ell_S(a + E_n) \subseteq c\ell_S E_k$, we have that $E_k + y \subseteq c\ell_S E_k$ as desired. Thus $x + y = \rho_y(x) \in c\ell_S(E_k + y) \subseteq c\ell_S E_k$. So $x + y \in E$ and E is a subsemigroup of S.

Now let $z \in I$ and let $k \in \mathbb{N}$. A similar argument shows that $E_k + z \subseteq c\ell_S I_k$ and $I_k + y \subseteq c\ell_S I_k$. Consequently $x+z \in c\ell_S(E_k+z) \subseteq c\ell_S I_k$ and $z+y \in c\ell_S(I_k+y) \subseteq c\ell_S I_k$ so $x + z \in I$ and $z + y \in I$. □

2.5 Lemma. *For all $n, k \in \mathbb{N}$, $V(n, k) \neq \emptyset$.*

Proof. We can form matrices in $\mathcal{M}$ which contain all possible rows of the form $(0\ 0\ \ldots\ 0\ 1\ u_{r+2}\ u_{r+3}\ \ldots\ u_{k+1}\ )$, where $r \in \{0, 1, \ldots, k-1\}$ and $u_i \in P(k)$ for each $i \in \{r+2, r+3, \ldots, k+1\}$ and there are r 0's preceding the first non-zero entry. There will be infinitely many matrices of this kind, because we can add on any number of additional rows. Thus we can choose a matrix A_t which contains all these rows and has $t > n$. If $\vec{\alpha}_t = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_k \\ v \end{pmatrix}$, then $v \in V(n, k)$, because the sums which occur in (*) are entries of $A_t \vec{\alpha}_t$ and so are in $Y_t \subseteq S_n$. □

2.6 Lemma. *Let $n, k \in \mathbb{N}$.*

(a) $V(n+1, k) \subseteq V(n, k)$.

(b) $V(n, k+1) \subseteq V(n, k)$.

Proof. (a) Since $S_{n+1} \subseteq S_n$, this is trivial.

(b) Suppose that $v \in V(n, k+1)$. Thus there exist $x_1, x_2, \ldots, x_{k+1} \in \mathbb{N}$ such that

$$x_{r+1} + \Sigma_{i=r+2}^{k+1} u_i x_i + u_{k+2} v \in S_n$$

for every $r \in \{0, 1, \ldots, k\}$ and every $u_i \in P(k+1)$, where $i \in \{r+2, r+3, \ldots, k+2\}$. Now these relations are satisfied for each $r \in \{0, 1, \ldots, k-1\}$ with $u_{k+1} = 0$ and with u_i chosen in $P(k)$ if $i \in \{r+2, r+3, \ldots, k\} \cup \{k+2\}$, because $P(k) \subseteq P(k+1)$. Hence $v \in V(n, k)$. □

2.7 Lemma. *Let A be a matrix with m columns which satisfies the first entries condition and has all its entries in F. Let $n, k \in \mathbb{N}$ with $k > m$. For each $v \in V(n, k)$, there exists $\vec{x} \in \mathbb{N}^m$ such that, for every entry y of $A\vec{x}$ and every $a \in F$, we have $y + av \in V(n, k-m)$.*

Proof. Since $v \in V(n, k)$, choose $z_1, z_2, \ldots, z_k \in \mathbb{N}$ such that $z_{r+1} + \Sigma_{i=r+2}^{k} u_i z_i + u_{k+1} v \in S_n$ for every $r \in \{0, 1, \ldots, k-1\}$ and every choice of $u_i \in P(k)$ for $i \in$

$\{r+2, r+3, \ldots, k+1\}$. Let $\vec{x} = \begin{pmatrix} z_{k-m+1} \\ z_{k-m+2} \\ \vdots \\ z_k \end{pmatrix}$, let y be an entry of $A\vec{x}$, and let $a \in F$.

Let $v' = y + av$. Choose $w_{k-m+1}, w_{k-m+2}, \ldots, w_k \in F$ such that $y = \Sigma_{i=k-m+1}^{k} w_i z_i$. Note that if $u \in P(k-m)$ and $w \in F$, then $uw \in P(k)$.

Let $r \in \{0, 1, \ldots, k-m-1\}$ and choose $u_i \in P(k-m)$ for each $i \in \{r+2, r+3, \ldots, k-m+1\}$. Then $z_{r+1} + \Sigma_{i=r+2}^{k-m} u_i z_i + u_{k-m+1} v' = z_{r+1} + \Sigma_{i=r+2}^{k-m} u_i z_i + \Sigma_{i=k-m+1}^{k} u_{k-m+1} w_i z_i + u_{k-m+1} av \in S_n$. □

2.8 Lemma. *Let $n, k \in \mathbb{N}$. For every $v \in V(n,k)$ there exists $s \in \mathbb{N}$ such that for all $n' > s$ and all $v' \in V(n',k)$, one has $v + v' \in V(n,k)$.*

Proof. Since $v \in V(n,k)$, choose $x_1, x_2, \ldots, x_k \in \mathbb{N}$ such that $x_{r+1} + \Sigma_{i=r+2}^{k} u_i x_i + u_{k+1} v \in S_n$ for every $r \in \{0, 1, \ldots, k-1\}$ and every $u_i \in P(k)$, where $i \in \{r+2, r+3, \ldots, k+1\}$.

We can choose $s \in \mathbb{N}$ such that all these sums as well as v are in $FS(\langle Y_t \rangle_{t=n}^{s})$, because the number of these sums is finite.

Suppose that $n' > s$ and that $v' \in V(n',k)$.

There exist $x'_1, x'_2, \ldots, x'_k \in \mathbb{N}$ such that $x'_{r+1} + \Sigma_{i=r+2}^{k} u_i x'_i + u_{k+1} v' \in S_{n'}$ for every $r \in \{0, 1, \ldots, k-1\}$ and every $u_i \in P(k)$, where $i \in \{r+2, r+3, \ldots, k+1\}$. Now the sum of a number in $FS(\langle Y_t \rangle_{t=n}^{s})$ and a number in $S_{n'}$ is in S_n. Thus, if we put $x''_j = x_j + x'_j$ for each $j \in \{1, 2, \ldots, k\}$, we have $x''_{r+1} + \Sigma_{i=r+2}^{k} u_i x''_i + u_{k+1}(v + v') \in S_n$ for every $r \in \{0, 1, 2, \ldots, k-1\}$ and every $u_i \in P(k)$, where $i \in \{r+2, r+3, \ldots k+1\}$. So $v + v' \in V(n',k)$. □

2.9 Definition. $V = \bigcap_n c\ell_{\beta\mathbb{N}} V(n,n)$.

2.10 Lemma. *V is a compact subsemigroup of $\beta\mathbb{N}$.*

Proof. By Lemma 2.8, for each $n \in \mathbb{N}$ and each $v \in V(n,n)$, there exists $n' \in \mathbb{N}$ such that $n' > n$ and $v + V(n',n) \subseteq V(n,n)$. This implies that $v + V(n',n') \subseteq V(n,n)$, because $V(n',n') \subseteq V(n',n)$ by Lemma 2.6(b). So the result follows from Lemma 2.4. □

The following theorem is the main result of this section.

2.11 Theorem. *Suppose that p is a minimal idempotent in V and that $B \in p$. Let $A \in \mathcal{M}$ be a matrix with m columns whose entries are all in F. Then there is some $\vec{x} \in \mathbb{N}^m$ such that all of the entries of $A\vec{x}$ are in B.*

Proof. We proceed by induction on m. If $m = 1$, A is simply a matrix with a single column all of whose entries are 1. In this case, it is trivial that the theorem is true. We shall assume that it is true for matrices with m columns and deduce that it holds for matrices with $m + 1$ columns.

So let $A' \in \mathcal{M}$ have $m+1$ columns and, say, $r+1$ rows, where all of the entries of A' are in F. We may presume that the last row of A' is $(0\ 0\ 0\ \ldots 0\ 1)$ and that all of the other rows of A' have a 1 somewhere in the first m columns. Let A be the upper left $r \times m$ corner of A'. Then

$$A' = \begin{pmatrix} A & \vec{u} \\ \vec{0} & 1 \end{pmatrix}$$

where $\vec{0}$ is a row with all 0's and $\vec{u} \in F^r$.

As in Lemma 2.3, define $B^* = \{b \in B : -b + B \in p\}$. Then for each $n \in \mathbb{N}$,

$$\bigcap_{s \in B^* \cap T_n} (-s + B^*) \in p$$

because each $-s + B^* \in p$ (by Lemma 2.3) and T_n is finite. For $n \in \mathbb{N}$, let

$$I_n = V(n,n)^r \cap \{A\vec{x} + v\vec{u} : \vec{x} \in \mathbb{N}^m \text{ and } v \in B^* \cap S_n \cap \bigcap_{s \in B^* \cap T_{n-1}} (-s + B^*)\}$$

and let

$$E_n = V(n,n)^r \cap \{A\vec{x} + v\vec{u} : \vec{x} \in \mathbb{N}^m \text{ and } v \in \{0\} \cup \big(B^* \cap S_n \cap \bigcap_{s \in B^* \cap T_{n-1}} (-s + B^*)\big)\}$$

(and agree that $B^* \cap S_1 \cap \bigcap_{s \in B^* \cap T_0} (-s + B^*) = B^* \cap S_1$).

We claim that each $I_n \neq \emptyset$. To see this, note that $V(n+m, n+m) \in p$ and so we can choose $v \in V(n+m, n+m) \cap B^* \cap \bigcap_{s \in B^* \cap T_{n-1}} (-s + B^*)$. Since $v \in V(n, n+m)$, we can choose $\vec{x} \in \mathbb{N}^m$ as guaranteed by Lemma 2.7. Then $A\vec{x} + v\vec{u} \in I_n$.

Let $(\beta\mathbb{N})^r$ have the product topology, let

$$I = \bigcap_{n=1}^{\infty} c\ell_{(\beta\mathbb{N})^r}\, I_n,$$

$$\text{and let } E = \bigcap_{n=1}^{\infty} c\ell_{(\beta\mathbb{N})^r}\, E_n\ .$$

Since, for each n, $E_n \subseteq V(n,n)^r$, one can routinely verify that $E \subseteq V^r$.

Now we claim that $\langle E_n \rangle_{n=1}^{\infty}$ and $\langle I_n \rangle_{n=1}^{\infty}$ satisfy the hypotheses of Lemma 2.4 with $S = (\beta\mathbb{N})^r$. Certainly $\mathbb{N}^r$ is contained in (in fact equal to) the topological centre of S and $I_n \subseteq E_n$ for each n. Further, since $T_{n-1} \subseteq T_n$, $S_{n+1} \subseteq S_n$, and $V(n+1, n+1) \subseteq V(n,n)$ (by Lemma 2.6) we have $I_{n+1} \subseteq I_n$ and $E_{n+1} \subseteq E_n$.

So let $k \in \mathbb{N}$ and let $\vec{b} \in E_k$. We need to show that there is some $n \in \mathbb{N}$ such that $\vec{b} + E_n \subseteq E_k$, $\vec{b} + I_n \subseteq I_k$, and if $\vec{b} \in I_k$, then $\vec{b} + E_n \subseteq I_k$.

Pick $\vec{x} \in \mathbb{N}^m$ and $v \in \{0\} \cup \big(B^* \cap S_k \cap \bigcap_{s \in B^* \cap T_{k-1}} (-s + B^*)\big)$ with $v \neq 0$ if $\vec{b} \in I_k$ such that $\vec{b} = A\vec{x} + v\vec{u} \in V(k,k)^r$. If $v = 0$, let $\ell = k$. Otherwise, pick $\ell \in \mathbb{N}$ such that $v \in FS(\langle Y_t \rangle_{t=k}^{\ell})$. Choose by Lemma 2.8 (applied r times) some $n > \ell$ such that for each $i \in \{1, 2, \ldots, r\}$ and each $z \in V(n,k)$, one has $b_i + z \in V(k,k)$.

Let $\vec{c} \in E_n$ and pick $\vec{y} \in \mathbb{N}^m$ and $w \in \{0\} \cup \big(B^* \cap S_n \cap \bigcap_{s \in B^* \cap T_{n-1}} (-s + B^*)\big)\}$, with $w \neq 0$ if $\vec{c} \in I_n$, such that $\vec{c} = A\vec{y} + w\vec{u} \in V(n,n)^r$.

Then $\vec{c} \in V(n,k)^r$ so $\vec{b} + \vec{c} \in V(k,k)^r$ and $\vec{b} + \vec{c} = A(\vec{x} + \vec{y}) + (v + w)\vec{u}$. Of course if $v = w = 0$ then $v + w = 0$ and if just one of v or w is 0 then trivially $v + w \in S_k \cap B^* \cap \bigcap_{s \in B^* \cap T_{k-1}} (-s + B^*)$. Assume then that neither v nor w is 0. Then $v \in FS(\langle Y_t \rangle_{t=k}^{\ell})$ and $w \in S_n$ so $v + w \in S_k$. Also, $v \in B^* \cap T_\ell \subseteq B^* \cap T_{n-1}$ so $v + w \in B^*$. To see that $v + w \in \bigcap_{s \in B^* \cap T_{k-1}} (-s + B^*)$, let $s \in B^* \cap T_{k-1}$. Then $s + v \in B^*$. Also $s \in T_{k-1}$ and $v \in FS(\langle Y_t \rangle_{t=k}^{\ell})$ and so $s + v \in T_\ell \subseteq T_{n-1}$. Thus $s + v + w \in B^*$ as required.

Define $\overline{p} = (p\, p\, \ldots\, p) \in V^r$. By [3, Corollary 1.2.6] $K(V^r) = K(V)^r$ so $\overline{p} \in K(V^r)$. We claim that $\overline{p} \in E$. To see this, let $n \in \mathbb{N}$ and let U be a neighbourhood of $\overline{p}$ in $\beta\mathbb{N}^r$. For each $i \in \{1, 2, \ldots, r\}$, pick $W_i \in p$ such that

$$\{q \in (\beta\mathbb{N})^r : \text{ for each } i \in \{1, 2, \ldots, r\},\ W_i \in q_i\} \subseteq U\ .$$

Let $W = \bigcap_{i=1}^{r} W_i$. Then $W \in p$ and $V(n,n) \in p$ so by the induction hypothesis, there is some $\vec{x} \in \mathbb{N}^m$ such that all entries of $A\vec{x}$ are in $W \cap V(n,n)$. Then $A\vec{x} \in U \cap E_n$ and so $\overline{p} \in E$ as claimed.

Thus $E \cap K(V^r) \neq \emptyset$ so by [3, Corollary 1.2.15], $K(E) = E \cap K(V^r)$ and thus $\overline{p} \in K(E)$. Since I is an ideal of E by Lemma 2.4, $K(E) \subseteq I$ and so $\overline{p} \in I$. Now $(c\ell_{\beta\mathbb{N}}\, B)^r$ is a neighbourhood of $\overline{p}$ and hence $(c\ell_{\beta\mathbb{N}}\, B)^r \cap I_1 \neq \emptyset$. So pick $\vec{b} \in (c\ell_{\beta\mathbb{N}}\, B)^r \cap I_1$ and pick $\vec{x} \in \mathbb{N}^m$ and $v \in S_1 \cap B^*$ such that $\vec{b} = A\vec{x} + v\vec{u}$. Let $\vec{y} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \\ v \end{pmatrix}$. Then any entry of $A'\vec{y}$ is either v or is an entry of $A\vec{x} + v\vec{u}$. In either case the entry is in B. $\square$

We conclude this section with a derivation of the key step in [12] where Deuber's Theorem was invoked in the process of proving the partition regularity of $(\mathcal{M}, \mathcal{P}, \mathcal{C})$-systems.

2.12 Corollary. *Let $M \subseteq \mathbb{N}$ be a set which is closed under the formation of finite products. Let $\langle C_t\rangle_1^\infty$ be an enumeration of all (m,p,c)-matrices for which $c \in M$. Suppose that C_t is an $\big(m(t),p(t),c(t)\big)$-matrix and that $\vec{\gamma}_t \in \mathbb{N}^{m(t)}$. Let Z_t denote the set of entries of $C_t\vec{\gamma}_t$. Then there is an ultrafilter $q \in \bigcap_n c\ell_{\beta\mathbb{N}} FS\langle Z_t\rangle_{t=n}^\infty$ with the following property: For every $t \in \mathbb{N}$ and every $B \in q$, there is some $\vec{x} \in \mathbb{N}^{m(t)}$ such that the entries of $C_t\vec{x}$ all lie in B.*

Proof. Let $C_t' = \frac{1}{c(t)}C_t$. Let $\mathcal{F}$ denote the set of all finite sets $F \subseteq \mathbb{Q} \cap [0,\infty)$ for which $\{0,1\} \subseteq F$. For each $F \in \mathcal{F}$, let $N_F = \{t \in \mathbb{N}$: all entries of C_t' are finite products of elements of $F\}$ and let $D_{F,n} = \{t \in N_F : t \geq n\}$.

We now show that for each $F \in \mathcal{F}$ there is an idempotent

$$q_F \in \textstyle\bigcap_{n=1}^\infty c\ell FS(\langle Z_t\rangle_{t\in D_{F,n}})$$

such that for every $t \in N_F$ and every $B \in q_F$ there is some $\vec{y} \in \mathbb{N}^{m(t)}$ such that all entries of $C_t'\vec{y}$ lie in B. To see this, let $F \in \mathcal{F}$ be given. Let $\langle\sigma(t)\rangle_{t=1}^\infty$ enumerate N_F in increasing order and for each t let $A_t = C'_{\sigma(t)}$. Then $\langle A_t\rangle_{t=1}^\infty$ enumerates $\mathcal{M}$ where $\mathcal{M}$ is as in Definition 2.2. (Recall that all of the notions defined in Definition 2.2 depend on the choice of F.) For $t \in \mathbb{N}$, let $\vec{\alpha}_t = c\big(\sigma(t)\big)\gamma_{\sigma(t)}$. Then for each t, $A_t\vec{\alpha}_t = C_t\vec{\gamma}_{\sigma(t)}$ so all entries of $A_t\vec{\alpha}_t$ are in $\mathbb{N}$.

Let $\langle Y_t\rangle_{t=1}^\infty$, $\langle S_n\rangle_{n=1}^\infty$, and $\langle\langle V(n,k)\rangle_{n=1}^\infty\rangle_{k=1}^\infty$ be as defined in Definition 2.2 in terms of F and α and let $V = \bigcap_{n=1}^\infty c\ell_{\beta\mathbb{N}}\ V(n,n)$. Pick by Theorem 2.11 an idempotent $q_F \in V$ such that for each $B \in q_F$ and each $t \in \mathbb{N}$, there is some $\vec{x} \in \mathbb{N}^{m(\sigma(t))}$ such that all entries of $A_t\vec{x}$ are in B.

For each n, $V(n,n) \subseteq S_n = FS(\langle Y_t\rangle_{t=n}^\infty) = FS(\langle Z_t\rangle_{t\in D_{F,\sigma(n)}})$ so

$$q_F \in \textstyle\bigcap_{n=1}^\infty c\ell FS(\langle Z_t\rangle_{t\in D_{F,n}})$$

as required.

Now direct $\mathcal{F}$ by inclusion and let q be a limit point of the net $\langle q_F\rangle_{F\in\mathcal{F}}$. To see that q is as required, let $t \in \mathbb{N}$ and let $B \in q$. Let H be the set of entries of C_t' and pick $F \supseteq H$ such that $B \in q_F$. Now q_F is idempotent so $\mathbb{N}c(t) \in q_F$. Pick $\vec{y} \in \mathbb{N}^{m(t)}$ such that all entries of $C_t'\vec{y}$ are in $B \cap \mathbb{N}c(t)$. All rows of the form $(0\ 0\ \ldots\ 0\ 1\ 0\ \ldots\ 0)$ are in C_t' so each entry of $\vec{y}$ is in $\mathbb{N}c(t)$. Let $\vec{x} = \frac{1}{c(t)}\vec{y}$. Then $\vec{x} \in \mathbb{N}^{m(t)}$ and the entries of $C_t\vec{x}$ all lie in B. □

2.13 Corollary. *Let $\langle C_t\rangle_{t=1}^\infty$ and $\langle Z_t\rangle_{t=1}^\infty$ be as described in Corollary 2.12. Whenever $FS(\langle Z_t\rangle_{t=1}^\infty)$ is expressed as the union of of a finite number of sets $D_1, D_2, \ldots, D_r$, there* *will be an* $i \in \{1, 2, \ldots, r\}$ *with the following property: for each* $t \in \mathbb{N}$*, there is some* $\vec{x} \in \mathbb{N}^{m(t)}$ *for which all the entries of* $C_t\vec{x}$ *are in* D_i*.*

Proof. Let q be the ultrafilter described in Corollary 2.12. Since $FS(\langle Y_t \rangle_{t=n}^{\infty}) \in q$, it follows that $D_i \in q$ for some $i \in \{1, 2, \ldots r\}$. □

3. Deuber's Theorem.

In this section, we derive Deuber's Theorem as a consequence of Theorem 2.11. Throughout this section, we shall assume that c is a given positive integer, and that $\langle Y_t \rangle_{t=1}^{\infty}$ is a sequence of (m, p, c^{α})-sets which contains an $(m, p, c^{\alpha})-$set for every choice of $m, p, \alpha \in \mathbb{N}$, if $p \geq c^{\alpha}$.

We first use a standard compactness argument to establish a finitary version of Corollary 2.13.

3.1 Lemma. *Let* $\langle Z_t \rangle_{t=1}^{\infty}$ *be defined as in Corollary* 2.12*. Let* A *be an* (m, p, c)*-matrix and let* $r \in \mathbb{N}$*. There is some* $w \in \mathbb{N}$ *such that whenever* $FS(\langle Z_s \rangle_{s=1}^{w}) = \bigcup_{i=1}^{r} D_i$*, there exist some* $i \in \{1, 2, \ldots, r\}$ *and some* $\vec{x} \in \mathbb{N}^m$ *such that* $A\vec{x}$ *has all its entries in* D_i*.*

Proof. Suppose the conclusion fails. Then for each $n \in \mathbb{N}$ we may choose some $\phi_n : FS(\langle Z_s \rangle_{s=1}^{n}) \longrightarrow \{1, 2, \ldots, r\}$ such that for no $\vec{x} \in \mathbb{N}^m$ and no $j \in \{1, 2, \ldots, r\}$ does one have all the entries of $A\vec{x}$ in $\phi_n^{-1}\{j\}$. For each n, let $T_n = FS(\langle Z_t \rangle_{s=1}^{n})$ and let $T = \bigcup_n T_n$.

We define $\tilde{\phi}_n : T \mapsto \{0, 1, 2, \ldots, r\}$ by stating that $\tilde{\phi}_n = \phi_n$ on T_n and that $\tilde{\phi}_n = \{0\}$ on $T \setminus T_n$. Let ϕ be a limit point of the sequence $\langle \tilde{\phi}_n \rangle_{n=1}^{\infty}$ in the compact space $\{0, 1, 2, \ldots, r\}^T$. By Corollary 2.13, there exists $\vec{x} \in \mathbb{N}^m$ for which all the entries of $A\vec{x} \subseteq \phi^{-1}\{j\}$ for some $j \in \{0, 1, 2, \ldots, r\}$. Now the entries of $A\vec{x}$ are contained in T_n for some $n \in \mathbb{N}$. Since T_n is finite, ϕ coincides with $\tilde{\phi}_{n'}$, and therefore with $\phi_{n'}$, on T_n for some $n' \in \mathbb{N}$ satisfying $n' > n$, a contradiction. □

For the remainder of this paper fix $m, p, c \in \mathbb{N}$ and let A be a given (m, p, c)-matrix.

3.2 Definition. (a) Let $\langle A_t \rangle_{t=1}^{\infty}$ be an enumeration of all (n, q, c^{α})-matrices, where n, q, α vary over $\mathbb{N}$, with A_t being an $(m(t), p(t), c^{\alpha(t)})$ matrix.

(b) Define $s(t)$ for $t \in \mathbb{N}$ inductively by $s(1) = 0$ and $s(t+1) = \Sigma_{i=1}^{t} m(i)$.

(c) For $i \in \mathbb{N}$, $M(i) = \max\{p(t) : s(t) < i\}$.

(d) For each $i \in \mathbb{N}$, we define t_i to be the integer for which $i \in \{s(t_i) + 1, s(t_i) + 2, \ldots, s(t_i + 1)\}$.

(e) We inductively choose a sequence $\langle x_i \rangle_{i=1}^{\infty}$ in $\mathbb{N}$, choosing $x_1 = 1$ and choosing x_{n+1} to satisfy $x_{n+1} > mpM(n+1)\Sigma_{i=1}^{n} x_i$.

(f) S will denote the set of all numbers in ω which can be expressed as sums of the form $\Sigma_{i=1}^{\infty} a_i x_i$, where $a_i \in \{0, 1, 2, \ldots, mpM(i)\}$ for each $i \in \mathbb{N}$.

In the following Lemma we construct a mapping with sufficient linearity to commute with A on part of its domain.

3.3 Lemma. *Given any sequence $\langle u_i \rangle_{i=1}^{\infty}$ in $\mathbb{N}$, we can define a mapping $\phi : S \mapsto [0, \infty)$ by stating that $\phi(z) = \Sigma_{i=1}^{\infty} c^{-\alpha(t_i)} a_i u_i$ if $z = \Sigma_{i=1}^{\infty} a_i x_i$ with $a_i \in \{0, 1, 2, \ldots, mpM(i)\}$ for each i. This mapping has the following property:*

Suppose that, for each $j \in \{1, 2, \ldots, m\}$, $z_j \in [0, \infty)$ satisfies $z_j = \Sigma_{k=1}^{\infty} b_{j,k} x_k$ with $b_{j,k} \in \{0, 1, 2, \ldots, M(k)\}$ for each $j \in \{1, 2, \ldots, m\}$ and each $k \in \mathbb{N}$. Let

$$\vec{z} = \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_m \end{pmatrix} \text{ and } \tilde{\phi}(\vec{z}) = \begin{pmatrix} \phi(z_1) \\ \phi(z_2) \\ \vdots \\ \phi(z_m) \end{pmatrix} .$$

Then $\phi\big((A\vec{z})_i\big) = \big(A\tilde{\phi}(\vec{z})\big)_i$ for each $i \in \{1, 2, \ldots, l\}$, where l denotes the number of rows in A.

Proof. We first show that ϕ is well defined, by showing that the expression of an integer in S in the form described is unique. Suppose then that $\Sigma_{i=1}^{\infty} a_i x_i$ and $\Sigma_{i=1}^{\infty} a_i' x_i$ are finite and equal, where $a_i, a_i' \in \{0, 1, 2, \ldots, mpM(i)\}$, but that the sequences $\langle a_i \rangle_{i=1}^{\infty}$ and $\langle a_i' \rangle_{i=1}^{\infty}$ are not identical. We also suppose that these sequences are chosen so that the number of non-zero terms that they contain is as small as possible subject to these assumptions.

Let $n = \max\{i \in \mathbb{N} : a_i \neq 0\}$ and $n' = \max\{i \in \mathbb{N} : a_i' \neq 0\}$. If $n' > n$, we have $a_{n'}' x_{n'} \geq x_{n'} > mpM(n')\Sigma_{i=1}^{n} x_i$. Hence $a_{n'}' x_{n'} > mpM(n')\Sigma_{i=1}^{n} x_i \geq \Sigma_{i=1}^{n} a_i x_i$, because $a_i \leq mpM(n')$ for each $i \in \{0, 1, 2, \ldots, n\}$. This contradiction shows that $n = n'$. The same argument then shows that $a_n = a_n'$; for $a_n' > a_n$ implies that $a_{n'} - a_n \geq 1$. Thus we can cancel $a_n x_n$ from the equation $\Sigma_{i=1}^{\infty} a_i x_i = \Sigma_{i=1}^{\infty} a_i' x_i$, contradicting our choice of the sequences $\langle a_i' \rangle_{i=1}^{\infty}$ and $\langle a_i \rangle_{i=1}^{\infty}$. So ϕ is well defined.

Let $a_{i,j}$ denote the entry in the i^{th} row and j^{th} column of the matrix A. We note that $a_{i,j} \leq p$ for every $i \in \{1, 2, \ldots, l\}$ and every $j \in \{1, 2, \ldots, m\}$. Thus $\Sigma_{j=1}^{m} a_{i,j} z_j = \Sigma_{k=1}^{\infty} \big(\Sigma_{j=1}^{m} a_{i,j} b_{j,k}\big) x_k$ with $\Sigma_{j=1}^{m} a_{i,j} b_{j,k} \leq mpM(k)$. So $\phi\big((A\vec{z})_i\big) = \phi\big(\Sigma_{j=1}^{m} a_{i,j} z_j\big) = \Sigma_{k=1}^{\infty} c^{-\alpha(t_k)} \big(\Sigma_{j=1}^{m} a_{i,j} b_{j,k}\big) u_k = \Sigma_{j=1}^{m} a_{i,j} \big(\Sigma_{k=1}^{\infty} c^{-\alpha(t_k)} b_{j,k} u_k\big) = \big(A\tilde{\phi}(\vec{z})\big)_i$. □

We are now in a position to derive Deuber's Theorem.

3.4 Theorem (Deuber). *Let $m, p, c, r \in \mathbb{N}$. There exist $n, q, \mu \in \mathbb{N}$ such that whenever D is an (n, q, c^μ)-set and $D = \bigcup_{k=1}^r D_k$, there exist $k \in \{1, 2, \ldots, r\}$ and an (m, p, c)-set C such that $C \subseteq D_k$.*

Proof. We put take Z_t to be the set of entries in $A_t\vec{\gamma}_t$ where $\vec{\gamma}_t = \begin{pmatrix} x_{s(t)+1} \\ x_{s(t)+2} \\ \vdots \\ x_{s(t+1)} \end{pmatrix}$. Note that, since $s(t+1) = s(t) + m(t)$, $\vec{\gamma}_t \in \mathbb{N}^{m(t)}$.

By Lemma 3.1 we can choose $w \in \mathbb{N}$ so that whenever $FS(\langle Z_t\rangle_{t=1}^w)$ is expressed as the union of r sets, one of these sets will contain all of the entries of $A\vec{z}$ for some $\vec{z} \in \mathbb{N}^m$.

Let $n = s(w+1)$ (which is $\Sigma_{i=1}^w m(i)$), let $\mu = \max\{\alpha(t) : t \in \{1, 2, \ldots, w\}\}$, and let $q = \max\{p(t)c^{\mu-\alpha(t)} : t \in \{1, 2, \ldots, w\}\}$. We shall show that n, q, and μ are as required.

We define a first entries matrix B with n columns as follows. For each $t \in \{1, 2, \ldots, w\}$, let $A'_t = \frac{1}{c^{\alpha(t)}}A_t$. The rows of B are all rows that can be written in block form as $\big(\theta(1)\ \theta(2)\ \ldots\ \theta(w)\big)$ where each $\theta(t)$ is either a row of A'_t or is a row of $m(t)$ 0's and and at least one $\theta(t)$ is not a row of 0's. For each $i \in \{1, 2, \cdots, n\}$, we put $v_i = c^{\alpha(t_i)}x_i$. Let $\vec{v} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}$ and observe that the set of entries of $B\vec{v}$ is precisely the set $FS(\langle Z_t\rangle_{t=1}^w)$.

Observe also that the rows of $c^\mu B$ are some of the rows of an (n, q, c^μ)-matrix. (Given a nonzero entry of B it is $a/c^{\alpha(t)}$ for some entry a of some A_t. Since $a \leq p(t)$, we have $c^\mu a/c^{\alpha(t)} \leq p(t)c^{\mu-\alpha(t)} \leq q$.)

Now let D be an (n, q, c^μ)-set and assume that $D = \bigcup_{k=1}^r D_k$. Since the rows of $c^\mu B$ are rows of an (n, q, c^μ)-matrix, choose $\vec{u} \in \mathbb{N}^n$ such that all of the entries of $c^\mu B\vec{u}$ are in D. Then for $i \leq n$ we have u_i is the i^{th} entry of $\vec{u}$. For $i > n$ let u_i be any member of $\mathbb{N}$. Let ϕ denote the mapping defined in Lemma 3.3 and let $b_{i,j}$ denote the entry in the i^{th} row and j^{th} column of B. Recall that we have defined t_j so that $j \in \{s(t_j) + 1, s(t_j) + 2, \ldots, s(t_j + 1)\}$. Given a row i and a column j of B, we know that $b_{i,j}$ is an entry of A'_{t_j} and so $c^{\alpha(t_j)}b_{i,j} \leq p(t_j)$ and $s(t_j) < j$ so $c^{\alpha(t_j)}b_{i,j} \in \{0, 1, \ldots, M(j)\}$.

We claim that ϕ maps the entries of $B\vec{v}$ into the entries of $B\vec{u}$. To see this, let i be a row of B. Then the i^{th} entry of $B\vec{v}$ is $\Sigma_{j=1}^n b_{i,j}v_j = \Sigma_{j=1}^n b_{i,j}c^{\alpha(t_j)}x_j$ and $\phi(\Sigma_{j=1}^n b_{i,j}c^{\alpha(t_j)}x_j) = \Sigma_{j=1}^n b_{i,j}u_j$. So the claim is established.

Since the set of entries of $B\vec{v}$ is $FS(\langle Z_t\rangle_{t=1}^w)$, we have

$$FS(\langle Z_t\rangle_{t=1}^w) \subseteq \bigcup_{k=1}^r \phi^{-1}[c^{-\mu}D_k]\ .$$

By the choice of w, pick $k \in \{1,2,\ldots,r\}$ and $\vec{z} \in \mathbb{N}^m$ such that the entries of $A\vec{z}$ are contained in $\phi^{-1}[c^{-\mu}D_k] \cap FS(\langle Z_t\rangle_{t=1}^w)$.

Now, given any entry z_h of $\vec{z}$, cz_h is an entry of $A\vec{z}$, and therefore an entry of $B\vec{v}$, and so for some i, $cz_h = \Sigma_{j=1}^n b_{i,j}v_j = \Sigma_{j=1}^n b_{i,j}c^{\alpha(t_j)}x_j$, where, as we have seen, $b_{i,j}c^{\alpha(t_j)} \in \{0,1,\ldots,M(j)\}$. Then $c\vec{z}$ satisfies the hypotheses of Lemma 3.3 and so, for each $i \in \{1,2,\ldots,l\}$, we have

$$\phi\big((Ac\vec{z})_i\big) = \big(A\tilde{\phi}(c\vec{z})\big)_i\ .$$

To conclude the proof, we show that all entries of $Ac^\mu\tilde{\phi}(\vec{z})$ are in D_k. So let $i \in \{1,2,\ldots,l\}$. Then $(A\vec{z})_i \in \phi^{-1}[c^{-\mu}D_k]$. So $c^\mu\phi\big((A\vec{z})_i\big) \in D_k$. Since

$$\begin{aligned} c^\mu\phi\big((A\vec{z})_i\big) &= c^{\mu-1}\phi\big((Ac\vec{z})_i\big) \\ &= c^{\mu-1}\phi\big(A\tilde{\phi}(c\vec{z})\big)_i \\ &= Ac^\mu\tilde{\phi}(\vec{z})_i\ , \end{aligned}$$

we have $Ac^\mu\tilde{\phi}(\vec{z})_i \in D_k$ as required. □

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