# Preserving integer images of matrices

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**Abstract**

Let $u, v \in \mathbb{N}$ and let $A$ be a $u \times v$ matrix with integer entries and rank $n$. In [2] it was shown that there is a $u \times 2v$ matrix $B$ with integer entries such that $\{A\vec{x} : \vec{x} \in \mathbb{Z}^v\} = \{B\vec{y} : \vec{y} \in \mathbb{N}^{2v}\}$ and in particular

$$\{A\vec{x} : \vec{x} \in \mathbb{Z}^v\} \cap \mathbb{N}^u = \{B\vec{y} : \vec{y} \in \mathbb{N}^{2v}\} \cap \mathbb{N}^u.$$

As a consequence, $A$ is weakly image partition regular if and only if $B$ is image partition regular. It was asked whether one could replace $B$ by a matrix with no more columns than $A$ in the second equation. In [1] this question was answered in the affirmative by producing a $u \times n$ matrix $B$ with $\{A\vec{x} : \vec{x} \in \mathbb{Z}^v\} \cap \mathbb{N}^u = \{B\vec{y} : \vec{y} \in \mathbb{N}^n\} \cap \mathbb{N}^u$. It is a consequence of elementary linear algebra that one cannot get a matrix $B$ with $n$ columns such that $\{A\vec{x} : \vec{x} \in \mathbb{Z}^v\} = \{B\vec{y} : \vec{y} \in \mathbb{N}^n\}$. We show in this paper that one can get such a matrix with $n + 1$ columns.

## 1 Introduction

The original motivation for the results of this paper comes from the study of partition regularity of matrices. That is, from the relationships among kernel partition regular matrices, image partition regular matrices, and weakly image partition regular matrices. These notions are not directly relevant to the results of this paper, so we refer the reader to Section 1 of [1] for an overview of these relations.

Our main results depend on Theorem 1.2 which says that, given a $u \times v$ matrix $A$ with integer entries and rank $n$, one can get a $u \times n$ matrix $C$ with integer entries such that for any abelian group $(G, +)$, $\{A\vec{x} : x \in G^v\} = \{C\vec{y} : \vec{y} \in G^n\}$, and in particular, $\{A\vec{x} : \vec{x} \in \mathbb{Z}^v\} = \{C\vec{y} : \vec{y} \in \mathbb{Z}^n\}$. (If one allowed the entries of $\vec{x}$ and $\vec{y}$ to come from $\mathbb{Q}$, this is a well known conclusion from linear algebra. The fact that one wants the entries to be integers comes from the

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motivation in terms of partition regularity.) The proof of Theorem 1.2 depends on the following lemma extending a result from [1].

**Lemma 1.1.** *Let* $u, v \in \mathbb{N} \setminus \{1\}$ *and let* $A$ *be a* $u \times v$ *matrix with entries from* $\mathbb{Z}$. *Let* $s \in \{1, 2, \ldots, u\}$ *and let* $t, r \in \{1, 2, \ldots, v\}$ *with* $t \neq r$ *and assume that* $a_{s,t} \neq 0$. *For* $i \in \{1, 2, \ldots, u\}$ *and for* $j \in \{1, 2, \ldots, v\} \setminus \{t, r\}$, *let* $b_{i,j} = a_{i,j}$. *Let* $w = \gcd(a_{s,t}, a_{s,r})$ *and pick* $p$ *and* $q$ *in* $\mathbb{Z}$ *such that* $w = pa_{s,t} + qa_{s,r}$. *For* $i \in \{1, 2, \ldots, u\}$, *let* $b_{i,t} = pa_{i,t} + qa_{i,r}$ *and let* $b_{i,r} = (a_{s,t}a_{i,r} - a_{s,r}a_{i,t})/w$. *Then*

*(1) the entries of* $B$ *are integers;*

*(2)* $b_{s,t} = w$ *and* $b_{s,r} = 0$;

*(3) if* $(G, +)$ *is any abelian group, then* $\{B\vec{x} : \vec{x} \in G^v\} = \{A\vec{k} : \vec{k} \in G^v\}$;

*(4) the matrix* $B$ *is obtainable from* $A$ *by at most four standard column operations;*

*(5) the matrices* $A$ *and* $B$ *have the same linear dependencies among their rows; and*

*(6) if* $u = v$, *then* $\det(B) = \det(A)$.

*Proof.* Except for conclusion (3), this is [1, Lemma 2.1]. We verify conclusion (3).

($\subseteq$) Let $\vec{x} \in G^v$. For $j \in \{1, 2, \ldots, v\} \setminus \{t, r\}$ (if any) let $k_j = x_j$. Let $k_t = px_t - \frac{a_{s,r}}{w}x_r$ and let $k_r = qx_t + \frac{a_{s,t}}{w}x_r$. Then $\vec{k} \in G^v$ and for each $i \in \{1, 2, \ldots, u\}$, a simple calculation establishes that $\sum_{j=1}^{v} a_{i,j}k_j = \sum_{j=1}^{v} b_{i,j}x_j$.

($\supseteq$) Let $\vec{k} \in G^v$ and let $D = \{1, 2, \ldots, v\} \setminus \{t, r\}$. For $j \in D$, if any, let $x_j = k_j$. Let $x_t = \frac{a_{s,t}}{w}k_t + \frac{a_{s,r}}{w}k_r$ and let $x_r = pk_r - qk_t$. Then $\vec{x} \in G^v$. Let $i \in \{1, 2, \ldots, u\}$. Then

$$
\begin{aligned}
\textstyle\sum_{j=1}^{v} b_{i,j}x_j &= b_{i,t}(\tfrac{a_{s,t}}{w}k_t + \tfrac{a_{s,r}}{w}k_r) + b_{i,r}(pk_r - qk_t) + \textstyle\sum_{j\in D} a_{i,j}k_j \\
&= (pa_{i,t} + qa_{i,r})(\tfrac{a_{s,t}}{w}k_t + \tfrac{a_{s,r}}{w}k_r) + (\tfrac{a_{s,t}a_{i,r}}{w} - \tfrac{a_{s,r}a_{i,t}}{w})(pk_r - qk_t) \\
&\quad + \textstyle\sum_{j\in D} a_{i,j}k_j \\
&= a_{i,t}\tfrac{pa_{s,t}+qa_{s,r}}{w}k_t + a_{i,r}\tfrac{pa_{s,t}+qa_{s,r}}{w}k_r + \textstyle\sum_{j\in D} a_{i,j}k_j \\
&= \textstyle\sum_{j=1}^{v} a_{i,j}k_j \,.
\end{aligned}
$$

□

The proof of Theorem 1.2 proceeds using Lemma 1.1 by a procedure analogous to Gaussian elimination. In fact, as long as one is reasonably liberal regarding how one defines Gaussian elimination, the matrix $C$ is the transpose of a matrix which could have been obtained from the transpose of $A$ by Gaussian elimination. But one would not normally expect the entries of a matrix obtained by Gaussian elimination to be integers.

**Theorem 1.2.** *Let $u, v \in \mathbb{N}\backslash\{1\}$, let $A$ be a $u \times v$ matrix with integer entries and rank $n$, and assume that the first $n$ rows of $A$ are linearly independent. There is a $u \times n$ matrix $C$ with integer entries such that the first $n$ rows of $C$ are linearly independent and for any abelian group $(G, +)$, $\{A\vec{x} : \vec{x} \in G^v\} = \{C\vec{y} : \vec{y} \in G^n\}$.*

*Proof.* Repeatedly apply Lemma 1.1, switching columns as need be, to obtain a $u \times v$ matrix $B$ with integer entries whose first $n$ rows and columns form a lower triangular matrix. At this stage, the last $v - n$ columns of $B$ are all $\vec{0}$ because the linear dependencies of the rows of $B$ are the same as the rows of $A$. If some entry of the last $v - n$ columns of $B$ were non zero, then there would be $n + 1$ linearly independent rows of $B$. Let $C$ be the $u \times n$ matrix consisting of the first $n$ columns of $B$. □

Note that the proof of Theorem 1.2 is constructive. We illustrate the calculations for a $4 \times 4$ matrix of rank 3.

Begin with

$$A = \begin{pmatrix} -12 & 30 & 4 & 14 \\ -6 & 15 & 4 & 5 \\ 1 & 0 & -1 & 2 \\ 2 & 1 & 5 & -2 \end{pmatrix}$$

We apply Lemma 1.1 with $s = t = 1$ and $r = 2$. Then $a_{s,t} = -12$ and $a_{s,r} = 30$ so $w = 6$. Taking $p = 2$ and $q = 1$ yields the following.

$$B = \begin{pmatrix} 6 & 0 & 4 & 14 \\ 3 & 0 & 4 & 5 \\ 2 & -5 & -1 & 2 \\ 5 & -12 & 5 & -2 \end{pmatrix}$$

Notice that the new column 2 does not depend on the choice of $p$ and $q$ and, of course, neither do the new columns 3 and 4. But the entries below row 1 of column 1 do depend on the choice of $p$ and $q$. If we had chosen $p = -3$ and $q = -1$, column 1 of $B$ would have been

$$\begin{pmatrix} 6 \\ 3 \\ -3 \\ -7 \end{pmatrix}.$$

Replacing $A$ by $B$ and applying Lemma 1.1 with $s = t = 1$ and $r = 3$, yields

$w = 2$. Taking $p = 1$ and $q = -1$ yields the following.

$$B = \begin{pmatrix} 2 & 0 & 0 & 14 \\ -1 & 0 & 6 & 5 \\ 3 & -5 & -7 & 2 \\ 0 & -12 & 5 & -2 \end{pmatrix}$$

Replacing $A$ by $B$ and applying Lemma 1.1 with $s = t = 1$ and $r = 4$, yields $w = 2$. Taking $p = -6$ and $q = 1$ yields the following.

$$B = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 11 & 0 & 6 & 12 \\ -16 & -5 & -7 & -19 \\ -2 & -12 & 5 & -2 \end{pmatrix}$$

Since we are working toward column echelon form, we interchange columns 2 and 3.

$$B = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 11 & 6 & 0 & 12 \\ -16 & -7 & -5 & -19 \\ -2 & 5 & -12 & -2 \end{pmatrix}$$

Replacing $A$ by $B$ and applying Lemma 1.1 with $s = t = 2$ and $r = 4$, yields $w = 6$. Taking $p = 1$ and $q = 0$ yields the following.

$$B = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 11 & 6 & 0 & 0 \\ -16 & -7 & -5 & -5 \\ -2 & 5 & -12 & -12 \end{pmatrix}$$

Replacing $A$ by $B$ and applying Lemma 1.1 with $s = t = 3$ and $r = 4$, yields $w = 5$. Taking $p = -1$ and $q = 0$ yields the following.

$$B = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 11 & 6 & 0 & 0 \\ -16 & -7 & 5 & 0 \\ -2 & 5 & 12 & 0 \end{pmatrix}$$

Notice that the original matrix $A$ and the final matrix $B$ satisfy $\vec{r}_4 = \frac{-107}{60}\vec{r}_1 + \frac{218}{60}\vec{r}_2 + \frac{144}{60}\vec{r}_3$, where $\vec{r}_i$ is row $i$.

Further, doing some elementary computations that one probably hasn't done since eighth grade, one sees that

(1) given any $\vec{x} \in G^4$, if $y_1 = -6x_1 + 15x_2 + 2x_3 + 7x_4$, $y_2 = 10x_1 - 25x_2 - 3x_3 - 12x_4$, and $y_3 = 5x_1 - 13x_2 - 2x_3 - 6x_4$, then $A\vec{x} = B\vec{y}$ and

(2) given any $\vec{y} \in G^3$ and letting $x_4$ be any member of $G$, if $x_1 = -11y_1 - 4y_2 - 5y_3 - x_4$, $x_2 = -5y_1 - 2y_2 - 2y_3 - x_4$, and $x_3 = 5y_1 + 3y_2 + x_4$, then $A\vec{x} = B\vec{y}$.

The point is that all of the coefficients are integers, even though the linear dependencies do not involve integers.

## 2 Preserving integer images

The first result is very simple and quite general. We will assume that a commutative cancellative semigroup $S$ is contained in its group of differences. The reason we deal with $S \setminus \{0\}$ rather than $S$ is that for the applications of partition regularity, one typically wants the entries of $\vec{y}$ to be nonzero. (Consider van der Waerden's Theorem with increment equal to 0.)

**Theorem 2.1.** *Let $u, v \in \mathbb{N}$, let $A$ be a $u \times v$ matrix with integer entries, let $(S, +)$ be a commutative cancellative semigroup such that $S \setminus \{0\}$ has at least $v+1$ members, and let $G = S - S$, its group of differences. There is a $u \times (v+1)$ matrix $B$ with integer entries such that $\{A\vec{k} : \vec{k} \in G^v\} = \{B\vec{x} : \vec{x} \in (S \setminus \{0\})^{v+1}\}$.*

*Proof.* For $i \in \{1, 2, \ldots, u\}$, let $b_{i,j} = a_{i,j}$ if $j \in \{1, 2, \ldots, v\}$ and let $b_{i,v+1} = -\sum_{j=1}^{v} a_{i,j}$.

($\subseteq$) Let $\vec{k} \in G^v$. For each $i \in \{1, 2, \ldots, v\}$, pick $a_i$ and $b_i$ in $S$ such that $k_i = a_i - b_i$. Let $t = \sum_{i=1}^{v} b_i$ and for $i \in \{1, 2, \ldots, v\}$, let $z_i = k_i + t$. Note that each $z_i \in S$. Pick $s \in S \setminus \{-t, -z_1, -z_2, \ldots, -z_v\}$, let $x_j = k_j + t + s$ for $j \in \{1, 2, \ldots, v\}$ and let $x_{v+1} = t + s$. Then $\vec{x} \in (S \setminus \{0\})^{v+1}$ and $B\vec{x} = A\vec{k}$.

($\supseteq$) Let $\vec{x} \in (S \setminus \{0\})^{v+1}$. For $j \in \{1, 2, \ldots, v\}$, let $k_j = x_j - x_{v+1}$. Then $A\vec{k} = B\vec{x}$. □

We do not know whether the requirement that $S \setminus \{0\}$ has at least $v + 1$ members is necessary in general.

**Conjecture 2.2.** *Let $v \in \mathbb{N}$, let $S = G = \mathbb{Z}_{v+1}$, and let $A$ be the $v \times v$ identity matrix. There does not exist a $v \times (v + 1)$ matrix $B$ with integer entries such that $\{A\vec{k} : \vec{k} \in G^v\} = \{B\vec{x} : \vec{x} \in (S \setminus \{0\})^{v+1}\}$.*

Note that the conjecture is trivially true if $v = 1$ or $v = 2$ since in those cases $|\{A\vec{k} : \vec{k} \in G^v\}| > |\{B\vec{x} : \vec{x} \in (S \setminus \{0\})^{v+1}\}|$. We can verify the conjecture for arbitrarily large values of $v$. Key to the proof is the fact that when $v + 1$ is a prime, $\mathbb{Z}_{v+1}$ is a field, so one can do all of the linear algebra that one is accustomed to doing. Note that when we speak of rank or inverse of a matrix, we are doing the computations in $\mathbb{Z}_{v+1}$. For example, let $A = \begin{pmatrix} 2 & 3 \\ 4 & 1 \end{pmatrix}$. As a matrix over $\mathbb{Q}$, the determinant of $A$ is $-10$ and its rank is 2. As a matrix over $\mathbb{Z}_5$, the determinant of $A$ is 0 and its rank is 1.

**Theorem 2.3.** *Let $v \in \mathbb{N}$ such that $v+1$ is prime, let $S = G = \mathbb{Z}_{v+1}$, and let $A$ be the $v \times v$ identity matrix. There does not exist a $v \times (v+1)$ matrix $B$ with integer entries such that $\{A\vec{k} : \vec{k} \in G^v\} = \{B\vec{x} : \vec{x} \in (S \setminus \{0\})^{v+1}\}$.*

*Proof.* Suppose we have such $B$. We may presume that the entries of $B$ come from $\mathbb{Z}_{v+1}$. Assume first that the rank of $B$ is $n < v$. We may presume that the first $n$ rows of $B$ are linearly independent over $\mathbb{Z}_{v+1}$ and that row $n+1$ of $B$ is a nontrivial linear combination of the first $n$ rows. Define $\vec{k} \in G^v$ by $k_i = 0$ if $i \in \{1, 2, \ldots, v\} \setminus \{n+1\}$ and $k_{n+1} = 1$. Then $\vec{k} \notin \{B\vec{x} : \vec{x} \in (S \setminus \{0\})^{v+1}\}$.

Now assume that the rank of $B$ is $v$. We may presume that the first $v$ columns of $B$ are linearly independent. Let $D$ consist of the first $v$ columns of $B$, so that $B = \begin{pmatrix} D & \vec{c}_{v+1} \end{pmatrix}$. Let $\vec{u} = D^{-1}\vec{c}_{v+1}$ and define $\vec{z} \in G^v$ by $z_i = i \cdot u_i$ for $i \in \{1, 2, \ldots, v\}$ and let $\vec{k} = D\vec{z}$.

We claim that $\vec{k} \notin \{B\vec{x} : \vec{x} \in (S \setminus \{0\})^{v+1}\}$. So suppose we have $\vec{x} \in (S \setminus \{0\})^{v+1}$ such that $B\vec{x} = \vec{k}$. Let $\vec{y} \in (S \setminus \{0\})^v$ consist of the first $v$ entries of $\vec{x}$. Then $D\vec{y} + \vec{c}_{v+1}x_{v+1} = \vec{k}$ so $\vec{y} + \vec{u}x_{v+1} = \vec{y} + D^{-1}\vec{c}_{v+1} = D^{-1}\vec{k} = \vec{z}$. Now $x_{v+1} = i$ for some $i \in \{1, 2, \ldots, v\}$ so $x_i + i \cdot u_i = z_i = i \cdot u_i$ and thus $x_i = 0$, a contradiction. $\square$

We illustrate the computations in the proof of Theorem 2.3 with an example for $v = 4$ so computations are in $\mathbb{Z}_5$. Let

$$B = \begin{pmatrix} 3 & 1 & 2 & 0 & 1 \\ 1 & 3 & 1 & 2 & 0 \\ 0 & 1 & 3 & 4 & 3 \\ 2 & 1 & 0 & 1 & 2 \end{pmatrix}. \text{ Then}$$

$$D = \begin{pmatrix} 3 & 1 & 2 & 0 \\ 1 & 3 & 1 & 2 \\ 0 & 1 & 3 & 4 \\ 2 & 1 & 0 & 1 \end{pmatrix}, \ \vec{c}_5 = \begin{pmatrix} 1 \\ 0 \\ 3 \\ 2 \end{pmatrix}, \text{ and } D^{-1} = \begin{pmatrix} 2 & 2 & 3 & 4 \\ 2 & 0 & 2 & 2 \\ 4 & 2 & 2 & 3 \\ 4 & 1 & 2 & 1 \end{pmatrix}.$$

Then

$$\vec{u} = D^{-1}\vec{c}_5 = \begin{pmatrix} 4 \\ 2 \\ 1 \\ 2 \end{pmatrix}, \ \vec{z} = \begin{pmatrix} 1 \cdot 4 \\ 2 \cdot 2 \\ 3 \cdot 1 \\ 4 \cdot 2 \end{pmatrix} = \begin{pmatrix} 4 \\ 4 \\ 3 \\ 3 \end{pmatrix}, \text{ and } \vec{k} = D\vec{z} = \begin{pmatrix} 2 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Thus if $\vec{x} \in (\mathbb{Z}_5 \setminus \{0\})^5$ and $B\vec{x} = \vec{k}$, one has

$$\begin{pmatrix} x_1 + 4x_5 \\ x_2 + 2x_5 \\ x_3 + x_5 \\ x_4 + 2x_5 \end{pmatrix} = \begin{pmatrix} 4 \\ 4 \\ 3 \\ 3 \end{pmatrix},$$

which is impossible.

**Question 2.4.** *Do there exist $v \in \mathbb{N}$, an abelian group $G$ with $v+1$ members, and a $v \times (v+1)$ matrix $B$ with integer entries such that $\{A\vec{k} : \vec{k} \in G^v\} = \{B\vec{x} : \vec{x} \in (G \setminus \{0\})^{v+1}\}$, where $A$ is the $v \times v$ identity matrix?*

**Corollary 2.5.** *Let $u, v \in \mathbb{N} \setminus \{1\}$ and let $A$ be a $u \times v$ matrix with integer entries and rank $n$. There is a $u \times (n+1)$ matrix $B$ with integer entries such that, if $(S, +)$ is any commutative cancellative semigroup such that $S \setminus \{0\}$ has at least $n+1$ members and $G$ is its group of differences, then $\{A\vec{x} : \vec{x} \in G^v\} = \{B\vec{y} : \vec{y} \in (S \setminus \{0\})^{n+1}\}$. In particular, $\{A\vec{x} : \vec{x} \in \mathbb{Z}^v\} = \{B\vec{y} : \vec{y} \in \mathbb{N}^{n+1}\}$.*

*Proof.* By Theorem 1.2, pick a $u \times n$ matrix $C$ with integer entries such that $\{A\vec{x} : \vec{x} \in G^v\} = \{C\vec{y} : \vec{y} \in G^n\}$. By Theorem 2.1, pick a $u \times (n+1)$ matrix $B$ with integer entries such that $\{C\vec{x} : \vec{x} \in G^n\} = \{B\vec{y} : \vec{y} \in (S \setminus \{0\})^{n+1}\}$. □

We conclude by showing that the $n+1$ columns of Corollary 2.5 are the best possible, even without the requirement that $B$ have integer entries.

**Theorem 2.6.** *Let $u, v \in \mathbb{N} \setminus \{1\}$ and let $A$ be a $u \times v$ matrix with integer entries and rank $n$. Assume that the first $n$ rows of $A$ are linearly independent. There does not exist a $u \times n$ matrix $B$ with rational entries such that $\{A\vec{x} : \vec{x} \in \mathbb{Z}^v\} = \{B\vec{y} : \vec{y} \in \mathbb{N}^n\}$.*

*Proof.* Suppose we have such. By Theorem 1.2, pick a $u \times n$ matrix $C$ with integer entries such that the first $n$ rows of $C$ are linearly independent and $\{A\vec{x} : \vec{x} \in \mathbb{Z}^v\} = \{C\vec{y} : \vec{y} \in \mathbb{Z}^n\}$. Let $C^*$ and $B^*$ be the first $n$ rows of $C$ and $B$ respectively. Then $\{C^*\vec{x} : \vec{x} \in \mathbb{Z}^n\} = \{B^*\vec{y} : \vec{y} \in \mathbb{N}^n\}$.

Case 1. The rank of $B^*$ is $n$. Then $C^*\vec{0} = B^*\vec{y}$ for some $\vec{y} \in \mathbb{N}^n$ while $\vec{0}$ is the unique member of $\mathbb{Q}^n$ solving the equation $B^*\vec{y} = \vec{0}$.

Case 2. The rank of $B^*$ is $r < n$. Then $\{C^*\vec{x} : \vec{x} \in \mathbb{Q}^n\} = \mathbb{Q}^n$ while $\{B^*\vec{y} : \vec{y} \in \mathbb{Q}^n\}$ is an $r$-dimensional subspace of $\mathbb{Q}^n$. Pick $\vec{x} \in \mathbb{Q}^n$ such that $C^*\vec{x} \in \mathbb{Q}^n \setminus \{B^*\vec{y} : \vec{y} \in \mathbb{Q}^n\}$. Pick $d \in \mathbb{N}$ such that $d\vec{x} \in \mathbb{Z}^n$. Then $C^*d\vec{x} \notin \{B^*\vec{y} : \vec{y} \in \mathbb{Q}^n\}$ so $C^*d\vec{x} \notin \{B^*\vec{y} : \vec{y} \in \mathbb{N}^n\}$. □

# References


[1] N. Hindman and K. Pleasant, *Preserving positive integer images of matrices*, New York J. Math. **22** (2016), 1021-1037.

[2] N. Hindman, I. Leader, and D. Strauss, *Duality for image and kernel partition regularity of infinite matrices*, J. Combinatorics, to appear. arXiv:1609.03225