This paper was published in *Topology Proceedings* **42** (2013), 107-119. To the best of my knowledge, this is the final version as it was submitted to the publisher. –NH

THE CENTER AND EXTENDED CENTER OF THE MAXIMAL GROUPS IN THE SMALLEST IDEAL OF $\beta\mathbb{N}$

NEIL HINDMAN AND DONA STRAUSS

Abstract. A good deal is known about the maximal groups in the smallest ideal $K(\beta\mathbb{N})$ of the compact right topological semigroup $(\beta\mathbb{N},+)$. For example they are pairwise isomorphic and highly non-commutative – they contain a copy of the free group on $2^{\mathfrak{c}}$ generators. If q is an idempotent in $K(\beta\mathbb{N})$, then $\mathbb{Z}+q$ is contained in the center of the maximal group $q+\beta\mathbb{N}+q$. We do not know whether that center is equal to $\mathbb{Z}+q$. In this paper we investigate the center of $q+\beta\mathbb{N}+q$ and the *extended center* consisting of all elements of $\beta\mathbb{N}$ that commute with every element of $q+\beta\mathbb{N}+q$. This extended center trivially includes all idempotents r of $\beta\mathbb{N}$ such that $q\leq r$ as well as elements of the form $n+r$ for such r and for $n\in\mathbb{Z}$. We show for example that if those are the only elements of the extended center, then there are no nontrivial continuous homomorphisms from $\beta\mathbb{N}$ to $\beta\mathbb{N}\setminus\mathbb{N}$. This would answer a long standing open question. We include several other open questions.

1. Introduction

Addition on the set $\mathbb{N}$ of positive integers extends to the Stone-Čech compactification $\beta\mathbb{N}$ of $\mathbb{N}$ making $(\beta\mathbb{N},+)$ a right topological semigroup (meaning that for each $p\in\beta\mathbb{N}$, the function $\rho_p:\beta\mathbb{N}\to\beta\mathbb{N}$ is continuous, where $\rho_p(q)=q+p$) with $\mathbb{N}$ contained in its topological center (meaning that for each $n\in\mathbb{N}$, the function $\lambda_n:\beta\mathbb{N}\to\beta\mathbb{N}$ is continuous, where $\lambda_n(q)=n+q$). As with any compact Hausdorff right topological

2010 *Mathematics Subject Classification.* Primary 54D80, 22A15; Secondary 54H13.

Key words and phrases. Stone-Čech compactification, center, extended center, smallest ideal, maximal groups.

The first author acknowledges support received from the National Science Foundation via Grants DMS-0852512 and DMS-1160566.

semigroup, $(\beta\mathbb{N},+)$ has a smallest two sided ideal

$$\begin{aligned} K(\beta\mathbb{N}) &= \bigcup\{L : L \text{ is a minimal left ideal of } \beta\mathbb{N}\} \\ &= \bigcup\{R : R \text{ is a minimal right ideal of } \beta\mathbb{N}\}\,. \end{aligned}$$

Any left ideal contains a minimal left ideal, which is closed, and any right ideal contains a minimal right ideal. If L is a minimal left ideal and R is a minimal right ideal, then $L \cap R$ is a group and $L \cap R = q + \beta\mathbb{N} + q$ where q is the unique idempotent in $L \cap R$. Any two such groups are isomorphic. If q and r are idempotents in the same minimal right ideal, then the restriction of ρ_r to $q + \beta\mathbb{N} + q$ is an isomorphism and a homeomorphism onto $r + \beta\mathbb{N} + r$.

The facts just mentioned about $K(\beta\mathbb{N})$ are true in any compact Hausdorff right topological semigroup. Many additional facts are known about $K(\beta\mathbb{N})$ that do not hold in all such semigroups. We know for example that there are $2^{\mathfrak{c}}$ minimal right ideals and $2^{\mathfrak{c}}$ minimal left ideals and the maximal groups in $K(\beta\mathbb{N})$ each contain a copy of the free group on $2^{\mathfrak{c}}$ generators. We also know that the center of $\beta\mathbb{Z}$ is $\mathbb{Z}$ so if q is an idempotent in $K(\beta\mathbb{N})$, then $\mathbb{Z} + q$ is contained in the center of $q + \beta\mathbb{N} + q$. We do not know whether the center of $q + \beta\mathbb{N} + q$ is equal to $\mathbb{Z} + q$. It is this question which is the primary focus of this paper.

We take the points of $\beta\mathbb{N}$ to be the ultrafilters on $\mathbb{N}$, identifying the principal ultrafilters with the points of $\mathbb{N}$. Given $A \subseteq \mathbb{N}$, the closure $\overline{A}$ of A in $\beta\mathbb{N}$ is $\{p \in \beta\mathbb{N} : A \in p\}$ and $\{\overline{A} : A \subseteq \mathbb{N}\}$ is a basis for the open sets of $\beta\mathbb{N}$. See [4] for an elementary introduction to the topology and the algebraic structure of βS where S is an infinite discrete semigroup, as well as for proofs of all of the facts mentioned in the paragraphs above. (The original references for these facts are [1], [2], [3], [5], [6], and [7].)

Definition 1.1. Let q be an idempotent in $K(\beta\mathbb{N})$. $G_q = q + \beta\mathbb{N} + q$ and $D_q = \{u \in \mathbb{N}^* : (\forall v \in G_q)(u + v = v + u)\}$.

Of course, the center $Z(G_q) = D_q \cap G_q$. We call D_q the *extended center* of G_q.

Definition 1.2. (1) $I = \bigcap_{n=1}^{\infty} \overline{n\mathbb{N}}$.
(2) $\mathbb{H} = \bigcap_{n=1}^{\infty} \overline{2^n\mathbb{N}}$.
(3) For $A \subseteq \beta\mathbb{N}$, $E(A) = \{q \in \beta\mathbb{N} : q + q = q\}$.

In Section 2 of this paper we present some basic results, including the fact that for any $q \in E(\beta\mathbb{N})$, $D_q \subseteq \mathbb{Z} + I$.

In Section 3 we investigate more deeply the structure of D_q, establishing the fact mentioned in the abstract that either there is no nontrivial continuous homomorphism from $\beta\mathbb{N}$ to $\mathbb{N}^* = \beta\mathbb{N} \setminus \mathbb{N}$, or there is a member of $D_q \cap I$ which is not an idempotent. We also include in this section a proof that D_q contains a decreasing sequence of idempotents of order type $(\omega+1)^*$, that is the reverse of $\omega+1$.

Section 4 consists primarily of a derivation of the fact that, if the center of G_q is not trivial, then G_q contains a copy of $\mathbb{Z}\times\mathbb{Z}$.

2. Basic Facts About the Extended Center

We begin by observing that the only elements in the extended center that are not in the center lie outside of the smallest ideal.

Theorem 2.1. *Let $q \in E\big(K(\beta\mathbb{N})\big)$. Then $Z(G_q) = D_q \cap K(\beta\mathbb{N})$.*

Proof. Trivially $Z(G_q) \subseteq D_q \cap K(\beta\mathbb{N})$. For the reverse inclusion, let $x \in D_q \cap K(\beta\mathbb{N})$. Since $q \in K(\beta\mathbb{N})$, $\beta\mathbb{N}+q$ is a minimal left ideal and $q+\beta\mathbb{N}$ is a minimal right ideal so $G_q = (\beta\mathbb{N}+q)\cap(q+\beta\mathbb{N})$. Thus if $x \notin G_q$, then either $x \notin \beta\mathbb{N}+q$ or $x \notin q+\beta\mathbb{N}$. So either x and q are in different minimal left ideals of $\beta\mathbb{N}$ or x and q are in different minimal right ideals of $\beta\mathbb{N}$. In either case, $x+q \neq q+x$. □

The idempotents of $\beta\mathbb{N}$ are partially ordered by the relation $\leq$, defined by $e \leq f$ if and only if $e = e+f = f+e$. By [4, Theorem 2.9] e is minimal with respect to this order if and only if $e \in K(\beta\mathbb{N})$. Further, given any non minimal idempotent e in $\beta\mathbb{N}$, by [4, Theorem 1.60] there is a minimal idempotent $q \in K(\beta\mathbb{N})$ with $q \leq e$. So the following lemma shows that for at least some $q \in K(\beta\mathbb{N})$, $D_q \neq Z(G_q)$.

Lemma 2.2. *Let $q \in E\big(K(\beta\mathbb{N})\big)$. Then $\{e \in E(\beta\mathbb{N}) : q \leq e\} = E(D_q)$.*

Proof. Let $e \in E(\beta\mathbb{N})$ such that $q \leq e$ and let $x \in G_q$. Then $e+x = e+q+x = q+x = x+q = x+q+e = x+e$ and so $e \in D_q$.

Conversely, let $e \in E(D_q)$. Since $e+q = q+e$, $e+q$ is an idempotent in the same minimal right ideal and the same minimal left ideal as q. So $e+q = q+e = q$ and $q \leq e$.

□

The remainder of this section will be devoted to a proof, as a consequence of a more general theorem, that $D_q \subseteq \mathbb{Z}+I$.

It is well known, and routine to verify, that each member of $\mathbb{N}$ has a unique factorial representation. That is, a representation of the form $\sum_{n\in H} a_n \cdot n!$ where H is a finite nonempty subset of $\mathbb{N}$ and for each $n \in H$, $a_n \in \{1,2,\ldots,n\}$.

Definition 2.3. Define $d : \mathbb{N} \to \times_{n=1}^{\infty}\{0,1,\ldots,n\}$ by, for each $y \in \mathbb{N}$, $y = \sum_{n=1}^{\infty} d(y)(n)\cdot n!$. For $y \in \mathbb{N}$, let $\operatorname{supp}_f(y) = \{n \in \mathbb{N} : d(y)(n) \neq 0\}$ and let $c(y) = |\operatorname{supp}_f(y)|$. Let $\widetilde{d} : \beta\mathbb{N} \to \times_{n=1}^{\infty}\{0,1,\ldots,n\}$ and $\widetilde{c} : \beta\mathbb{N} \to \beta\mathbb{N}$ be the continuous extensions of d and c respectively.

Lemma 2.4. *Let $x \in \mathbb{N}^*$. If $x \notin \mathbb{Z} + I$, then*

(1) $\{n \in \mathbb{N} : \widetilde{d}(x)(n) \neq 0\}$ *is infinite,*
(2) $\{n \in \mathbb{N} : \widetilde{d}(x)(n) < n\}$ *is infinite, and*
(3) $\{n \in \mathbb{N} :$ *either* $0 < \widetilde{d}(x)(n) < n$ *or both* $\widetilde{d}(x)(n) = n$ *and* $\widetilde{d}(x)(n+1) = 0\}$ *is infinite.*

Proof. (1) Suppose that $\{n \in \mathbb{N} : \widetilde{d}(x)(n) \neq 0\}$ is finite and let $k = \max\{n \in \mathbb{N} : \widetilde{d}(x)(n) \neq 0\}$. Let $m = \sum_{n=1}^{k} \widetilde{d}(x)(n) \cdot n!$. We claim that $x \in m + I$. To see this, let $l > k$. To see that $x \in -m + \mathbb{N}l!$, pick $A \in x$ such that $\widetilde{d}(x)[\overline{A}\,] \subseteq \times_{n=1}^{l} \pi_n^{-1}[\{\widetilde{d}(x)(n)\}]$ and let $y \in A$. Pick $j > l$ such that $j! > y$. Then

$$\begin{aligned} y - m &= \textstyle\sum_{n=l+1}^{j} d(y)(n) \cdot n! + \sum_{n=1}^{l} \widetilde{d}(x)(n) \cdot n! - \sum_{n=1}^{k} \widetilde{d}(x)(n) \cdot n! \\ &= \textstyle\sum_{n=l+1}^{j} d(y)(n) \cdot n! \end{aligned}$$

since $\widetilde{d}(x)(n) \cdot n! = 0$ for $n > k$.

(2) Suppose that $\{n \in \mathbb{N} : \widetilde{d}(x)(n) < n\}$ is finite and pick $k \in \mathbb{N}$ such that for all $n > k$, $\widetilde{d}(x)(n) = n$. Let $m = 1 + \sum_{n=1}^{k} \big(n - \widetilde{d}(x)(n)\big) \cdot n!$ We claim that $x \in -m + I$. To see this, let $l > k$. To see that $x \in m + \mathbb{N}l!$, pick $A \in x$ such that $\widetilde{d}(x)[\overline{A}\,] \subseteq \times_{n=1}^{l} \pi_n^{-1}[\{\widetilde{d}(x)(n)\}]$ and let $y \in A$. Pick $j > l$ such that $j! > y$. Then $m + y - 1 = \sum_{n=1}^{k} \big(n - \widetilde{d}(x)(n)\big) \cdot n! + \sum_{n=1}^{l} \widetilde{d}(x)(n) \cdot n! + \sum_{n=l+1}^{j} d(y)(n) \cdot n! = \sum_{n=l+1}^{j} d(y)(n) \cdot n! + \sum_{n=1}^{l} n \cdot n!$ and so $(l+1)!$ divides $m + y$.

(3) Assume that $\{n \in \mathbb{N} : 0 < \widetilde{d}(x)(n) < n\}$ is finite. Pick k such that for all $n > k$, $\widetilde{d}(x)(n) \in \{0, n\}$. Then by (1) and (2), both $\{n \in \mathbb{N} : \widetilde{d}(x)(n) = 0\}$ and $\{n \in \mathbb{N} : \widetilde{d}(x)(n) = n\}$ are infinite and consequently $\{n \in \mathbb{N} : \widetilde{d}(x)(n) = n$ and $\widetilde{d}(x)(n+1) = 0\}$ is infinite. □

Lemma 2.5. *Let $x \in \mathbb{N}^*$ and let $q, r \in I$. If $q + x + r \in \mathbb{Z} + I$, then $x \in \mathbb{Z} + I$.*

Proof. Assume that $m \in \mathbb{Z}$, $z \in I$, and $q + x + r = m + z$. Given any $n \in \mathbb{N}$, $\{k \in \mathbb{N} : k \equiv m \pmod{n!}\} \in q + x + r$ and $\{k \in \mathbb{N} : k \equiv 0 \pmod{n!}\} \in q \cap r$, so $\{k \in \mathbb{N} : k \equiv m \pmod{n!}\} \in x$. □

Lemma 2.6. *Let $q \in E\big(K(\beta\mathbb{N})\big)$ and let $x \in Z(G_q)$. Then $x \in \mathbb{Z} + I$.*

Proof. Suppose that $x \notin \mathbb{Z} + I$ and let

$$\begin{aligned} E = \{n \in \mathbb{N} : \quad & \text{either } 0 < \widetilde{d}(x)(n) < n \text{ or both} \\ & \widetilde{d}(x)(n) = n \text{ and } \widetilde{d}(x)(n+1) = 0\}\,. \end{aligned}$$

Then by Lemma 2.4(3), E is infinite so pick $p \in \mathbb{N}^*$ such that $\{n! : n \in E\} \in p$. We shall show that

(a) $\widetilde{c}(x+p+q) = \widetilde{c}(x+q)+1$ and
(b) $\widetilde{c}(q+p+x) = \widetilde{c}(q+x)$.

This will suffice because (using the fact that $x+q = q+x = x$) we will then have that $\widetilde{c}(x+q+p+q) = \widetilde{c}(x+p+q) = \widetilde{c}(x+q)+1 = \widetilde{c}(x)+1 \neq \widetilde{c}(x) = \widetilde{c}(q+x) = \widetilde{c}(q+p+x) = \widetilde{c}(q+p+q+x)$.

To verify (a), it suffices that $\widetilde{c}\circ\rho_{p+q}$ and $\rho_1\circ\widetilde{c}\circ\rho_q$ agree on $\mathbb{N}$, so let $y\in\mathbb{N}$ be given. To see that $\widetilde{c}(y+p+q) = \widetilde{c}(y+q)+1$, it suffices that $\widetilde{c}\circ\lambda_y\circ\rho_q$ is constantly equal to $\widetilde{c}(y+q)+1$ on $\{n! : n\in\mathbb{N}$ and $n > \max\mathrm{supp}_f(y)\}$, so let $n\in\mathbb{N}$ such that $n>\max\mathrm{supp}_f(y)$. To see that $\widetilde{c}(y+n!+q) = \widetilde{c}(y+q)+1$, it suffices that $\widetilde{c}\circ\lambda_{y+n!}$ and $\rho_1\circ\widetilde{c}\circ\lambda_y$ agree on $\mathbb{N}(n+1)!$, so let $z\in\mathbb{N}(n+1)!$. Then $\mathrm{supp}_f(y+n!+z) = \mathrm{supp}_f(y)\cup\{n\}\cup\mathrm{supp}_f(z)$ and $\mathrm{supp}_f(y+z) = \mathrm{supp}_f(y)\cup\mathrm{supp}_f(z)$.

To verify (b), it suffices that $\widetilde{c}\circ\rho_{p+x}$ and $\widetilde{c}\circ\rho_x$ agree on $\mathbb{N}$ so let $y\in\mathbb{N}$ and pick $a\in E$ such that $a>\max\mathrm{supp}_f(y)$. To see that $\widetilde{c}(y+p+x) = \widetilde{c}(y+x)$ it suffices that $\widetilde{c}\circ\lambda_y\circ\rho_x$ is constantly equal to $\widetilde{c}(y+x)$ on $\{n! : n\in E$ and $n>a+2\}$ so let $b\in E$ such that $b>a+2$. Pick $A\in x$ such that $\widetilde{d}[\,\overline{A}\,]\subseteq\bigcap_{n=1}^{b+1}\pi_n^{-1}[\{\widetilde{d}(x)(n)\}]$. To see that $\widetilde{c}(y+b!+x) = \widetilde{c}(y+x)$, it suffices that $\widetilde{c}\circ\lambda_{y+b!}$ agrees with $\widetilde{c}\circ\lambda_y$ on A, so let $z\in A$. We need to show that $c(y+b!+z) = c(y+z)$.

Pick $l>b+1$ such that $l!>z$. Then $z = \sum_{n=b+2}^{l} d(z)(n)\cdot n! + \sum_{n=1}^{b+1}\widetilde{d}(x)(n)\cdot n!$. Since $b>a+2>a>\max\mathrm{supp}_f(y)$, there is no carrying beyond position $a+1$ when the factorial representations of z and y are added. (Either $\widetilde{d}(x)(a)<n$, in which case there is no carrying beyond position a, or $\widetilde{d}(x)(a) = n$ and $\widetilde{d}(x)(a+1) = 0$.) Thus $z+y = \sum_{n=b+2}^{l} d(z)(n)\cdot n! + \sum_{n=a+2}^{b+1}\widetilde{d}(x)(n)\cdot n! + \sum_{n=1}^{a+1} d(z+y)(n)\cdot n!$.

Assume first that $0<\widetilde{d}(x)(b)<n$. Then

$$\begin{aligned}&y+b!+z = \textstyle\sum_{n=b+2}^{l} d(z)(n)\cdot n! + \widetilde{d}(x)(b+1)\cdot(b+1)! +\\ &(\widetilde{d}(x)(b)+1)\cdot b! + \textstyle\sum_{n=a+2}^{b-1}\widetilde{d}(x)(n)\cdot n! + \sum_{n=1}^{a+1} d(z+y)(n)\cdot n!\,,\end{aligned}$$

so $\mathrm{supp}_f(y+b!+z) = \mathrm{supp}_f(y+z)$.

Now assume that $\widetilde{d}(x)(b) = n$ and $\widetilde{d}(x)(b+1) = 0$. Then $y+b!+z =$

$$\textstyle\sum_{n=b+2}^{l} d(z)(n)\cdot n! + (b+1)! + \sum_{n=a+2}^{b-1}\widetilde{d}(x)(n)\cdot n! + \sum_{n=1}^{a+1} d(z+y)(n)\cdot n!\,,$$

so $\mathrm{supp}_f(y+b!+z) = (\mathrm{supp}_f(y+z)\setminus\{b\})\cup\{b+1\}$. □

Theorem 2.7. *Let $u,v,x\in\beta\mathbb{N}$. If x commutes with every member of $u+\beta\mathbb{N}+v$, then either $x\in\mathbb{N}$ or $x\in\mathbb{Z}+I$.*

Proof. Assume that $x\notin\mathbb{N}$. Pick a minimal right ideal R and a minimal left ideal L of $\beta\mathbb{N}$ such that $R\subseteq u+\beta\mathbb{N}$ and $L\subseteq\beta\mathbb{N}+v$. Let q be the identity of $R\cap L$. Then $q+\beta\mathbb{N}+q\subseteq u+\beta\mathbb{N}+v$ and so x commutes with every member of $q+\beta\mathbb{N}+q$ and consequently so does $q+x+q$. (Let

$p \in q + \beta\mathbb{N} + q$. Then $p + q = q + p = p$ so $q + x + q + p = q + x + p + q = q + p + x + q = p + q + x + q$.) By Lemma 2.6, $q + x + q \in \mathbb{Z} + I$ and so by Lemma 2.5, $x \in \mathbb{Z} + I$. □

The following corollary is an immediate consequence of Theorem 2.7.

Corollary 2.8. *Let $q \in E\big(K(\beta\mathbb{N})\big)$. Then $D_q \subseteq \mathbb{Z} + I$.*

3. The Structure of the Extended Center

Section 1.7 of [4] has a large number of results whose hypothesis asserts the existence of a minimal left ideal with an idempotent. The following theorem puts all of those results at our disposal.

Theorem 3.1. *Let $q \in E\big(K(\beta\mathbb{N})\big)$. Then D_q is a semigroup and $D_q \cap G_q$ is both a minimal left ideal of D_q with an idempotent and a minimal right ideal of D_q.*

Proof. Trivially D_q is a semigroup. To see that $D_q \cap G_q$ is an ideal of D_q, let $x \in D_q \cap G_q$ and let $y \in D_q$. Then $y + x = y + q + x = q + y + x = q + y + x + q \in G_q$ and $x + y = x + q + y = x + y + q = q + x + y + q \in G_q$. Since $D_q \cap G_q = Z(G_q)$, $D_q \cap G_q$ is a group and is therefore a minimal left ideal and a minimal right ideal and has an idempotent. □

Among the consequences of the existence of a minimal left ideal in a semigroup is the fact that the smallest ideal exists.

Corollary 3.2. *Let $q \in E\big(K(\beta\mathbb{N})\big)$. Then $D_q \cap G_q = K(D_q)$.*

Proof. Since $K(D_q)$ is the union of all of the minimal left ideals of D_q, we have by Theorem 3.1 that $D_q \cap G_q \subseteq K(D_q)$. Since $D_q \cap G_q$ is a two sided ideal of D_q we have that $K(D_q) \subseteq D_q \cap G_q$. □

Lemma 3.3. *Let $q \in E\big(K(\beta\mathbb{N})\big)$ and let $x \in D_q$. There is some $y \in D_q \cap G_q$ such that $y + x = x + y = q$.*

Proof. Since D_q contains a minimal left ideal with an idempotent, we have by [4, Corollary 1.47 and Theorem 1.56] that $D_q + x$ contains a minimal left ideal of D_q with an idempotent, and this idempotent is in $K(D_q) = D_q \cap G_q$. Since q is the only idempotent in G_q, $q \in D_q + x$. Pick $w \in D_q$ such that $q = w + x$. Let $y = q + w + q$. Then $y \in D_q \cap G_q$. Also, $y + x = q + w + q + x = q + w + x + q = q + q + q = q$. Since $y \in D_q$, we also have that $x + y = q$. □

Theorem 3.4. *Let $q \in E\big(K(\beta\mathbb{N})\big)$ and let $x \in D_q$. Then $G_q = x + G_q + x = x + G_q = G_q + x$.*

Proof. Pick by Lemma 3.3 some $y \in D_q \cap G_q$ such that $y + x = x + y = q$. Since $x \in D_q$, we have that $x + G_q = G_q + x$. We shall show that $G_q \subseteq x + G_q + x \subseteq x + G_q \subseteq G_q$. Let $w \in G_q$. Then $w = q + w + q = x + y + w + y + x \in x + G_q + x$.

To see that $x + G_q + x \subseteq x + G_q$, let $w \in G_q$ and pick $z \in G_q$ such that $z + y = w$. Then $x + w + x = x + z + y + x = x + z + q = x + z$.

To see that $x + G_q \subseteq G_q$, let $w \in G_q$ and pick $z \in G_q$ such that $y + z = w$. Then $x + w = x + y + z = q + z = z$. □

Corollary 3.5. *Let $q \in E\big(K(\beta\mathbb{N})\big)$. For any distinct $x, y \in D_q$, $x \in \beta\mathbb{N} + y$ or $y \in \beta\mathbb{N} + x$.*

Proof. By Theorem 3.4, $q \in (G_q + x) \cap (G_q + y)$. So our claim follows from [4, Corollary 6.21]. □

Corollary 3.6. *Let $q \in E\big(K(\beta\mathbb{N})\big)$. If M is a G_δ subset of $\mathbb{N}^*$, then $D_q \cap M$ is nowhere dense in M. In particular, $D_q \cap I$ is nowhere dense in I and $D_q \cap \mathbb{H}$ is nowhere dense in $\mathbb{H}$.*

Proof. We first observe that $\mathbb{N}^* \setminus (\mathbb{N}^* + \mathbb{N}^*)$ contains a dense open subset U of $\mathbb{N}^*$. This follows from the fact that, if $p \in \mathbb{N}^*$ and $B \in p$, we can choose a sequence $\langle x_n \rangle_{n=1}^{\infty}$ contained in B such that $x_{n+1} - x_n > n$ for every $n \in \mathbb{N}$. So, if $A = \{x_n : n \in \mathbb{N}\}$, $\overline{A} \subseteq \overline{B}$ and, by [4, Exercise 4.1.7], $\overline{A} \cap (\mathbb{N}^* + \mathbb{N}^*) = \emptyset$.

If there exists an element $x \in D_q \cap (\mathbb{N}^* \setminus (\mathbb{N}^* + \mathbb{N}^*))$, then, by Corollary 3.5, for any element $y \in D_q \cap (\mathbb{N}^* \setminus (\mathbb{N}^* + \mathbb{N}^*))$, $x \in \mathbb{N} + y$ or $y \in \mathbb{N} + x$. So $D_q \cap (\mathbb{N}^* \setminus (\mathbb{N}^* + \mathbb{N}^*)) = \mathbb{Z} + x$. Since $\mathbb{Z} + x$ is countable, $\mathbb{Z} + x$ is nowhere dense in $\mathbb{N}^*$ ([4, Corollary 3.37]). Put $V = U$ if no such element x exists; otherwise put $V = U \setminus cl(\mathbb{Z} + x)$. Then V is a dense open subset of $\mathbb{N}^*$ and $V \cap D_q = \emptyset$.

It follows from [4, Theorem 3.36] that $\mathrm{int}_{\mathbb{N}^*}(M)$ is dense in M. So $V \cap M$ is a dense open subset of M disjoint from D_q. □

Given $q \in E\big(K(\beta\mathbb{N})\big)$, we know $\big(\mathbb{Z} + E(D_q)\big) \subseteq D_q$ and we know, by Lemma 2.2 that $E(D_q) = \{e \in E(\beta\mathbb{N}) : q \leq e\}$. So the only things that we know are in $D_q \cap I$ are the idempotents above q. It is a longstanding open problem as to whether there are any nontrivial continuous homomorphisms from $\beta\mathbb{N}$ to $\mathbb{N}^*$. The following theorem connects our lack of knowledge about these two issues.

Theorem 3.7. *If for all $q \in E\big(K(\beta\mathbb{N})\big)$, $D_q \cap I \subseteq E(\beta\mathbb{N})$, then there is no nontrivial continuous homomorphism from $\beta\mathbb{N}$ to $\mathbb{N}^*$.*

Proof. Let $q \in E\big(K(\beta\mathbb{N})\big)$ and suppose that $D_q \cap I \subseteq E(\beta\mathbb{N})$ and that there is a nontrivial continuous homomorphism from $\beta\mathbb{N}$ to $\mathbb{N}^*$. By [4, Corollary 10.20], pick $e \in E(\beta\mathbb{N})$ and $p \neq e$ such that $p + p = p + e =$

$e+p=e$. Pick $q \in E\big(K(\beta\mathbb{N})\big)$ such that $q \leq e$. Then $p+q=p+e+q=e+q=q=q+e=q+e+p=q+p$. We claim that $p \in D_q$, so let $w \in G_q$. Then $p+w=p+q+w=q+w=w=w+q=w+q+p=w+p$. Thus $p \in D_q$. Since $p+q=q$ we also have that $p \in I$. But p is not an idempotent. □

We do not know whether there are any maximal idempotents in $\beta\mathbb{N}$ or even whether there are maximal idempotents in $K(\beta\mathbb{N})$, so as far as we know, it is possible that for some $q \in E\big(K(\beta\mathbb{N})\big)$, the extended center D_q of G_q is equal to the center of G_q. We shall show in Theorem 3.9 that for many $q \in E\big(K(\beta\mathbb{N})\big)$, $D_q \cap c\ell K(\beta\mathbb{N})$ contains an infinite decreasing chain of idempotents. As a corollary, we obtain the fact that $c\ell K(\beta\mathbb{N})$ contains a decreasing sequence of idempotents of reverse order type $\omega+1$. (It was previously known that it contains such a sequence of reverse order type ω.)

Lemma 3.8. *Let R be a minimal right ideal of $\beta\mathbb{N}$. There is an injective sequence $\langle q_n\rangle_{n=1}^{\infty}$ of idempotents in R such that, if p is an accumulation point of $\langle q_n\rangle_{n=1}^{\infty}$, then $p \notin \mathbb{Z}^* + \mathbb{Z}^*$. In particular any accumulation point of $\langle q_n\rangle_{n=1}^{\infty}$ is right cancelable in $\beta\mathbb{Z}$.*

Proof. Pick an injective sequence $\langle v_n\rangle_{n=1}^{\infty}$ in

$$\{2^n : n \in \mathbb{N}\}^* = \overline{\{2^n : n \in \mathbb{N}\}} \setminus \mathbb{N}\,.$$

We claim that $(\beta\mathbb{N}+v_n) \cap (\beta\mathbb{N}+v_m) = \emptyset$ when $n \neq m$. To see this, define $\phi : \mathbb{N} \to \omega$ by $\phi(n) = \max\big(\mathrm{supp}(n)\big)$, where $\mathrm{supp}(n)$ is the binary support of n. That is, $n = \sum_{t \in \mathrm{supp}(n)} 2^t$. Let $\widetilde{\phi} : \beta\mathbb{N} \to \beta\omega$ be the continuous extension of ϕ. By [4, Exercise 3.4.1] $\widetilde{\phi}$ is injective on $\overline{\{2^n : n \in \mathbb{N}\}}$ and by [4, Lemma 6.8], for each $n \in \mathbb{N}$, $\widetilde{\phi}[\beta\mathbb{N}+v_n] = \{\,\widetilde{\phi}(v_n)\,\}$, so the claim is established.

For each $n \in \mathbb{N}$ choose an idempotent $q_n \in R \cap (\beta\mathbb{N}+v_n)$ and note that $q_n \neq q_m$ if $n \neq m$. Let p be an accumulation point of $\langle q_n\rangle_{n=1}^{\infty}$ and suppose that $p = x+y$ for some $x, y \in \mathbb{Z}^*$. By [4, Exercise 4.3.5], $y \in \mathbb{N}^*$. Note that there is at most one $n \in \mathbb{Z}$ such that $n+y \in \mathbb{H}$. (If $n<m$ and $2^k > m-n$, then $(-n+2^k\mathbb{N}) \cap (-m+2^k\mathbb{N}) = \emptyset$.) Let $X = \{n \in \mathbb{Z} : n+y \notin \mathbb{H}\}$. Then $X \in x$. If $n \neq m$, we have that $\widetilde{\phi}(q_n) = \widetilde{\phi}(v_n) \neq \widetilde{\phi}(v_m) = \widetilde{\phi}(q_m)$, so there are at most three values of $n \in \mathbb{N}$ for which

$$\widetilde{\phi}(q_n) \in \{\widetilde{\phi}(y)-1, \widetilde{\phi}(y), \widetilde{\phi}(y)+1\}\,.$$

Let $M = \big\{n \in \mathbb{N} : \widetilde{\phi}(q_n) \notin \{\widetilde{\phi}(y)-1, \widetilde{\phi}(y), \widetilde{\phi}(y)+1\}\big\}$. Then

$$p \in c\ell\{q_n : n \in M\} \cap c\ell(X+y)$$

so by [4, Theorem 3.40], either there is some $n \in X$ such that $n+y \in c\ell\{q_n : n \in M\}$ or there is some $n \in M$ such that $q_n \in c\ell(X+y) = \overline{X}+y$.

Suppose first that we have $n \in X$ such that $n + y \in c\ell\{q_n : n \in M\}$. By [4, Lemma 6.8], $\{q_n : n \in M\} \subseteq \mathbb{H}$, so $n + y \in \mathbb{H}$, contradicting the fact that $n \in X$.

Now assume that we have $n \in M$ such that $q_n \in \overline{X} + y$ and pick $z \in \overline{X}$ such that $q_n = z + y$. Then $\widetilde{\phi}(q_n) \notin \{\widetilde{\phi}(y) - 1\widetilde{\phi}(y), \widetilde{\phi}(y) + 1\}$ so pick $A \in \widetilde{\phi}(q_n)$ such that $\mathbb{N} \setminus A \in \widetilde{\phi}(y) - 1$, $\mathbb{N} \setminus A \in \widetilde{\phi}(y)$, and $\mathbb{N} \setminus A \in \widetilde{\phi}(y) + 1$. Pick $B \in z$ such that $\widetilde{\phi}[\overline{B} + y] \subseteq \overline{A}$ and pick $k \in B$. Then $\widetilde{\phi}(k + y) \in \overline{A}$ so pick $C \in y$ such that $\widetilde{\phi}[k + \overline{C}\,] \subseteq \overline{A}$. Since $\mathbb{N} \setminus A \in \widetilde{\phi}(y) - 1$, $\mathbb{N} \setminus A \in \widetilde{\phi}(y)$, and $\mathbb{N} \setminus A \in \widetilde{\phi}(y) + 1$, pick $D \in y$ such that $\widetilde{\phi}[\overline{D}\,] - 1 \subseteq \overline{\mathbb{N} \setminus A}$, $\widetilde{\phi}[\overline{D}\,] \subseteq \overline{\mathbb{N} \setminus A}$, and $\widetilde{\phi}[\overline{D}\,] + 1 \subseteq \overline{\mathbb{N} \setminus A}$. Pick $r \in C \cap D$ such that $r > k$. Then $\phi(k + r) = \phi(r) - 1$, $\phi(k + r) = \phi(r)$, or $\phi(k + r) = \phi(r) + 1$. Since $\phi(k + r) \in A$, this says that $\phi(r) - 1 \in A$, $\phi(r) \in A$, or $\phi(r) + 1 \in A$, a contradiction.

The "in particular" conclusion follows from [4, Theorem 8.18]. $\square$

Theorem 3.9. *Let R be a minimal right ideal of $\beta\mathbb{N}$. There is a decreasing sequence $\langle p_n \rangle_{n=1}^{\infty}$ of idempotents in $c\ell K(\beta\mathbb{N})$ such that*

$$|\{q \in E(R) : \{p_n : n \in \mathbb{N}\} \subseteq D_q\}| = 2^{\mathfrak{c}}\,.$$

Proof. Choose a sequence $\langle q_n \rangle_{n=1}^{\infty}$ as guaranteed by Lemma 3.8 and pick an accumulation point x of this sequence. Then x is right cancelable in $\beta\mathbb{Z}$. Since each q_n is in R and each idempotent in R is a right identity for R, we have that for each $n \in \mathbb{N}$ and each $p \in E(R)$, $q_n + p = p$, and consequently for each $p \in E(R)$, $x + p = p$. Let $M = \bigcap\{C \subseteq \beta\mathbb{Z} : C$ is a compact subsemigroup of $\beta\mathbb{Z}$ and $x \in C\}$. Note that $M \subseteq \beta\mathbb{N}$. For each $p \in E(R)$, $\{z \in \beta\mathbb{N} : z + p = p\}$ is a compact subsemigroup of $\beta\mathbb{Z}$ with x as a member, so we have that for all $z \in M$ and all $p \in E(R)$, $z + p = p$.

By [4, Corollary 8.54], pick a decreasing sequence $\langle p_n \rangle_{n=1}^{\infty}$ in M and let w be a cluster point of $\langle p_n \rangle_{n=1}^{\infty}$. By [4, Lemma 9.22], w is right cancelable in $\beta\mathbb{Z}$ and for each $n \in \mathbb{N}$, $w \in \beta\mathbb{Z} + p_n$. By [4, Theorem 6.56], $\beta\mathbb{N} + w$ contains $2^{\mathfrak{c}}$ pairwise disjoint left ideals. Let L be one of these and pick an idempotent $q \in R \cap L$. To complete the proof, we show that for each $n \in \mathbb{N}$, $q \leq p_n$ (so by Corollary 2.2, $\{p_n : n \in \mathbb{N}\} \subseteq D_q$). Let $n \in \mathbb{N}$. Then $L \subseteq \beta\mathbb{N} + w \subseteq \beta\mathbb{Z} + p_n$ so $q + p_n = q$. Also, $p_n \in M$ and $q \in E(R)$, so $p_n + q = q$. $\square$

Corollary 3.10. *There exist decreasing chains of idempotents in $c\ell K(\beta\mathbb{N})$ of reverse order type $\omega + 1$.*

Proof. Pick a minimal right ideal R of $\beta\mathbb{N}$, pick a sequence $\langle p_n \rangle_{n=1}^{\infty}$ as guaranteed by Theorem 3.9, and pick $q \in E(R)$ such that

$$\{p_n : n \in \mathbb{N}\} \subseteq D_q\,.$$

By Corollary 2.2, given $n \in \mathbb{N}$, $q \leq p_n$ and since $p_{n+1} < p_n$, $q < p_n$. $\square$

4. Copies of $\mathbb{Z} \times \mathbb{Z}$ in G_q

We know, of course, that if $q \in E\big(K(\beta\mathbb{N})\big)$, then the center of G_q contains $\mathbb{Z} + q$. We show in this section that if it is not equal to $\mathbb{Z} + q$, then G_q contains an algebraic copy of $\mathbb{Z} \times \mathbb{Z}$.

Definition 4.1. Let $k \in \mathbb{N}$, let $B_1, B_2, \ldots, B_k$ be pairwise disjoint infinite subsets of ω, and let $m, x \in \mathbb{N}$.

(a) $\mathrm{supp}(x)$ is the binary support of x.
(b) $c_{B_1}(x) = |\mathrm{supp}(x) \cap B_1|$.
(c) $c_{B_1,\ldots,B_k}(x) =$
$|\{(i_1, i_2, \ldots, i_k) \in \big(\mathrm{supp}(x)\big)^k : i_1 < \ldots < i_k \text{ and each } i_t \in B_t\}|$.
(d) $c_{B_1,m}(x) \in \mathbb{Z}_m$ and $c_{B_1,m}(x) \equiv c_{B_1}(x) \pmod{m}$.
(e) $c_{B_1,\ldots,B_k,m}(x) \in \mathbb{Z}_m$ and $c_{B_1,\ldots,B_k,m}(x) \equiv c_{B_1,\ldots,B_k}(x) \pmod{m}$.

Lemma 4.2. *Let $u, v \in \mathbb{H}$, let $k \in \mathbb{N}$, let $B_1, B_2, \ldots, B_k$ be pairwise disjoint infinite subsets of ω, and let $m \in \mathbb{N}$.*

(1) $\widetilde{c}_{B_1,m}(u + v) = \widetilde{c}_{B_1,m}(u) + \widetilde{c}_{B_1,m}(v)$.
(2) *If $k > 1$, then* $\widetilde{c}_{B_1,\ldots,B_k,m}(u+v) = \widetilde{c}_{B_1,\ldots,B_k,m}(u) + \widetilde{c}_{B_1,\ldots,B_k,m}(v) + \sum_{t=1}^{k-1} \widetilde{c}_{B_1,\ldots,B_t,m}(u) \cdot \widetilde{c}_{B_{t+1},\ldots,B_k,m}(v)$.

Proof. (1) It suffices that $\widetilde{c}_{B_1,m} \circ \rho_v$ and $\rho_{\widetilde{c}_{B_1,m}(v)} \circ \widetilde{c}_{B_1,m}$ agree on $\mathbb{N}$, so let $x \in \mathbb{N}$. Let $k = \max \mathrm{supp}(x) + 1$. It suffices to observe that $\widetilde{c}_{B_1,m} \circ \lambda_x$ and $\lambda_{c_{B_1,m}(x)} \circ \widetilde{c}_{B_1,m}$ agree on $\mathbb{N}2^k$.

(2) Note that singletons are open in $\mathbb{Z}_m$. Pick $C \in u$ such that for all $x \in C$, $\widetilde{c}_{B_1,\ldots,B_k,m}(x + v) = \widetilde{c}_{B_1,\ldots,B_k,m}(u + v)$ and for $t \in \{1, 2, \ldots, k\}$, $\widetilde{c}_{B_1,\ldots,B_t,m}(x) = \widetilde{c}_{B_1,\ldots,B_t,m}(u)$. Pick $x \in C$ and let $l = \max \mathrm{supp}(x) + 1$. Pick $D \in v$ such that for all $y \in D$, $c_{B_1,\ldots,B_k,m}(x+y) = \widetilde{c}_{B_1,\ldots,B_k,m}(x+v)$ and for $t \in \{1, 2, \ldots, k-1\}$, $c_{B_{t+1},\ldots,B_k,m}(y) = \widetilde{c}_{B_{t+1},\ldots,B_k,m}(v)$. Pick $y \in D \cap \mathbb{N}2^l$. Then $c_{B_1,\ldots,B_k,m}(x + y) = c_{B_1,\ldots,B_k,m}(x) + c_{B_1,\ldots,B_k,m}(y) + \sum_{t=1}^{k-1} c_{B_1,\ldots,B_t,m}(x) \cdot c_{B_{t+1},\ldots,B_k,m}(y)$. $\square$

Lemma 4.3. *Let $q \in E\big(K(\beta\mathbb{N})\big)$, let $k \in \mathbb{N}$, let $B_1, B_2, \ldots, B_k$ be pairwise disjoint infinite subsets of ω, and let $m \in \mathbb{N}$. Then $\widetilde{c}_{B_1,\ldots,B_k,m}(q) = 0$.*

Proof. This follows immediately by induction on k from Lemma 4.2. $\square$

Lemma 4.4. *Let $q \in E\big(K(\beta\mathbb{N})\big)$, let $k \in \mathbb{N}$, let $B_1, B_2, \ldots, B_k$ be pairwise disjoint infinite subsets of ω, let $m \in \mathbb{N}$, and let $u \in \mathbb{H} \cap D_q$. Then $\widetilde{c}_{B_1,\ldots,B_k,m}(u) = 0$.*

Proof. We show first that it suffices to show this under the additional assumption that $\mathbb{N} \setminus \bigcup_{i=1}^{k} B_i$ is infinite. Suppose we have done this and let B_k' and B_k'' be disjoint infinite subsets of B_k with $B_k' \cup B_k'' = B_k$. Note that for all $x \in \mathbb{N}$, $c_{B_1,\ldots,B_k,m}(x) = c_{B_1,\ldots,B_k',m}(x) + c_{B_1,\ldots,B_k'',m}(x)$ so $\widetilde{c}_{B_1,\ldots,B_k,m}(u) = \widetilde{c}_{B_1,\ldots,B_k',m}(u) + \widetilde{c}_{B_1,\ldots,B_k'',m}(u) = 0 + 0$.

So assume that $B_{k+1}\mathbb{N} \setminus \bigcup_{i=1}^{k} B_i$ is infinite. Pick $p \in \{2^n : n \in B_{k+1}\}^*$. Note that for all $n \in B_{k+1}$, $c_{B_{k+1},m}(2^n) = 1$, and if $t \in \{1, 2, \ldots, k\}$, then $c_{B_1,\ldots,B_t,m}(2^n) = c_{B_t,\ldots,B_{k+1},m}(2^n) = 0$. Therefore $\widetilde{c}_{B_{k+1},m}(p) = 1$, and if $t \in \{1, 2, \ldots, k\}$, then $\widetilde{c}_{B_1,\ldots,B_t,m}(p) = \widetilde{c}_{B_t,\ldots,B_{k+1},m}(p) = 0$. Since all terms of the expansions given in Lemma 4.2 except one involve q, and are therefore 0, we have that $\widetilde{c}_{B_{k+1},m}(p+q) = 1$, and if $t \in \{1, 2, \ldots, k\}$, then $\widetilde{c}_{B_1,\ldots,B_t,m}(p+q) = \widetilde{c}_{B_t,\ldots,B_{k+1},m}(p+q) = 0$ and $\widetilde{c}_{B_{k+1},m}(q+p) = 1$, and if $t \in \{1, 2, \ldots, k\}$, then $\widetilde{c}_{B_1,\ldots,B_t,m}(q+p) = \widetilde{c}_{B_t,\ldots,B_{k+1},m}(q+p) = 0$.

Next note that

$$\begin{aligned}
&\widetilde{c}_{B_1,\ldots,B_{k+1},m}(q+u+p+q) = \\
&\widetilde{c}_{B_1,\ldots,B_{k+1},m}(q) + \widetilde{c}_{B_1,\ldots,B_{k+1},m}(u+p+q) + \\
&\textstyle\sum_{t=1}^{k} \widetilde{c}_{B_1,\ldots,B_t,m}(q) \cdot \widetilde{c}_{B_{t+1},\ldots,B_{k+1},m}(u+p+q) = \\
&\widetilde{c}_{B_1,\ldots,B_{k+1},m}(u+p+q) \text{ and} \\
&\widetilde{c}_{B_1,\ldots,B_{k+1},m}(q+p+u+q) = \\
&\widetilde{c}_{B_1,\ldots,B_{k+1},m}(q+p+u) + \widetilde{c}_{B_1,\ldots,B_{k+1},m}(q) + \\
&\textstyle\sum_{t=1}^{k} \widetilde{c}_{B_1,\ldots,B_t,m}(q+p+u) \cdot \widetilde{c}_{B_{t+1},\ldots,B_{k+1},m}(q) = \\
&\widetilde{c}_{B_1,\ldots,B_{k+1},m}(q+p+u)\,.
\end{aligned}$$

Since

$$\begin{aligned}
\widetilde{c}_{B_1,\ldots,B_{k+1},m}(q+u+p+q) &= \widetilde{c}_{B_1,\ldots,B_{k+1},m}(u+q+p+q) \\
&= \widetilde{c}_{B_1,\ldots,B_{k+1},m}(q+p+q+u) \\
&= \widetilde{c}_{B_1,\ldots,B_{k+1},m}(q+p+u+q)
\end{aligned}$$

we therefore have that $\widetilde{c}_{B_1,\ldots,B_{k+1},m}(u+p+q) = \widetilde{c}_{B_1,\ldots,B_{k+1},m}(q+p+u)$.

Now

$$\begin{aligned}
\widetilde{c}_{B_1,\ldots,B_{k+1},m}(u+p+q) &= \widetilde{c}_{B_1,\ldots,B_{k+1},m}(u) + \widetilde{c}_{B_1,\ldots,B_{k+1},m}(p+q) \\
&\quad + \textstyle\sum_{t=1}^{k} \widetilde{c}_{B_1,\ldots,B_t,m}(u) \cdot \widetilde{c}_{B_{t+1},\ldots,B_{k+1},m}(p+q) \\
&= \widetilde{c}_{B_1,\ldots,B_{k+1},m}(u) + \widetilde{c}_{B_1,\ldots,B_k,m}(u) \text{ and} \\
\widetilde{c}_{B_1,\ldots,B_{k+1},m}(q+p+u) &= \widetilde{c}_{B_1,\ldots,B_{k+1},m}(q+p) + \widetilde{c}_{B_1,\ldots,B_{k+1},m}(u) \\
&\quad + \textstyle\sum_{t=1}^{k} \widetilde{c}_{B_1,\ldots,B_t,m}(q+p) \cdot \widetilde{c}_{B_{t+1},\ldots,B_{k+1},m}(u) \\
&= \widetilde{c}_{B_1,\ldots,B_{k+1},m}(u)\,.
\end{aligned}$$

Consequently $\widetilde{c}_{B_1,\ldots,B_{k+1},m}(u) + \widetilde{c}_{B_1,\ldots,B_k,m}(u) = \widetilde{c}_{B_1,\ldots,B_{k+1},m}(u)$ so $\widetilde{c}_{B_1,\ldots,B_k,m}(u) = 0$. □

Lemma 4.5. *Let $q \in E\big(K(\beta\mathbb{N})\big)$, let $p \in \{2^n : n \in \mathbb{N}\}^*$, and let $\psi_p : \mathbb{Z} \to G_q$ be the homomorphism such that $\psi_p(1) = q+p+q$. Then for all $n \in \mathbb{Z} \setminus \{0\}$, $\psi_p(n) \notin D_q$.*

Proof. Pick infinite $B \subseteq \mathbb{N}$ such that $\{2^n : n \in B\} \in p$. Then for each $m \in \mathbb{N} \setminus \{1\}$, $\widetilde{c_{B,m}}(q+p+1) = 1$ so for all $n \in \mathbb{N}$ and all $m > n$,

$\widetilde{c_{B,m}}\big(\psi_p(n)\big) = n$ and thus $\psi_p(n) \notin D_q$ by Lemma 4.4. Now $D_q \cap G_q$ is a group, so if $n \in \mathbb{N}$ and $\psi_p(-n) \in D_q$, so is $\psi_p(n)$. $\square$

Theorem 4.6. *Let $q \in E\big(K(\beta\mathbb{N})\big)$, let $p \in \{2^n : n \in \mathbb{N}\}^*$, and let $\psi_p : \mathbb{Z} \to G_q$ be the homomorphism such that $\psi_p(1) = q + p + q$. Assume that $u \in \mathbb{H} \cap G_q \cap D_q \setminus \{q\}$ and let $\varphi : \mathbb{Z} \to G_q$ be the homomorphism such that $\varphi(1) = u$. Define $\tau : \mathbb{Z} \times \mathbb{Z} \to G_q$ by $\tau(m,n) = \varphi(m) + \psi_p(n)$. Then τ is an injective homomorphism.*

Proof. Given (m,n) and (k,l) in $\mathbb{Z} \times \mathbb{Z}$, one has that $\tau\big((m,n) + (k,l)\big) = \tau(m,n) + \tau(k,l)$ because $\varphi(k) \in D_q$. Now assume that $(m,n) \in \mathbb{Z} \times \mathbb{Z}$ and $\tau(m,n) = q$. Then $\varphi(m) + \psi_p(n) = q$ so $\varphi(m) = \psi_p(-n)$ and thus $\psi_p(-n) \in D_q$ so that $n = 0$ by Lemma 4.5. Therefore $\varphi(m) = q$. By Zelenyuk's Theorem [8] (or see [4, Theorem 7.17]), $\beta\mathbb{N}$ contains no notrivial finite groups. If one had $m \neq 0$, then $\varphi[\mathbb{Z}]$ would be a nontrivial finite group, so $m = 0$. $\square$

Corollary 4.7. *Let $q \in E\big(K(\beta\mathbb{N})\big)$. If the center of G_q is not equal to $\mathbb{Z} + q$, then G_q contains an algebraic copy of $\mathbb{Z} \times \mathbb{Z}$.*

Proof. Assume we have $x \in Z(G_q) \setminus (\mathbb{Z} + q)$. Then by Lemma 2.6, $x \in \mathbb{Z} + I$ so pick $n \in \mathbb{Z}$ and $u \in I$ such that $x = n + u$. Then $u \in \mathbb{H} \cap G_q \cap D_q \setminus \{q\}$. Pick any $p \in \{2^n : n \in \mathbb{N}\}^*$. Define τ as in Theorem 4.6. Then τ is an injective homomorphism. $\square$

We conclude by listing some of the tantalising open questions that have arisen in the study of the center and extended center of G_q.

Questions 4.8.
(1) *Let $q \in E\big(K(\beta\mathbb{N})\big)$. Does $Z(G_q) = \mathbb{Z} + q$?*
(2) *Let $q \in E\big(K(\beta\mathbb{N})\big)$. Is $D_q \subseteq \mathbb{Z} + E(\beta\mathbb{N})$?*
(3) *Does there exist $q \in E\big(K(\beta\mathbb{N})\big)$ for which $E(D_q)$ is finite?*
(4) *Does there exist $q \in E\big(K(\beta\mathbb{N})\big)$ for which $E(D_q)$ is uncountable?*
(5) *Let $q_1, q_2 \in E\big(K(\beta\mathbb{N})\big)$. Are D_{q_1} and D_{q_2} isomorphic?*

References

[1] J. Baker and P. Milnes, *The ideal structure of the Stone-Čech compactification of a group*, Math. Proc. Cambr. Phil. Soc. **82** (1977), 401-409.

[2] C. Chou, *On a geometric property of the set of invariant means on a group*, Proc. Amer. Math. Soc. **30** (1971), 296-302.

[3] N. Hindman and J. Pym, *Free groups and semigroups in $\beta\mathbb{N}$*, Semigroup Forum **30** (1984), 177-193.

[4] N. Hindman and D. Strauss, *Algebra in the Stone-Čech compactification: theory and applications, 2nd edition*, de Gruyter, Berlin, 2012.

[5] D. Rees, *On semi-groups*, Math. Proc. Cambr. Phil. Soc. **36** (1940), 387-400.

[6] W. Ruppert, *Rechstopologische Halbgruppen*, J. Reine Angew. Math. **261** (1973), 123-133.

[7] A. Suschkewitsch, *Über die endlichen Gruppen ohne das Gesetz der eindeutigen Umkehrbarkeit*, Math. Annalen **99** (1928), 30-50.

[8] Y. Zelenyuk, *Finite groups in $\beta\mathbb{N}$ are trivial*, Semigroup Forum **55** (1997), 131-132.

Department of Mathematics, Howard University, Washington, DC 20059, USA

E-mail address: nhindman@aol.com

Department of Pure Mathematics, University of Leeds, Leeds LS2 9J2, UK

E-mail address: d.strauss@hull.ac.uk