

THIS PAPER WAS PUBLISHED IN *FUNDAMENTA MATHEMATICAE* **252** (2021), 121-145. TO THE BEST OF MY KNOWLEDGE, THIS IS THE FINAL VERSION AS IT WAS SUBMITTED TO THE PUBLISHER. –NH

## FACTORIZING A MINIMAL ULTRAFILTER INTO A THICK PART AND A SYNDETTIC PART

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ABSTRACT. Let  $S$  be an infinite discrete semigroup. The operation on  $S$  extends uniquely to its Stone-Čech compactification  $\beta S$ , making  $\beta S$  a compact right topological semigroup with  $S$  contained in its topological center. As such,  $\beta S$  has a smallest two-sided ideal,  $K(\beta S)$ . An ultrafilter  $p$  on  $S$  is *minimal* if and only if  $p \in K(\beta S)$ .

We show that any minimal ultrafilter  $p$  factors into a thick part and a syndetic part. That is, there exist filters  $\mathcal{F}$  and  $\mathcal{G}$  such that  $\mathcal{F}$  consists only of thick sets,  $\mathcal{G}$  consists only of syndetic sets, and  $p$  is the unique ultrafilter containing  $\mathcal{F} \cup \mathcal{G}$ .

Letting  $L = \widehat{\mathcal{F}}$  and  $C = \widehat{\mathcal{G}}$ , the sets of ultrafilters containing  $\mathcal{F}$  and  $\mathcal{G}$  respectively, we have that  $L$  is a minimal left ideal of  $\beta S$ ,  $C$  meets every minimal left ideal of  $\beta S$  in exactly one point, and  $L \cap C = \{p\}$ . We show further that  $K(\beta S)$  can be partitioned into relatively closed sets, each of which meets each minimal left ideal in exactly one point.

With some weak cancellation assumptions on  $S$ , we prove also that for each minimal ultrafilter  $p$ ,  $S^* \setminus \{p\}$  is not normal. In particular, if  $p$  is a member of either of the disjoint sets  $K(\beta\mathbb{N}, +)$  or  $K(\beta\mathbb{N}, \cdot)$ , then  $\mathbb{N}^* \setminus \{p\}$  is not normal.

### 1. INTRODUCTION

Throughout this paper  $S$  will denote an infinite discrete semigroup with operation  $\cdot$ . The Stone-Čech compactification  $\beta S$  of  $S$  is the set of ultrafilters on  $S$ , with the principal ultrafilters being identified with the points of  $S$ . We let  $S^* = \beta S \setminus S$ . The operation  $\cdot$  extends to  $\beta S$  so that  $(\beta S, \cdot)$  is a right topological semigroup, meaning that for each  $p \in \beta S$ , the function  $\rho_p$  defined by  $\rho_p(q) = q \cdot p$  is continuous, with  $S$  contained in the topological center, meaning that for each  $x \in S$ , the function  $\lambda_x$  defined by  $\lambda_x(q) = x \cdot q$  is continuous. Given  $p, q \in \beta S$  and  $A \subseteq S$ , we have  $A \in p \cdot q$  if and only if  $\{x \in S : x^{-1}A \in q\} \in p$ , where  $x^{-1}A = \{y \in S : x \cdot y \in A\}$ .

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2010 *Mathematics Subject Classification*. Primary: 54D35, 54D80, 22A15 Secondary: 06E15, 03E05.

*Key words and phrases*. Stone-Čech compactification, minimal ultrafilters, thick sets, syndetic sets, minimal left/right ideals, butterfly and non-normality points.

As does any compact Hausdorff right topological semigroup,  $\beta S$  has a smallest two sided ideal,  $K(\beta S)$ . According to the structure theorem [7, Theorem 1.64]), we have

$$\begin{aligned} K(\beta S) &= \bigcup \{L \subseteq \beta S : L \text{ is a minimal left ideal}\} \\ &= \bigcup \{R \subseteq \beta S : R \text{ is a minimal right ideal}\}, \end{aligned}$$

where each of these unions is a disjoint union. The minimal left ideals are closed while the minimal right ideals are usually not closed. Furthermore, if  $L$  is a minimal left ideal and  $R$  is a minimal right ideal then

- $L \cap R = R \cdot L \neq \emptyset$ ;
- $L \cap R$  is a group, and it contains exactly one element of the set  $E(R)$  of idempotents in  $R$ , namely the identity of the group.

In fact, the structure theorem says more than this, but this summary is sufficient for what follows. Furthermore,

- If  $G = L \cap R$ , then the map  $(p, e) \mapsto p \cdot e$  is a bijection  $G \times E(R) \rightarrow R$ .

This last assertion follows from [7, Theorem 2.11(b)], which asserts that if  $L'$  is a minimal left ideal of  $\beta S$  and  $e$  is the identity of  $L' \cap R$ , then the restriction of  $\rho_e$  to  $G = L \cap R$  is an isomorphism and a homeomorphism onto  $L' \cap R$ . The idempotent ultrafilters in  $K(\beta S)$  are called *minimal idempotents* and the elements of  $K(\beta S)$  are called *minimal ultrafilters*.

We will show in Theorem 2.1 that if  $L$  is a minimal left ideal of  $\beta S$ ,  $R$  is a minimal right ideal,  $p \in L \cap R$ , and  $C = \overline{p \cdot E(R)}$ , then  $L \cap C = \{p\}$  and  $C$  meets each each minimal left ideal in exactly one point. Further,  $\{q \cdot E(R') : R' \text{ is a minimal right ideal and } q \in L \cap R'\}$  partitions  $K(\beta S)$

into relatively closed sets. The fact that the partition elements are closed in  $K(\beta S)$  can be seen as a topological addition to the (algebraic) structure theorem described above. Particularly, in the final bullet point, our result shows that the given bijection has at least one nice topological property: the images of the “vertical sections”  $\{p\} \times E(R)$  of  $G \times E(R)$ , namely the sets of the form  $p \cdot E(R)$ , are closed in  $R$ . (Note that the images of horizontal sections are also closed in  $R$ , but this is not difficult to prove; it follows from the fact that minimal left ideals of  $\beta S$  are closed.)

Closed subsets of  $\beta S$  correspond naturally to filters on  $S$ . For a filter  $\mathcal{F}$  on  $S$ , let  $\widehat{\mathcal{F}} = \{p \in \beta S : \mathcal{F} \subseteq p\} = \bigcap \{\overline{A} : A \in \mathcal{F}\}$ . Given any nonempty subset  $X$  of  $\beta S$ ,  $\bigcap X$  is a filter, and if  $\mathcal{F} = \bigcap X$ , then  $\widehat{\mathcal{F}} = \overline{X}$ . In terms of filters, our results show that every minimal ultrafilter  $p$  on  $S$  can be “factored” into two filters  $\mathcal{F}$  and  $\mathcal{G}$ , where  $\mathcal{F}$  consists entirely of thick sets

and  $\mathcal{G}$  consists entirely of syndetic sets. The ultrafilter  $p$  factors into  $\mathcal{F}$  and  $\mathcal{G}$  in the sense that  $p$  is the filter generated by  $\mathcal{F} \cup \mathcal{G}$ .

One immediate consequence of this factorization is that every minimal ultrafilter  $p$  on  $\mathbb{N}$  is a butterfly point of  $\mathbb{N}^*$ . (When we refer to a minimal ultrafilter on  $\mathbb{N}$  without specifying the operation, we mean a member of  $K(\beta\mathbb{N}, +)$ .) Recall that a *butterfly point* of a space  $X$  is a point  $p$  such that, for some  $A, B \subseteq X \setminus \{p\}$ , we have  $\overline{A} \cap \overline{B} = \{p\}$ . It is an open problem whether every point of  $\mathbb{N}^*$  is a butterfly point (e.g., it is "classic problem IX" in Peter Nyikos's *Classic Problems in Topology* series [10]).

With a little more work, we show that every minimal ultrafilter  $p$  is a *non-normality point* of  $\mathbb{N}^*$ , which means that  $\mathbb{N}^* \setminus \{p\}$  is not normal. It is a longstanding open problem whether every point of  $\mathbb{N}^*$  is a non-normality point (e.g., it is problem 3 on Jan van Mill's list of open problems in [9]). This problem is closely related to the one mentioned in the previous paragraph, because every non-normality point is also a butterfly point. It is known that the answer to both problems is consistently positive: for example, CH implies that every point of  $\mathbb{N}^*$  is a non-normality point. (This is due to Rajagopalan [11] and Warren [12] independently.) It is also known that, using only ZFC, at least some points of  $\mathbb{N}^*$  are non-normality points: for example, this holds when  $p$  is not Rudin-Frolík minimal [2]. Our results add to the list of known non-normality points of  $\mathbb{N}^*$ .

The result that minimal ultrafilters are butterfly points (respectively, non-normality points) will be proved in a general setting: it holds in  $S^*$  whenever  $S$  satisfies certain cancellation properties. Under the additional assumption that  $S$  is countable, we also prove that for a minimal right ideal  $R$  of  $\beta S$  and any minimal ultrafilter  $p \in R$ , the spaces  $E(R)$  and  $p \cdot E(R)$  are  $P$ -spaces and not Borel in  $\beta S$ .

## 2. CLOSED TRANSVERSALS AND FACTORIZING A MINIMAL ULTRAFILTER

In this section we establish results that do not require any cancellation assumptions about  $S$ , beginning by producing closed transversals for the set of minimal left ideals. (By a *transversal* for this set, we mean a set which meets each minimal left ideal in exactly one point.)

A fact that we will use repeatedly is that if  $R$  is a minimal right ideal of  $\beta S$  and  $e \in E(R)$ , then  $e$  is a left identity for  $R$ , which means that  $e \cdot p = p$  for all  $p \in R$ . (In particular, if  $e, f \in E(R)$  then  $e \cdot f = f$  and  $f \cdot e = e$ .) To see this, note that  $e \cdot \beta S$  is a right ideal contained in  $R$ , so  $e \cdot \beta S = R$

by minimality. Thus  $p \in R$  implies  $p = e \cdot q$  for some  $q \in \beta S$ , so that  $p = e \cdot q = e \cdot e \cdot q = e \cdot p$ .

**Theorem 2.1.** *Let  $L$  and  $R$  be minimal left and right ideals of  $\beta S$ , respectively, and let  $p \in L \cap R$ . Then*

$$L \cap \overline{p \cdot E(R)} = \{p\}.$$

Furthermore,

(1) *If  $L'$  is any minimal left ideal of  $\beta S$ , then*

$$L' \cap \overline{p \cdot E(R)} = \{p \cdot e\}$$

*where  $e$  is the (unique) idempotent contained in  $L' \cap R$ . In particular,  $\overline{p \cdot E(R)}$  meets every minimal left ideal in exactly one point.*

(2)  *$\{q \cdot E(R) : q \in L \cap R\}$  is a partition of  $R$  into relatively closed sets (i.e., they are closed in  $R$ ), and*

$$\{q \cdot E(R') : R' \text{ is a minimal right ideal and } q \in L \cap R'\}$$

*is a partition of  $K(\beta S)$  into relatively closed sets.*

*Proof.* Let  $p$  be a minimal ultrafilter in  $\beta S$ , let  $L$  and  $R$  denote the minimal left and right ideals of  $\beta S$ , respectively, that contain  $p$ , and let  $f$  be the identity of  $L \cap R$ . Then  $p = p \cdot f$  so  $p \in L \cap \overline{p \cdot E(R)}$ .

Suppose  $q \in L \cap \overline{p \cdot E(R)}$ . We will show that  $p = q$ . Let  $R'$  denote the minimal right ideal of  $\beta S$  containing  $q$ . For each  $e \in E(R)$ , we have  $e \cdot f = f$  so  $p \cdot e \cdot f = p \cdot f = p$ . Thus the function  $\rho_f$  is constant on the set  $p \cdot E(R)$ , with value  $p$ . But  $\rho_f$  is continuous on all of  $\beta S$ , so this means that  $\rho_f$  is constant on  $\overline{p \cdot E(R)}$  with value  $p$ . In particular,  $q \cdot f = p$ . Because  $R'$  is a right ideal containing  $q$  we have  $q \cdot f \in R'$ ; but  $p \in R$ , so it follows that  $R' = R$ . Thus  $q$  and  $p$  are both members of the group  $L \cap R$ . As  $f$  is the identity element of this group,  $q \cdot f = p$  implies  $q = p$ , as desired, completing the proof that  $L \cap \overline{p \cdot E(R)} = \{p\}$ .

To prove (1), suppose  $L'$  is any minimal left ideal of  $\beta S$ , and let  $e$  denote the identity element of the group  $L' \cap R$ . Then  $p \cdot e \in L' \cap R$  and  $e \cdot E(R) = E(R)$  so by what we proved in the paragraph above,  $\{p \cdot e\} = L' \cap \overline{p \cdot e \cdot E(R)} = L' \cap \overline{p \cdot E(R)}$ .

To prove (2), let  $G = L \cap R$  and let  $h : G \times E(R) \rightarrow R$  be the function  $h(q, e) = q \cdot e$ . We noted in the introduction that  $h$  is a bijection, which implies that  $\{q \cdot E(R) : q \in G\}$  is a partition of  $R$ , which implies that  $\{q \cdot E(R') : R' \text{ is a minimal right ideal and } q \in L \cap R'\}$  is a partition of  $K(\beta S)$ . Finally, all sets of the form  $q \cdot E(R)$  are closed in  $K(\beta S)$ , because

any point of  $(K(\beta S) \cap \overline{q \cdot E(R)}) \setminus q \cdot E(R)$  would be a member of some minimal left ideal, and this contradicts (1).  $\square$

Given a set  $X$ , we let  $\mathcal{P}_f(X)$  be the set of finite nonempty subsets of  $X$ . A subset  $A$  of  $S$  is called

- *thick* if for each  $F \in \mathcal{P}_f(S)$ , there exists  $x \in S$  such that  $Fx \subseteq A$ , or, equivalently, if the collection of all sets of the form  $\{s^{-1}A : s \in S\}$  has the finite intersection property.
- *syndetic* if there is some  $F \in \mathcal{P}_f(S)$  such that  $S = \bigcup_{s \in F} s^{-1}A$ .

Notice that if  $A$  is thick and  $B$  is syndetic, then  $A \cap B \neq \emptyset$ . (To see this, pick  $F \in \mathcal{P}_f(S)$  such that  $S = \bigcup_{s \in F} s^{-1}B$  and pick  $x \in S$  such that  $Fx \subseteq A$ . Pick  $s \in F$  such that  $sx \in B$ . Then  $sx \in A \cap B$ .)

For the semigroup  $(\mathbb{N}, +)$ ,  $A \subseteq \mathbb{N}$  is thick if and only if it contains arbitrarily long intervals, and is syndetic if and only if it has bounded gaps, which means that there is some  $k \in \mathbb{N}$  such that every interval of length  $k$  contains a point of  $A$ .

Let  $\Theta$  denote the family of thick subsets of  $S$ , and let  $\Sigma$  denote the family of syndetic subsets of  $S$ . These two families of sets are dual to each other, in the following sense, which follows immediately from the definitions.

**Lemma 2.2.** *A set is thick if and only if its complement fails to be syndetic, and it is syndetic if and only if its complement fails to be thick.*

The families  $\Theta$  and  $\Sigma$  are related to  $K(\beta S)$  by the following lemma.

**Lemma 2.3.** *If  $A \subseteq S$  then*

- (1)  *$A \in \Theta$  if and only if  $\overline{A}$  contains a minimal left ideal of  $\beta S$ .*
- (2)  *$A \in \Sigma$  if and only if  $\overline{A}$  meets every minimal left ideal of  $\beta S$ .*

*Proof.* This is part of [1, Theorem 2.9], or see [7, Theorem 4.48].  $\square$

Let us say that a filter  $\mathcal{F}$  on  $S$  is  $\Theta$ -*maximal* if  $\mathcal{F} \subseteq \Theta$ , and if every filter properly extending  $\mathcal{F}$  contains some set not in  $\Theta$ . Similarly, let us say that a filter  $\mathcal{G}$  on  $S$  is  $\Sigma$ -*maximal* if  $\mathcal{G} \subseteq \Sigma$ , and if every filter properly extending  $\mathcal{G}$  contains some set not in  $\Sigma$ . The existence of  $\Theta$ -maximal filters and  $\Sigma$ -maximal filters is ensured by Zorn's Lemma.

Note that  $\Theta$ -maximal filters on  $\mathbb{N}$  are never ultrafilters: for example, they will contain neither the set of even numbers nor the set of odd numbers. Neither are  $\Sigma$ -maximal ultrafilters on  $\mathbb{N}$  ever maximal. In fact, if one identifies subsets of  $\mathbb{N}$  with points of the Cantor space via characteristic functions, then one can show that  $\Sigma$  is a meager, measure-zero subset of the Cantor

space. Hence every  $\Sigma$ -maximal filter on  $\mathbb{N}$  is also meager and null; in this sense, these filters are very far from being ultrafilters.

**Lemma 2.4.** *A filter  $\mathcal{F}$  on  $S$  is a  $\Theta$ -maximal filter if and only if  $\widehat{\mathcal{F}}$  is a minimal left ideal of  $\beta S$ .*

*Proof.* This is [4, Proposition 3.2]. □

**Lemma 2.5.** *Let  $\mathcal{F}$  be a filter on  $S$ . Then*

- (1)  $\mathcal{F} \subseteq \Theta$  if and only if  $\widehat{\mathcal{F}}$  contains a minimal left ideal.
- (2)  $\mathcal{F} \subseteq \Sigma$  if and only if  $\widehat{\mathcal{F}}$  meets every minimal left ideal.

*Proof.* If  $\mathcal{F} \subseteq \Theta$ , then, by an application of Zorn's Lemma,  $\mathcal{F}$  can be extended to a  $\Theta$ -maximal filter  $\mathcal{G}$ . But then  $\widehat{\mathcal{F}} \supseteq \widehat{\mathcal{G}}$ , so  $\widehat{\mathcal{F}}$  contains a minimal left ideal by Lemma 2.4. This proves the “only if” direction of (1).

If  $\mathcal{F} \not\subseteq \Theta$ , then there is some  $A \in \mathcal{F} \setminus \Theta$ . But then  $\overline{A} \supseteq \widehat{\mathcal{F}}$ , so  $\widehat{\mathcal{F}}$  contains no minimal left ideals by Lemma 2.3. This proves the “if” direction of (1).

The “if” direction of (2) is proved just as it was for (1). Supposing  $\mathcal{F} \not\subseteq \Sigma$ , there is some  $A \in \mathcal{F} \setminus \Sigma$ . But then  $\overline{A} \supseteq \widehat{\mathcal{F}}$ , so  $\widehat{\mathcal{F}}$  fails to meet some minimal left ideal by Lemma 2.3.

For the “only if” direction of (2), suppose  $\mathcal{F} \subseteq \Sigma$  and let  $L$  be any minimal left ideal.  $L$  is closed in  $\beta S$ , hence compact, and  $\{\overline{A} \cap L : A \in \mathcal{F}\}$  is a collection of closed subsets of  $L$  with the finite intersection property (by Lemma 2.3, because  $\mathcal{F} \subseteq \Sigma$ ). Thus

$$\widehat{\mathcal{F}} \cap L = \left( \bigcap \{ \overline{A} : A \in \mathcal{F} \} \right) \cap L = \bigcap \{ \overline{A} \cap L : A \in \mathcal{F} \} \neq \emptyset$$

by compactness. As  $L$  was arbitrary,  $\widehat{\mathcal{F}}$  meets every minimal left ideal. □

In light of this lemma, one might hope that the  $\Sigma$ -maximal filters correspond precisely to closed transversals for the set of minimal left ideals, in the same way that  $\Theta$ -maximal filters correspond to the minimal left ideals themselves. We show in Section 4 below that this is at least consistently not the case. However, the transversals that we found in Theorem 2.1 do all correspond to  $\Sigma$ -maximal filters:

**Lemma 2.6.** *Let  $R$  be a minimal right ideal of  $\beta S$ , let  $p \in R$ , and let  $\mathcal{G} = \bigcap (p \cdot E(R))$ . Then  $\mathcal{G}$  is a  $\Sigma$ -maximal filter on  $S$ .*

*Proof.* By Theorem 2.1  $\widehat{\mathcal{G}} = \overline{p \cdot E(R)}$  meets every minimal left ideal, so by Lemma 2.5,  $\mathcal{G} \subseteq \Sigma$ . Now suppose we have a filter  $\mathcal{H} \subseteq \Sigma$  such that  $\mathcal{G} \subsetneq \mathcal{H}$  and pick  $A \in \mathcal{H} \setminus \mathcal{G}$ . Since  $A \notin \mathcal{G}$ , pick  $f \in E(R)$  such that  $A \notin p \cdot f$ . Let  $L = \beta S \cdot f$ . By Lemma 2.5,  $\widehat{\mathcal{H}} \cap L \neq \emptyset$  so pick  $q \in \widehat{\mathcal{H}} \cap L$ . Since  $\widehat{\mathcal{H}} \subseteq \widehat{\mathcal{G}}$ ,  $q \in \overline{p \cdot E(R)}$ .

Since  $A \notin p \cdot f$ ,  $q \notin p \cdot f$ , contradicting Theorem 2.1(1).  $\square$

**Theorem 2.7.** *Let  $p$  be a minimal ultrafilter on  $S$ . Then there exist a  $\Theta$ -maximal filter  $\mathcal{F}$  and a  $\Sigma$ -maximal filter  $\mathcal{G}$  such that  $p$  is the ultrafilter generated by  $\mathcal{F} \cup \mathcal{G}$ . Specifically, if  $L$  and  $R$  are respectively the minimal left and right ideals of  $\beta S$  containing  $p$ , then  $\mathcal{F} = \bigcap L$  and  $\mathcal{G} = \bigcap (p \cdot E(R))$  are two such filters.*

Moreover,  $\mathcal{F}$  is the only  $\Theta$ -maximal filter contained in  $p$ , and

$$\mathcal{F} = \{A \in p : s^{-1}A \in p \text{ for all } s \in S\}.$$

*Proof.* Let  $p$  be a minimal ultrafilter on  $S$ . Let  $L$  and  $R$  denote respectively the minimal left and right ideals of  $\beta S$  containing  $p$ . Let  $\mathcal{F} = \bigcap L$  and let  $\mathcal{G} = \bigcap (p \cdot E(R))$ .

$\mathcal{F}$  is  $\Theta$ -maximal by Lemma 2.4 and  $\mathcal{G}$  is  $\Sigma$ -maximal by Lemma 2.6. Since any thick set meets any syndetic set,  $\mathcal{F} \cup \mathcal{G}$  generates a filter  $\mathcal{U}$ . Then  $\emptyset \neq \widehat{\mathcal{U}} \subseteq \widehat{\mathcal{F}} \cap \widehat{\mathcal{G}} = L \cap \overline{p \cdot E(R)} = \{p\}$  by Theorem 2.1. Hence  $\widehat{\mathcal{U}} = \{p\}$ , and this means  $\mathcal{U} = p$ .

To prove the ‘‘moreover’’ assertion of the theorem, suppose  $\mathcal{F}'$  is any  $\Theta$ -maximal filter contained in  $p$ . Then  $p \in \widehat{\mathcal{F}'}$ , and  $\widehat{\mathcal{F}'}$  is a minimal left ideal by Lemma 2.4. This implies  $\widehat{\mathcal{F}'} = L$ , because the minimal left ideals of  $\beta S$  are disjoint.  $\widehat{\mathcal{F}'} = L = \widehat{\mathcal{F}}$  implies  $\mathcal{F}' = \mathcal{F}$ , so  $\mathcal{F}$  is the only  $\Theta$ -maximal filter contained in  $p$ .

It remains to show  $\mathcal{F} = \{A \in p : s^{-1}A \in p \text{ for all } s \in S\}$ . Let

$$\mathcal{H} = \{A \subseteq S : s^{-1}A \in p \text{ for all } s \in S\}.$$

By [7, Theorem 6.18],  $\widehat{\mathcal{H}} = \beta S \cdot p = L$ . Since  $p \in L$ ,

$$\mathcal{H} = \{A \in p : s^{-1}A \in p \text{ for all } s \in S\}.$$

Since  $\widehat{\mathcal{H}} = L = \widehat{\mathcal{F}}$ ,  $\mathcal{H} = \mathcal{F}$ .  $\square$

While a minimal ultrafilter contains exactly one  $\Theta$ -maximal filter by the previous theorem, we see now that it may contain more than one  $\Sigma$ -maximal filter.

**Theorem 2.8.** *There is a minimal ultrafilter on  $\mathbb{N}$  that contains more than one  $\Sigma$ -maximal filter.*

*Proof.* Let  $E$  be the set of even numbers, let  $O$  be the set of odd numbers, let  $A = \bigcup_{n=0}^{\infty} \{2^{2n}, 2^{2n} + 1, 2^{2n} + 2, \dots, 2^{2n+1} - 1\}$ , and let  $B = (E \cap A) \cup (O \setminus A)$ . Then  $B$  has no gaps longer than 2, so  $B$  is syndetic. By a routine application of Zorn’s Lemma, there is a  $\Sigma$ -maximal filter  $\mathcal{H}$  such that  $B \in \mathcal{H}$ . Let  $L$  be

a minimal left ideal. Then  $\widehat{\mathcal{H}} \cap L \neq \emptyset$  by Lemma 2.5; thus there is a minimal ultrafilter  $p \in \widehat{\mathcal{H}} \cap L$ .

Let  $R$  be the minimal right ideal with  $p \in R$  and let  $\mathcal{G} = \bigcap (p + E(R))$ . By Theorem 2.7,  $\mathcal{G}$  is  $\Sigma$ -maximal and  $\mathcal{G} \subseteq p$ . We claim that  $\mathcal{G} \neq \mathcal{H}$ . To see this, note that  $E(R) \subseteq E^*$  so if  $p \in E^*$ , then  $p + E(R) \subseteq E^*$  so  $E \in \mathcal{G}$ . If  $p \in O^*$ , then  $p + E(R) \subseteq O^*$  so  $O \in \mathcal{G}$ . But  $B \in \mathcal{H}$  and neither  $B \cap E$  nor  $B \cap O$  is syndetic so neither  $E$  nor  $O$  is a member of  $\mathcal{H}$ .  $\square$

To end this section, we will demonstrate a technique for building  $\Sigma$ -maximal filters on  $\mathbb{N}$  that offers some control over the filter obtained (more control, anyway, than is given by Zorn's Lemma). This accomplishes two things. One is to demonstrate that there are many closed transversals for the set of minimal left ideals other than the ones of the form  $\overline{p \cdot E(R)}$ . The second is an improvement on Theorem 2.8: we will show that every minimal ultrafilter on  $\mathbb{N}$  contains more than one  $\Sigma$ -maximal filter.

**Lemma 2.9.** *Let  $n \in \mathbb{N}$  and assume  $\langle X_i \rangle_{i=1}^n$  is a sequence of pairwise disjoint subsets of  $S$  such that  $(\forall G \in \mathcal{P}_f(S))(\exists H \in \mathcal{P}_f(S))(\forall x \in S)(\exists y \in S)(\exists i \in \{1, 2, \dots, n\})(Gy \subseteq (Hx \cap X_i))$ . Then for each minimal left ideal  $L$  of  $\beta S$ , there is some  $i \in \{1, 2, \dots, n\}$  such that  $L \subseteq \overline{X_i}$ .*

*Proof.* Let  $L$  be a minimal left ideal of  $\beta S$ . Aiming for a contradiction, suppose that for each  $i \in \{1, 2, \dots, n\}$ ,  $L \setminus \overline{X_i} \neq \emptyset$ . Let  $\mathcal{F} = \bigcap L$ . By Lemma 2.4,  $\mathcal{F}$  is  $\Theta$ -maximal. We claim that for each  $i$ , there exists  $B_i \in \mathcal{F}$  such that  $B_i \cap X_i$  is not thick. If  $L \cap \overline{X_i} = \emptyset$ , one may let  $B_i = S \setminus X_i$ . If  $L \cap \overline{X_i} \neq \emptyset$ , then  $\mathcal{G}_i = \bigcap (L \cap \overline{X_i}) = \{C \subseteq S : (\exists B \in \mathcal{F})(B \cap X_i \subseteq C)\}$  is a filter properly containing  $\mathcal{F}$ . (The containment is proper because  $L \setminus \overline{X_i} \neq \emptyset$  so  $X_i \in \mathcal{G}_i \setminus \mathcal{F}$ .) Hence one may pick  $B_i \in \mathcal{F}$  such that  $B_i \cap X_i$  is not thick.

For each  $j \in \{1, 2, \dots, n\}$ , let  $D_j = (\bigcap_{i=1}^n B_i) \cap X_j$ . Then for  $j \in \{1, 2, \dots, n\}$ ,  $D_j$  is not thick so pick  $G_j \in \mathcal{P}_f(S)$  such that for all  $y \in S$ ,  $G_j y \not\subseteq D_j$ . Let  $G = \bigcup_{j=1}^n G_j$  and pick  $H \in \mathcal{P}_f(S)$  as guaranteed by the hypothesis. Now  $\bigcap_{i=1}^n B_i \in \mathcal{F}$ , so in particular  $\bigcap_{i=1}^n B_i$  is thick. Pick  $x \in S$  such that  $Hx \subseteq \bigcap_{i=1}^n B_i$ . Pick  $y \in S$  and  $j \in \{1, 2, \dots, n\}$  such that  $G_j y \subseteq (Hx \cap X_j)$ . Then  $G_j y \subseteq D_j$ , a contradiction.  $\square$

Note that, as a consequence of the following theorem, for each  $n \in \mathbb{N}$ , there is a partition of  $K(\beta\mathbb{N})$  into  $n$  sets, each clopen in  $K(\beta\mathbb{N})$ , so that every minimal left ideal is contained in one cell of the partition.

**Theorem 2.10.** *Let  $n \in \mathbb{N}$  and let  $\langle Z_j \rangle_{j=1}^n$  be a partition of  $\mathbb{N}$ . Let  $\langle I_t \rangle_{t=1}^\infty$  be a partition of  $\mathbb{N}$  into intervals such that  $\lim_{t \rightarrow \infty} |I_t| = \infty$ . For  $j \in \{1, 2, \dots, n\}$ ,*



let  $X_j = \bigcup_{t \in Z_j} I_t$ . Then for each for each minimal left ideal  $L$  of  $\beta\mathbb{N}$ , there exists  $j \in \{1, 2, \dots, n\}$  such that  $L \subseteq \overline{X_j}$ . If  $Z_j$  is infinite, then  $\overline{X_j}$  contains a minimal left ideal of  $\beta\mathbb{N}$ .

*Proof.* We may presume that for each  $t \in \mathbb{N}$ ,  $\max I_t + 1 = \min I_{t+1}$ . If  $Z_j$  is infinite, then  $X_j$  is thick, so the second conclusion is immediate. To establish the first conclusion we invoke Lemma 2.9. That is, we show that

$$\begin{aligned} & (\forall G \in \mathcal{P}_f(\mathbb{N})) (\exists H \in \mathcal{P}_f(\mathbb{N})) (\forall x \in \mathbb{N}) (\exists y \in \mathbb{N}) \\ & (\exists i \in \{1, 2, \dots, n\}) (G + y \subseteq (H + x) \cap X_i). \end{aligned}$$

So let  $G \in \mathcal{P}_f(\mathbb{N})$  and let  $k = \max G$ . Pick  $M \in \mathbb{N}$  such that for all  $r \geq M$ , the length of  $I_r$  is at least  $k$ . Let  $m = \max I_M$  and let  $H = \{1, 2, \dots, m + k\}$ . Note that  $m \geq k$ . Let  $x \in \mathbb{N}$ . Pick the largest  $r$  such that  $z = \max I_r \leq x + k + m$ . Note that  $r \geq M$ , so that the length of  $I_r$  and the length of  $I_{r+1}$  are both at least  $k$ . Thus  $\{z - k + 1, z - k + 2, \dots, z\} \subseteq I_r$  and  $\{z + 1, z + 2, \dots, z + k\} \subseteq I_{r+1}$ . If  $z - k \geq x$ , let  $y = z - k$  so that  $G + y \subseteq \{y + 1, y + 2, \dots, y + k\} = \{y + 1, y + 2, \dots, z\}$ . Now  $\{y + 1, y + 2, \dots, z\} \subseteq \{x + 1, x + 2, \dots, x + m + k\}$  because  $x \leq y$  and  $z \leq x + m + k$ , and  $\{y + 1, y + 2, \dots, z\} \subseteq I_r$  because  $y + 1 = z - k + 1$ . Thus  $G + y \subseteq \{x + 1, x + 2, \dots, x + m + k\} \cap I_r \subseteq (H + x) \cap I_r$ . If instead  $z - k < x$ , let  $y = \max\{x, z\}$ . If  $x \leq z$  then  $z - k < x \leq z$  and so  $G + y \subseteq \{y + 1, y + 2, \dots, y + k\} = \{z + 1, z + 2, \dots, z + k\} \subseteq I_{r+1} \cap \{x + 1, x + 2, \dots, x + 2k\} \subseteq (H + x) \cap I_{r+1}$ . If  $z < x$  then our choice of  $r$  guarantees  $\max I_{r+1} > x + k + m$ , so that  $G + y \subseteq \{y + 1, y + 2, \dots, y + k\} = \{x + 1, x + 2, \dots, x + k\} \subseteq (x + H) \cap I_{r+1}$ .  $\square$

We remark that if  $S$  is the free semigroup on a finite alphabet (where the operation  $\cdot$  is concatenation), if  $n \in \mathbb{N}$ ,  $X_j$  is as in Theorem 2.10 for  $j \in \{1, 2, \dots, n\}$ , and  $Y_j = \{w \in S : \text{the length of } w \text{ is in } X_j\}$ , then each minimal left ideal of  $\beta S$  is contained in  $\overline{Y_j}$  for some  $j \in \{1, 2, \dots, n\}$ . We leave the details to the reader.

**Theorem 2.11.** *Let  $\mathcal{R}$  be a finite set of minimal right ideals of  $\beta\mathbb{N}$ . There is a  $\Sigma$ -maximal filter  $\mathcal{G}$  on  $\mathbb{N}$  such that  $\widehat{\mathcal{G}}$  is a closed transversal for the minimal left ideals of  $\beta\mathbb{N}$ ,  $K(\beta\mathbb{N}) \cap \widehat{\mathcal{G}} \subseteq \bigcup \mathcal{R}$ , and  $\widehat{\mathcal{G}} \cap R \neq \emptyset$  for every  $R \in \mathcal{R}$ . Furthermore, if  $p$  is any minimal ultrafilter contained in one of the members of  $\mathcal{R}$ , then we may find such a filter  $\mathcal{G}$  with  $p \in \widehat{\mathcal{G}}$ .*

*Proof.* Enumerate  $\mathcal{R}$  as  $\langle R_i \rangle_{i=1}^n$ , and fix  $p \in R_1$ . Let  $\langle X_j \rangle_{j=1}^n$  be as in Theorem 2.10, assuming that each  $Z_j$  is infinite. Without loss of generality (by relabelling the  $Z_j$  if necessary) we may assume that  $p \in \overline{X_1}$ . Let

$$\mathcal{G} = \bigcap \left( ((p + E(R_1)) \cap \overline{X_1}) \cup \bigcup_{i=2}^n (E(R_i) \cap \overline{X_i}) \right).$$

We show first that if  $L$  is a minimal left ideal of  $\beta\mathbb{N}$ ,  $i \in \{1, 2, \dots, n\}$ , and  $L \subseteq \overline{X_i}$ , then either

- $i = 1$  and  $\widehat{\mathcal{G}} \cap L = \{p + f\}$ , where  $f$  is the identity of  $L \cap R_1$ , or
- $i > 1$  and  $\widehat{\mathcal{G}} \cap L = \{f\}$ , where  $f$  is the identity of  $L \cap R_i$ .

This will establish that  $\widehat{\mathcal{G}}$  is a transversal for the minimal left ideals of  $\beta\mathbb{N}$  and that  $p \in \widehat{\mathcal{G}}$ . It will also establish that  $\widehat{\mathcal{G}} \cap R_i \neq \emptyset$  for each  $i \in \{1, 2, \dots, n\}$ , because each  $X_i$  is thick, which implies that for each  $i$  there is some minimal left ideal  $L$  with  $L \subseteq \overline{X_i}$ .

Observe that

$$\begin{aligned}
 (*) \quad \widehat{\mathcal{G}} &= \overline{((p + E(R_1)) \cap \overline{X_1}) \cup \bigcup_{i=2}^n (E(R_i) \cap \overline{X_i})} \\
 &= \overline{(p + E(R_1)) \cap \overline{X_1}} \cup \bigcup_{i=2}^n \overline{E(R_i) \cap \overline{X_i}} \\
 &= \overline{(p + E(R_1)) \cap \overline{X_1}} \cup \bigcup_{i=2}^n \overline{E(R_i) \cap \overline{X_i}}.
 \end{aligned}$$

(The third line follows from the second because the  $\overline{X_i}$  are not only closed, but clopen.)

For the first bullet point, suppose  $i = 1$ , let  $L \subseteq \overline{X_1}$  be a minimal left ideal, and let  $f$  be the identity of  $L \cap R_1$ . By Theorem 2.1,  $L \cap \overline{p + E(R_1)} = \{p + f\}$ . Since  $L \subseteq \overline{X_1}$ , and since  $\overline{X_j} \cap \overline{X_1} = \emptyset$  for  $j \neq 1$ , (\*) implies that  $\widehat{\mathcal{G}} \cap L = \overline{p + E(R_1)} \cap L = \{p + f\}$ .

For the second bullet point, suppose  $i \neq 1$ , let  $L \subseteq \overline{X_i}$  be a minimal left ideal, and let  $f$  be the identity of  $L \cap R_i$ . By Theorem 2.1,  $L \cap \overline{E(R_i)} = \{f\}$ . Since  $L \subseteq \overline{X_i}$ , and since  $\overline{X_j} \cap \overline{X_i} = \emptyset$  for  $j \neq i$ , (\*) implies that  $\widehat{\mathcal{G}} \cap L = \overline{E(R_i)} \cap L = \{f\}$ .

$\widehat{\mathcal{G}}$  meets every minimal left ideal, so  $\mathcal{G} \subseteq \Sigma$  by Lemma 2.5. To finish the proof, we must show that  $\mathcal{G}$  is  $\Sigma$ -maximal. Aiming for a contradiction, suppose that  $\mathcal{H}$  is a filter contained in  $\Sigma$  which properly contains  $\mathcal{G}$  and pick  $A \in \mathcal{H} \setminus \mathcal{G}$ . Since  $A \notin \mathcal{G}$ , pick

$$f \in ((p + E(R_1)) \cap \overline{X_1}) \cup \bigcup_{i=2}^n (E(R_i) \cap \overline{X_i})$$

such that  $A \notin f$ . Either

- $f \in (p + E(R_1)) \cap \overline{X_1}$ , or
- $f \in E(R_j) \cap \overline{X_j}$  for some  $j \neq 1$ .

In either case, let  $L = \beta\mathbb{N} + f$ . By Lemma 2.5,  $L \cap \widehat{\mathcal{H}} \neq \emptyset$  so pick  $q \in L \cap \widehat{\mathcal{H}}$ . Since  $A \in q$  we have  $q \neq f$ . But  $q \in L \cap \widehat{\mathcal{H}} \subseteq L \cap \widehat{\mathcal{G}} = \{f\}$ , a contradiction.  $\square$

**Corollary 2.12.** *Every minimal ultrafilter on  $\mathbb{N}$  contains more than one  $\Sigma$ -maximal filter.*

*Proof.* Let  $p$  be a minimal ultrafilter, let  $R$  be the minimal right ideal containing  $p$ , and let  $R'$  be any other minimal right ideal. By Theorem 2.7,  $\mathcal{G} = \bigcap(p + E(R))$  is a  $\Sigma$ -maximal filter contained in  $p$ . By the previous theorem, there is a  $\Sigma$ -maximal filter  $\mathcal{H}$  contained in  $p$  such that  $\widehat{\mathcal{H}} \cap R' \neq \emptyset$ . Theorem 2.1 implies that  $\widehat{\mathcal{G}} \cap K(\beta\mathbb{N}) \subseteq p + E(R) \subseteq R$ , so that  $\widehat{\mathcal{G}} \cap R' = \emptyset$ . Thus  $\mathcal{G} \neq \mathcal{H}$ .  $\square$

### 3. TOPOLOGY IN $K(\beta S)$

A set  $F \subseteq S$  is a *left solution set* (respectively a *right solution set*) if and only if there exist  $a, b \in S$  such that  $F = \{x \in S : ax = b\}$  (respectively  $F = \{x \in S : xa = b\}$ ). If every left solution set and every right solution set is finite, then  $S$  is called *weakly cancellative*. If  $|S| = \kappa$  and the union of fewer than  $\kappa$  solution sets (left or right) always has cardinality less than  $\kappa$ , then  $S$  is called *very weakly cancellative*. Of course, if  $\kappa = \omega$ , then “weakly cancellative” and “very weakly cancellative” mean the same thing. We let  $U(S)$  denote the set of uniform ultrafilters on  $S$ . By [7, Lemma 6.34.3], if  $S$  is very weakly cancellative, then  $U(S)$  is an ideal of  $\beta S$ .

The easy results of the following lemma do not appear to have been written down before.

**Lemma 3.1.** *Assume that  $S$  is very weakly cancellative. Then  $K(\beta S) = K(U(S))$ , the minimal left ideals of  $\beta S$  and  $U(S)$  are the same, and the minimal right ideals of  $\beta S$  and  $U(S)$  are the same. If  $L$  is a minimal left ideal of  $\beta S$  and  $p \in L$ , then  $L = \beta S \cdot p = S^* \cdot p = U(S) \cdot p$ .*

*Proof.* Since  $U(S)$  is an ideal of  $\beta S$ ,  $K(\beta S) \subseteq U(S)$  and thus by [7, Theorem 1.65],  $K(U(S)) = K(\beta S)$ .

Let  $T$  be a minimal left ideal of  $U(S)$ . Since  $U(S)$  is a left ideal of  $\beta S$ , by [7, Lemma 1.43(c)],  $T$  is a minimal left ideal of  $\beta S$ . Then  $L \subseteq K(\beta S) = K(U(S)) \subseteq U(S)$ . Thus  $L$  is a left ideal of  $U(S)$  so pick a minimal left ideal  $T$  of  $U(S)$  such that  $T \subseteq L$ . As we just saw,  $T$  is a left ideal of  $\beta S$  so  $T = L$ .

The arguments in the paragraph above were completely algebraic, so by a left-right switch, we have that the minimal right ideals of  $\beta S$  and  $U(S)$  are the same.

Finally, let  $L$  be a minimal left ideal of  $\beta S$  and let  $p \in L$ . Then  $\beta S \cdot p$  is a left ideal of  $\beta S$  contained in  $L$ , so  $L = \beta S \cdot p$ . Also,  $U(S) \cdot p$  is a left ideal of  $U(S)$  contained in  $L$ , so  $L = U(S) \cdot p$ . Thus  $L = U(S) \cdot p \subseteq S^* \cdot p \subseteq \beta S \cdot p = L$ .  $\square$

Note the similarity of this lemma with [7, Theorems 4.36 and 4.37], which state that  $S$  is weakly cancellative if and only if  $S^*$  is an ideal of  $\beta S$ , in which case  $K(S^*) = K(\beta S)$ , and a set is a minimal left ideal (respectively, minimal right ideal) for  $S^*$  if and only if it is a minimal left ideal (respectively, minimal right ideal) for  $\beta S$ .

Note that very weak cancellativity does not imply  $S^*$  is an ideal of  $\beta S$  (because this is equivalent to weak cancellativity); in general, it may not even be a sub-semigroup of  $\beta S$ . However, as an immediate corollary to Lemma 3.1, if  $S$  is very weakly cancellative then  $K(\beta S) \subseteq S^*$ .

Most of the results of this section (all except for the next four, Lemma 3.2 through Corollary 3.5) assume that  $S$  is very weakly cancellative. Under the additional assumption that there is a uniform, finite bound on  $|\{x \in S : xa = a\}|$  for each  $a \in S$ , we establish that if  $p$  is a minimal ultrafilter on  $S$ , then

- $p$  is a butterfly point of  $S^*$  and, furthermore,
- $S^* \setminus \{p\}$  is not normal.

Under the additional assumption that  $S$  is countable, we also show that if  $R$  is the minimal right ideal containing  $p$ , then

- $p \cdot E(R)$  is a  $P$ -space, and
- $p \cdot E(R)$  is not Borel in  $\beta S$ .

Note in particular that these results apply to the semigroups  $(\mathbb{N}, +)$  and  $(\mathbb{N}, \cdot)$ , which are easily seen to satisfy all of the above assumptions. The proofs proceed by extracting the topological content of Theorem 2.7, which provides a canonical (and useful) basis for the space  $p \cdot E(R)$ .

**Lemma 3.2.** *Let  $R$  be a minimal right ideal of  $\beta S$  and let  $p \in R$ . If  $q \in p \cdot E(R)$ , then  $q \cdot E(R) = p \cdot E(R)$ .*

*Proof.* Assume that  $q \in p \cdot E(R)$  and pick  $e \in E(R)$  such that  $q = p \cdot e$ . Given any  $f \in E(R)$ ,  $q \cdot f = p \cdot e \cdot f = p \cdot f$  so  $q \cdot E(R) \subseteq p \cdot E(R)$ . Now let  $L$  be the minimal left ideal with  $p \in L$  and let  $f$  be the identity of  $L \cap R$ . Then  $q \cdot f = p \cdot f = p$  so  $p \in q \cdot E(R)$  so, as above,  $p \cdot E(R) \subseteq q \cdot E(R)$ .  $\square$

For  $a \in S$ , let  $\text{Fix}(a) = \{x \in S : xa = a\}$ . Several results in this section use the hypothesis that there is a uniform, finite bound on the size of the sets  $\text{Fix}(a)$ . The left-right switch of [6, Theorem 4.11] shows that this assumption is strictly weaker than the assertion that there is a finite bound on the size of right solution sets.

**Lemma 3.3.** *Let  $k \in \mathbb{N}$  and assume that for each  $a \in S$ ,  $|\text{Fix}(a)| \leq k$ . Then for each  $p \in S^*$ ,  $|\{x \in S : x \cdot p = p\}| \leq k$ .*

*Proof.* Let  $p \in S^*$  and suppose we have distinct  $x_1, x_2, \dots, x_{k+1}$  in  $S$  such that  $x_i p = p$  for each  $i$ . For  $i \in \{1, 2, \dots, k+1\}$ , let  $D_i = \{a \in S : x_i a = a\}$ . Since each  $\lambda_{x_i}$  is continuous, we have by [7, Theorem 3.35] that each  $D_i \in p$ . Pick  $a \in \bigcap_{i=1}^{k+1} D_i$ . Then  $\{x_1, x_2, \dots, x_{k+1}\} \subseteq \text{Fix}(a)$ , a contradiction.  $\square$

Let us note that the conclusion of this lemma is not a consequence of very weak cancellativity, or even of weak cancellativity. The semigroup  $(\mathbb{N}, \vee)$ , where  $a \vee b = \max\{a, b\}$ , is weakly cancellative, but  $n \vee p = p$  for every  $p \in \mathbb{N}^*$  and  $n \in \mathbb{N}$ .

**Theorem 3.4.** *Let  $p$  be a minimal ultrafilter on  $S$ , and let  $L$  and  $R$  denote the minimal left and right ideals of  $\beta S$  containing  $p$ . Then*

$$\{A^* \cap B^* : L \subseteq A^* \text{ and } p \cdot E(R) \subseteq B^*\}$$

*is a local basis for  $p$  in  $S^*$ .*

*Proof.* Let  $\mathcal{F} = \bigcap L$  and  $\mathcal{G} = \bigcap (p \cdot E(R))$ . By Theorem 2.7,  $p$  is the filter generated by  $\mathcal{F} \cup \mathcal{G}$ . That is,  $p = \{C \subseteq S : (\exists A \in \mathcal{F})(\exists B \in \mathcal{G})(A \cap B \subseteq C)\}$ . Thus, given  $C \in p$ , pick  $A \in \mathcal{F}$  and  $B \in \mathcal{G}$  such that  $A \cap B \subseteq C$ . Then  $p \in A^* \cap B^* \subseteq C^*$ .  $\square$

**Corollary 3.5.**

- (1) *If  $L$  is a minimal left ideal of  $\beta S$ , then  $\{B^* \cap L : B \in \Sigma\}$  is a basis for  $L$ .*
- (2) *If  $R$  is a minimal right ideal and  $p \in R$ , then  $\{A^* \cap p \cdot E(R) : A \in \Theta\}$  is a basis for  $p \cdot E(R)$ .*

*Proof.* To establish (1), assume that  $L$  is a minimal left ideal, let  $C \subseteq S$  such that  $C^* \cap L \neq \emptyset$ , and pick  $p \in C^* \cap L$ . Let  $R$  be the minimal right ideal with  $p \in R$ . By Theorem 3.4 pick  $A, B \subseteq S$  such that  $L \subseteq A^*$ ,  $p \cdot E(R) \subseteq B^*$ , and  $p \in A^* \cap B^* \subseteq C^*$ . By Theorem 2.1,  $\overline{p \cdot E(R)}$  meets every minimal left ideal so by Lemma 2.3,  $B \in \Sigma$ . Also  $C^* \cap L \supseteq A^* \cap B^* \cap L = B^* \cap L$ .

To establish (2), let  $R$  be a minimal right ideal and let  $p \in R$ . Let  $C \subseteq S$  such that  $C^* \cap (p \cdot E(R)) \neq \emptyset$ , and pick  $q \in C^* \cap (p \cdot E(R))$ . Let  $L$  be the minimal left ideal with  $q \in L$ . Then  $q \cdot E(R) = p \cdot E(R)$  by Lemma 3.2 so by Theorem 3.4 pick  $A, B \subseteq S$  such that  $L \subseteq A^*$ ,  $q \cdot E(R) \subseteq B^*$ , and  $q \in A^* \cap B^* \subseteq C^*$ . Since  $L \subseteq A^*$ ,  $A \in \Theta$  by Lemma 2.3, and  $C^* \cap (p \cdot E(R)) \supseteq A^* \cap B^* \cap (p \cdot E(R)) = A^* \cap (p \cdot E(R))$ .  $\square$

**Lemma 3.6.** *Assume  $S$  is very weakly cancellative and  $|S| = \kappa$ . Let  $A$  be a thick subset of  $S$  and let  $F \subseteq S$  such that  $|F| < \kappa$ . Then  $A \setminus F$  is thick.*

*Proof.* Pick a minimal left ideal  $L$  of  $\beta S$  such that  $L \subseteq \overline{A}$ . As we have noted,  $U(S)$  is an ideal of  $\beta S$ , which implies that  $L \subseteq U(S)$ . Now  $\overline{F} \cap U(S) = \emptyset$  so  $L \subseteq \overline{A} \setminus \overline{F} = \overline{A \setminus F}$ .  $\square$

**Lemma 3.7.** *Assume that  $S$  is very weakly cancellative,  $|S| = \kappa$ , and  $A$  is a thick subset of  $S$ . Then  $A$  contains  $\kappa$  pairwise disjoint thick sets.*

*Proof.* Enumerate  $\kappa \times \kappa$  as  $\langle \delta(\sigma), \tau(\sigma) \rangle_{\sigma < \kappa}$  and enumerate  $\mathcal{P}_f(S)$  as  $\langle F_\iota \rangle_{\iota < \kappa}$ . We inductively choose  $\langle x_\sigma \rangle_{\sigma < \kappa}$  such that for  $\sigma < \kappa$ ,  $F_{\delta(\sigma)} \cdot x_\sigma \subseteq A$  and for  $\mu < \sigma < \kappa$ ,  $F_{\delta(\sigma)} \cdot x_\sigma \cap F_{\delta(\mu)} \cdot x_\mu = \emptyset$ . Having chosen  $\langle x_\mu \rangle_{\mu < \sigma}$ , let  $H = \bigcup_{\mu < \sigma} F_{\delta(\mu)} \cdot x_\mu$ . Then  $|H| < \kappa$  so by Lemma 3.6,  $A \setminus H$  is thick, so we may pick  $x_\sigma$  with  $F_{\delta(\sigma)} \cdot x_\sigma \subseteq A \setminus H$ . Having chosen  $\langle x_\sigma \rangle_{\sigma < \kappa}$ , for each  $\eta < \kappa$ , let  $A_\eta = \bigcup \{F_{\delta(\sigma)} \cdot x_\sigma : \tau(\sigma) = \eta\}$ . Then  $\langle A_\eta \rangle_{\eta < \kappa}$  is a sequence of  $\kappa$  pairwise disjoint thick subsets of  $A$ .  $\square$

**Lemma 3.8.** *Suppose  $S$  is very weakly cancellative and let  $L$  and  $R$  be minimal left and right ideals of  $\beta S$ .*

- (1) *If there is a uniform, finite bound on  $|\text{Fix}(a)|$  for  $a \in S$ , then  $L$  has no isolated points in the topology inherited from  $S^*$ .*
- (2) *If  $p \in R$ , then  $p \cdot E(R)$  has no isolated points in the topology it inherits from  $S^*$ .*

*Proof.* (1) Suppose that  $q$  is an isolated point of  $L$ . Pick  $A \in q$  such that  $A^* \cap L = \{q\}$ . Now  $L$  is a minimal left ideal of  $\beta S$  and  $q \in L = S^* \cdot q$  by Lemma 3.1, so pick  $r \in S^*$  such that  $q = r \cdot q$ . Then  $\{x \in S : x^{-1}A \in q\} \in r$ . Let  $F = \{x \in S : x \cdot q = q\}$ . Then  $F$  is finite by Lemma 3.3, so pick  $x \in S \setminus F$  such that  $x^{-1}A \in q$ . Then  $A \in x \cdot q$  so  $x \cdot q \in A^* \cap L$  and  $x \cdot q \neq q$ , a contradiction.

(2) Suppose  $q \in p \cdot E(R)$ , and let  $U$  be a neighborhood of  $q$  in  $p \cdot E(R)$ . By Corollary 3.5, there is some thick set  $A$  such that  $A^* \cap (p \cdot E(R)) \subseteq U$ . By combining Lemma 3.7 with Lemma 2.3,  $A^*$  contains more than one minimal left ideal. Each minimal left ideal contains a point of  $p \cdot E(R)$  by Theorem 2.1, so this shows that  $A^*$ , hence  $U$ , contains more than one point of  $p \cdot E(R)$ .  $\square$

Let us note that weak cancellativity alone is not enough to prove Lemma 3.8(1). The semigroup  $(\mathbb{N}, \vee)$  is weakly cancellative, but  $q \vee p = p$  for every  $p, q \in \mathbb{N}^*$ . This means that  $\{p\}$  is a minimal left ideal for every  $p \in \mathbb{N}^*$ .

**Theorem 3.9.** *Suppose  $S$  is very weakly cancellative and has a uniform, finite bound on  $|\text{Fix}(a)|$  for  $a \in S$ . Then every minimal ultrafilter on  $S$  is a butterfly point of  $S^*$ .*

*Proof.* Let  $L$  and  $R$  be the minimal left and right ideals containing  $p$ . Theorem 2.1 asserts that  $\{p\} = L \cap \overline{p \cdot E(R)}$ . Neither  $L$  nor  $\overline{p \cdot E(R)}$  has any isolated points by Lemma 3.8, so this makes  $p$  a butterfly point.  $\square$

We have included the proof of Theorem 3.9 because of its naturalness and simplicity. But we prove next a stronger result that supersedes Theorem 3.9 by showing, under the same assumptions, that every minimal ultrafilter is a non-normality point of  $S^*$ .

**Lemma 3.10.** *Let  $S$  be a very weakly cancellative semigroup with  $|S| = \kappa$ , and let  $\mathcal{U}$  be a collection of open subsets of  $S^*$  with  $|\mathcal{U}| \leq \kappa$ . If  $\bigcap \mathcal{U}$  contains a minimal left ideal of  $\beta S$ , then  $\bigcap \mathcal{U}$  contains  $2^{2^\kappa}$  distinct minimal left ideals of  $\beta S$ .*

*Proof.* Let  $L$  be a minimal left ideal of  $\beta S$  with  $L \subseteq \bigcap \mathcal{U}$ . We claim that for each  $U \in \mathcal{U}$ , there exists  $B_U \subseteq S$  such that  $L \subseteq B_U^* \subseteq U$ . Let  $U \in \mathcal{U}$ . For each  $p \in L$  pick  $C_p \in p$  such that  $C_p^* \subseteq U$ . Using the compactness of  $L$ , pick a finite  $F \subseteq L$  such that  $L \subseteq \bigcup_{p \in F} C_p^*$ , and let  $B_U = \bigcup_{p \in F} C_p$ . Then  $B_U^* = \bigcup_{p \in F} C_p^* \subseteq U$ , as claimed. Let

$$\mathcal{B} = \left\{ \bigcap \mathcal{F} : \mathcal{F} \in \mathcal{P}_f(\{B_U : U \in \mathcal{U}\}) \right\}.$$

Observe that  $\mathcal{B}$  is a set of at most  $\kappa$  subsets of  $S$ ,  $L \subseteq \bigcap_{B \in \mathcal{B}} B^* \subseteq \bigcap \mathcal{U}$ , and  $\mathcal{B}$  is closed under finite intersections.

Enumerate  $S$  as  $\langle s_\sigma : \sigma < \kappa \rangle$  and enumerate  $\mathcal{B} \times \mathcal{P}_f(S)$  as  $\langle D_\sigma : \sigma < \kappa \rangle$ . For  $\sigma < \kappa$ , let  $E_\sigma = \bigcap_{s \in F} s^{-1}B$ , where  $(B, F) = D_\sigma$ .

We claim that  $|E_\sigma| = \kappa$  for each  $\sigma < \kappa$ . To see this, let  $p \in L$ , let  $\sigma < \kappa$ , and let  $(B, F) = D_\sigma$ . For each  $s \in F$ ,  $s \cdot p \in L \subseteq \overline{B}$ , which implies that  $s^{-1}B \in p$ , which implies that  $E_\sigma \in p$ . From this and [7, Lemma 6.34.3], it follows that  $|E_\sigma| = \kappa$ .

We now construct a sequence of elements of  $S$  by transfinite recursion. To begin, pick  $t_0 \in E_0$ . Given  $0 < \mu < \kappa$ , assume we have chosen  $\langle t_\sigma : \sigma < \mu \rangle$  already such that

- (1) if  $\sigma < \mu$ , then  $t_\sigma \in E_\sigma$ ,
- (2) if  $\sigma < \delta < \mu$ , then  $t_\sigma \neq t_\delta$ , and
- (3) if  $\sigma < \mu$ ,  $\eta < \sigma$ ,  $\nu < \sigma$ , and  $\tau < \sigma$ , then  $s_\eta \cdot t_\nu \neq s_\tau \cdot t_\sigma$ .

Given  $\eta < \mu$ ,  $\nu < \mu$ , and  $\tau < \mu$ , let  $A_{\eta, \nu, \tau} = \{t \in S : s_\eta \cdot t_\nu = s_\tau \cdot t\}$ . Then each  $A_{\eta, \nu, \tau}$  is a left solution set, so  $|\bigcup_{\eta < \mu} \bigcup_{\nu < \mu} \bigcup_{\tau < \mu} A_{\eta, \nu, \tau}| < \kappa$ . Pick

$$t_\mu \in E_\mu \setminus (\{t_\sigma : \sigma < \mu\} \cup \bigcup_{\eta < \mu} \bigcup_{\nu < \mu} \bigcup_{\tau < \mu} A_{\eta, \nu, \tau}).$$

The three hypotheses are again satisfied at the next stage of the recursion, and this completes the construction of our sequence  $\langle t_\sigma : \sigma < \kappa \rangle$ .

**Claim.** If  $p$  and  $q$  are distinct uniform ultrafilters on  $T = \{t_\sigma : \sigma < \kappa\}$ , then  $\beta S \cdot p \cap \beta S \cdot q = \emptyset$ .

*Proof of claim.* Assume  $P$  and  $Q$  are disjoint subsets of  $T$ , with  $P \in p$  and  $Q \in q$ . Then we claim that

$$\beta S \cdot p \subseteq \overline{\{s_\eta \cdot t_\sigma : t_\sigma \in P \text{ and } \eta < \sigma\}}.$$

To see this, it suffices to show that  $S \cdot p \subseteq \overline{\{s_\eta \cdot t_\sigma : t_\sigma \in P \text{ and } \eta < \sigma\}}$ . Let  $s_\nu \in S$ . As  $p$  is uniform,  $\{t_\sigma : t_\sigma \in P \text{ and } \nu < \sigma\} \in p$ , so that  $s_\nu \cdot \{t_\sigma : t_\sigma \in P \text{ and } \nu < \sigma\} \in s_\nu \cdot p$  and  $s_\nu \cdot \{t_\sigma : t_\sigma \in P \text{ and } \nu < \sigma\} \subseteq \{s_\eta \cdot t_\sigma : t_\sigma \in P \text{ and } \eta < \sigma\}$ . Similarly,

$$\beta S \cdot q \subseteq \overline{\{s_\nu \cdot t_\delta : t_\delta \in Q \text{ and } \nu < \delta\}}.$$

Because  $\{s_\eta \cdot t_\sigma : t_\sigma \in P \text{ and } \eta < \sigma\} \cap \{s_\nu \cdot t_\delta : t_\delta \in Q \text{ and } \nu < \delta\} = \emptyset$  by construction, we have that  $\beta S \cdot p \cap \beta S \cdot q = \emptyset$ , as desired.  $\square$

Consider the relation on  $\{D_\sigma : \sigma < \kappa\}$  defined by

$$D_\sigma \prec D_\tau \quad \text{if and only if} \quad \pi_1(D_\tau) \subseteq \pi_1(D_\sigma) \quad \text{and} \quad \pi_2(D_\sigma) \subseteq \pi_2(D_\tau)$$

where, as usual,  $\pi_1(B, F) = B$  and  $\pi_2(B, F) = F$ . Observe that, by our choice of  $\mathcal{B}$  and the definition of the  $D_\sigma$ , any finitely many members of  $\{D_\sigma : \sigma < \kappa\}$  have a common upper bound with respect to  $\prec$ . In other words,  $\{D_\sigma : \sigma < \kappa\}$  is directed by  $\prec$ .

For each  $\sigma < \kappa$ , let  $T_\sigma = \{t_\tau : D_\sigma \prec D_\tau\}$ . We claim that  $\{T_\sigma : \sigma < \kappa\}$  has the  $\kappa$ -uniform finite intersection property. (This means that the intersection of finitely many of the  $T_\sigma$  has size  $\kappa$ .) To see this, first observe that each  $T_\sigma$  has size  $\kappa$ , because if  $T_\sigma = (B, F)$  then for any  $s \in S \setminus F$ ,  $D_\sigma \prec (B, F \cup \{s\})$ . Then, if  $H \in \mathcal{P}_f(\kappa)$ , pick  $\tau$  such that  $D_\sigma \prec D_\tau$  for each  $\sigma \in H$ , and observe that  $|\bigcap_{\sigma \in H} T_\sigma| \geq |T_\tau| = \kappa$ .

By [7, Theorem 3.62], there are  $2^{2^\kappa}$  distinct uniform ultrafilters on  $S$  containing  $\{T_\sigma : \sigma < \kappa\}$ . For each such ultrafilter  $p$ , let  $L_p$  denote a minimal left ideal contained in  $\beta S \cdot p$ . (One must exist, because  $\beta S \cdot p$  is a left ideal.) If  $p \neq q$ , then  $L_p \neq L_q$ , because  $\beta S \cdot p$  and  $\beta S \cdot q$  are disjoint by the claim above. To complete the proof of the theorem, we will show that each such  $L_p$  is contained in  $\bigcap \mathcal{U}$ .

Let  $p$  be a uniform ultrafilter on  $S$  containing  $\{T_\sigma : \sigma < \kappa\}$ . If  $B \in \mathcal{B}$  and  $s \in S$ , then pick  $\sigma < \kappa$  such that  $D_\sigma = (B, \{s\})$ , and observe that  $T_\sigma \in p$ . If  $\tau < \kappa$  and  $D_\sigma \prec D_\tau = (C, F)$ , then  $t_\tau \in E_\tau = \bigcap_{r \in F} r^{-1}C \subseteq s^{-1}B$ , so that  $s \cdot p \in \overline{B}$ . Because  $s$  and  $B$  were arbitrary, this shows that  $S \cdot p \subseteq \overline{B}$  for all  $B \in \mathcal{B}$ . By the continuity of  $\rho_p$ , this implies  $\beta S \cdot p \subseteq \overline{B}$  for all  $B \in \mathcal{B}$ . Thus

$$L_p \subseteq \beta S \cdot p \subseteq \bigcap \{\overline{B} : B \in \mathcal{B}\} \subseteq \bigcap \mathcal{U},$$



completing the proof of the lemma.  $\square$

**Theorem 3.11.** *Let  $S$  be a very weakly cancellative semigroup with  $|S| = \kappa$ , and assume there is a uniform, finite bound on  $|\text{Fix}(a)|$  for  $a \in S$ . Then for every minimal ultrafilter  $p$  on  $S$ ,  $S^* \setminus \{p\}$  is not normal.*

*Proof.* Let  $L$  and  $R$  be the minimal left and right ideals respectively with  $p \in L \cap R$ . Let  $e$  be the identity of  $L \cap R$ . Let  $C = \overline{p \cdot E(R)}$ . We claim that  $L \setminus \{p\}$  and  $C \setminus \{p\}$  cannot be separated by open sets in  $S^* \setminus \{p\}$ . Suppose instead that we have open subsets  $U$  and  $V$  of  $S^*$  such that  $L \setminus \{p\} \subseteq U$ ,  $C \setminus \{p\} \subseteq V$ , and  $U \cap V \subseteq \{p\}$ . Let  $D = \{s \in S : s \cdot p \neq p\}$  and observe that, by Lemma 3.3,  $S \setminus D$  is finite.

For each  $s \in D$ , let  $W_s = S^* \setminus s \cdot C$ . Now fix  $s \in D$ . Because  $\lambda_s$  is continuous,  $W_s$  is open in  $S^*$ . We claim also that  $p \in W_s$ . We know that  $p \cdot E(R) \cdot e = \{p \cdot e\} = \{p\}$ . Because  $\rho_e(p \cdot E(R)) = \{p\}$ , and  $\rho_e$  is continuous on all of  $\beta S$ , we have

$$C \cdot e = \rho_e(\overline{p \cdot E(R)}) = \overline{\rho_e(p \cdot E(R))} = \{p\}.$$

If  $p \in s \cdot C$ , then  $p = p \cdot e \in s \cdot C \cdot e = \{s \cdot p\}$ , so  $p = s \cdot p$ . This contradicts the assumption that  $s \in D$ , so we may conclude that  $W_s$  is a neighborhood of  $p$ . Hence  $L \subseteq U \cup W_s$ . Furthermore, because  $s$  was an arbitrary element of  $D$ ,  $L \subseteq \bigcap_{s \in D} (U \cup W_s)$ .

By Lemma 3.10, there is a minimal left ideal  $L'$  of  $\beta S$  such that  $L' \neq L$  and  $L' \subseteq \bigcap_{s \in D} (U \cup W_s)$ . Let  $f$  be the identity of  $L' \cap R$ . Now  $p \in L = S^* \cdot p$  so  $p = q \cdot p$  for some  $q \in S^*$ . Since  $q \in S^*$ ,  $D \in q$  and so  $p \in \overline{D \cdot p}$ . Therefore  $p \cdot f = \rho_f(p) \in \rho_f(\overline{D \cdot p}) = \overline{D \cdot p \cdot f}$ .

We claim  $L' \cap \bigcap_{s \in D} W_s \cap (D \cdot p \cdot f) = \emptyset$ . Suppose instead that  $s \in D$  and  $s \cdot p \cdot f \in L' \cap \bigcap_{t \in D} W_t$ . Then, in particular,  $s \cdot p \cdot f \in W_s$ . But  $p \cdot f \in C$ , so  $s \cdot p \cdot f \in s \cdot C = S^* \setminus W_s$ , a contradiction.

As  $p \cdot f \in \overline{D \cdot p \cdot f}$  and  $L' \cap \bigcap_{s \in D} W_s \cap (D \cdot p \cdot f) = \emptyset$ , we have that  $L' \cap \bigcap_{s \in D} W_s$  is not a neighborhood of  $p \cdot f$  in  $L'$ .

Now  $p \cdot f \in C \setminus \{p\} \subseteq V$  and  $V$  is open in  $S^*$ , so  $L' \cap V$  is a neighborhood of  $p \cdot f$  in  $L'$ . Therefore  $L' \cap V$  cannot be contained in  $L' \cap \bigcap_{s \in D} W_s$ . Pick  $q \in L' \cap V \setminus (L' \cap \bigcap_{s \in D} W_s)$ , and pick  $s \in D$  such that  $q \notin W_s$ . Because  $L' \subseteq U \cup \bigcap_{t \in D} W_t$ , we must have  $q \in U \cap V$  and  $q \notin W_s$ . But  $q \notin W_s$  implies  $q \neq p$ , so this shows that  $U \cap V \not\subseteq \{p\}$ , as desired.  $\square$

**Corollary 3.12.** *Let  $p \in K(\beta\mathbb{N}, +)$ . Then  $\mathbb{N}^* \setminus \{p\}$  is not normal.*

*Proof.*  $(\mathbb{N}, +)$  is cancellative and for  $a \in \mathbb{N}$ ,  $\{x \in \mathbb{N} : x + a = a\} = \emptyset$ .  $\square$

**Corollary 3.13.** *Let  $p \in K(\beta\mathbb{N}, \cdot)$ . Then  $\mathbb{N}^* \setminus \{p\}$  is not normal.*

*Proof.*  $(\mathbb{N}, \cdot)$  is cancellative and for  $a \in \mathbb{N}$ ,  $\{x \in \mathbb{N} : xa = a\} = \{1\}$ .  $\square$

$K(\beta\mathbb{N}, +) \cap K(\beta\mathbb{N}, \cdot) = \emptyset$  by [7, Corollary 13.15], so the results of Corollaries 3.12 and 3.13 do not overlap.

Let us note that while Theorems 3.9 and 3.11 are stated for the space  $S^*$ , the conclusions also hold when  $S^*$  is replaced with either  $\beta S$  or  $U(S)$ . For  $\beta S$ , this can be deduced directly from Theorems 3.9 and 3.11 (using the fact that if  $p$  is a butterfly/non-normality point a closed subspace of  $X$ , then it is one in  $X$  too). For  $U(S)$ , it follows from making a few trivial modifications to the proofs presented already.

We turn now to the last two results of this section, which give two curious topological properties of the spaces of the form  $p \cdot E(R)$ . These results are proved under the extra assumption that  $S$  is countable.

Recall that  $x$  is a  $P$ -point of a space  $X$  if every countable intersection of neighborhoods of  $x$  is a neighborhood of  $x$ .  $X$  is a  $P$ -space if all its points are  $P$ -points or, equivalently, if countable intersections of open sets are open.

**Theorem 3.14.** *Suppose  $S$  is countable, weakly cancellative, and has a uniform, finite bound on  $|\text{Fix}(a)|$  for  $a \in S$ . Let  $R$  be a minimal right ideal of  $S^*$  and let  $r \in R$ . Then every  $q \in r \cdot E(R)$  is a  $P$ -point of  $\overline{r \cdot E(R)}$ . In particular,  $r \cdot E(R)$  is a  $P$ -space.*

*Proof.* Let  $q \in r \cdot E(R)$ , and let  $U_1, U_2, U_3, \dots$  be open neighborhoods of  $q$  in  $C = \overline{r \cdot E(R)}$ . Let  $L$  denote the minimal left ideal of  $S^*$  containing  $q$ .

Setting  $K_n = C \setminus U_n$  for every  $n \in \mathbb{N}$ , we have

$$L \cap \overline{\bigcup_{n \in \mathbb{N}} K_n} \subseteq L \cap C = \{q\}$$

by Theorem 2.1. Let  $\{d_n : n \in \mathbb{N}\}$  be a countable dense subset of  $L$  that does not contain  $q$ . (Such a set exists because  $L$  is separable, as  $L = \overline{S \cdot q}$ ,  $\{x \in S : x \cdot q = q\}$  is finite by Lemma 3.3, and  $L$  has no isolated points by Lemma 3.8). Taking  $A = \bigcup_{n \in \mathbb{N}} K_n$  and  $B = \{d_n : n \in \mathbb{N}\}$ ,  $A$  and  $B$  are  $\sigma$ -compact subsets of  $\beta\mathbb{N}$  such that  $\overline{A} \cap B = \emptyset$  and  $\overline{B} \cap A = \emptyset$ . By Theorem 3.40 in [7], this implies  $\overline{A} \cap \overline{B} = \emptyset$ . As  $q \in \overline{B}$ , we have  $q \notin \overline{A}$ . Taking complements, this implies  $q$  is in the interior of  $\bigcap_{n \in \mathbb{N}} U_n$ . This shows  $q$  is a  $P$ -point of  $C$ .  $\square$

**Corollary 3.15.** *Suppose  $S$  is countable, weakly cancellative, and has a uniform, finite bound on  $|\text{Fix}(a)|$  for  $a \in S$ . Let  $R$  be a minimal right ideal of  $S^*$ . Then  $E(R)$  is not Borel in  $\beta S$ , nor is  $q \cdot E(R)$  for any  $q \in R$ .*

*Proof.* Since  $q \cdot E(R) = E(R)$  if  $q \in E(R)$ , it suffices to establish the second conclusion. By (a very special case of) Lemma 3.10, there are  $2^c$  minimal left

ideals in  $\beta S$  and  $q \cdot E(R)$  meets each of them, so  $|q \cdot E(R)| = 2^{\mathfrak{c}}$ . Theorem 3.14 implies that any compact subset of  $q \cdot E(R)$  is finite (because every subspace of a  $P$ -space is a  $P$ -space, but infinite compact spaces are never  $P$ -spaces by [5, Exercise 4K1]). Applying Lemma 3.1 from [8], this implies that  $q \cdot E(R)$  is not Borel in  $\beta S$ . (Lemma 3.1 of [8] says that any Borel subset of  $\beta \mathbb{N}$  is the union of at most  $\mathfrak{c}$  compact sets, but the proof only uses the fact that  $\mathbb{N}$  is countable.)  $\square$

Once again, we note that weak cancellativity alone is not enough to prove Theorem 3.14 or Corollary 3.15. The semigroup  $(\mathbb{N}, \vee)$  is weakly cancellative, but it has a single minimal right ideal, namely  $\mathbb{N}^*$  itself, and  $E(\mathbb{N}^*) = \mathbb{N}^*$ . Clearly  $\mathbb{N}^*$  is not a  $P$ -space, and it is Borel in  $\beta \mathbb{N}$ .

#### 4. A NEGATIVE RESULT

In this final section, we address the natural question of whether, given a  $\Theta$ -maximal filter  $\mathcal{F}$  and a  $\Sigma$ -maximal filter  $\mathcal{G}$ , their union  $\mathcal{F} \cup \mathcal{G}$  must generate an ultrafilter. We show that it is consistent with ZFC that the answer is negative. More precisely, we will use the hypothesis  $\mathfrak{p} = \mathfrak{c}$  (a weak form of Martin's Axiom) to construct a  $\Theta$ -maximal filter  $\mathcal{F}$  on  $\mathbb{N}$  and a  $\Sigma$ -maximal filter  $\mathcal{G}$  on  $\mathbb{N}$  such that  $\mathcal{F} \cup \mathcal{G}$  does not generate an ultrafilter.

In light of Lemma 2.4, the assertion that some such  $\mathcal{F}$  and  $\mathcal{G}$  exist is equivalent to the assertion that there is a minimal left ideal  $L$  and a  $\Sigma$ -maximal filter  $\mathcal{G}$  such that  $L \cap \widehat{\mathcal{G}}$  contains more than one point. In other words, it is equivalent to the assertion that not every  $\Sigma$ -maximal filter corresponds to a closed transversal for the minimal left ideals.

The hypothesis  $\mathfrak{p} = \mathfrak{c}$  is used indirectly in order to invoke a result from [3] to prove Lemma 4.2 below. In order to keep this section relatively self-contained, we also include a (short) derivation of Lemma 4.2 from CH. This latter hypothesis is stronger (i.e., CH implies  $\mathfrak{p} = \mathfrak{c}$ ) so that proving the result from  $\mathfrak{p} = \mathfrak{c}$  is “better” in some sense. But either hypothesis is, of course, adequate to establish consistency with ZFC, and the reader who wishes to do so may ignore any further mention of  $\mathfrak{p}$  and  $\mathfrak{c}$  and read this section as a self-contained proof carried out in ZFC + CH.

**Lemma 4.1.** *For every thick set  $A \subseteq \mathbb{N}$ , there is a thick  $B \subseteq A$  such that  $A \setminus B$  is also thick.*

*Proof.* Suppose  $A$  is thick. Then we may find a sequence  $I_0, I_1, I_2, I_3 \dots$  of pairwise disjoint intervals contained in  $A$  such that  $\lim_{n \rightarrow \infty} |I_n| = \infty$ .

Taking  $B = \bigcup \{I_n : n \text{ is even}\}$ , it is clear that  $B$  is thick and that  $A \setminus B \supseteq \bigcup \{I_n : n \text{ is odd}\}$  is also thick.  $\square$

**Lemma 4.2.** *Assume  $\mathfrak{p} = \mathfrak{c}$  (or CH). If  $\alpha$  is an ordinal with  $\alpha < \mathfrak{c}$  and  $\langle A_\beta : \beta < \alpha \rangle$  is a sequence of thick subsets of  $\mathbb{N}$ , well-ordered in type  $\alpha$ , such that  $A_\beta^* \supseteq A_\gamma^*$  whenever  $\beta < \gamma$ , then there is a thick set  $A_\alpha$  such that  $A_\beta^* \supseteq A_\alpha^*$  for all  $\beta < \alpha$ .*

*Proof.* This follows from [3, Theorem 3.4].

More precisely, in [3] the cardinal number  $\mathfrak{t}_\Theta$  is defined to be the least cardinal  $\kappa$  such that the conclusion of the present lemma is true for all  $\alpha < \kappa$ . Thus the present lemma can be rephrased as follows: *if  $\mathfrak{p} = \mathfrak{c}$ , then  $\mathfrak{t}_\Theta = \mathfrak{c}$ .* But Theorem 3.4 in [3] asserts that  $\mathfrak{t}_\Theta = \mathfrak{t}$ , and it is known that  $\mathfrak{p} \leq \mathfrak{t} \leq \mathfrak{c}$ . Hence  $\mathfrak{p} = \mathfrak{c}$  implies  $\mathfrak{t}_\Theta = \mathfrak{c}$ , as claimed.  $\square$

*Proof of Lemma 4.2 from CH.* If  $\alpha = \delta + 1$ , let  $A_\alpha = A_\delta$ . So assume  $\alpha$  is a (nonzero) limit ordinal. We claim that for each  $F \in \mathcal{P}_f(\alpha)$ ,  $\bigcap_{\delta \in F} A_\delta$  is thick. For such  $F$ , let  $\gamma = \max F$ . Then  $A_\gamma^* \subseteq \bigcap_{\delta \in F} A_\delta^* = (\bigcap_{\delta \in F} A_\delta)^*$  so  $G = A_\gamma \setminus \bigcap_{\delta \in F} A_\delta$  is finite. Since  $A_\gamma$  is thick, by Lemma 3.6,  $A_\gamma \setminus G$  is thick and  $A_\gamma \setminus G \subseteq \bigcap_{\delta \in F} A_\delta$ .

Now  $\alpha$  is countable, by CH. Thus we may enumerate  $\{A_\delta : \delta < \alpha\}$  as  $\langle B_n \rangle_{n=1}^\infty$ . For each  $n$ , let  $C_n = \bigcap_{t=1}^n B_t$ . Then each  $C_n$  is thick. For  $n \in \mathbb{N}$ , pick  $x_n \in \mathbb{N}$  such that  $\{x_n + 1, x_n + 2, \dots, x_n + n\} \subseteq C_n$  and let  $A_\alpha = \bigcup_{n=1}^\infty \{x_n + 1, x_n + 2, \dots, x_n + n\}$ . Then  $A_\alpha$  is thick. Given  $\delta < \alpha$ , pick  $n \in \mathbb{N}$  such that  $A_\delta = B_n$ . Then  $A_\alpha \setminus A_\delta \subseteq \bigcup_{t=1}^{n-1} \{x_t + 1, x_t + 2, \dots, x_t + t\}$ , so  $A_\alpha^* \subseteq A_\delta^*$ .  $\square$

**Lemma 4.3.** *Assuming  $\mathfrak{p} = \mathfrak{c}$  (or CH), there is a sequence  $\langle B_\alpha : \alpha < \mathfrak{c} \rangle$  of thick subsets of  $\mathbb{N}$  such that*

- if  $\beta < \alpha$ , then  $B_\alpha \setminus B_\beta$  is finite; i.e.,  $B_\alpha \subseteq^* B_\beta$ .
- if  $\beta < \alpha$ , then  $B_\beta \setminus B_\alpha$  is thick.
- $\{B_\alpha : \alpha < \mathfrak{c}\}$  is a basis for a  $\Theta$ -maximal filter on  $\mathbb{N}$ .

*Proof.* Fix an enumeration  $\{A_\alpha : \alpha < \mathfrak{c}\}$  of  $\Theta$ . The sequence  $\langle B_\alpha : \alpha < \mathfrak{c} \rangle$  is constructed via transfinite recursion. For the base stage of the recursion, set  $B_0 = \mathbb{N}$ .

At the successor stage  $\alpha + 1$  of the recursion, assuming  $B_\alpha$  has already been defined, there are two cases. If  $B_\alpha \cap A_\alpha \notin \Theta$ , then choose  $B_{\alpha+1}$  to be any thick subset of  $B_\alpha$  such that  $B_\alpha \setminus B_{\alpha+1}$  is thick. (This is possible by Lemma 4.1.) If  $B_\alpha \cap A_\alpha \in \Theta$ , then let  $B_{\alpha+1}$  be some thick set contained in  $B_\alpha \cap A_\alpha$  with the property that  $B_\alpha \setminus B_{\alpha+1}$  is also thick. (Again, this is possible by Lemma 4.1.)

If  $\alpha$  is a limit ordinal with  $\alpha < \mathfrak{c}$ , then at stage  $\alpha$  of the recursion we will have a sequence  $\langle B_\beta : \beta < \alpha \rangle$  of thick sets such that  $B_\beta^* \supseteq B_\gamma^*$  whenever  $\beta < \gamma$ . By Lemma 4.2, there is a thick set  $B_\alpha$  such that  $B_\beta^* \supseteq B_\alpha^*$  for all  $\beta < \alpha$ . This completes the recursion.

It is clear that  $\langle B_\alpha : \alpha < \mathfrak{c} \rangle$  is a sequence of thick sets, and it follows easily from our construction that if  $\beta < \alpha$  then  $B_\alpha \setminus B_\beta$  is finite and  $B_\beta \setminus B_\alpha \supseteq^* B_\beta \setminus B_{\beta+1}$  is thick. It remains to check that  $\{B_\alpha : \alpha < \mathfrak{c}\}$  is a basis for a  $\Theta$ -maximal filter on  $\mathbb{N}$ .

Let  $\mathcal{F} = \{X \subseteq \mathbb{N} : X \supseteq^* B_\alpha \text{ for some } \alpha < \mathfrak{c}\}$ . Then  $\mathcal{F}$  is a filter. We shall show that in fact  $\mathcal{F} = \{X \subseteq \mathbb{N} : X \supseteq B_\alpha \text{ for some } \alpha < \mathfrak{c}\}$  and consequently  $\{B_\alpha : \alpha < \mathfrak{c}\}$  is a basis for  $\mathcal{F}$ . One inclusion is trivial. Let  $X \in \mathcal{F}$  and pick  $\alpha < \mathfrak{c}$  such that  $X \supseteq^* B_\alpha$ . Let  $F = B_\alpha \setminus X$ . Then  $F$  is finite so  $B_\alpha \setminus F$  is thick so  $B_\alpha \setminus F = A_\delta$  for some  $\delta < \mathfrak{c}$ . Then  $B_\alpha \cap B_\delta$  is thick so  $A_\delta \cap B_\delta$  is thick. Then by construction,  $B_{\delta+1} \subseteq A_\delta \cap B_\delta \subseteq A_\delta \subseteq X$ .

If  $A \in \Theta$ , then  $A = A_\alpha$  for some  $\alpha < \mathfrak{c}$ . At stage  $\alpha + 1$  of our recursion, we ensured that either  $B_{\alpha+1} \cap A_\alpha \notin \Theta$  or that  $B_{\alpha+1} \subseteq A_\alpha$  (which implies  $A_\alpha \in \mathcal{F}$ ). Thus  $A \in \Theta$  implies that either  $A \in \mathcal{F}$  or  $A \cap B \notin \Theta$  for some  $B \in \mathcal{F}$ . Thus no proper extension of  $\mathcal{F}$  contains only thick sets, which means that  $\mathcal{F}$  is a  $\Theta$ -maximal filter.  $\square$

**Theorem 4.4.** *Assuming  $\mathfrak{p} = \mathfrak{c}$  (or CH), there is a  $\Theta$ -maximal filter  $\mathcal{F}$  on  $\mathbb{N}$  such that*

- (1) *for each  $n > 0$ , there is a  $\Sigma$ -maximal filter  $\mathcal{G}$  on  $\mathbb{N}$  such that  $\mathcal{F} \cup \mathcal{G}$  extends to exactly  $n$  distinct ultrafilters.*
- (2) *there is a  $\Sigma$ -maximal filter  $\mathcal{G}$  on  $\mathbb{N}$  such that  $\mathcal{F} \cup \mathcal{G}$  extends to  $2^{2^{\aleph_0}}$  distinct ultrafilters.*

*Proof.* Let  $\langle B_\alpha : \alpha < \mathfrak{c} \rangle$  be a sequence of thick subsets of  $\mathbb{N}$  having the properties described in Lemma 4.3, and let  $\mathcal{F}$  be the  $\Theta$ -maximal filter generated by  $\{B_\alpha : \alpha < \mathfrak{c}\}$ . For each  $\alpha < \mathfrak{c}$ , because  $B_\alpha \setminus B_{\alpha+1}$  is thick there is some  $Z_\alpha \subseteq B_\alpha \setminus B_{\alpha+1}$  such that  $Z_\alpha$  is an infinite union of increasingly long intervals. Note that if  $\alpha \neq \beta$ , then  $Z_\alpha \cap Z_\beta$  is finite (because if say  $\beta < \alpha$ , then  $Z_\beta \cap B_{\beta+1} = \emptyset$  and  $B_{\beta+1} \supseteq^* B_\alpha \supseteq Z_\alpha$ .)

By Lemma 2.4,  $\widehat{\mathcal{F}}$  is a minimal left ideal of  $\beta\mathbb{N}$  so there are  $2^{2^{\aleph_0}}$  ultrafilters extending  $\mathcal{F}$  by [7, Theorem 6.9].

To prove (1), fix  $n > 0$  and let  $p_0, p_1, \dots, p_{n-1}$  be any  $n$  distinct ultrafilters extending  $\mathcal{F}$ . We shall find a  $\Sigma$ -maximal filter  $\mathcal{G}$  such that the set of all ultrafilters extending  $\mathcal{F} \cup \mathcal{G}$  is precisely  $\{p_0, p_1, \dots, p_{n-1}\}$ . For each  $k < n$ , let  $\mathcal{G}_k$  be a  $\Sigma$ -maximal filter such that  $\mathcal{F} \cup \mathcal{G}_k$  generates the ultrafilter  $p_k$ . (Some such  $\mathcal{G}_k$  exists by Theorem 2.7.)

For every ordinal  $\alpha$ , let  $\text{int}_n(\alpha)$  denote the integer part of  $\alpha$  modulo  $n$ : i.e.,  $\text{int}_n(\alpha)$  is equal to the unique  $k \in \{0, 1, \dots, n-1\}$  with the property that  $\alpha = n \cdot \beta + k$  for some ordinal  $\beta$ .

Given an  $n$ -tuple  $\vec{A} = \langle A_0, A_1, \dots, A_{n-1} \rangle \in \mathcal{G}_0 \times \mathcal{G}_1 \times \dots \times \mathcal{G}_{n-1}$  and a finite  $F \subseteq \mathfrak{c}$ , define

$$Y(F, \vec{A}) = \bigcup \{Z_\alpha \cap A_{\text{int}_n(\alpha)} : \alpha \in F\} \cup \left( \bigcup_{k < n} A_k \setminus \bigcup \{Z_\alpha : \alpha \in F\} \right).$$

We claim that each set of this form is syndetic. To see this, fix some tuple  $\vec{A} = \langle A_0, A_1, \dots, A_{n-1} \rangle \in \mathcal{G}_0 \times \mathcal{G}_1 \times \dots \times \mathcal{G}_{n-1}$  and some finite  $F \subseteq \mathfrak{c}$ . For each  $k < n$ , the set  $A_k$  is syndetic, so there is some  $m_k \in \mathbb{N}$  such that every interval in  $\mathbb{N}$  of length at least  $m_k$  contains a point of  $A_k$ . Let  $m = \max\{m_0, m_1, \dots, m_{n-1}\}$ . Recall that if  $\alpha \neq \beta$  then  $Z_\alpha \cap Z_\beta$  is finite, and that each  $Z_\alpha$  is a union of intervals of increasing length. Hence there is some  $N \in \mathbb{N}$  such that on  $[N, \infty)$ , the  $Z_\alpha$ , for  $\alpha \in F$ , are pairwise disjoint, and each  $Z_\alpha$  consists of intervals all of length at least  $m$ . Now suppose  $I \subseteq [N, \infty)$  is an interval of length at least  $3m$ . Then either  $I$  contains  $m$  consecutive points from some particular  $Z_\alpha$  for  $\alpha \in F$ , in which case  $I$  contains a point of  $Z_\alpha \cap A_{\text{int}_n(\alpha)}$ , or else  $I$  contains  $m$  consecutive points not in any  $Z_\alpha$ , which case  $I$  contains a point of  $\bigcup_{k < n} A_k \setminus \bigcup \{Z_\alpha : \alpha \in F\}$ . Either way,  $I$  contains a point of  $Y(F, \vec{A})$ . Thus  $Y(F, \vec{A})$  meets every interval in  $\mathbb{N}$  of length at least  $3m + N$ , so  $Y(F, \vec{A})$  is syndetic as claimed.

If  $\vec{A} = \langle A_0, \dots, A_{n-1} \rangle$  and  $\vec{B} = \langle B_0, \dots, B_{n-1} \rangle$  are both in  $\mathcal{G}_0 \times \dots \times \mathcal{G}_{n-1}$  and if  $B_k \subseteq A_k$  for every  $k < n$ , then it is clear that  $Y(F, \vec{B}) \subseteq Y(F, \vec{A})$  for any fixed finite  $F \subseteq \mathfrak{c}$ . Similarly, if  $\vec{A} \in \mathcal{G}_0 \times \dots \times \mathcal{G}_{n-1}$  is fixed then for any finite  $F, G \subseteq \mathfrak{c}$ ,  $F \subseteq G$  implies  $Y(G, \vec{A}) \subseteq^* Y(F, \vec{A})$  because  $Y(G, \vec{A}) \setminus Y(F, \vec{A}) \subseteq \bigcup_{\alpha \in G \setminus F} \bigcup_{\delta \in F} (Z_\alpha \cap Z_\delta)$ . Thus, if for any tuples  $\vec{A}, \vec{B} \in \mathcal{G}_0 \times \dots \times \mathcal{G}_{n-1}$  we denote by  $\vec{A} \wedge \vec{B}$  the tuple obtained by taking intersections coordinate-wise, we have

$$Y(F, \vec{A}) \cap Y(G, \vec{B}) \supseteq^* Y(F \cup G, \vec{A} \wedge \vec{B})$$

whenever  $F, G$  are finite subsets of  $\mathfrak{c}$  and  $\vec{A}, \vec{B} \in \mathcal{G}_0 \times \dots \times \mathcal{G}_{n-1}$ . Hence

$$\mathcal{H}_0 = \{X \subseteq \mathbb{N} : \text{there exist finite } F \subseteq \mathfrak{c} \text{ and } \vec{A} \in \mathcal{G}_0 \times \mathcal{G}_1 \times \dots \times \mathcal{G}_{n-1} \\ \text{such that } Y(F, \vec{A}) \subseteq^* X\}$$

is a filter. By the previous paragraph,  $\mathcal{H}_0 \subseteq \Sigma$ . Using Zorn's Lemma, extend  $\mathcal{H}_0$  to a  $\Sigma$ -maximal filter  $\mathcal{G}$  on  $\mathbb{N}$ .

Because  $\mathcal{F} \subseteq \Theta$  and  $\mathcal{G} \subseteq \Sigma$ ,  $A \cap B \neq \emptyset$  for every  $A \in \mathcal{F}$  and  $B \in \mathcal{G}$ . Thus  $\mathcal{F} \cup \mathcal{G}$  generates a filter. To finish the proof of (1), we must show that there are exactly  $n$  ultrafilters extending this filter, namely  $p_0, p_1, \dots, p_{n-1}$ .

Fix  $k < n$ , and suppose (aiming for a contradiction) that  $p_k$  does not extend the filter generated by  $\mathcal{F} \cup \mathcal{G}$ . Then there are some  $A \in \mathcal{G}$ ,  $B \in \mathcal{F}$ , and  $C \in p_k$  such that  $A \cap B \cap C = \emptyset$ . Because  $p_k$  is the filter generated by  $\mathcal{F} \cup \mathcal{G}_k$ , there are some  $D \in \mathcal{F}$  and  $E \in \mathcal{G}_k$  such that  $D \cap E \subseteq C$  and (consequently)  $A \cap B \cap D \cap E = \emptyset$ . Because  $B, D \in \mathcal{F}$  and  $\mathcal{F}$  is a filter,  $B \cap D \in \mathcal{F}$ . Because  $\{B_\alpha : \alpha < \mathfrak{c}\}$  is a filter base for  $\mathcal{F}$ , there is some  $\alpha < \mathfrak{c}$  such that  $B_\alpha \subseteq B \cap D$ . Let  $\beta \geq \alpha$  with  $\text{int}_n(\beta) = k$ . Observe that  $\beta \geq \alpha$  implies  $B_\beta \subseteq^* B_\alpha$ , so

$$B_\beta \cap A \cap E \subseteq^* B_\alpha \cap A \cap E \subseteq B \cap D \cap A \cap E = \emptyset$$

or, in other words,  $B_\beta \cap A \cap E$  is finite. Let  $\vec{E}_k = \langle \mathbb{N}, \dots, \mathbb{N}, E, \mathbb{N}, \dots, \mathbb{N} \rangle$  be the  $n$ -tuple that has  $\mathbb{N}$  in every coordinate except the  $k^{\text{th}}$  coordinate, where it has  $E$  instead. Observe that

$$Y(\{\beta\}, \vec{E}_k) = (Z_\beta \cap E) \cup (\mathbb{N} \setminus Z_\beta).$$

This implies  $Z_\beta \cap Y(\{\beta\}, \vec{E}_k) = Z_\beta \cap E \subseteq B_\beta \cap E$ . Hence

$$Z_\beta \cap Y(\{\beta\}, \vec{E}_k) \cap A \subseteq B_\beta \cap E \cap A.$$

On the one hand,  $B_\beta \cap E \cap A$  is finite. On the other,  $A$  and  $Y(\{\beta\}, \vec{E}_k)$  are both in  $\mathcal{G}$ , so  $A \cap Y(\{\beta\}, \vec{E}_k) \in \mathcal{G}$  and in particular,  $A \cap Y(\{\beta\}, \vec{E}_k)$  is syndetic. Thus the intersection of the thick set  $Z_\beta$  with the syndetic set  $A \cap Y(\{\beta\}, \vec{E}_k)$  is finite. This is a contradiction. Thus  $p_k$  extends the filter generated by  $\mathcal{F} \cup \mathcal{G}$ .

To finish the proof of (1), it remains to show that if  $q$  is an ultrafilter extending  $\mathcal{F} \cup \mathcal{G}$ , then  $q = p_k$  for some  $k < n$ . Let  $q$  be an ultrafilter extending  $\mathcal{F}$ , and suppose  $q \notin \{p_0, p_1, \dots, p_{n-1}\}$ . We shall show that  $q$  does not extend  $\mathcal{G}$ . For every  $k < n$ ,  $q \neq p_k$  implies there is some  $C_k \in p_k$  such that  $C_k \notin q$ . Because  $p_k$  is the unique ultrafilter extending  $\mathcal{F} \cup \mathcal{G}_k$ , there are  $B \in \mathcal{F}$  and  $A_k \in \mathcal{G}_k$  such that  $A_k \cap B \subseteq C_k \notin q$ . But  $B \in \mathcal{F}$  and  $q \supseteq \mathcal{F}$ , so  $B \in q$ . Thus  $A_k \notin q$ . Because this holds for every  $k < n$ , and because  $q$  is an ultrafilter, we have  $\bigcup_{k < n} A_k \notin q$ . However, if  $\vec{A} = \langle A_1, A_2, \dots, A_{n-1} \rangle$  then  $Y(\emptyset, \vec{A}) = \bigcup_{k < n} A_k \in \mathcal{H}_0 \subseteq \mathcal{G}$ . Thus  $q$  does not extend  $\mathcal{G}$ .

This finishes the proof of (1), and in fact establishes something slightly stronger than stated in the theorem: any finite subset of  $\widehat{\mathcal{F}}$  may be obtained as the set of ultrafilters extending  $\mathcal{F} \cup \mathcal{G}$  for some  $\Sigma$ -maximal filter  $\mathcal{G}$  on  $\mathbb{N}$ .

The proof of (2) is similar to the proof of (1), but with a few key differences (some of which make the proof easier). The main difference is that rather than choosing beforehand the set of ultrafilters that will extend  $\mathcal{F} \cup \mathcal{G}$ , we choose an infinite collection of disjoint syndetic sets, and define  $\mathcal{G}$  so that  $\mathcal{F} \cup \mathcal{G}$  must contain at least one ultrafilter containing each of these sets.

Let  $\langle B_\alpha: \alpha < \mathfrak{c} \rangle$  and  $\langle Z_\alpha: \beta < \mathfrak{c} \rangle$  be as above. For each  $n \geq 0$ , define

$$E_n = \{k \cdot 2^n: k \in \mathbb{N} \text{ is odd}\}$$

and observe that  $\{E_n: n \geq 0\}$  is a partition of  $\mathbb{N}$  into pairwise disjoint syndetic sets.

For every ordinal  $\alpha$ , let  $\text{int}(\alpha)$  denote the integer part of  $\alpha$ : i.e., if  $\alpha < \omega$  then  $\text{int}(\alpha) = \alpha$ , and if  $\alpha \geq \omega$  then  $\text{int}(\alpha)$  denotes the unique  $n \in \omega$  such that  $\alpha = \lambda + n$  for some limit ordinal  $\lambda$ . Given some finite  $F \subseteq \mathfrak{c}$ , define

$$Y(F) = \bigcup \{Z_\alpha \cap E_{\text{int}(\alpha)}: \alpha \in F\} \cup (\mathbb{N} \setminus \bigcup \{Z_\alpha: \alpha \in F\}).$$

We claim that each set of this form is syndetic. To see this, fix some finite  $F \subseteq \mathfrak{c}$ . Let  $M = \max \{\text{int}(\alpha): \alpha \in F\}$ . Recall that if  $\alpha \neq \beta$  then  $Z_\alpha \cap Z_\beta$  is finite, and that each  $Z_\alpha$  is a union of intervals of increasing length. Hence there is some  $N \in \mathbb{N}$  such that on  $[N, \infty)$ , the  $Z_\alpha$ , for  $\alpha \in F$ , are pairwise disjoint, and each  $Z_\alpha$  consists of intervals all of length at least  $2^{M+1}$ . Now suppose  $I \subseteq [N, \infty)$  is an interval of length at least  $2^{M+2}$ . Then either  $I$  contains a point not in any  $Z_\alpha$  for  $\alpha \in F$ , or else  $I$  contains  $2^{M+1}$  consecutive points from some particular  $Z_\alpha$  for  $\alpha \in F$ , and because  $M \geq \text{int}(\alpha)$  this implies  $I$  contains a point of  $Z_\alpha \cap E_{\text{int}(\alpha)}$ . Either way,  $I$  contains a point of  $Y(F)$ . Thus  $Y(F)$  meets every interval in  $\mathbb{N}$  of length at least  $2^{M+2} + N$ , so  $Y(F)$  is syndetic as claimed.

If  $F, G \subseteq \mathfrak{c}$  are finite and  $F \subseteq G$ , then  $Y(F) \supseteq^* Y(G)$  because  $Y(G) \setminus Y(F) \subseteq \bigcup_{\alpha \in G \setminus F} \bigcup_{\delta \in F} (Z_\alpha \cap Z_\delta)$ . Therefore, if  $F, G \subseteq \mathfrak{c}$  are finite, then  $Y(F \cup G) \subseteq^* Y(F) \cap Y(G)$ . This shows that

$$\mathcal{G}_0 = \{X \subseteq \mathbb{N}: X \supseteq^* Y(F) \text{ for some finite } F \subseteq \mathfrak{c}\}$$

is a filter, and by the previous paragraph,  $\mathcal{G}_0 \subseteq \Sigma$ . Using Zorn's Lemma, extend  $\mathcal{G}_0$  to a  $\Sigma$ -maximal filter  $\mathcal{G}$  on  $\mathbb{N}$ .

To finish the proof of the theorem, we must show that there are  $2^{2^{\aleph_0}}$  distinct ultrafilters extending  $\mathcal{F} \cup \mathcal{G}$ . In fact, it suffices to show that for any  $n \geq 0$ ,  $\mathcal{F} \cup \mathcal{G} \cup \{E_n\}$  generates a filter. This suffices because it shows that  $\mathcal{F} \cup \mathcal{G}$  can be extended to infinitely many distinct ultrafilters on  $\mathbb{N}$ , and it is known that if a filter on  $\mathbb{N}$  extends to infinitely many distinct ultrafilters, then it extends to  $2^{2^{\aleph_0}}$  distinct ultrafilters [7, Theorem 3.59].

Fix  $n \geq 0$ , and suppose that  $\mathcal{F} \cup \mathcal{G} \cup \{E_n\}$  does not generate a filter. This means that there is some  $B \in \mathcal{F}$  and  $A \in \mathcal{G}$  such that  $A \cap B \cap E_n = \emptyset$ . Because  $\{B_\alpha: \alpha < \mathfrak{c}\}$  is a filter base for  $\mathcal{F}$ , there is some  $\alpha < \mathfrak{c}$  such that  $B_\alpha \subseteq B$ , which implies  $B_\alpha \cap A \cap E_n = \emptyset$ . Fix  $\beta \geq \alpha$  with  $\text{int}(\beta) = n$ . Observe



that  $\beta \geq \alpha$  implies  $B_\beta \subseteq^* B_\alpha$ , so

$$B_\beta \cap A \cap E_n \subseteq^* B_\alpha \cap A \cap E_n = \emptyset$$

or, in other words,  $B_\beta \cap A \cap E_n$  is finite. Observe that

$$Y(\{\beta\}) = (Z_\beta \cap E_n) \cup (\mathbb{N} \setminus Z_\beta)$$

which implies  $Z_\beta \cap Y(\{\beta\}) = Z_\beta \cap E_n \subseteq B_\beta \cap E_n$ . Hence

$$Z_\beta \cap Y(\{\beta\}) \cap A \subseteq B_\beta \cap A \cap E_n.$$

On the one hand,  $B_\beta \cap A \cap E_n$  is finite. On the other,  $A$  and  $Y(\{\beta\})$  are both in  $\mathcal{G}$ , so  $A \cap Y(\{\beta\}) \in \mathcal{G}$  and in particular,  $A \cap Y(\{\beta\})$  is syndetic. Thus the intersection of the thick set  $Z_\beta$  with the syndetic set  $A \cap Y(\{\beta\})$  is finite. This is a contradiction, so  $\mathcal{F} \cup \mathcal{G} \cup \{E_n\}$  generates a filter.  $\square$

The proof of this theorem raises two questions. The first (and perhaps obvious) question is whether the hypothesis  $\mathfrak{p} = \mathfrak{c}$  is really necessary.

**Question.** Is it consistent that for every  $\Theta$ -maximal filter  $\mathcal{F}$  and every  $\Sigma$ -maximal filter  $\mathcal{G}$ , there is a unique ultrafilter extending  $\mathcal{F} \cup \mathcal{G}$ ?

The second question is whether our use of a “special” filter  $\mathcal{F}$  in the proof was really necessary. For all we know, it may be true that (in ZFC alone) there is some  $\Theta$ -maximal filter  $\mathcal{F}$  such that the join of  $\mathcal{F}$  with any  $\Sigma$ -maximal filter is an ultrafilter.

**Question.** Is there some  $\Theta$ -maximal ultrafilter  $\mathcal{F}$  such that  $\mathcal{F} \cup \mathcal{G}$  generates an ultrafilter for every  $\Sigma$ -maximal filter  $\mathcal{G}$ ? Is it consistent that there is such a filter?

**Acknowledgements.** Both authors wish to thank Dona Strauss for her insightful comments on an earlier draft of this paper, and for providing us with the proofs of Lemma 3.10 and Theorem 3.11, allowing us to remove some extraneous hypotheses from these results. The authors also thank the referee for a careful and thorough report and for suggesting significant improvements in the presentation of Section 4.

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