This paper was published in *Discrete Math.* **242** (2002), 115-144. As published there were gaps in the proofs that (a) implies (d) and that (b) implies (c) in Theorem 2.10. These gaps have been filled here. Other than that, to the best of my knowledge, this is the final version as it was submitted to the publisher. – NH

Image Partition Regular Matrices – Bounded Solutions and Preservation of Largeness

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Abstract

A $u \times v$ matrix A is *image partition regular* provided that, whenever $\mathbb{N}$ is finitely colored, there is some $\vec{x} \in \mathbb{N}^v$ with all entries of $A\vec{x}$ monochrome. Image partition regular matrices are a natural way of representing some of the classic theorems of Ramsey Theory, including theorems of Hilbert, Schur, and van der Waerden.

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We present here some new characterizations and consequences of image partition regularity and investigate some issues raised by these. One of our characterizations is that the image partition regular matrices are precisely those that preserve a certain notion of largeness ("central sets") – we examine what happens for other well known notions of largeness. Another property of image partition regular matrices is that (except in trivial cases) the entries of $A\vec{x}$ may be chosen to be distinct – we investigate when we may choose the entries to be "close together" or "far apart" in various senses.

1 Introduction

Consider the following classical theorems of Ramsey Theory. (We take $\mathbb{N} = \{1, 2, 3, \ldots\}$ and $\omega = \mathbb{N} \cup \{0\}$.)

Theorem 1.1 (Hilbert) *Let $r \in \mathbb{N}$ and let $\mathbb{N} = \bigcup_{i=1}^{r} D_i$. For each $m \in \mathbb{N}$ there exist $i \in \{1, 2, \ldots, r\}$, $a \in \mathbb{N}$, and a sequence $\langle x_n \rangle_{n=1}^{m}$ in $\mathbb{N}$ such that for each nonempty $F \subseteq \{1, 2, \ldots, m\}$, $a + \sum_{t \in F} x_t \in D_i$.*

Proof. [9]. $\square$

We are using here "partition" terminology (at least if one assumes that $D_i \cap D_j = \emptyset$ when $i \neq j$). In the alternative "coloring" terminology which we will also use on occasion, the following theorem would say: "Whenever $\mathbb{N}$ is finitely colored, there must exist $x, y \in \mathbb{N}$ with $\{x, y, x+y\}$ monochrome."

Theorem 1.2 (Schur) *Let $r \in \mathbb{N}$ and let $\mathbb{N} = \bigcup_{i=1}^{r} D_i$. There exist $i \in \{1, 2, \ldots, r\}$ and x and y in ben with $\{x, y, x + y\} \subseteq D_i$.*

Proof. [15]. $\square$

Theorem 1.3 (van der Waerden) *Let $r \in \mathbb{N}$ and let $\mathbb{N} = \bigcup_{i=1}^{r} D_i$. For each $l \in \mathbb{N}$ there exist $i \in \{1, 2, \ldots, r\}$ and $a, d \in \mathbb{N}$ such that $\{a, a+d, \ldots, a+ld\} \subseteq D_i$.*

Proof. [16]. $\square$

Consider also the matrices

$$A = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}, \quad \text{and} \quad C = \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \\ 1 & 5 \\ 1 & 6 \end{pmatrix}.$$

The case $m = 3$ of Theorem 1.1 is the assertion that whenever $r \in \mathbb{N}$ and $\mathbb{N} = \bigcup_{i=1}^{r} D_i$, there exist $i \in \{1, 2, \ldots, r\}$ and $\vec{x} \in \mathbb{N}^4$ such that $A\vec{x} \in D_i^7$. (We use the notation $\vec{x}$ for both row vectors and column vectors, expecting the reader to rely on the context to tell which is intended.) Theorem 1.2 is the assertion that whenever $r \in \mathbb{N}$ and $\mathbb{N} = \bigcup_{i=1}^{r} D_i$, there exist $i \in \{1, 2, \ldots, r\}$ and $\vec{x} \in \mathbb{N}^2$ such that $B\vec{x} \in D_i^3$. The case $l = 6$ of Theorem 1.3 is the assertion that whenever $r \in \mathbb{N}$ and $\mathbb{N} = \bigcup_{i=1}^{r} D_i$, there exist $i \in \{1, 2, \ldots, r\}$ and $\vec{x} \in \mathbb{N}^2$ such that $C\vec{x} \in D_i^7$. That is, each instance of Theorems 1.1, 1.2, and 1.3, is the assertion that a particular matrix is *image partition regular*.

Definition 1.4 Let $u, v \in \mathbb{N}$ and let A be a $u \times v$ matrix with entries from $\mathbb{Q}$. The matrix A is *image partition regular* if and only if whenever $r \in \mathbb{N}$ and $\mathbb{N} = \bigcup_{i=1}^{r} D_i$, there exist $i \in \{1, 2, \ldots, r\}$ and $\vec{x} \in \mathbb{N}^v$ such that $A\vec{x} \in D_i^u$.

Notice that in each case the correspondence is natural. No great amount of thought is required to produce the matrix corresponding to the particular theorem. Consider by way of contrast the notion of *kernel partition regularity*. (The terminology in both cases, based on the interpretation of A as a map from $\mathbb{Q}^v$ to $\mathbb{Q}^u$, was suggested by W. Deuber.)

Definition 1.5 Let $u, v \in \mathbb{N}$ and let A be a $u \times v$ matrix with entries from $\mathbb{Q}$. The matrix A is *kernel partition regular* if and only if whenever $r \in \mathbb{N}$ and $\mathbb{N} = \bigcup_{i=1}^{r} D_i$, there exist $i \in \{1, 2, \ldots, r\}$ and $\vec{x} \in D_i^v$ such that $A\vec{x} = \vec{0}$.

In a justly celebrated result, R. Rado obtained in 1933 a combinatorial characterization of kernel partition regular matrices. This characterization (which we shall present below) can be used to establish each of Theorems 1.1, 1.2, and 1.3.

Schur's Theorem asks that there exist $a, b, c \in D_i$ such that $a = x$, $b = y$, and $c = x + y$. That is, there must exist a, b, and c in D_i such that

$a + b = c$. Consequently, Theorem 1.2 is precisely the assertion that the matrix $\begin{pmatrix} 1 & 1 & -1 \end{pmatrix}$ is kernel partition regular.

We remark that, although van der Waerden's Theorem is naturally stated in terms of image partition regularity, one needs to be careful when stating it in terms of kernel partition regularity. For example, if one has just $l+1$ variables $y_1, y_2, \ldots, y_{l+1}$ and the equations stating that $y_{i+1} - y_i = y_{i+2} - y_{i+1}$ for all $i \in \{1, 2, \ldots, l-1\}$, then this allows the trivial solution in which all y_i's are equal. So van der Waerden's Theorem is not equivalent to the statement that matrices of the form

$$\begin{pmatrix} 1 & -1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & -1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & -1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & -1 \end{pmatrix}$$

are kernel partition regular. However, a stronger version of van der Waerden's Theorem, in which the differences $y_{i+1} - y_i$ are required to have the same color as the y_i's, is equivalent to the statement that matrices of the form

$$\begin{pmatrix} 1 & 1 & -1 & 0 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 1 & -1 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & 1 & -1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & 0 & 0 & 0 & \cdots & -1 \end{pmatrix}$$

are kernel partition regular.

The reader is invited to try to represent the case $m = 2$ of Theorem 1.1 by equations.

There is thus a natural interest in determining those matrices that are image partition regular. Further, one of the major problems in this area was solved using certain image partition regular matrices. That is, W. Deuber [4] showed that those sets that always contained solutions to every kernel partition regular matrix were themselves partition regular using (m, p, c) sets, which are the images of certain image partition regular matrices.

The determination of those matrices that are image partition regular was finally accomplished in 1993 [10], when several equivalent characterizations of the problem were given, including some that made the problem effectively computable (see later). Some of these characterizations involved converting the problem into one about kernel partition regular matrices, thereby allowing the use of Rado's Theorem.

Definition 1.6 Let $u, v \in \mathbb{N}$, let A be a $u \times v$ matrix with entries from $\mathbb{Q}$, and let $\overrightarrow{c_1}, \overrightarrow{c_2}, \ldots, \overrightarrow{c_v}$ be the columns of A. The matrix A satisfies the *columns condition* if and only if there exist $m \in \mathbb{N}$ and $I_1, I_2, \ldots, I_m$ such that

(1) $\{I_1, I_2, \ldots, I_m\}$ is a partition of $\{1, 2, \ldots, v\}$,
(2) $\sum_{i \in I_1} \overrightarrow{c_i} = \vec{0}$, and
(3) if $m > 1$ and $t \in \{2, 3, \ldots, m\}$, then $\sum_{i \in I_t} \overrightarrow{c_i}$ is a linear combination of $\{\overrightarrow{c_i} : i \in \bigcup_{j=1}^{t-1} I_j\}$.

It was shown by Rado that A is kernel partition regular if and only if it satisfies the columns condition.

Here, for example, is one of the characterizations of a $u \times v$ image partition regular matrix A presented in Theorem 2.10:
There exist $b_1, b_2, \ldots, b_v \in \mathbb{Q}^+$ such that the matrix

$$N = \begin{pmatrix} b_1 & 0 & 0 & \cdots & 0 \\ 0 & b_2 & 0 & \cdots & 0 \\ 0 & 0 & b_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & b_v \\ & & A & & \end{pmatrix}$$

is image partition regular.
That is, we can insist that certain multiples of the x_i's are themselves the same color as the image vector.

Several characterizations of image partition regular matrices involve the notion of a "first entries matrix", a concept based on Deuber's (m, p, c) sets. We follow here, and elsewhere, the custom of denoting the entries of a matrix by the lower case letter corresponding to the upper case letter denoting the matrix. Thus $a_{i,j}$ denotes the entry of the matrix A in the i^{th} row and j^{th} column. Similarly, x_i denotes the i^{th} entry of the vector $\vec{x}$.

Definition 1.7 Let A be a $u \times v$ matrix with rational entries. Then A is a *first entries matrix* if and only if

(1) no row of A is $\vec{0}$,
(2) the first nonzero entry of each row is positive, and

(3) the first nonzero entries of any two rows are equal if they occur in the same column.

If A is a first entries matrix and d is the first nonzero entry of some row of A, then d is called a *first entry* of A.

Thus the "first entries" of A are those numbers that actually occur as the first nonzero entry of some row of A. For example,

$$A = \begin{pmatrix} 1 & -2 & 0 & 3 & 2 \\ 0 & 0 & 5 & -1 & 2 \\ 0 & 0 & 5 & 3 & 0 \\ 0 & 0 & 0 & 0 & 7 \end{pmatrix},$$

is a first entries matrix with first entries 1, 5 and 7.

We remark that it is not hard to show that the columns condition (Definition 1.6) is equivalent to the statement that $AB = \mathbf{O}$ for some first entries matrix B.

We shall show that all first entry matrices are image partition regular. This implies Theorems 1.1, 1.2 and 1.3, because we have seen that each of these classical theorems is equivalent to the statement that a certain first entries matrix is image partition regular.

Some of the characterizations of image partition regularity that we shall give involve "central" sets. Central sets were introduced by Furstenberg [5] and defined in terms of notions of topological dynamics. These sets enjoy very strong combinatorial properties. (See [5, Proposition 8.21] or [12, Chapter 14].) They have a nice characterization in terms of the algebraic structure of $\beta\mathbb{N}$, the Stone-Čech compactification of $\mathbb{N}$. We shall present this characterization below, after introducing the necessary background information.

Let $(S, +)$ be an infinite discrete semigroup. We take the points of βS to be the ultrafilters on S, the principal ultrafilters being identified with the points of S. Given a set $A \subseteq S$, $\overline{A} = \{p \in \beta S : A \in p\}$. The set $\{\overline{A} : A \subseteq S\}$ is a basis for the open sets (as well as a basis for the closed sets) of βS.

There is a natural extension of the operation $+$ of S to βS, making βS a compact right topological semigroup with S contained in its topological center. This says that for each $p \in \beta S$ the function $\rho_p : \beta S \to \beta S$ is continuous and for each $x \in S$, the function $\lambda_x : \beta S \to \beta S$ is continuous, where $\rho_p(q) = q + p$ and $\lambda_x(q) = x + q$. Given $p, q \in \beta S$ and $A \subseteq S$, one has that $A \in p + q$ if and only if $\{x \in S : -x + A \in q\} \in p$, where

$-x + A = \{y \in S : x + y \in A\}$. See [12] for an elementary introduction to the semigroup βS.

We are denoting the operation of the semigroup by $+$ because we shall be almost exclusively concerned with the semigroups $(\mathbb{N}, +)$, $(\mathbb{Z}, +)$, $(\mathbb{Q}, +)$, and $(\mathbb{N}^v, +)$. The reader should be cautioned, however, that while the semigroup $(S, +)$ may very well be commutative, the semigroup $(\beta S, +)$ almost never is. (See [12, Theorem 4.27].)

Any compact Hausdorff right topological semigroup $(T, +)$ has a smallest two sided ideal $K(T)$ which is the union of all of the minimal left ideals of T, each of which is closed [12, Theorem 2.8] and any compact right topological semigroup contains idempotents. Since the minimal left ideals are themselves compact right topological semigroups, this says in particular that there are idempotents in the smallest ideal. There is a partial ordering of the idempotents of T determined by $p \leq q$ if and only if $p = p + q = q + p$. An idempotent p is minimal with respect to this order if and only if $p \in K(T)$ [12, Theorem 1.59]. Such an idempotent is called simply "minimal"

Definition 1.8 Let $(S, +)$ be an infinite discrete semigroup. A set $A \subseteq S$ is *central* if and only if there is some minimal idempotent p such that $A \in p$.

See [12, Theorem 19.27] for a proof of the equivalence of the definition above with the original dynamical definition. (In [7, Proposition 4.6] S. Glasner anticipated this result by showing that, if S is a countable abelian group, then a subset of S is central as defined above if and only if it satisfies conditions similar to Furstenberg's dynamical definition of "central".)

Central sets are interesting combinatorial objects because of the fact (a part of Theorem 2.10 below) that they contain images of any image partition regular matrix, because any finite partition of $\mathbb{N}$ is guaranteed to have one cell which is central, and because they satisfy the *Central Sets Theorem* [5, Proposition 8.21] (or see [12, Theorem 14.11]), which guarantees the existence of elaborate combinatorial structures in any central set.

In Section 2 we shall present most of the previously known characterizations of image partition regular matrices, as well as several new ones. Two of the new ones are of special interest to us, namely statements (n) and (c) of Theorem 2.10.

Consider the matrix $A = \begin{pmatrix} 1 & 3 \\ 2 & 2 \end{pmatrix}$. This matrix is trivially image partition regular because any vector $\vec{x}$ with $x_1 = x_2$ necessarily has all entries of $A\vec{x}$ monochrome with respect to any coloring. Theorem 2.10(n) tells us that it also is nontrivially image partition regular. Namely, given any finite coloring there must exist $x_1 \neq x_2$ with the entries of $A\vec{x}$ monochrome and distinct. This fact raises two natural questions. Under what conditions can we force the ratios between entries of $\vec{x}$ to be bounded? And, under what conditions can we force the ratios between entries of $\vec{x}$ to be unbounded? We investigate these questions in Section 3. The second question turns out to be especially interesting because of its answer, namely exactly when A is a first entries matrix. (Recall that, in the guise of Deuber's (m, p, c) sets, these matrices have long played a role in the theory of partition regularity of matrices.)

Theorem 2.10(c) says that, whenever C is a central set and A is a $u \times v$ image partition regular matrix, not only is there some $\vec{x} \in \mathbb{N}^v$ such that $A\vec{x} \in C^u$, but in fact $\{\vec{x} \in \mathbb{N}^v : \text{such that } A\vec{x} \in C^u\}$ is central in $\mathbb{N}^v$. This characterization is one of a growing body (see [6], [2], and [1]) of results in which, given as input a suitably large set, one obtains a correspondingly large set of "good" results. We investigate this phenomenon further in Section 4.

Throughout this paper, we shall use $\mathbb{Q}^+$ for the set of positive rational numbers.

2 Characterizations of Image Partition Regular Matrices

We present in this section several characterizations of finite image partition regular matrices.

In Theorem 2.10, we give several characterizations of image partition regular matrices. Some of the equivalences are proved in [10]. Nevertheless, we provide proofs of several of these equivalences rather than just referring the reader to [10], because the proofs are fairly short and it may be more helpful to the reader to be given a proof.

We shall need the following fact, which is well known but is not mentioned in [12]. Notice that in this lemma, the notation $\alpha \cdot p$ refers to multiplication in $(\beta\mathbb{Q}_d, \cdot)$, where $\mathbb{Q}_d$ denotes the rationals with the discrete topology. In particular, if $\alpha \in \mathbb{N}$, $\alpha \cdot p$ is not the sum of p with itself α times (which is simply p, if p is idempotent).

Lemma 2.1 *Let p be a minimal idempotent in $(\beta\mathbb{N}, +)$ and let $\alpha \in \mathbb{Q}$ with $\alpha > 0$. Then $\alpha \cdot p$ is also a minimal idempotent in $\beta\mathbb{N}$. Consequently, if C is central in $(\mathbb{N}, +)$, then so is $(\alpha C) \cap \mathbb{N}$.*

Proof. The function $l_\alpha : \mathbb{N} \to \mathbb{Q}$ defined by $l_\alpha(x) = \alpha \cdot x$ is a homomorphism, hence so is its continuous extension $\widetilde{l_\alpha} : \beta\mathbb{N} \to \beta\mathbb{Q}_d$ by [12, Corollary 4.22]. Further $\alpha \cdot p = \widetilde{l_\alpha}(p)$. Thus $\alpha \cdot p$ is an idempotent and $\alpha \cdot p \in \widetilde{l_\alpha}[K(\beta\mathbb{N})] = K(\,\overline{\alpha\mathbb{N}}\,)$. (The latter equality holds by [12, Exercise 1.7.3].) Assume that $\alpha = \frac{a}{b}$ with $a, b \in \mathbb{N}$. Then $b\mathbb{N} \subseteq \alpha^{-1}a\mathbb{N}$ and thus $a\mathbb{N} \in \alpha \cdot p$ because $b\mathbb{N} \in p$ by [12, Lemma 6.6]. In particular, $\alpha \cdot p \in \beta\mathbb{N}$. Also $\alpha \cdot p \in K(\,\overline{\alpha\mathbb{N}}\,) \cap \overline{a\mathbb{N}}$ and $a\mathbb{N} \subseteq \alpha\mathbb{N}$ and consequently $K(\,\overline{a\mathbb{N}}\,) = K(\,\overline{\alpha\mathbb{N}}\,) \cap \overline{a\mathbb{N}}$ by [12, Theorem 1.65]. Since every idempotent in $\beta\mathbb{N}$ is in $\overline{a\mathbb{N}}$ by [12, Lemma 6.6], we have that $\overline{a\mathbb{N}} \cap K(\beta\mathbb{N}) \neq \emptyset$ and consequently $K(\,\overline{a\mathbb{N}}\,) = \overline{a\mathbb{N}} \cap K(\beta\mathbb{N})$, again by [12, Theorem 1.65]. Thus $(\alpha \cdot p) \in K(\beta\mathbb{N})$ as required.

For the second assertion, let C be central in $(\mathbb{N}, +)$ and pick a minimal idempotent p with $C \in p$. Then $\alpha C \cap \mathbb{N} \in \alpha \cdot p$. $\square$

Lemma 2.2 *Let A be a $u \times v$ matrix with entries from $\mathbb{Z}$, define $\varphi : \mathbb{N}^v \to \mathbb{Z}^u$ by $\varphi(\vec{x}) = A\vec{x}$, and let $\widetilde{\varphi} : \beta(\mathbb{N}^v) \to (\beta\mathbb{Z})^u$ be its continuous extension. Let p be a minimal idempotent in $\beta\mathbb{N}$ with the property that for every $C \in p$ there exists $\vec{x} \in \mathbb{N}^v$ such that $A\vec{x} \in C^u$ and let $\overline{p} = (p, p, \ldots, p)^T$. Then there is a minimal idempotent $q \in \beta(\mathbb{N}^v)$ such that $\widetilde{\varphi}(q) = \overline{p}$.*

Proof. By [12, Exercise 4.3.5 and Theorem 1.65] $p \in K(\beta\mathbb{Z})$ and so by [12, Theorem 2.23], $\overline{p} \in K((\beta\mathbb{Z})^u)$. By [12, Corollary 4.22], $\widetilde{\varphi} : \beta(\mathbb{N}^v) \to (\beta\mathbb{Z})^u$ is a homomorphism.

We claim that $\overline{p} \in \widetilde{\varphi}[\beta(\mathbb{N}^v)]$ so suppose instead that $\overline{p} \notin \widetilde{\varphi}[\beta(\mathbb{N}^v)]$, which is closed, and pick a neighborhood U of $\overline{p}$ such that $U \cap \widetilde{\varphi}[\beta(\mathbb{N}^v)] = \emptyset$. Pick $D \in p$ such that $\overline{D}^u \subseteq U$ and pick $\vec{x} \in \mathbb{N}^v$ such that $A\vec{x} \in D^u$. Then $\varphi(\vec{x}) \in U \cap \widetilde{\varphi}[\beta(\mathbb{N}^v)]$, a contradiction.

Let $M = \{q \in \beta(\mathbb{N}^v) : \widetilde{\varphi}(q) = \overline{p}\}$. Then M is a compact subsemigroup of $\beta(\mathbb{N}^v)$, so pick an idempotent $w \in M$ by [12, Theorem 2.5]. By [12, Theorem 1.60], pick a minimal idempotent $q \in \beta(\mathbb{N}^v)$ with $q \leq w$. Since $\widetilde{\varphi}$ is a homomorphism, $\widetilde{\varphi}(q) \leq \widetilde{\varphi}(w) = \overline{p}$ so, since $\overline{p}$ is minimal in $(\beta\mathbb{Z})^u$, we have that $\widetilde{\varphi}(q) = \overline{p}$. $\square$

The following coloring is very important to us.

Lemma 2.3 *Let $\epsilon > 0$. There is a finite coloring of $\mathbb{N}$ such that, if y and z are positive integers with the same color and $y > z$, then either $\dfrac{y}{z} < 1 + \epsilon$ or $\dfrac{y}{z} > \dfrac{1}{\epsilon}$.*

Proof. Choose $\alpha \in (1, 1+\epsilon)$ and $r \in \mathbb{N}$ satisfying $r > 1 + \log_\alpha \frac{1}{\epsilon}$. For each $i \in \{0, 1, 2, \ldots, r-1\}$, let $P_i = \{n \in \mathbb{N} : \lfloor \log_\alpha n \rfloor \equiv i (\bmod r)\}$. Let $i \in \{1, 2, \ldots, r\}$ and let $y, z \in P_i$ with $y > z$. Then $\lfloor \log_\alpha y \rfloor \geq \lfloor \log_\alpha z \rfloor$.

If $\lfloor \log_\alpha y \rfloor > \lfloor \log_\alpha z \rfloor$, then $\lfloor \log_\alpha y \rfloor \geq \lfloor \log_\alpha z \rfloor + r$ and thus $y > z \cdot \alpha^{r-1} > z \cdot \dfrac{1}{\epsilon}$. If $\lfloor \log_\alpha y \rfloor = \lfloor \log_\alpha z \rfloor$, then $y < \alpha \cdot z < (1+\epsilon) \cdot z$. □

Lemma 2.4 *Let A be a $u \times v$ image partition regular matrix over $\mathbb{Q}$. There exist $m \in \mathbb{N}$ and a partition $\{I_1, I_2, \ldots, I_m\}$ of $\{1, 2, \ldots, u\}$ with the following property: for every $\epsilon > 0$, there exists $\vec{x} \in \mathbb{N}^v$ such that $\vec{y} = A\vec{x} \in \mathbb{N}^u$ and, if $i \in I_r$ and $j \in I_s$, then $1 - \epsilon < \dfrac{y_j}{y_i} < 1 + \epsilon$ if $r = s$ and $\dfrac{y_j}{y_i} < \epsilon$ if $r < s$.*

Proof. Suppose that $0 < \epsilon < \frac{1}{4}$. Choose a coloring of $\mathbb{N}$ guaranteed by Lemma 2.3 and a vector $\vec{x} \in \mathbb{N}^v$ for which the entries of $\vec{y} = A\vec{x}$ are monochrome positive integers. We define a relation $\approx$ on $\{1, 2, \ldots, u\}$ by putting $i \approx j$ if and only if $1 - \epsilon < \dfrac{y_j}{y_i} < 1 + \epsilon$. Since $\epsilon < (1-\epsilon)^2 < (1+\epsilon)^2 < \frac{1}{\epsilon}$, it is easy to verify that this is an equivalence relation. It therefore defines a partition $\mathcal{P}(\epsilon) = \{I_1(\epsilon), I_2(\epsilon), \ldots, I_{m(\epsilon)}(\epsilon)\}$ of $\{1, 2, \ldots, u\}$. We can arrange the sets in this partition so that, if $y_i \in I_r(\epsilon)$, $y_j \in I_s(\epsilon)$, and $r < s$, then $y_j < y_i$ and so $\dfrac{y_j}{y_i} < \epsilon$. Since there are only finitely many partitions of $\{1, 2, \ldots, u\}$, by the pigeon hole principle, there is an infinite sequence of values of ϵ converging to 0 for which the partitions $\mathcal{P}(\epsilon)$ are all the same. □

Lemma 2.5 *Let $\vec{c}_1, \vec{c}_2, \ldots, \vec{c}_m$ be vectors in $\mathbb{Q}^v$. Suppose that the equation $\vec{0} = \sum_{i=1}^m x_i \vec{c}_i$ holds for real numbers $x_1, x_2, \ldots, x_m$. Then we also have $\vec{0} = \sum_{i=1}^m r_i \vec{c}_i$, for rational numbers $r_1, r_2, \ldots, r_m$, with the property that, for every $i \in \{1, 2, \ldots, m\}$, $r_i > 0$ if $x_i > 0$, $r_i < 0$ if $x_i < 0$, and $r_i = 0$ if $x_i = 0$.*

Proof. Let $P = \{i \in \{1, 2, \ldots, m\} : x_i > 0\}$ and $Q = \{i \in \{1, 2, \ldots, m\} : x_i < 0\}$. Let B denote the row reduced echelon matrix obtained by applying elementary row operations to the matrix whose columns are the vectors $\vec{c}_i$ with $i \in P \cup Q$. Let I denote the set of pivot columns, that is the values of $i \in P \cup Q$ for which the i^{th} column of B contains the first nonzero entry of some row, and let $J = (P \cup Q) \backslash I$. If $J = \emptyset$, then the only solution to $\vec{0} = \sum_{i \in P \cup Q} y_i \vec{c}_i$ has each $y_i = 0$. But then $P = Q = \emptyset$ and we are done. So assume that $J \neq \emptyset$. Then the equation $\vec{0} = \sum_{i \in P \cup Q} y_i \vec{c}_i$ holds if and only if, for every $i \in I$, $y_i = -\sum_{j \in J} b_{i,j} y_j$.

We are assuming that there exists $\langle y_j\rangle_{j\in J}\in\mathbb{R}^J$, such that $y_j>0$ if $j\in J\cap P$, $y_j<0$ if $j\in J\cap Q$, $-\sum_{j\in J}b_{i,j}y_j>0$ if $i\in I\cap P$ and $-\sum_{j\in J}b_{i,j}y_j<0$ if $i\in I\cap Q$. These inequalities define a neighborhood of $\langle x_j\rangle_{j\in J}$ in $\mathbb{R}^J$, and this contains an element $\langle r_j\rangle_{j\in J}$ of $\mathbb{Q}^J$. For $i\in I$, let $r_i=-\sum_{j\in J}b_{i,j}r_j$ and for $i\in\{1,2,\ldots,m\}\backslash(P\cup Q)$, let $r_i=0$. We then have $\vec{0}=\sum_{i=1}^m r_i\vec{c}_i$, where $r_i>0$ if $i\in P$, $r_i<0$ if $i\in Q$, and $r_i=0$ if $i\notin P\cup Q$. $\Box$

Corollary 2.6 . *Let $\vec{d},\vec{c}_1,\vec{c}_2,\ldots,\vec{c}_m$ be vectors in $\mathbb{Q}^v$. Suppose that the equation $\vec{d}=\sum_{i=1}^m x_i\vec{c}_i$ holds for real numbers $x_1,x_2,\ldots,x_m$. Then we also have $\vec{d}=\sum_{i=1}^m r_i\vec{c}_i$, for rational numbers $r_1,r_2,\ldots,r_m$, with the property that, for every $i\in\{1,2,\ldots,m\}$, $r_i>0$ if $x_i>0$, $r_i<0$ if $x_i<0$, and $r_i=0$ if $x_i=0$.*

Proof. This follows by applyng Lemma 2.5 to the equation $\vec{0}=-\vec{d}+\sum_{i=1}^m x_i\vec{c}_i$. $\Box$

Lemma 2.7 *Let $\vec{c}_1,\vec{c}_2,\ldots,\vec{c}_m$ be vectors in $\mathbb{Q}^v$ and let $P,Q\subseteq\{1,2,\ldots,m\}$ be disjoint. Let $C=\{x_1\vec{c}_1+x_2\vec{c}_2+\ldots+x_m\vec{c}_m:$ each $x_i\in\mathbb{R}$, $x_i\geq 0$ if $i\in P$, and $x_i\leq 0$ if $i\in Q\}$. Then C is closed in $\mathbb{R}^v$.*

Proof. This was proved in [12, Lemma 15.23] in the case in which $Q=\emptyset$. The general case follows from this one, by replacing each $\vec{c}_j$ by $-\vec{c}_j$ if $j\in Q$, replacing P by $P\cup Q$ and Q by $\emptyset$. $\Box$

Lemma 2.8 *Let $u,v\in\mathbb{N}$, let A be a $u\times v$ first entries matrix with entries from $\mathbb{Q}$, and let C be a central subset of $\mathbb{N}$. Then there exists $\vec{x}\in\mathbb{N}^v$ for which $A\vec{x}\in C^u$.*

Proof. This is an immediate consequence of [10, Theorem 2.11]. (Or see [12, Theorem 15.5] replacing the first sentence of the proof with: "If 0 were a minimal idempotent, then $\beta S=0+\beta S=\beta S+0$ would be a minimal left ideal and a minimal right ideal, hence a group by Theorem 1.61. In particular, S would be cancellative so by Corollary 4.33, S^* would be a left ideal properly contained in βS, a contradiction. Thus we may presume that $0\notin C$.") $\Box$

We now give a proof of Rado's Theorem.

Theorem 2.9 (Rado) *Let $u, v \in \mathbb{N}$ and let A be a $u \times v$ matrix with entries from $\mathbb{Q}$. The matrix A is kernel partition regular if and only if A satisfies the columns condition.*

Proof. We have observed that A satisfies the columns condition if and only if $AB = \mathbf{O}$ for some first entries matrix B.

Suppose that A is kernel partition regular. By Lemma 2.3, for each $\epsilon > 0$, there exists $\vec{y}(\epsilon) \in \mathbb{N}^v$ such that $A\vec{y}(\epsilon) = \vec{0}$ and, for each $i, j \in \{1, 2, \cdots, v\}$, $y(\epsilon)_i \geq y(\epsilon)_j$ implies that $\frac{y(\epsilon)_i}{y(\epsilon)_j} < 1 + \epsilon$ or $\frac{y(\epsilon)_i}{y(\epsilon)_j} > \frac{1}{\epsilon}$. By the argument used in the proof of Lemma 2.4, there is a sequence $\langle \epsilon_n \rangle_{n=1}^{\infty}$ of positive numbers converging to 0 and a partition $\{I_1, I_2, \cdots I_m\}$ of $\{1, 2, \cdots, v\}$ with the following property: for every $n \in \mathbb{N}$, $i \in I_r$ and $j \in I_s$ implies that $1 - \epsilon_n < \frac{y(\epsilon_n)_j}{y(\epsilon_n)_i} < 1 + \epsilon_n$ if $r = s$ and $\frac{y(\epsilon_n)_j}{y(\epsilon_n)_i} < \epsilon_n$ if $r < s$.

For each $j \in \{1, 2, \cdots, m\}$, choose $k \in I_j$. For each $n \in \mathbb{N}$ and each $i \in \{1, 2, \cdots, v\}$, put $(z(j,n))_i = \frac{(y(\epsilon_n))_i}{(y(\epsilon_n))_k}$. We observe that $A\vec{z}(j,n) = \vec{0}$, that $1 - \epsilon_n < (z(j,n))_i < 1 + \epsilon_n$ if $i \in I_j$ and that $(z(j,n))_i < \epsilon_n$ if $i \in \bigcup_{l>j} I_l$.

Let $\vec{e}_i$ denote the i^{th} unit vector in $\mathbb{Q}^v$. Put $\vec{v}(j,n) = \sum_{i \in \bigcup_{l \geq j} I_l} (z(j,n))_i \vec{e}_i$. Since $A\vec{z}(j,n) = \vec{0}$, it follows that $A\vec{v}(j,n) \in -AL(j)$, where $L(j)$ denotes the linear span of $\{\vec{e}_i : i \in \bigcup_{l<j} I_l\}$ in $\mathbb{Q}^{|\bigcup_{l<j} I_l|}$. (We put $L(1) = \{\vec{0}\}$.) Now $\vec{v}(j,n) \to \sum_{i \in I_j} \vec{e}_i$ as $n \to \infty$. Since $-AL(j)$ is closed, we have $A(\sum_{i \in I_j} \vec{e}_j + \vec{w}(j)) = \vec{0}$ for some $\vec{w}(j) \in L(j)$. Let $\vec{w}(j) = \sum_{i \in \bigcup_{l<j} I_l} x_{ij} \vec{e}_i$. If B denotes the $v \times m$ matrix defined by

$$b_{ij} = \begin{cases} x_{ij} & \text{if } i \in \bigcup_{l<j} I_l \\ 1 & \text{if } i \in I_j \\ 0 & \text{if } i \in \bigcup_{l>j} I_l \end{cases},$$

then B is a first entries matrix for which $AB = \mathbf{O}$.

Conversely, suppose that $AB = \mathbf{O}$ for some first entries matrix B. By [12, Lemma 15.14], we may suppose that the entries of B are in ω. Since B is image partition regular, by Lemma 2.8, it follows immediately that A is kernel partition regular. $\square$

The equivalence of statements (h), (i), and (l) with statement (a) of the following theorem was established in [10, Theorem 3.1] and the equivalence of statement (g) was established in [12, Theorem 15.24]. The others are new. Notice that statement (i) can be used to check whether a given matrix is image partition regular. (See the conclusion of [10] for a discussion of this and two other effectively computable characterizations.)

We let $\mathbb{Q}^+ = \{x \in \mathbb{Q} : x > 0\}$.

Theorem 2.10 *Let $u, v \in \mathbb{N}$ and let A be a $u \times v$ matrix with entries from $\mathbb{Q}$. The following statements are equivalent.*

(a) *A is image partition regular.*

(b) *For every central set C in $\mathbb{N}$, there exists $\vec{x} \in \mathbb{N}^v$ such that $A\vec{x} \in C^u$.*

(c) *For every central set C in $\mathbb{N}$, $\{\vec{x} \in \mathbb{N}^v : A\vec{x} \in C^u\}$ is central in $\mathbb{N}^v$.*

(d) *There exist $m \in \mathbb{N}$, a $v \times m$ matrix G with non-negative rational entries and no row equal to $\vec{0}$, and a $u \times m$ first entries matrix B, with non-negative entries and all its first entries equal to 1, such that $AG = B$.*

(e) *There exist $m \in \mathbb{N}$, a $v \times m$ matrix G with non-negative rational entries and no row equal to $\vec{0}$, and a $u \times m$ first entries matrix B, with all its first entries equal to 1, such that $AG = B$.*

(f) *There exist $m \in \mathbb{N}$, a $v \times m$ matrix G with entries from ω and no row equal to $\vec{0}$, a $u \times m$ first entries matrix B with entries from ω, and $c \in \mathbb{N}$ such that c is the only first entry of B and $AG = B$.*

(g) *There exist $m \in \mathbb{N}$, a $u \times m$ first entries matrix B with all entries from ω, and $c \in \mathbb{N}$ such that c is the only first entry of B and for each $\vec{y} \in \mathbb{N}^m$ there exists $\vec{x} \in \mathbb{N}^v$ such that $A\vec{x} = B\vec{y}$.*

(h) *There exist $m \in \mathbb{N}$ and a $u \times m$ first entries matrix B such that for each $\vec{y} \in \mathbb{N}^m$ there exists $\vec{x} \in \mathbb{N}^v$ such that $A\vec{x} = B\vec{y}$.*

(i) *There exist $t_1, t_2, \ldots, t_v \in \mathbb{Q}^+$ such that the matrix*

$$M = \begin{pmatrix} t_1a_{1,1} & t_2a_{1,2} & t_3a_{1,3} & \cdots & t_va_{1,v} & -1 & 0 & 0 & \cdots & 0 \\ t_1a_{2,1} & t_2a_{2,2} & t_3a_{2,3} & \cdots & t_va_{2,v} & 0 & -1 & 0 & \cdots & 0 \\ t_1a_{3,1} & t_2a_{3,2} & t_3a_{3,3} & \cdots & t_va_{3,v} & 0 & 0 & -1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ t_1a_{u,1} & t_2a_{u,2} & t_3a_{u,3} & \cdots & t_va_{u,v} & 0 & 0 & 0 & \cdots & -1 \end{pmatrix}$$

is kernel partition regular.

(j) *There exist* $t_1, t_2, \ldots, t_v \in \mathbb{Q}^+$ *such that the matrix*

$$P = \begin{pmatrix} 1 & 0 & 0 & \ldots & 0 \\ 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 1 \\ t_1 a_{1,1} & t_2 a_{1,2} & t_3 a_{1,3} & \ldots & t_v a_{1,v} \\ t_1 a_{2,1} & t_2 a_{2,2} & t_3 a_{2,3} & \ldots & t_v a_{2,v} \\ t_1 a_{3,1} & t_2 a_{3,2} & t_3 a_{3,3} & \ldots & t_v a_{3,v} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ t_1 a_{u,1} & t_2 a_{u,2} & t_3 a_{u,3} & \ldots & t_v a_{u,v} \end{pmatrix}$$

is image partition regular.

(k) *There exist* $b_1, b_2, \ldots, b_v \in \mathbb{Q}^+$ *such that the matrix*

$$N = \begin{pmatrix} b_1 & 0 & 0 & \cdots & 0 \\ 0 & b_2 & 0 & \cdots & 0 \\ 0 & 0 & b_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & b_v \\ & & A & & \end{pmatrix}$$

is image partition regular.

(l) *For each* $\vec{r} \in \mathbb{Q}^v \setminus \{\vec{0}\}$ *there exists* $b \in \mathbb{Q} \setminus \{0\}$ *such that*

$$\begin{pmatrix} b\vec{r} \\ A \end{pmatrix}$$

is image partition regular.

(m) *Whenever* $m \in \mathbb{N}$, $\phi_1, \phi_2, \ldots, \phi_m$ *are nonzero linear mappings from* $\mathbb{Q}^v$ *to* $\mathbb{Q}$, *there exists* $\vec{b} \in \mathbb{Q}^m$ *such that, whenever* C *is central in* $\mathbb{N}$, *there exists* $\vec{x} \in \mathbb{N}^v$ *for which* $A\vec{x} \in C^u$ *and, for each* $i \in \{1, 2, \ldots, m\}$, $b_i \phi_i(\vec{x}) \in C$, *and in particular* $\phi_i(\vec{x}) \neq 0$.

(n) *For every central set* C *in* $\mathbb{N}$, *there exists* $\vec{x} \in \mathbb{N}^v$ *such that* $\vec{y} = A\vec{x} \in C^u$, *all entries of* $\vec{x}$ *are distinct, and for all* $i, j \in \{1, 2, \ldots, u\}$, *if rows* i *and* j *of* A *are unequal, then* $y_i \neq y_j$.

Proof. We show first that statements (a), (b), (c), (d), (e), (f), (g) and (h) are equivalent.

$(a) \Rightarrow (d)$. Let $\vec{c}_1, \vec{c}_2, \ldots, \vec{c}_v$ denote the columns of A and let $\vec{e}_i$ denote the i^{th} unit vector in $\mathbb{R}^u$. Let $\{I_1, I_2, \ldots, I_m\}$ be the partition of $\{1, 2, \ldots, u\}$ guaranteed by Lemma 2.4. We claim that for each $k \in \{1, 2, \ldots, m\}$,

$$\textstyle\sum_{n\in I_k} \vec{e}_n \in c\ell\{\sum_{j=1}^{v} \alpha_j\vec{c}_j - \sum_{i=1}^{k-1}\sum_{n\in I_i} \delta_n\vec{e}_n : \text{ each } \alpha_j > 0 \text{ and each } \delta_n > 0\}\,.$$

To see this, let $k \in \{1, 2, \ldots, m\}$ and let $\epsilon > 0$. Choose $\vec{x} \in \mathbb{N}^v$ such that $\vec{y} = A\vec{x} \in \mathbb{N}^u$ and, if $i \in I_r$ and $j \in I_s$, then $1-\epsilon < \dfrac{y_j}{y_i} < 1+\epsilon$ if $r = s$ and $\dfrac{y_j}{y_i} < \epsilon$ if $r < s$. Pick $l \in I_k$. For $j \in \{1, 2, \ldots, v\}$, let $\alpha_j = \dfrac{x_j}{y_l}$, noting that $\alpha_j > 0$. For $n \in \bigcup_{i=1}^{k-1} I_i$, let $\delta_n = \dfrac{y_n}{y_l}$. Then $\sum_{j=1}^{v} \alpha_j\vec{c}_j - \sum_{i=1}^{k-1}\sum_{n\in I_i} \delta_n\vec{e}_n - \sum_{n\in I_k} \vec{e}_n = \vec{z}$ where

$$z_n = \begin{cases} \dfrac{y_n}{y_l} & \text{if } n \in \bigcup_{i=k+1}^{m} I_i \\ \dfrac{y_n}{y_l} - 1 & \text{if } n \in I_k \\ 0 & \text{if } n \in \bigcup_{i=1}^{k-1} I_i \end{cases}.$$

In particular, $|z_n| < \epsilon$ for each $n \in \{1, 2, \ldots, u\}$. Thus, by Corollary 2.6 and Lemma 2.7, we may pick nonnegative $g_{j,k} \in \mathbb{Q}$ for $j \in \{1, 2, \ldots, v\}$ and nonnegative $b_{n,k}$ for $n \in \bigcup_{i=1}^{k-1} I_i$ such that $\sum_{n\in I_k} \vec{e}_n = \sum_{j=1}^{v} g_{j,k}\vec{c}_j - \sum_{i=1}^{k-1}\sum_{n\in I_i} b_{n,k}\vec{e}_n$. For $n \in I_k$, let $b_{n,k} = 1$ and for $n \in \bigcup_{i=k+1}^{m} I_i$, let $b_{n,k} = 0$.

We have thus defined a $v \times m$ matrix G with nonnegative rational entries and a $u \times m$ first entries matrix B with all first entries equal to 1 such that $AG = B$. It may happen that G has some row equal to $\vec{0}$. In this case, pick $\vec{c} \in \mathbb{N}^v$ such that the entries of $A\vec{c}$ are all positive (which one may do since A is image partition regular). Letting $G' = \begin{pmatrix} G & \vec{c} \end{pmatrix}$ and $B' = AG'$ we have that B' is a first entries matrix with all first entries equal to 1.

$(d) \Rightarrow (e)$. This is trivial.

$(d) \Rightarrow (f)$. Assume that (d) holds. We may suppose that the entries of G and AG are integers, because this can be achieved by multiplying G by a suitable positive integer. The first entries of B are then all equal.

$(f) \Rightarrow (g)$. Let B and G be as guaranteed by (f). Given $\vec{y} \in \mathbb{N}^m$, let $\vec{x} = G\vec{y}$.

$(e) \Rightarrow (h)$ This follows immediately from the observation that we can choose $n \in \mathbb{N}$ so that the entries of nG are in ω and that nB is then a first entries matrix.

$(g) \Rightarrow (h)$. This is trivial.

$(h) \Rightarrow (b)$. Let B be as guaranteed by (h) and let C be a central set in $\mathbb{N}$. Pick by Lemma 2.8 some $\vec{y} \in \mathbb{N}^m$ such that $B\vec{y} \in C^u$, and pick $\vec{x}$ such that $A\vec{x} = B\vec{y}$.

$(b) \Rightarrow (c)$. Pick $d \in \mathbb{N}$ such that all entries of dA are in $\mathbb{Z}$. We claim that for every central set C in $\mathbb{N}$, there exists $\vec{x} \in \mathbb{N}^v$ such that $dA\vec{x} \in C^u$. By Lemma 2.1 $(\frac{1}{d}C \cap \mathbb{N})$ is central, so pick $\vec{x} \in \mathbb{N}^v$ such that $A\vec{x} \in (\frac{1}{d}C \cap \mathbb{N})^u$. Then $dA\vec{x} \in C^u$.

Let C be a central subset of $\mathbb{N}$ and pick a minimal idempotent $p \in \beta\mathbb{N}$ such that $C \in p$. Define $\varphi : \mathbb{N}^v \to \mathbb{Z}^u$ by $\varphi(\,\vec{x}\,) = dA\vec{x}$ and let $\widetilde{\varphi} : \beta(\mathbb{N}^v) \to (\beta\mathbb{Z})^u$ be its continuous extension. Now dp is a minimal idempotent by Lemma 2.1. Define $\overline{dp} = (dp, dp, \ldots, dp)^T$ and pick by Lemma 2.2, a minimal idempotent $q \in \beta(\mathbb{N}^v)$ such that $\widetilde{\varphi}(q) = \overline{dp}$. Now $\times_{i=1}^u \overline{dC}$ is a neighborhood of $\overline{dp}$ so pick $B \in q$ such that $\widetilde{\varphi}[\,\overline{B}\,] \subseteq \times_{i=1}^u \overline{dC}$. Then $B \subseteq \{\vec{x} \in \mathbb{N}^v : A\vec{x} \in C^u\}$, so $\{\vec{x} \in \mathbb{N}^v : A\vec{x} \in C^u\}$ is central in $\mathbb{N}^v$.

$(c) \Rightarrow (a)$. This is immediate because some cell of any finite partition of $\mathbb{N}$ must be central.

Now we establish that statements (i), (j), and (k) are equivalent to each of the statements (a) through (h).

$(f) \Rightarrow (i)$. For each $i \in \{1, 2, \ldots, v\}$, let k_i be the first nonzero entry in row i of G, let $s_i = \dfrac{c}{k_i}$ and $t_i = \dfrac{1}{s_i}$. Let

$$S = \begin{pmatrix} s_1 & 0 & \cdots & 0 \\ 0 & s_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & s_v \end{pmatrix}$$

and let I be the $u \times u$ identity matrix. Then $M = (\, AS^{-1} \quad -I\,)$ and

$$M \begin{pmatrix} SG \\ B \end{pmatrix} = B - B = \mathbf{O}\,.$$

Also SG is a first entries matrix with all first entries equal to c and so $\begin{pmatrix} SG \\ B \end{pmatrix}$ is a $(u+v) \times m$ first entries matrix. To see that M is kernel partition regular, let $r \in \mathbb{N}$ and let $\mathbb{N} = \bigcup_{i=1}^{r} C_i$. Pick $i \in \{1, 2, \ldots, r\}$ such that C_i is central and pick by Lemma 2.8 some $\vec{x} \in \mathbb{N}^m$ such that

$$\vec{y} = \begin{pmatrix} SG \\ B \end{pmatrix} \vec{x} \in C_i^{u+v} .$$

Then $M\vec{y} = \vec{0}$.

$(i) \Rightarrow (j)$. Let

$$B = \begin{pmatrix} t_1 a_{1,1} & t_2 a_{1,2} & \ldots & t_v a_{1,v} \\ t_1 a_{2,1} & t_2 a_{2,2} & \ldots & t_v a_{2,v} \\ t_1 a_{3,1} & t_2 a_{3,2} & \ldots & t_v a_{3,v} \\ \vdots & \vdots & \ddots & \vdots \\ t_1 a_{u,1} & t_2 a_{u,2} & \ldots & t_v a_{u,v} \end{pmatrix}$$

and let I_u and I_v be the $u \times u$ and $v \times v$ identity matrices respectively. Then $P = \begin{pmatrix} I_v \\ B \end{pmatrix}$ and $M = \begin{pmatrix} B & -I_u \end{pmatrix}$. To see that P is image partition regular, let $\mathbb{N}$ be finitely colored and pick $\vec{z} \in \mathbb{N}^{u+v}$ such that $M\vec{z} = \vec{0}$ and the entries of $\vec{z}$ are monochrome. Let $\vec{x} \in \mathbb{N}^v$ and $\vec{y} \in \mathbb{N}^u$ such that $\vec{z} = \begin{pmatrix} \vec{x} \\ \vec{y} \end{pmatrix}$. Then $\vec{0} = M\vec{z} = B\vec{x} - \vec{y}$ and so $P\vec{x} = \begin{pmatrix} \vec{x} \\ \vec{y} \end{pmatrix} = \vec{z}$.

$(j) \Rightarrow (k)$. For each $i \in \{1, 2, \ldots, v\}$, let $b_i = \dfrac{1}{t_i}$ and let

$$S = \begin{pmatrix} t_1 & 0 & \cdots & 0 \\ 0 & t_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & t_v \end{pmatrix} .$$

Then $P = NS$. Pick $d \in \mathbb{N}$ such that $\{dt_1, dt_2, \ldots, dt_v\} \subseteq \mathbb{N}$. We show that statement (b) holds for N. Let C be central in $\mathbb{N}$ and pick a minimal idempotent $p \in \beta\mathbb{N}$ such that $C \in p$. By [12, Lemma 6.6], $\mathbb{N}d \in p$ so $C \cap \mathbb{N}d$ is central. We have already shown that statement (a) implies statement (b), so statement (b) holds for the matrix P. Pick $x \in \mathbb{N}^v$ such that $P\vec{x} \in (C \cap \mathbb{N}d)^{u+v}$. Then the entries of $\vec{x}$ are the first v entries of $P\vec{x}$, hence are multiples of d. Therefore $\vec{y} = S\vec{x} \in \mathbb{N}^v$ and $N\vec{y} = P\vec{x} \in C^{u+v}$.

$(k) \Rightarrow (a)$. This is trivial.

Finally we show that statements (l), (m), and (n) are equivalent to each of the statements (a) through (k).

$(e) \Rightarrow (l)$. If $\vec{r}G \neq \vec{0}$, we can choose b so that the first entry of $b\vec{r}G$ is 1. If $\vec{r}G = \vec{0}$, we can choose $\vec{c} \in \mathbb{N}^v$ such that $\vec{r} \cdot \vec{c} \neq \vec{0}$ and add $\vec{c}$ to G as a new final column. In this case, we choose b so that $b\vec{r} \cdot \vec{c} = 1$. In either case, $\begin{pmatrix} b\vec{r} \\ A \end{pmatrix} G$ is a first entries matrix with all first entries equal to 1 and so statement (e) holds for $\begin{pmatrix} b\vec{r} \\ A \end{pmatrix}$.

$(l) \Rightarrow (m)$. For each $i \in \{1, 2, \ldots, m\}$, there exists $\vec{r_i} \in \mathbb{Q}^v \backslash \{\vec{0}\}$ such that $\phi_i(\vec{x}) = \vec{r_i} \cdot \vec{x}$ for all $\vec{x} \in \mathbb{Q}^v$. By applying statement (l) m times in succession (using the fact that at each stage the new matrix satisfies (l) because (a) implies (l)), we can choose $b_1, b_2, \ldots, b_m \in \mathbb{Q}$ for which the matrix

$$\begin{pmatrix} b_1\vec{r}_1 \\ b_1\vec{r}_2 \\ \vdots \\ b_m\vec{r}_m \\ A \end{pmatrix}$$

is image partition regular. The conclusion then follows from the fact that every image partition regular matrix satisfies statement (b).

$(m) \Rightarrow (n)$. We may presume that A has no repeated rows so that the conclusion regarding $\vec{y}$ becomes the statement that all entries of $\vec{y}$ are distinct. For $i \neq j$ in $\{1, 2, \ldots, v\}$, let $\overrightarrow{\phi_{i,j}}$ be the linear mapping from $\mathbb{Q}^v$ to $\mathbb{Q}$ taking $\vec{x}$ to $x_i - x_j$. For $i \neq j$ in $\{1, 2, \ldots, u\}$, let $\overrightarrow{\psi_{i,j}}$ be the linear mapping from $\mathbb{Q}^v$ to $\mathbb{Q}$ taking $\vec{x}$ to $\sum_{t=1}^{v}(a_{i,t} - a_{j,t}) \cdot x_t$. Applying statement (m) to the set $\{\phi_{i,j} : i \neq j \text{ in } \{1, 2, \ldots, v\}\} \cup \{\psi_{i,j} : i \neq j \text{ in } \{1, 2, \ldots, u\}\}$, we reach the desired conclusion.

$(n) \Rightarrow (b)$. This is trivial. $\square$

As illustrated by the proofs of statements (m) and (n) of Theorem 2.10, the fact, guaranteed by statement (l), that finite image partition regular matrices can be almost arbitrarily extended is very useful. Another important property, originally established by W. Deuber in [4] in terms of first entries matrices, is given by the following corollary.

Corollary 2.11 *Let A and B be finite image partition regular matrices. Then the matrix*

$$\begin{pmatrix} A & \mathbf{O} \\ \mathbf{O} & B \end{pmatrix}$$

is also image partition regular.

Proof. Let C be a central set in $\mathbb{N}$. Pick $\vec{x}$ and $\vec{y}$ with entries from $\mathbb{N}$ such that all entries of $A\vec{x}$ and all entries of $B\vec{y}$ are in C. Then all entries of

$$\begin{pmatrix} A & \mathbf{O} \\ \mathbf{O} & B \end{pmatrix} \begin{pmatrix} \vec{x} \\ \vec{y} \end{pmatrix}$$

are in C. □

3 Bounded and Unbounded Ratios in Solutions

Recall that we have seen in Theorem 2.10(n) that for any $u \times v$ image partition regular matrix A and any finite coloring of $\mathbb{N}$, there must be $\vec{x} \in \mathbb{N}^v$ with the entries of $A\vec{x}$ monochrome and with the entries of $\vec{x}$ distinct. We note now, that in fact one can require that the gaps between the entries of $\vec{x}$ and the gaps between the entries of $A\vec{x}$ be as large as we please.

Theorem 3.1 *Let A be a $u \times v$ image partition regular matrix with entries from $\mathbb{Q}$ such that no two rows of A are identical. There exist permutations σ of $\{1, 2, \ldots, u\}$ and τ of $\{1, 2, \ldots, v\}$ with the property that for every $k \in \mathbb{N}$ and every finite coloring of $\mathbb{N}$, there exists $\vec{x} \in \mathbb{N}^v$ such that*

(i) all entries of $\vec{y} = A\vec{x}$ are monochrome,

(ii) for $i \in \{1, 2, \ldots, v-1\}$, $x_{\tau(i)} > k + x_{\tau(i+1)}$, and

(iii) for $i \in \{1, 2, \ldots, u-1\}$, $y_{\sigma(i)} > k + y_{\sigma(i+1)}$.

Proof. The conclusions are trivial if $u = 1$ so we assume that $u \geq 2$. Let $w = \binom{u}{2} + \binom{v}{2}$. We apply Theorem 2.10($l$) a total of w times to produce a $(u+w) \times v$ image partition regular matrix B such that

(1) the first u rows of B are the rows of A,

(2) for each pair $i < j$ of members of $\{1, 2, \ldots, v\}$ there exist $b \in \mathbb{Q} \setminus \{0\}$ and a row of B consisting of all 0's except for a b in column i and a $-b$ in column j, and
(3) for each pair $i < j$ of members of $\{1, 2, \ldots, u\}$ there exist $c \in \mathbb{Q} \setminus \{0\}$ and a row of B which is c times the difference between rows i and j of A.

We define the permutations as follows. Pick $\vec{x} \in \mathbb{N}^v$ such that $\vec{y} = B\vec{x} \in \mathbb{N}^{u+w}$. Let σ and τ be the permutations of $\{1, 2, \ldots, u\}$ and $\{1, 2, \ldots, v\}$ such that $x_{\tau(1)} > x_{\tau(2)} > \ldots > x_{\tau(v)}$ and $y_{\sigma(1)} > y_{\sigma(2)} > \ldots > y_{\sigma(u)}$. Notice that the permutations chosen are independent of the choice of $\vec{x}$. (If for example the row of B guaranteed by (2) for $\{1, 3\}$ has a positive b in column 1 and $-b$ in column 3, then $bx_1 - bx_3 \in \mathbb{N}$ so $x_1 > x_3$.)

Pick $m \in \mathbb{N}$ such that for each pair $\{i, j\}$ of distinct members of $\{1, 2, \ldots, v\}$, if b is the number guaranteed for $\{i, j\}$ by (2), then $|b| \leq m$.

Now, let $k \in \mathbb{N}$ and a finite coloring γ of $\mathbb{N}$ be given. Define a coloring γ' of $\mathbb{N}$ by agreeing that $\gamma'(a) = \gamma'(b)$ if and only if either $a = b \leq mk$ or $a > mk$, $b > mk$, $\gamma(a) = \gamma(b)$, and $a \equiv b \pmod{k+1}$. Pick $\vec{x} \in \mathbb{N}^v$ such that all entries of $\vec{y} = B\vec{x}$ are monochrome with respect to γ'. We know that $y_{\sigma(1)} > y_{\sigma(2)}$ and consequently it must be the case that for each $i \in \{1, 2, \ldots, u+w\}$, $y_i > mk$.

We have that for all $i \in \{1, 2, \ldots, u-1\}$ $y_{\sigma(i)} > y_{\sigma(i+1)}$ and $y_{\sigma(i)} \equiv y_{\sigma(i+1)} \pmod{k+1}$ and thus $y_{\sigma(i)} > k + y_{\sigma(i+1)}$.

Finally, let $i \in \{1, 2, \ldots, u-1\}$ and pick $b \in \mathbb{Q} \setminus \{0\}$ and a row t of B with all zero entries except for a b in column $\tau(i)$ and a $-b$ in column $\tau(i+1)$. Then $mk < y_t = b(x_{\tau(i)} - x_{\tau(i+1)}) \leq m(x_{\tau(i)} - x_{\tau(i+1)})$ and thus $x_{\tau(i)} > k + x_{\tau(i+1)}$ as required. $\square$

Theorem 3.1 raises its own questions. Namely, how large (or small) can we insist the ratios between $x_{\tau(i)}$ and $x_{\tau(i+1)}$ be? And how large (or small) can we insist the ratios between $y_{\sigma(i)}$ and $y_{\sigma(i+1)}$ be? We address now the first of these questions.

First entries matrices have been important for some time in the theory of partition regularity of matrices. In [4], Deuber showed that any subset of $\mathbb{N}$ contains solutions to all kernel partition regular matrices if and only if it contains images of all first entries matrices. He showed further that the property of containing images of all first entries matrices is partition regular, thereby answering the old question of Rado's about the property of containing solutions to all kernel partition regular matrices.

It was therefore not surprising when image partition regular matrices were characterized as those intimately related to first entries matrices. (See Theorem 2.10(d), (e), (f), (g), and (h).) In the following theorem, we see that it is not just an intimate relation, but that first entries matrices are precisely the answer to our question.

Theorem 3.2 *Let A be a $u \times v$ matrix with entries from $\mathbb{Q}$. The following statements are equivalent.*

(a) For any $k \in \mathbb{N}$ and any finite coloring of $\mathbb{N}$, there exists $\vec{x} \in \mathbb{N}^v$ with all entries of $A\vec{x}$ monochrome and $x_i > kx_{i+1}$ for every $i \in \{1, 2, \ldots, v-1\}$.

(b) A is a first entries matrix.

Proof. (a) $\Rightarrow$ (b). Suppose that the first nonzero entries of the l^{th} and m^{th} rows of A both occur in the j^{th} column. For every $k \in \mathbb{N}$, there exists $\vec{x}$ in $\mathbb{N}^v$ such that $\vec{y} = A\vec{x} \in \mathbb{N}^u$ and $x_i > kx_{i+1}$ for all $i \in \{1, 2, \ldots, v-1\}$. By choosing k large enough, we can ensure that $\dfrac{y_l}{x_j}$ is arbitrarily close to $a_{l,j}$. So $a_{l,j} > 0$ and, similarly, $a_{m,j} > 0$. Now $\dfrac{y_l}{y_m}$ is arbitrarily close to $\dfrac{a_{l,j}}{a_{m,j}}$. By Lemma 2.3, we may also suppose that $\dfrac{y_l}{y_m}$ is arbitrarily close to 1. (Given $\epsilon > 0$ with $\epsilon < \min\left\{\dfrac{a_{l,j}}{a_{m,j}}, \dfrac{a_{m,j}}{a_{l,j}}\right\}$, we may suppose that $\frac{y_l}{y_m}, \frac{y_m}{y_l} < \frac{1}{\epsilon}$. Then $\frac{y_l}{y_m}, \frac{y_m}{y_l} < 1 + \epsilon$.) $\dfrac{1}{1+\epsilon} < \dfrac{y_l}{y_m} < 1 + \epsilon$.) Thus $a_{l,j} = a_{m,j}$.

(b) $\Rightarrow$ (a). Let $d_1, d_2, \ldots, d_v \in \mathbb{Q}^+$ be chosen so that d_j is equal to the first entry of any row of A whose first entry is in the j^{th} column. Let $k \in \mathbb{N}$ and a coloring γ of $\mathbb{N}$ be given. Let

$$B = \begin{pmatrix} d_1 & -kd_2 & 0 & \cdots & 0 & 0 \\ 0 & d_2 & -kd_3 & \cdots & 0 & 0 \\ 0 & 0 & d_3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & d_{v-1} & -kd_v \\ & & & A & & \end{pmatrix}$$

and note that B is a first entries matrix, hence image partition regular. Pick $\vec{x} \in \mathbb{N}^v$ such that the entries of $B\vec{x}$ are monochrome with respect to γ. Then for each $i \in \{1, 2, \ldots, v-1\}$, $d_i x_i - kd_i x_{i+1} > 0$. □

The idea in the proof that (b) implies (a) of forcing the inequality $d_i x_i - kd_i x_{i+1} > 0$ by requiring that $d_i x_i - kd_i x_{i+1}$ be in the same cell of a coloring of $\mathbb{N}$ has been previously used in [11].

We now turn our attention to the other side of the coin and ask when we can guarantee the existence of a bound k so that whenever $\mathbb{N}$ is finitely colored there must exist $\vec{x}$ with the entries of $A\vec{x}$ monochrome and the ratio betweem entries of $\vec{x}$ bounded by k. There is a trivial sufficient condition.

Lemma 3.3 *Let A be a $u \times v$ matrix with entries from $\mathbb{Q}$ and assume that there exists $\vec{x} \in (\mathbb{Q}^+)^v$ such that $A\vec{x} = \vec{1}$, where $\vec{1}$ is the vector in $\mathbb{N}^u$ with all entries equal to 1. Then there exists $k \in \mathbb{N}$ such that whenever $\mathbb{N}$ is finitely colored there must exist $\vec{x} \in \mathbb{N}^v$ with the entries of $A\vec{x}$ monochrome and $\dfrac{x_i}{x_j} \le k$ whenever $i, j \in \{1, 2, \ldots, v\}$.*

Proof. Pick $\vec{x} \in (\mathbb{Q}^+)^v$ such that $A\vec{x} = \vec{1}$ and let

$$k = \max\left\{\frac{x_i}{x_j} : i, j \in \{1, 2, \ldots, v\}\right\}.$$

Pick $d \in \mathbb{N}$ such that $d\vec{x} \in \mathbb{N}^v$. Then the entries of $A(d\vec{x})$ are monochrome with respect to *any* coloring of $\mathbb{N}$. $\square$

If all entries of A are nonnegative, we shall see in Corollary 3.5 that this trivial condition is also necessary.

The condition of Lemma 3.3 is not necessary for an arbitrary matrix to guarantee monochrome solutions with bounded ratios, as may be seen by considering the matrix $A = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$. If $A\vec{x} = \vec{1}$, then $x_2 = 0$. On the other hand, given any finite coloring and any $\delta > 1$, there exist $\vec{x} \in \mathbb{N}^2$ with the entries of $A\vec{x}$ monochrome and $x_2 < x_1 < \delta x_2$. To see this, pick any color class with infinitely many members and pick y_1 and $y_2 > \frac{\delta}{\delta - 1} y_1$ in the same color class. Let $x_1 = y_2$ and $x_2 = y_2 - y_1$.

One of the general characterizations involves a version of the columns condition.

Theorem 3.4 *Let A be a $u \times v$ matrix with entries from $\mathbb{Q}$. The following statements are equivalent.*

(a) *There exists $k \in \mathbb{N}$ such that whenever $\mathbb{N}$ is finitely colored there must exist $\vec{x} \in \mathbb{N}^v$ with the entries of $A\vec{x}$ monochrome and $\dfrac{x_i}{x_j} \leq k$ whenever $i, j \in \{1, 2, \ldots, v\}$.*

(b) *For some $n \in \mathbb{N}$, there is a $v \times n$ matrix B over $\mathbb{Q}$, with all its entries non-negative and all those in the first column positive, for which AB is a first entries matrix.*

(c) *There exist $t_1, t_2, \ldots, t_v \in \mathbb{Q}^+$ such that the matrix*

$$M = \begin{pmatrix} t_1 a_{1,1} & t_2 a_{1,2} & t_3 a_{1,3} & \cdots & t_v a_{1,v} & -1 & 0 & 0 & \cdots & 0 \\ t_1 a_{2,1} & t_2 a_{2,2} & t_3 a_{2,3} & \cdots & t_v a_{2,v} & 0 & -1 & 0 & \cdots & 0 \\ t_1 a_{3,1} & t_2 a_{3,2} & t_3 a_{3,3} & \cdots & t_v a_{3,v} & 0 & 0 & -1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ t_1 a_{u,1} & t_2 a_{u,2} & t_3 a_{u,3} & \cdots & t_v a_{u,v} & 0 & 0 & 0 & \cdots & -1 \end{pmatrix}$$

satisfies the columns condition (Definition 1.6) with $\{1, 2, \ldots, v\} \subseteq I_1$.

Proof. $(a) \Rightarrow (b)$. By Lemma thAa, for every $\epsilon > 0$, we can choose $\vec{x} \in \mathbb{N}^v$ so that $\vec{y} = A\vec{x} \in \mathbb{N}^u$, $\dfrac{x_i}{x_j} \leq k$ for all $i, j \in \{1, 2, \ldots, v\}$ and, for every $s, t \in \{1, 2, \ldots, u\}$ such that $y_s < y_t$, either $\dfrac{y_t}{y_s} < 1 + \epsilon$ or $\dfrac{y_t}{y_s} > \dfrac{1}{\epsilon}$. (Of course, $\vec{x}$ and $\vec{y}$ depend on ϵ. We do not indicate this in the notation, in order to avoid equations that would be quite cumbersome.) Choose $m \in \{1, 2, \ldots, u\}$ such that $y_m = \max\{y_i : i \in \{1, 2, \ldots, u\}\}$. For each $i \in \{1, 2, \ldots, u\}$, we have $\sum_{j=1}^v a_{i,j} \dfrac{x_j}{x_1} = \dfrac{y_m}{x_1} \dfrac{y_i}{y_m}$. We note that, for every $i \in \{1, 2, \ldots, u\}$, $1 - \epsilon < \dfrac{y_i}{y_m} \leq 1$ or $\dfrac{y_i}{y_m} < \epsilon$. Furthermore, $\dfrac{1}{k} < \dfrac{x_j}{x_1} < k$ for every $j \in \{1, 2, \ldots, v\}$ and $\dfrac{y_m}{x_1} < k \sum_{j=1}^{v} |a_{m,j}|$. So the numbers $\dfrac{y_m}{x_1}$ have an upper bound independent of ϵ. We can choose a sequence of values of ϵ converging to 0 for which the numbers $\dfrac{x_j}{x_1}$ converge to a limit d_j, which is necessarily positive; and the values $\dfrac{y_m}{x_1}$ converge to a limit $d \geq 0$. We may also suppose that the numbers $\dfrac{y_i}{y_m}$ converge to a limit $w_i \in \{0, 1\}$. Let $\vec{c}_1, \vec{c}_2, \ldots, \vec{c}_v$ denote the columns of A. Then $\sum_{j=1}^v d_j \vec{c}_j = d\vec{w}$ for some $\vec{w} \in \{0, 1\}^u$. By Lemma 2.5, we have $\sum_{j=1}^v r_j \vec{c}_j = r\vec{w}$, where each $r_j \in \mathbb{Q}^+$ and $r \in \mathbb{Q} \cap [0, \infty)$. By Theorem

2.10(d), there exist $m \in \mathbb{N}$ and a $v \times m$ matrix G with non-negative rational entries for which AG is a first entries matrix. Let $\vec{b} = (\, r_1 \quad r_2 \quad \dots \quad r_v \,)^T$. If $B = (\, \vec{b} \quad G \,)$, then AB is a first entries matrix.

(b) $\Rightarrow$ (a). Choose $d \in \mathbb{N}$ such that all entries of $G = dB$ are in $\mathbb{Z}$. Then AG is a first entries matrix. Pick $r \geq 2$ such that for all $t \in \{1, 2, \dots, v\}$ and all $j \in \{1, 2, \dots, m\}$, $g_{t,j} \leq \dfrac{r}{4} \cdot g_{t,1}$ and pick $k \in \mathbb{N}$ such that $k > \dfrac{3}{2} \cdot \dfrac{g_{t,1}}{g_{n,1}}$ for every $t, n \in \{1, 2, \dots, v\}$.

Let $\mathbb{N}$ be finitely colored, and pick by Theorem 3.2 $\vec{x} \in \mathbb{N}^m$ such that the entries of $AG\vec{x}$ are monochrome and for all $i \in \{1, 2, \dots, m-1\}$, $x_i > r \cdot x_{i+1}$. Let $\vec{z} = G\vec{x}$ and let $t, n \in \{1, 2, \dots, v\}$. Then

$$\begin{aligned} z_t &= \textstyle\sum_{j=1}^{m} g_{t,j} \cdot x_j \\ &\leq g_{t,1} \cdot x_1 + \textstyle\sum_{j=2}^{m} \dfrac{r}{4} \cdot g_{t,1} \cdot \dfrac{1}{r^{j-1}} \cdot x_1 \\ &= g_{t,1} \cdot x_1 \cdot \left(1 + \dfrac{1}{4} \displaystyle\sum_{j=2}^{m} \dfrac{1}{r^{j-2}} \right) \\ &\leq g_{t,1} \cdot x_1 \cdot \dfrac{3}{2} . \end{aligned}$$

Now $z_n \geq g_{n,1} \cdot x_1$. Thus $\dfrac{z_t}{z_n} \leq \dfrac{3}{2} \cdot \dfrac{g_{t,1}}{g_{n,1}} \leq k$.

(b) $\Rightarrow$ (c). We note that statement (c) is equivalent to stating that there exists $m \in \mathbb{N}$ such that $(\, AT \quad -I \,)\, M = \mathbf{O}$, for some diagonal $v \times v$ matrix T with positive rational diagonal entries and some first entries $(u + v) \times m$ matrix M, for which all the first v entries in the first column are positive.

We may suppose that all the first entries of AB are equal to 1, since this could be achieved by multiplying each column of B by a suitable positive rational number. We can choose a diagonal $v \times v$ matrix S, with positive rational diagonal entries equal to the reciprocals of the entries in the first column of B, for which SB is a first entries matrix with all its first entries equal to 1. Then $(\, AS^{-1} \quad -I \,) \begin{pmatrix} SB \\ AB \end{pmatrix} = \mathbf{O}$, where I denotes the $u \times u$ identity matrix. Now $\begin{pmatrix} SB \\ AB \end{pmatrix}$ is a first entries matrix with all of the first v entries in the first column equal to 1. So statement (c) follows, with $t_1, t_2, \dots, t_v$ denoting the diagonal entries of S^{-1}.

(c) $\Rightarrow$ (b). We may suppose that all the entries of M are non-negative, because we can replace M by ME, where E is an $m \times m$ matrix for which

$e_{i,j} = \lambda^{j-i}$ if $i \leq j$ and $e_{i,j} = 0$ if $i > j$, where λ is chosen to be a suitably large rational number (see [12, Lemma 15.14]). Then (b) follows immediately by writing $M = \begin{pmatrix} B \\ C \end{pmatrix}$, where B is a $v \times m$ matrix and C a $u \times m$ matrix. $\square$

Corollary 3.5 *Let A be a $u \times v$ matrix with entries from $\mathbb{Q}$ and all entries nonnegative. The following statements are equivalent.*

(a) There exists $\vec{x} \in (\mathbb{Q}^+)^v$ such that $A\vec{x} = \vec{1}$.

(b) There exists $k \in \mathbb{N}$ such that whenever $\mathbb{N}$ is finitely colored there must exist $\vec{x} \in \mathbb{N}^v$ with the entries of $A\vec{x}$ monochrome and $\frac{x_i}{x_j} \leq k$ whenever $i, j \in \{1, 2, \ldots, v\}$.

Proof. The fact that (a) implies (b) is a consequence of Lemma 3.3.

$(b) \Rightarrow (a)$. Since the entries of A are nonnegative, this is an immediate consequence of Theorem 3.4. $\square$

We now turn our attention to the question of bounded and unbounded ratios in the image. Notice the difference between the condition of statement (a) in the following theorem and that in Corollary 3.5, where the entries of $\vec{x}$ are required to be positive. In particular, as a consequence of Corollary 3.5 and Theorem 3.6, we have that the matrix $A = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$ has, for any finite coloring of $\mathbb{N}$, monochrome solutions $A\vec{x} = \vec{y}$ with $\frac{y_1}{y_2}$ as close to 1 as desired, but for any k there is a coloring so that any monochrome solution has $\frac{x_1}{x_2} > k$ or $\frac{x_2}{x_1} > k$.

Theorem 3.6 *Let A be a $u \times v$ matrix over $\mathbb{Q}$. The following statements are equivalent.*

(a) There exists $\vec{z} \in \mathbb{Q}^v$ such that $A\vec{z} = \vec{1}$ and for each $i \in \{1, 2, \ldots, v\}$, $z_i \geq 0$.

(b) There exists $m \in \mathbb{N}$ such that, for every $p \in K(\beta\mathbb{N})$ for which $m\mathbb{N} \in p$, every $P \in p$, and every $\epsilon > 0$, there exists $\vec{x} \in \mathbb{N}^v$ such that $\vec{y} = A\vec{x} \in P^u$ and for all $i, j \in \{1, 2, \ldots, u\}$, $\frac{y_i}{y_j} < 1 + \epsilon$.

(c) For every central set C in $\mathbb{N}$ and every $\epsilon > 0$, there exists $\vec{x} \in \mathbb{N}^v$ such that $\vec{y} = A\vec{x} \in C^u$ and for all $i, j \in \{1, 2, \ldots, u\}$, $\frac{y_i}{y_j} < 1 + \epsilon$.

(d) *For every finite coloring of* $\mathbb{N}$ *and every* $\epsilon > 0$*, there exists* $\vec{x} \in \mathbb{N}^v$ *such that* $\vec{y} = A\vec{x} \in \mathbb{N}^u$*, the entries of* $\vec{y}$ *are monochrome and, for all* $i, j \in \{1, 2, \ldots, u\}$*,* $\dfrac{y_i}{y_j} < 1 + \epsilon$*.*

(e) *For every* $\epsilon > 0$*, there exists* $\vec{x} \in \mathbb{N}^v$ *such that* $\vec{y} = A\vec{x} \in \mathbb{N}^u$ *and, for all* $i, j \in \{1, 2, \ldots, u\}$*,* $\dfrac{y_i}{y_j} < 1 + \epsilon$*.*

(f) *There exists* $k, m \in \mathbb{N}$ *such that, for every* p *in the smallest ideal of* $\beta\mathbb{N}$ *for which* $m\mathbb{N} \in p$ *and every* $P \in p$*, there exists* $\vec{x} \in \mathbb{N}^v$ *such that* $\vec{y} = A\vec{x} \in P^u$ *and, for all* $i, j \in \{1, 2, \ldots, u\}$*,* $\dfrac{y_i}{y_j} < k$*.*

(g) *There exists* $k \in \mathbb{N}$ *such that, for every finite coloring of* $\mathbb{N}$*, there exists* $\vec{x} \in \mathbb{N}^v$ *such that* $\vec{y} = A\vec{x} \in \mathbb{N}^u$*, the entries of* $\vec{y}$ *are monochrome, and for all* $i, j \in \{1, 2, \ldots, u\}$*,* $\dfrac{y_i}{y_j} < k$*.*

Proof. $(a) \Rightarrow (b)$. Pick $\vec{z} \in \mathbb{Q}^v$ such that $A\vec{z} = \vec{1}$ and for each $i \in \{1, 2, \ldots, v\}$, $z_i \geq 0$. Pick $m \in \mathbb{N}$ such that $m\vec{z} \in \omega^v$, let $p \in K(\beta\mathbb{N})$ such that $m\mathbb{N} \in p$, let $P \in p$, and let $0 < \epsilon < 1$. Pick $\vec{s} \in \mathbb{N}^v$ such that all entries of $A\vec{s}$ are in $\mathbb{Z}$ and let $\vec{t} = A\vec{s}$. Let

$$I = \{(rt_1+nm, rt_2+nm, \ldots, rt_u+nm) : r, n \in \mathbb{N} \text{ and } nm > \frac{4r|t_i|}{\epsilon} \text{ for each } i\}$$

and let $E = I \cup \{(nm, nm, \ldots, nm) : n \in \mathbb{N}\}$. Then E is a subsemigroup of $\mathbb{N}^u$ and I is an ideal of E. Thus $c\ell_{(\beta\mathbb{N})^u} E$ is a subsemigroup of $(\beta\mathbb{N})^u$ and $c\ell_{(\beta\mathbb{N})^u} I$ is an ideal of $c\ell_{(\beta\mathbb{N})^u} E$ by [12, Theorems 2.22 and 4.17].

Let $\overline{p} = (p, p, \ldots, p)$. Then since $m\mathbb{N} \in p$ and $(m\mathbb{N})^u \subseteq E$, we have $\overline{p} \in c\ell_{(\beta\mathbb{N})^u} E$. By [12, Theorem 2.23], $K\Big((\beta\mathbb{N})^u\Big) = \Big(K(\beta\mathbb{N})\Big)^u$ and so $\overline{p} \in K\Big((\beta\mathbb{N})^u\Big) \cap c\ell_{(\beta\mathbb{N})^u} E$ and thus by [12, Theorem 1.65], $\overline{p} \in K(c\ell_{(\beta\mathbb{N})^u} E)$ and therefore $\overline{p} \in c\ell_{(\beta\mathbb{N})^u} I$. Thus one may choose $r, n \in \mathbb{N}$ with $nm > \dfrac{4r|t_i|}{\epsilon}$ for each $i \in \{1, 2, \ldots, u\}$ such that $rt_i + nm \in P$ for each i.

Let $\vec{x} = r\vec{s} + nm\vec{z}$ and let $\vec{y} = A\vec{x}$. Then $\vec{x} \in \mathbb{N}^v$ and for each $i \in \{1, 2, \ldots, u\}$, $y_i = rt_i + nm$. Let $i, j \in \{1, 2, \ldots, u\}$. Then $rt_i + nm < nm \cdot (1 + \dfrac{\epsilon}{4})$ and $rt_j + nm > nm \cdot (1 - \dfrac{\epsilon}{4})$ and so $\dfrac{rt_i + nm}{rt_j + nm} < \dfrac{1 + \frac{\epsilon}{4}}{1 - \frac{\epsilon}{4}} < 1 + \epsilon$.

$(b) \Rightarrow (c)$. Pick a minimal idempotent p in $\beta\mathbb{N}$ such that $C \in p$. By [12, Lemma 6.6], for every $m \in \mathbb{N}$, $m\mathbb{N} \in p$.

It is obvious that (c) implies (d), (d) implies (e), (b) implies (f) and (f) implies (g). By Lemma 2.3, (g) implies (e).

To complete the proof, we shall show that (e) implies (a). Let $\vec{c}_1, \vec{c}_2, \ldots, \vec{c}_v$ be the columns of A. By Lemmas 2.5 and 2.7 it suffices to show that $\vec{0} \in c\ell\Big\{z_0\vec{1} + \sum_{j=1}^{v} z_j\vec{c}_j : z_0, z_1, \ldots, z_v \in \mathbb{R}, z_0 < 0$, and $z_j \geq 0$ for $j \in \{1, 2, \ldots, v\}\Big\}$. (For then $-z_0\vec{1} = \sum_{j=1}^{v} z_j\vec{c}_j$ for some $z_0, z_1, \ldots, z_v \in \mathbb{Q}$ with $z_0 < 0$ and $z_j \geq 0$ for $j \in \{1, 2, \ldots, v\}$.) To see this, let $0 < \epsilon < 0$ and pick $\vec{x} \in \mathbb{N}^v$ such that $\vec{y} = A\vec{x} \in \mathbb{N}^u$ and, for all $i, j \in \{1, 2, \ldots, u\}$, $\dfrac{y_i}{y_j} < 1 + \epsilon$. Let $z_0 = -1$ and for $j \in \{1, 2, \ldots, v\}$, let $z_j = \dfrac{x_j}{y_1}$. Then for each $i \in \{1, 2, \ldots, u\}$ we have $-1 + \sum_{j=1}^{v} z_j a_{i,j} = -1 + \dfrac{y_i}{y_1}$ and $1 - \epsilon < \dfrac{1}{1+\epsilon} < \dfrac{y_i}{y_1} < 1 + \epsilon$. □

Theorem 3.7 *Let A be a $u \times v$ matrix over $\mathbb{Q}$. The following statements are equivalent:*

(a) For every $k \in \mathbb{N}$, there exists $\vec{x} \in \mathbb{N}^v$ such that $\vec{y} = A\vec{x} \in \mathbb{N}^u$ and, for every $i \in \{1, 2, \ldots, u-1\}$, $y_i > ky_{i+1}$.

(b) There exists a $v \times u$ matrix B over $\mathbb{Q}$ with non-negative entries, such that AB is an upper triangular matrix with positive diagonal entries.

(c) For every $k \in \mathbb{N}$ and every finite coloring of $\mathbb{N}$, there exists $\vec{x} \in \mathbb{N}^v$ such that $\vec{y} = A\vec{x} \in \mathbb{N}^u$, the entries of $\vec{y}$ are monochrome, and $y_i > ky_{i+1}$ for every $i \in \{1, 2, \ldots, u-1\}$.

Proof. $(a) \Rightarrow (b)$. Given $\epsilon > 0$, choose $k \in \mathbb{N}$ such that $\frac{1}{k} < \epsilon$ and let $\vec{x}$ have the properties guaranteed by (a). For every $i, m \in \{1, 2, \ldots, u\}$, we have $\sum_{j=1}^{v} a_{i,j}\frac{x_j}{y_m} = \frac{y_i}{y_m}$. Note that $\frac{y_i}{y_m} = 1$ if $i = m$ and $\frac{y_i}{y_m} < \epsilon$ if $i > m$. Thus, if $\vec{e}_m$ denotes the m^{th} unit vector in $\mathbb{R}^u$, every neighborhood of $\vec{e}_m$ contains a vector which is a linear combination of the columns of A and the vectors $\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_{m-1}$, with the coefficients of the columns of A being positive. It follows from Corollary 2.6 and Lemma 2.7 that there exists $\vec{b}_m \in \mathbb{Q}^v$, with non-negative entries, such that the i^{th} entry of $A\vec{b}_m$ is 1 if $i = m$ and 0 if $i > m$. If B denotes the matrix with columns $\vec{b}_1, \vec{b}_2, \ldots, \vec{b}_u$, then AB is an upper triangular matrix with all diagonal entries equal to 1.

$(b) \Rightarrow (c)$. Let $k \in \mathbb{N}$ and let a finite coloring of $\mathbb{N}$ be given. We can multiply B by a suitable positive integer to obtain a matrix C, with entries in ω, for which $AC = D$ is a first entries matrix in which, for each $i \in$

$\{1,2,\ldots,u\}$, the first entry of the i^{th} row occurs in the i^{th} column. We may suppose that C has no row equal to $\vec{0}$, because we can add any column with entries in $\mathbb{N}$ to C as a new final column. Let w denote the number of columns of D (so $w=u$ or $w=u+1$). Choose ϵ satisfying $0<\epsilon<\dfrac{1}{k+1}\min\Big\{d_{i,i}: i\in\{1,2,\ldots,u\}\Big\}$ and then choose $m\in\mathbb{N}$ such that $\sum_{j=1}^{w}|d_{i,j}|<m\epsilon$ for every $i\in\{1,2,\ldots,u\}$. By Theorem 3.2, there exists $\vec{x}\in\mathbb{N}^{w}$ such that $x_i>mx_{i+1}$ for every $i\in\{1,2,\ldots,w-1\}$ and $\vec{y}=D\vec{x}\in\mathbb{N}^{u}$ has monochrome entries. For each $i\in\{1,2,\ldots,u-1\}$, one has that $\sum_{j=i+1}^{w}|d_{i,j}|\cdot x_j<x_i\cdot\epsilon$ and

$$\frac{y_i}{y_{i+1}}=\frac{\sum_{j=i}^{w}d_{ij}x_j}{\sum_{j=i+1}^{w}d_{i+1,j}x_j}$$

so that $\dfrac{y_i}{y_{i+1}}>\dfrac{d_{i,i}-\epsilon}{\epsilon}>k$. Thus (c) follows from the observation that $D\vec{x}=A(C\vec{x})$ and $C\vec{x}\in\mathbb{N}^{v}$.

This completes the proof, since (c) obviously implies (a). $\square$

In Corollary 3.5 and Theorem 3.6 we had conditions involving $\vec{1}$ being in the positive and nonnegative spans of the columns of A respectively. In the following theorem we explore the implications of $\vec{1}$ being in the range of A.

Theorem 3.8 *Let A be a $u\times v$ image partition regular matrix over $\mathbb{Q}$. The following are equivalent:*

(a) *There exists $\vec{s}\in\mathbb{Q}^{v}$ such that $A\vec{s}=\vec{1}$.*

(b) *There exists $l\in\mathbb{N}$ such that, if p is in the smallest ideal of $\beta\mathbb{N}$ and $l\mathbb{N}\in p$, then, for every $P\in p$, there exists $\vec{x}\in\mathbb{N}^{v}$ such that $A\vec{x}\in P^{u}$.*

(c) *There exists $k\in\mathbb{N}$ such that, given any finite coloring of $\mathbb{N}$, there exists $\vec{z}\in\mathbb{Z}^{v}$ such that the entries of $\vec{y}=A\vec{z}$ are monochrome positive integers and satisfy $\dfrac{y_i}{y_j}<k$ for every $i,j\in\{1,2,\ldots,u\}$.*

(d) *Given any $\epsilon>0$ and any finite coloring of $\mathbb{N}$, there exists $\vec{z}\in\mathbb{Z}^{v}$ such that the entries of $\vec{y}=A\vec{z}$ are monochrome positive integers and satisfy $\dfrac{y_i}{y_j}<1+\epsilon$ for every $i,j\in\{1,2,\ldots,u\}$.*

(e) *Given any $\epsilon>0$, there exists $\vec{z}\in\mathbb{Z}^{v}$ such that $\vec{y}=A\vec{z}\in\mathbb{N}^{u}$ and $\frac{y_i}{y_j}<1+\epsilon$ for every $i,j\in\{1,2,\ldots,u\}$.*

Proof. We first show that (a) and (b) are equivalent.

$(a) \Rightarrow (b)$. We can choose $\vec{z} \in \mathbb{Z}^v$ and $l \in \mathbb{N}$ such that $A\vec{z} = \vec{l}$, where $\vec{l} = (\,l \quad l \quad \cdots \quad l\,)^T \in \mathbb{N}^u$. Suppose that p is in the smallest ideal of $\beta\mathbb{N}$ and that $l\mathbb{N} \in p$. Let $P \in p$. There exists a minimal idempotent $q \in \beta\mathbb{N}$ such that $p = p+q$ [12, Theorem 2.8 and Lemma 1.30]. Let $P' = \{n \in P : -n+P \in q\}$. Then $P' \in p$. Since $l\mathbb{N} \in p$ we can choose $m \in \mathbb{N}$ such that $lm \in P'$. Let $Q = -lm + P \in q$. By Theorem 2.10(b), there exists $\vec{x} \in \mathbb{N}^v$ such that $A\vec{x} \in Q^v$ and $x_i + mz_i > 0$ for every $i \in \{1, 2, \ldots, v\}$. (The fact that the entries of $\vec{x}$ can be chosen to be arbitrarily large follows from Lemma 2.2. For every $r \in \mathbb{N}$, $\{\vec{x} \in \mathbb{N}^v : x_i > r$ for all $i \in \{1, 2, \ldots, v\}\}$ is a member of every idempotent in $\beta(\mathbb{N}^v)$.) Then $m\vec{z} + \vec{x} \in \mathbb{N}^v$ and $A(m\vec{z} + \vec{x}) = m\vec{l} + A\vec{x} \in P^u$.

$(b) \Rightarrow (a)$. We may suppose that the entries of A are in $\mathbb{Z}$, as we could replace A by nA for a suitable $n \in \mathbb{N}$.

Suppose that $\vec{1} \notin \{A\vec{x} : \vec{x} \in \mathbb{Q}^v\}$. Then there exists $\vec{u} \in \mathbb{Z}^v$ such that $\vec{u} \cdot A\vec{x} = 0$ for every $x \in \mathbb{Q}^v$, but $\vec{u} \cdot \vec{1} \neq 0$.

Choose a prime number r satisfying $r > l$ and $r > |l\vec{u} \cdot \vec{1}|$. Let q be a minimal idempotent in $\beta\mathbb{N}$ and let $p = l + q$. Then $P = \{n \in \mathbb{N} : n \equiv l \,(\text{mod } r)\} \in p$ (by [12, Lemma 6.6]). It follows from (b) that there exists $\vec{x} \in \mathbb{Z}^v$ such that $A\vec{x} \in P^u$ and hence that $A\vec{x} = l\vec{1}$ in $\mathbb{Z}_r$. This is a contradiction, as $\vec{u} \cdot A\vec{x} = 0$ in $\mathbb{Z}_r$, but $\vec{u} \cdot l\vec{1} \neq 0$ in $\mathbb{Z}_r$.

We now show that (c), (d) and (e) are equivalent to (a) and (b). It is immediate from Lemma 1.3 that (c) and (d) are equivalent, and it is obvious that (d) implies (e). Now (e) implies that $\vec{1}$ is in the closure of the range of A and therefore in the range of A, since linear subspaces of $\mathbb{Q}^u$ are closed. So (e) implies (a). It is trivial that (a) implies (e). $\square$

4 Preserving Large Sets

In this section we investigate questions raised by the characterization in Theorem 2.10(c). There are several notions of largeness that make sense in any semigroup. The notion of "central" sets is one of these. Among the others are the notions of "syndetic", "piecewise syndetic", "IP", and "Δ" sets.

Definition 4.1 Let $(S, +)$ be a semigroup and let $B \subseteq S$.

(a) The set B is *syndetic* if and only if there exists some $G \in \mathcal{P}_f(S) = \{H \subseteq S : H$ is finite and nonempty$\}$ such that $S = \bigcup_{t \in G} -t + B$.

(b) The set B is *piecewise syndetic* if and only if there exists some $G \in \mathcal{P}_f(S)$ such that for every $F \in \mathcal{P}_f(S)$ there exists $x \in S$ such that $F + x \subseteq \bigcup_{t \in G} -t + B$.

(c) The set B is an *IP set* if and only if there exists a sequence $\langle x_n \rangle_{n=1}^{\infty}$ in S such that $FS(\langle x_n \rangle_{n=1}^{\infty}) \subseteq B$, where $FS(\langle x_n \rangle_{n=1}^{\infty}) = \{\sum_{n \in F} x_n : F \in \mathcal{P}_f(\mathbb{N})\}$ and the sums are taken in increasing order of indices.

(d) The set B is a Δ *set* if and only if there exists a sequence $\langle s_n \rangle_{n=1}^{\infty}$ in S such that for every $n, m \in \mathbb{N}$ with $n < m$, $s_m \in (s_n + B)$.

Notice that, if S can be embedded in a group G, B is a Δ set if and only if there is a sequence $\langle s_n \rangle_{n=1}^{\infty}$ such that $\{-s_n + s_m : m, n \in \mathbb{N} \text{ and } n < m\} \subseteq B$. Notice also that any IP set is a Δ set. (Given $\langle x_n \rangle_{n=1}^{\infty}$ with $FS(\langle x_n \rangle_{n=1}^{\infty}) \subseteq B$ and given $n \in \mathbb{N}$, let $s_n = \sum_{t=1}^{n} x_t$.)

Given any property $\mathcal{E}$ of subsets of a set X, there is a dual property $\mathcal{E}^*$ defined by specifying that a subset B of X is an $\mathcal{E}^*$ set if and only if $B \cap A \neq \emptyset$ for every $\mathcal{E}$ set A.

Definition 4.2 Let $(S, +)$ be a semigroup and let $B \subseteq S$. Then B is a *central** set if and only if $B \cap A \neq \emptyset$ for every central set A in S. Also, B is a *PS** set if and only if $B \cap A \neq \emptyset$ for every piecewise syndetic set A in S, B is an *IP** set if and only if $B \cap A \neq \emptyset$ for every IP set A in S, B is a *syndetic** set if and only if $B \cap A \neq \emptyset$ for every syndetic set A in S, and B is a Δ^* set if and only if $B \cap A \neq \emptyset$ for every Δ set A in S.

The concept of "syndetic*" is more commonly referred to as "thick", and we shall follow this practice.

The Δ sets and Δ^* sets are interesting because they arise as sets of recurrence, which in turn have significant combinatorial properties. (See [5].) The other notions discussed above have simple, and useful, algebraic characterizations in terms of βS.

Lemma 4.3 *Let $(S, +)$ be a semigroup and let $B \subseteq S$.*

(a) A is piecewise syndetic if and only if $\overline{A} \cap K(\beta S) \neq \emptyset$.

(b) A is IP if and only if there is some idempotent of βS in $\overline{A}$.

(c) A is syndetic if and only if for every left ideal L of βS, $\overline{A} \cap L \neq \emptyset$.

(d) A is central if and only if there is some minimal idempotent of βS in $\overline{A}$.

(e) A is central if and only if every minimal idempotent of βS is in $\overline{A}$.*

(f) A is thick if and only if $\overline{A}$ contains a left ideal of βS.

(g) A is IP if and only if every idempotent of βS is in $\overline{A}$.*

(h) A is PS if and only if $K(\beta S) \subseteq \overline{A}$.*

Proof. Statement (a) is [12, Theorem 4.40], (b) is [12, Theorem 5.12], (c) is [3, Theorem 2.9(d)], and (d) is the definition of central. Statements (e), (f), (g), and (h) follow easily from statements (d), (c), (b), and (a) respectively. □

As a consequence of Lemma 4.3, and the observation already made that any IP set is a Δ set, we see that the pattern of implications given below holds.

$$\begin{tikzcd}
\Delta^* \arrow[d] & & & \\
\text{IP}^* \arrow[dr] & & \text{PS}^* \arrow[dl] \arrow[d] & \\
& \text{central}^* \arrow[dl] \arrow[d] & \text{thick=syndetic}^* \arrow[dl] & \\
\text{syndetic} \arrow[d] & \text{central} \arrow[dl] \arrow[dr] & & \\
\text{piecewise syndetic} & & \text{IP} \arrow[d] & \\
& & \Delta &
\end{tikzcd}$$

It is easy to produce examples in $(\mathbb{N}, +)$ showing that none of the missing implications is valid in general. (Or see [1] for an explicit list of examples.)

The following theorem relates image partition regular matrices and piecewise syndetic sets.

Theorem 4.4 *Let A be a $u \times v$ image partition regular matrix over $\mathbb{Q}$. The following statements are equivalent:*

(a) For every piecewise syndetic subset P of $\mathbb{N}$, there exists $\vec{x} \in \mathbb{N}^v$ such that $A\vec{x} \in P^u$;

(b) There exists $\vec{z} \in \mathbb{Z}^v$ such that $A\vec{z} = \vec{1}$, where $\vec{1}$ denotes the vector in $\mathbb{N}^u$ whose entries are all equal to 1.

Proof. This follows easily from Lemma 4.3(a) and the proof of $(a) \Leftrightarrow (b)$ in Theorem 3.8, taking $l = 1$. $\square$

In [2] it was shown that if "large" meant any of "Δ", "IP", "central", "central*", "IP*", or "Δ*", and B is a large subset of $\mathbb{N}$, then for every positive $\alpha \in \mathbb{R}$ and every $\gamma \in \mathbb{R}$ with $0 < \gamma < 1$, $\{\lfloor \alpha n + \gamma \rfloor : n \in B\}$ is also large (in the same sense). In [6] it was shown that if B is a piecewise syndetic subset of $\mathbb{Z}$, $l \in \mathbb{N}$, and $AP_l = \{(a, a+d, \ldots, a+(l-1)d) : a, d \in \mathbb{Z}\}$, the group of length l arithmetic progressions (including the constant ones), then $B^l \cap AP_l$ is piecewise syndetic in AP_l. In [1] a systematic study of this latter phenomenon was undertaken. These results apply in the current context in terms of when the set of images contained in a given set is large among the set of all images.

Theorem 4.5 *Let A be a $u \times v$ matrix with entries from $\mathbb{Q}$, let $I = \{A\vec{x} : \vec{x} \in \mathbb{N}^v\} \cap \mathbb{N}^u$, and let $C \subseteq \mathbb{N}$.*

(a) If $I \neq \emptyset$, "large" is any of "IP", "Δ*", "PS*", or "central*", and C is large in $\mathbb{N}$, then $I \cap C^u$ is large in I.*

(b) If $\vec{1} \in I$, "large" is any of "piecewise syndetic", "central", or "thick", and C is large in $\mathbb{N}$, then $I \cap C^u$ is large in I.

Proof. (a) For IP* and Δ*, [1, Corollary 2.3] requires only that I be a subsemigroup of $\mathbb{N}^u$. For PS* and central*, [1, Corollary 2.7] requires in addition that for each $i \in \{1, 2, \ldots, u\}$, the i^{th} projection $\pi_i[I]$ be piecewise syndetic in $\mathbb{N}$. This trivially holds because, if $d \in \pi_i[I]$, then $d\mathbb{N} \subseteq \pi_i[I]$.

(b) Letting $E = I$, we have that

$$\left\{ \begin{pmatrix} a \\ a \\ \vdots \\ a \end{pmatrix} : a \in \mathbb{N} \right\} \subseteq E$$

so that [1, Theorem 3.7] applies. $\square$

Notice that the hypothesis of Theorem 4.5(b) differs from that of Corollary 3.5 which asked that $\vec{1}$ be in the positive rational span of the columns of A. If "large" is "piecewise syndetic", some such distinction is necessary as can be seen by considering the 1×1 matrix $A = (2)$ and the piecewise syndetic subset $C = 2\mathbb{N} + 1$ of $\mathbb{N}$. For "central" and "thick" the proof of [1, Theorem

3.7] can be modified to show that the assumption that $\vec{1}$ is in the positive rational span of the columns of A is enough.

We shall be concerned for the rest of this section with establishing analogues of Theorem 2.10(c) for the other notions of largeness. That is, we wish to determine conditions that guarantee that if a set C is "large" in $\mathbb{N}$, then $\{\vec{x} \in \mathbb{N}^v : A\vec{x} \in C^u\}$ is "large" in $\mathbb{N}^v$.

Lemma 4.6 *Let A be a $u \times v$ matrix with entries from $\mathbb{Z}$, define $\varphi : \mathbb{N}^v \to \mathbb{Z}^u$ by $\varphi(\vec{x}) = A\vec{x}$, and let $\widetilde{\varphi} : \beta(\mathbb{N}^v) \to (\beta\mathbb{Z})^u$ be its continuous extension. Then $\widetilde{\varphi}$ is a homomorphism and $K((\beta\mathbb{N})^u) = (K(\beta\mathbb{N}))^u$.*

(a) *If there exists $\vec{x} \in \mathbb{N}^v$ such that $A\vec{x} \in \mathbb{N}^u$, then $\widetilde{\varphi}[\beta(\mathbb{N}^v)] \cap K((\beta\mathbb{N})^u) \neq \emptyset$.*

(b) *If for all $\vec{x} \in \mathbb{N}^v$, $A\vec{x} \in \mathbb{N}^u$, then $\widetilde{\varphi}[K(\beta(\mathbb{N}^v))] \subseteq K((\beta\mathbb{N})^u)$.*

Proof. By [12, Corollary 4.22] we have that $\widetilde{\varphi}$ is a homomorphism, and by [12, Theorem 2.23] $K((\beta\mathbb{N})^u) = (K(\beta\mathbb{N}))^u$.

(a). Since $\widetilde{\varphi}[\beta(\mathbb{N}^v)] = c\ell\{A\vec{x} : \vec{x} \in \mathbb{N}^v\}$, we need to show that $c\ell\{A\vec{x} : \vec{x} \in \mathbb{N}^v\} \cap (K(\beta\mathbb{N}))^u \neq \emptyset$.

Pick $\vec{x} \in \mathbb{N}^v$ such that $A\vec{x} = \vec{y} \in \mathbb{N}^u$. Pick any minimal idempotent p in $\beta\mathbb{N}$. Then by Lemma 2.1 $\vec{p} = \vec{y}p \in (K(\beta\mathbb{N}))^u$. To see that $\vec{p} \in c\ell\{A\vec{z} : \vec{z} \in \mathbb{N}^v\}$, let U be a neighborhood of $\vec{p}$ and for each $i \in \{1, 2, \ldots, u\}$, pick $D_i \in p$ such that $\times_{i=1}^u \overline{y_i D_i} \subseteq U$. Pick $a \in \bigcap_{i=1}^u D_i$. Then $A(a\vec{x}) = a\vec{y} \in U$.

(b). By part (a), $\widetilde{\varphi}[\beta(\mathbb{N}^v)] \cap K((\beta\mathbb{N})^u) \neq \emptyset$, so by [12, Theorem 1.65], $K(\widetilde{\varphi}[\beta(\mathbb{N}^v)]) = \widetilde{\varphi}[\beta(\mathbb{N}^v)] \cap K((\beta\mathbb{N})^u)$ (because $\widetilde{\varphi}[\beta(\mathbb{N}^v)] \subseteq (\beta\mathbb{N})^u$). Also by [12, Exercise 1.7.3], $K(\widetilde{\varphi}[\beta(\mathbb{N}^v)]) = \widetilde{\varphi}[K(\beta(\mathbb{N}^v))]$. □

Theorem 4.7 *Let A be a $u \times v$ matrix with entries from $\mathbb{Q}$ and assume that for all $\vec{x} \in \mathbb{N}^v$, every entry of $A\vec{x}$ is positive. If "large" is any of "IP*", "Δ*", or "central*", and C is large in $\mathbb{N}$, then $W = \{\vec{x} \in \mathbb{N}^v : A\vec{x} \in C^u\}$ is large in $\mathbb{N}^v$.*

Proof. We show first that it suffices to prove the theorem under the additional assumption that all entries of A are in $\mathbb{Z}$. Indeed, suppose we have done so and pick $d \in \mathbb{N}$ such that all entries of dA are in $\mathbb{Z}$. We claim that dC is large in $\mathbb{N}$, which we check individually.

Assume first that "large" is "IP*" and let a sequence $\langle x_n \rangle_{n=1}^\infty$ in $\mathbb{N}$ be given. By [12, Theorem 5.14 and Lemma 6.6] pick a sum subsystem $\langle y_n \rangle_{n=1}^\infty$ of $\langle x_n \rangle_{n=1}^\infty$ such that $d | y_n$ for each n. Pick $a \in C \cap FS(\langle \frac{y_n}{d} \rangle_{n=1}^\infty)$. Then $da \in dC \cap FS(\langle x_n \rangle_{n=1}^\infty)$.

Next assume that "large" is "Δ^*". Let a Δ set B in $\mathbb{N}$ be given and choose a sequence $\langle s_n\rangle_{n=1}^{\infty}$ such that for every $n, m \in \mathbb{N}$ with $n < m$, $s_m \in (s_n + B)$. (In particular, for each $n < m$, $s_n < s_m$.) By passing to a subsequence, we may presume that for each $n < m$, $s_n \equiv s_m \,(\mathrm{mod}\ d)$. Pick $i \in \{1, 2, \ldots, d\}$ such that for each $n \in \mathbb{N}$, $s_n + i \equiv 0 \,(\mathrm{mod}\ d)$. Then $\langle \frac{s_n+i}{d}\rangle_{n=1}^{\infty}$ is a sequence in $\mathbb{N}$ so pick $n < m$ such that $\frac{s_m+i}{d} - \frac{s_n+i}{d} \in C$. Then $s_m - s_n \in dC$.

Finally assume that "large" is "central*". Let p be a minimal idempotent in $\beta\mathbb{N}$. By Lemma 2.1, $\frac{1}{d}p$ is a minimal idempotent so $C \in \frac{1}{d}p$ and consequently $dC \in p$.

Since we have established that dC is large, we have that $\{\vec{x} \in \mathbb{N}^v : dA\vec{x} \in (dC)^u\}$ is large, and $\{\vec{x} \in \mathbb{N}^v : dA\vec{x} \in (dC)^u\} = \{\vec{x} \in \mathbb{N}^v : A\vec{x} \in C^u\}$.

Thus we assume that all entries of A are in $\mathbb{Z}$. Define $\varphi : \mathbb{N}^v \to \mathbb{N}^u$ by $\varphi(\vec{x}) = A\vec{x}$, and let $\widetilde{\varphi} : \beta(\mathbb{N}^v) \to (\beta\mathbb{N})^u$ be its continuous extension.

Assume first that "large" is "IP*". Let p be an idempotent in $\beta(\mathbb{N}^v)$. We need to show that $W \in p$. Since $\widetilde{\varphi}$ is a homomorphism, $\widetilde{\varphi}(p)$ is an idempotent in $(\beta\mathbb{N})^u$ and so $\overline{C}^u$ is a neighborhood of $\widetilde{\phi}(p)$, and hence $W \in p$ as required.

Next assume that "large" is "Δ^*". Let B be a Δ set in $\mathbb{N}^v$ and pick a sequence $\langle \vec{x}^{(n)}\rangle_{n=1}^{\infty}$ in $\mathbb{N}^v$ such that for every $n, m \in \mathbb{N}$ with $n < m$, $\vec{x}^{(m)} \in (\vec{x}^{(n)} + B)$. In particular, for each $n < m$, we have $\vec{x}^{(m)} - \vec{x}^{(n)} \in \mathbb{N}^v$. We need to show that there exists $n < m$ such that $\vec{x}^{(m)} - \vec{x}^{(n)} \in W$.

For each $n \in \mathbb{N}$, let $\vec{y}^{(n)} = A\vec{x}^{(n)}$. Notice that, for $n < m$ we have that all entries of $A(\vec{x}^{(m)} - \vec{x}^{(n)})$ are positive, and consequently each entry of $\vec{y}^{(m)}$ is larger than the corresponding entry of $\vec{y}^{(n)}$. By Ramsey's Theorem [14] (or see [8, Theorem 1.5] or [12, Theorem 18.2]), pick an infinite subset D_1 of $\mathbb{N}$ such that for all $n < m$ in D_1, $y_1^{(m)} - y_1^{(n)} \in C$ or for all $n < m$ in D_1, $y_1^{(m)} - y_1^{(n)} \in \mathbb{N}\backslash C$. Since C is a Δ^* set, the latter alternative is impossible, so the former must hold. Inductively, given $i \in \{1, 2, \ldots, u-1\}$, choose by Ramsey's Theorem an infinite subset D_{i+1} of D_i such that for all $n < m$ in D_{i+1}, $y_{i+1}^{(m)} - y_{i+1}^{(n)} \in C$. Having chosen D_u pick $n < m$ in D_u. Then $\vec{x}^{(m)} - \vec{x}^{(n)} \in W$.

Finally assume that "large" is "central*", and let p be a minimal idempotent in $\beta(\mathbb{N}^v)$. By Lemma 4.6, $\widetilde{\varphi}(p)$ is an idempotent and $\widetilde{\varphi}(p) \in (K(\beta\mathbb{N}))^u$ so that $\widetilde{\varphi}(p) \in \overline{C}^u$. $\square$

The requirement in Theorem 4.7 that for all $\vec{x} \in \mathbb{N}^v$, every entry of $A\vec{x}$ be positive may not be omitted. To see this, consider the 1×2 matrix $A = (\,1 \quad -1\,)$ and let $C = \mathbb{N}$. Then $W = \{\vec{x} \in \mathbb{N}^2 : A\vec{x} \in \mathbb{N}\} = \{\vec{x} \in \mathbb{N}^2 : x_1 > x_2\}$. It is routine to check that $\mathbb{N}^2\backslash W$ is thick and thus W is not syndetic, and hence is not central*, PS*, IP*, or Δ*. (The fact that W is not PS* tells us that this requirement cannot be eliminated from the following theorem either.)

Theorem 4.8 *Let A be a $u \times v$ matrix with entries from $\mathbb{Z}$ and assume that for all $\vec{x} \in \mathbb{N}^v$, every entry of $A\vec{x} \in \mathbb{N}^u$. If C is PS* in $\mathbb{N}$, then $W = \{\vec{x} \in \mathbb{N}^v : A\vec{x} \in C^u\}$ is PS* in $\mathbb{N}^v$.*

Proof. Define $\varphi : \mathbb{N}^v \to \mathbb{N}^u$ by $\varphi(\vec{x}) = A\vec{x}$, and let $\widetilde{\varphi} : \beta(\mathbb{N}^v) \to (\beta\mathbb{N})^u$ be its continuous extension. Let $p \in K(\beta(\mathbb{N}^v))$. By Lemma 4.6, $\widetilde{\varphi}(p) \in (K(\beta\mathbb{N}))^u$ and thus $\widetilde{\varphi}(p) \in \overline{C}^u$ and thus $W \in p$ as required. $\square$

The requirement that A have entries from $\mathbb{Z}$ cannot be reduced to a requirement that entries come from $\mathbb{Q}$ in Theorem 4.8, as can be seen by considering the 1×1 matrix $A = (\frac{1}{2})$ and $C = \mathbb{N}$. In this case one has $W = \{x \in \mathbb{N} : Ax \in \mathbb{N}\} = 2\mathbb{N}$ which is neither PS* nor thick. (The latter fact shows that this requirement cannot be eliminated from Theorem 4.10 as well.)

The following is another well known fact that we cannot find in [12].

Lemma 4.9 *Let $p \in K(\beta\mathbb{N})$. Then $\beta\mathbb{Z} + p = \beta\mathbb{N} + p$.*

Proof. By [12, Corollary 4.33], $\mathbb{N}^*$ is an ideal of $\beta\mathbb{N}$ so $K(\beta\mathbb{N}) \subseteq \mathbb{N}^*$. Therefore, by [12, Theorem 1.65], $K(\mathbb{N}^*) = K(\beta\mathbb{N})$. Since $p \in K(\mathbb{N}^*)$, $p \in \mathbb{N}^* + p$, so pick $q \in \mathbb{N}^*$ such that $p = q + p$. To see that $\beta\mathbb{Z} + p \subseteq \beta\mathbb{N} + p$, let $r \in \beta\mathbb{Z}$. By [12, Exercise 4.3.5], $\mathbb{N}^*$ is a left ideal of $\beta\mathbb{Z}$ so $r + q \in \beta\mathbb{N}$. Thus $r + p = r + q + p \in \beta\mathbb{N} + p$. $\square$

Theorem 4.10 *Let A be a $u \times v$ image partition regular matrix with entries from $\mathbb{Z}$. If C is thick in $\mathbb{N}$, then $W = \{\vec{x} \in \mathbb{N}^v : A\vec{x} \in C^u\}$ is thick in $\mathbb{N}^v$.*

Proof. Since C is thick, pick a left ideal L of $\beta\mathbb{N}$ such that $L \subseteq \overline{C}$. Pick by [12, Corollary 2.6] a minimal idempotent $p \in L$. Define $\varphi : \mathbb{N}^v \to \mathbb{Z}^u$ by $\varphi(\vec{x}) = A\vec{x}$, and let $\widetilde{\varphi} : \beta(\mathbb{N}^v) \to (\beta\mathbb{Z})^u$ be its continuous extension. Let $\overline{p} = (p, p, \ldots, p)^T$ and pick by Lemma 2.2 a minimal idempotent $q \in \beta(\mathbb{N}^v)$ such that $\widetilde{\varphi}(q) = \overline{p}$. (By Theorem 2.10($b$), p satisfies the hypotheses of Lemma 2.2.)

We claim that $\widetilde{\varphi}[\beta(\mathbb{N}^v) + q] \subseteq \overline{C}^u$ so that $\beta(\mathbb{N}^v) + q \subseteq \overline{W}$ as required. To this end, let $r \in \beta(\mathbb{N}^v)$ and let $i \in \{1, 2, \ldots, u\}$. Then $\pi_i \circ \widetilde{\varphi}(r + q) = \pi_i(\widetilde{\varphi}(r)) + p \in \beta\mathbb{Z} + p$. By Lemma 4.9, $\beta\mathbb{Z} + p = \beta\mathbb{N} + p \subseteq L \subseteq \overline{C}$. $\square$

The requirement in Theorem 4.10 that A be image partition regular cannot be reduced to requiring that its entries come from $\mathbb{Z}$ or even from $\mathbb{N}$ as can be seen by considering $A = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $C = \{2^{2n} + i : n \in \mathbb{N}$ and $i \in \{1, 2, \ldots, n\}\}$. Then C is thick in $\mathbb{N}$ and $\{x \in \mathbb{N} : Ax \in C^2\} = \emptyset$.

Finally, we observe that no analogous results are available for the notions "syndetic" or "piecewise syndetic" (considering $A = (2)$ and $C = 2\mathbb{N} + 1$) nor for the notions "IP" or "Δ" (considering $A = \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{pmatrix}$ and $C = FS(\langle 2^{2n} \rangle_{n=1}^{\infty})$.)

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