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Forbidden Distances in the Rationals and the Reals

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Abstract. Our main aim in this paper is to show that there is a partition of the reals into finitely many classes with "many" forbidden distances, in the following sense: for each positive real x there is a natural number n such that no two points in the same class are at distance x/n. In fact, more generally, given any infinite set $\{c_n : n < \omega\}$ of positive rationals, there is a partition of the reals into three classes such that for each positive real x there is some n such that no two points in the same class are at distance $c_n x$. This result is motivated by some questions in partition regularity.

1. Introduction

Let the reals be partitioned into finitely many classes: $\mathbb{R} = \bigcup_{i=1}^{k} C_i$. We say that a positive real x is a *forbidden distance* for this partition if, for each i, no two points of C_i are at distance x. Our main aim in this paper is to show that there exists a finite partition of the reals, in fact $\mathbb{R} = C_1 \cup C_2 \cup C_3$, such that there are many forbidden distances. More precisely, for each positive real x there is a positive integer n with x/n being a forbidden distance.

Note that such a partition certainly cannot be into measurable pieces, since any set S of positive measure has the property that all sufficiently small distances occur in S. (This is trivial if S is an open set, and measurable sets may be closely approximated by open sets. For the details of this well known fact, see almost any book on measure theory, or alternatively see the proof of Theorem 3.3.)

In contrast to measurability questions, our partition property makes perfect sense for the rationals instead of the reals, and indeed our first task will be to find such a partition for the rationals.

If we write $c_n = 1/n$, the above problem asks for a forbidden distance of the form $c_n x$, for each positive x. There is nothing special about this particular sequence. Indeed,

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the argument we give for $c_n = 1/n$ also works whenever c_n tends to zero. In fact, the result holds whenever $\{c_n : n < \omega\}$ is any infinite set of rationals (although the proof in the case that $\{c_n : n < \omega\}$ is unbounded is different).

Although these questions are natural in their own right, our main motivation for studying them comes from partition regularity, as we now describe. The reader who is not interested in Ramsey Theory may skip the rest of the Introduction.

Let A be an $m \times n$ matrix with rational entries. We say that A is *kernel partition regular* if for every finite coloring of the natural numbers there is a monochromatic vector $x \in \mathbb{N}^n$ with $Ax = 0$. In other words, A is kernel partition regular if for every positive integer k, and every function $c : \mathbb{N} \to \{1, 2, \ldots, k\}$, there is a vector $x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{N}^n$ with $c(x_1) = \ldots = c(x_n)$ such that $Ax = 0$. We may also speak of the 'system of equations $Ax = 0$' being kernel partition regular.

Many of the classical results of Ramsey Theory may naturally be considered as statements about kernel partition regularity. For example, Schur's Theorem [18], that in any finite coloring of the natural numbers we may solve $x + y = z$ in one color class, is precisely the assertion that the 1×3 matrix $\begin{pmatrix} 1 & 1 & -1 \end{pmatrix}$ is kernel partition regular.

The $m \times n$ kernel partition regular matrices were characterized by Rado [16] – see [9] or [3] for more information.

In the infinite case, things are far less well understood. If A is an infinite matrix, with rational entries and only finitely many non-zero entries in each row, we say as before that A is *kernel partition regular* if whenever $\mathbb{N}$ is finitely colored there is a monochromatic vector x with $Ax = 0$. See [10] for a discussion of the small amount that is known about infinite kernel partition regular matrices.

The above is just concerned with colorings of $\mathbb{N}$, so it is natural to ask how the notion of kernel partition regularity depends on the 'ambient space'. We say that a finite or infinite matrix A with rational entries is *kernel partition regular* over $\mathbb{Z}$ (respectively $\mathbb{Q}$, $\mathbb{R}$) if whenever the set $\mathbb{Z} \setminus \{0\}$ (respectively $\mathbb{Q} \setminus \{0\}$, $\mathbb{R} \setminus \{0\}$) is finitely colored there is a monochromatic vector x with $Ax = 0$. How do these notions differ from each other?

In the integers, nothing changes. Indeed, if a matrix is kernel partition regular (over $\mathbb{N}$) then it is certainly kernel partition regular over $\mathbb{Z}$. Conversely, if a matrix has a bad k-coloring over $\mathbb{N}$ then it has a bad $2k$-coloring over $\mathbb{Z}$: we just copy the coloring from the positive to the negative integers, but using a new set of k colors.

But for the rationals, we do not know what happens. In the finite case, the notions are the same (and remain the same for the reals as well – this was proved by Rado [17]), but for the infinite case the question is open. The question at the start of this paper, in the rationals, is the result of perhaps the most obvious way to build a system that is kernel partition regular over $\mathbb{Q}$ but not over $\mathbb{N}$. The system of equations is:

$$x_1 - y_1 = t\,,\; x_2 - y_2 = t/2\,,\; x_3 - y_3 = t/3, \dots .$$

This is clearly not kernel partition regular over $\mathbb{N}$, as there is not even a solution to these equations over $\mathbb{N}$. One might hope that it has sufficient 'freedom' that it would be kernel partition regular over $\mathbb{Q}$. However, if we ignore the color of t then it will be seen that we arrive precisely at the opening question of this paper: this is where the problem comes from.

Turning to the reals, it is known that there is a system that is kernel partition regular over $\mathbb{R}$ but not over $\mathbb{N}$ – this was proved in [10]. The system is given by:

$$x_1 - x_2 = y_1\,,\; x_2 - x_3 = y_2\,,\; x_3 - x_4 = y_3, \dots .$$

However, we feel that the system mentioned in the previous paragraph is very natural, and the question of whether or not it is kernel partition regular over $\mathbb{R}$ leads to the main result of the paper. Thus, for Ramsey Theory, the main consequence of the result at the start of this paper is that the system

$$x_1 - y_1 = t, x_2 - y_2 = t/2, x_3 - y_3 = t/3, \dots$$

is not kernel partition regular over $\mathbb{Q}$ or $\mathbb{R}$.

The plan of the paper is as follows. Our main results about the existence of bad colorings are proved in Section 2. Then, in Section 3, we turn our attention to colorings that are measurable, or have the property of Baire. It turns out that our partition regularity questions *do* have affirmative answers in these cases.

Let us say a brief word about the Continuum Hypothesis. As will be seen below, our problem is very closely related to some countable partitions of $\mathbb{R}$. Although we are trying to find a finite partition of $\mathbb{R}$, our problem certainly involves a countable partition of the positive reals, since for each x we must specify an n for which x/n will be a forbidden distance, and this corresponds to a countable partition: for each n we take the set of those x for which x/n is a forbidden distance. If these sets have linear dependence over $\mathbb{Q}$, then there are substantial problems to overcome (as the reader who examines the second half of Section 2 will realize). Now, it was proved by Erdős and Kakutani [6] that the reals may be partitioned into countably many independent sets if and only if the Continuum Hypothesis holds, so one might imagine that the Continuum

Hypothesis is somehow needed. But this, curiously, is not correct: we wish to stress that our proof does *not* assume the Continuum Hypothesis.

While on the subject of countable partitions, let us mention that our result does not seem to have anything to do with results like the beautiful theorem of Kunen [13] that (necessarily assuming the Continuum Hypothesis) there is a countable partition of $\mathbb{R}^n$ with each distance occurring at most once within any one cell. (From this it is easy to forbid any countable set of distances, such as the rational distances, within any cell simply by assigning troublesome points to their own cells.) One could say that Kunen's result has a far larger space but far fewer forbidden distances.

We write $\omega = \mathbb{N} \cup \{0\}$ for the first infinite ordinal, $\mathbb{R}^+ = \{x \in \mathbb{R} : x > 0\}$, and $\mathbb{Q}^+ = \{x \in \mathbb{Q} : x > 0\}$. Given a set X we write $\mathcal{P}_f(X)$ for the set of finite nonempty subsets of X. Also $\mathbb{T}$ will denote the unit circle $\mathbb{R}/\mathbb{Z}$ and $\pi : \mathbb{R} \to \mathbb{T}$ will denote the canonical homomorphism.

2. Forbidding Distances in $\mathbb{Q}$ and $\mathbb{R}$

We shall be concerned in this section with the following problem. Let $\{c_n : n < \omega\}$ be a set of positive rationals and let $\mathbb{R} \setminus \{0\}$ be finitely colored. Must there exist z and sequences $\langle x_n \rangle_{n<\omega}$ and $\langle y_n \rangle_{n<\omega}$ such that $\{z\} \cup \{x_n : n < \omega\} \cup \{y_n : n < \omega\}$ is monochrome and for each $n < \omega$, $x_n - y_n = c_n z$? Equivalently, setting

$$A = \begin{pmatrix} c_0 & 1 & -1 & 0 & 0 & 0 & 0 & \ldots \\ c_1 & 0 & 0 & 1 & -1 & 0 & 0 & \ldots \\ c_2 & 0 & 0 & 0 & 0 & 1 & -1 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

is A kernel partition regular over $\mathbb{R} \setminus \{0\}$? Notice that if $\{c_n : n \in \omega\}$ is finite, then A is certainly kernel partition regular over $\mathbb{N}$ (for example, because its rows are the rows of a finite matrix that has Rado's columns property – see [16]).

We shall see in Theorem 2.9 that, with this trivial exception, the answer is "no" even for three colors and even without the requirement that z be the same color as the x_n's and y_n's.

Of course the corresponding assertion about colorings of $\mathbb{Q} \setminus \{0\}$ is an immediate consequence of this result. However, we shall present the proof for $\mathbb{Q}$ separately because it is significantly simpler, while containing some of the ideas we shall need later on. We write $\overline{C}$ for the closure of a set C.

2.1 Lemma. *Let $\langle F_n \rangle_{n<\omega}$ be a sequence in $\mathcal{P}_f(\mathbb{Q} \setminus \{0\})$, let $\{c_n : n < \omega\}$ be an infinite subset of $\mathbb{Q}^+$ and, for each $n \in \omega$, let (α_n, β_n) be a non-empty open interval in $[0, 1]$.*

Then there is a function $\gamma : \omega \to \omega$ such that for all $n \in \omega$,

(†) *for all $z_0 \in F_0, z_1 \in F_1, \ldots, z_n \in F_n$ and all $a_0, a_1, \ldots, a_n \in \mathbb{R}$,*
$(\exists s \in \mathbb{R})(\forall t \in \{0, 1, \ldots, n\})\big(\pi(a_t + s \cdot c_{\gamma(t)} \cdot z_t) \in \pi[(\alpha_t, \beta_t)]\big)$.

Furthermore, if $\{c_n : n < \omega\}$ is bounded, then (†) holds with $s \in \mathbb{N}$ while if $\{c_n : n < \omega\}$ is unbounded, then (†) holds with $0 < s < 1$.

Proof. Case 1. $\{c_n : n < \omega\}$ is bounded. We define the denominator $d(q)$ of $q \in \mathbb{Q}$ to be the smallest positive integer such that $q \cdot d(q) \in \mathbb{Z}$. We observe that

(∗) if $a, \alpha, \beta \in \mathbb{R}$ with $\alpha < \beta$, $y \in \mathbb{Q} \setminus \{0\}$, and $d(y) > \frac{1}{\beta - \alpha}$, then there exists $s \in \mathbb{N}$ such that $\pi(a + sy) \in \pi[(\alpha, \beta)]$.

To see this, let $y = \frac{k}{b}$ where $k \in \mathbb{Z}$ and $b = d(y)$. Then $\frac{1}{b} < \beta - \alpha$ so pick $m \in \mathbb{N}$ such that $\pi(a + \frac{m}{b}) \in \pi[(\alpha, \beta)]$. Since k and b are relatively prime we may pick $s \in \{1, 2, \ldots, b\}$ and $t \in \mathbb{Z}$ such that $k \cdot s = m + b \cdot t$. (See [12, Theorem 57].) Then $a + s \cdot y = a + \frac{m}{b} + t$ so $\pi(a + s \cdot y) = \pi(a + \frac{m}{b})$.

Let I be a bounded real interval such that $c_n \in I$ for every $n \in \omega$. We observe that, for any $k \in \mathbb{N}$ and any $z \in \mathbb{Q} \setminus \{0\}$, $\{q \in I \cap \mathbb{Q} : d(q \cdot z) \leq k\}$ is finite. Thus we can choose $\gamma(0) \in \omega$ such that $d(c_{\gamma(0)} \cdot z) > \frac{1}{\beta_0 - \alpha_0}$ for every $z \in F_0$. Then by (∗), for each $z \in F_0$ and each $a_0 \in \mathbb{R}$, there exists $s \in \mathbb{N}$ such that $\pi(a_0 + s \cdot c_{\gamma(0)} \cdot z) \in \pi[(\alpha_0, \beta_0)]$.

Now suppose that $n \in \omega$ and that we have defined $\gamma(t)$ for $t \in \{0, 1, \ldots, n\}$ so that our claim holds for n. We can choose $u \in \mathbb{N}$ so that $u \cdot c_{\gamma(t)} \cdot z \in \mathbb{Z}$ for every $t \in \{0, 1, \ldots, n\}$ and every $z \in F_t$. We can then choose $\gamma(n+1) \in \omega$ so that $d(u \cdot c_{\gamma(n+1)} \cdot z) > \frac{1}{\beta_{n+1} - \alpha_{n+1}}$ for every $z \in F_{n+1}$.

Let $z_0 \in F_0, z_1 \in F_1, \ldots, z_{n+1} \in F_{n+1}$ and let $a_0, a_1, \ldots, a_{n+1} \in \mathbb{R}$. By our inductive assumption, pick $r \in \mathbb{N}$ so that, for every $t \in \{0, 1, \ldots, n\}$, we have $\pi(a_t + r \cdot c_{\gamma(t)} \cdot z_t) \in \pi[(\alpha_t, \beta_t)]$. We can then choose by (∗) (with $a = a_{n+1} + r \cdot c_{\gamma(n+1)} \cdot z_{n+1}$ and $y = u \cdot c_{\gamma(n+1)} \cdot z_{n+1}$) some $v \in \mathbb{N}$ such that

$$\pi(a_{n+1} + r \cdot c_{\gamma(n+1)} \cdot z_{n+1} + v \cdot u \cdot c_{\gamma(n+1)} \cdot z_{n+1}) \in \pi[(\alpha_{n+1}, \beta_{n+1})]\,.$$

If $s = r + v \cdot u$, we have $\pi(a_t + s \cdot c_{\gamma(t)} \cdot z_t) \in \pi[(\alpha_m, \beta_m)]$ for every $t \in \{0, 1, \ldots, n+1\}$.

Case 2. $\{c_n : n < \omega\}$ is unbounded. We first choose $\gamma(0)$ so that $|\gamma(0) \cdot z| > 1$ for every $z \in F_0$. Then (†) holds for $n = 0$ with $s \in (0, 1)$, because, for every $a \in \mathbb{R}$ and every $z \in F_0$, the mapping $s \mapsto \pi(a + s \cdot c_{\gamma(0)} \cdot z)$ from $(0, 1)$ to $\mathbb{T}$ is surjective.

We then assume that $n \in \omega$ and that $\gamma(0), \gamma(1), \ldots, \gamma(n)$ have been chosen so that (†) holds for n with $s \in (0, 1)$. For $z_0 \in F_0, z_1 \in F_1, \ldots, z_n \in F_n$ let $\vec{z} = (z_0, z_1, \ldots, z_n)$

and define $\psi_{\vec{z}} : [0,1]^{n+1} \times (0,1) \to \mathbb{T}^{n+1}$ by $\big(\psi_{\vec{z}}(\vec{b},s)\big)_t = \pi(b_t + s \cdot c_{\gamma(t)} \cdot z_t)$ for $\vec{b} \in [0,1]^{n+1}$, $s \in (0,1)$, and $t \in \{0,1,\ldots,n\}$. Then $\psi_{\vec{z}}$ is continuous. Further, given $\vec{b} \in [0,1]^{n+1}$ we have by assumption some $s \in (0,1)$ such that $\psi_{\vec{z}}(\vec{b},s) \in \prod_{t=0}^{n} \pi[(\alpha_t,\beta_t)]$. So pick a neighborhood $U_{\vec{b},\vec{z}}$ of $\vec{b}$ in $[0,1]^{n+1}$ and a non-empty open interval $I_{\vec{b},\vec{z}} \subseteq (0,1)$ such that

$$(\forall t \in \{0,1,\ldots,n\})\big(\pi(a_t + s \cdot c_{\gamma(t)} \cdot z_t) \in \pi[(\alpha_t,\beta_t)]\big)$$

whenever $\vec{a} = \langle a_0, a_1, \cdots, a_n\rangle \in U_{\vec{b},\vec{z}}$ and $s \in I_{\vec{b},\vec{z}}$.

For each $\vec{z} \in \prod_{t=0}^{n} F_t$, there is a finite subset $H_{\vec{z}}$ of $[0,1]^{n+1}$ such that $[0,1]^{n+1} \subseteq \bigcup_{\vec{b}\in H_{\vec{z}}} U_{\vec{b},\vec{z}}$. We choose $\gamma(n+1)$ so that the interval $c_{\gamma(n+1)} \cdot I_{\vec{b},\vec{z}} \cdot v$ has length greater than 1 whenever $\vec{z} \in \prod_{t=0}^{n} F_t$, $\vec{b} \in H_{\vec{z}}$, and $v \in F_{n+1}$.

Now let $a_t \in [0,1]$ and $z_t \in F_t$ for each $t \in \{0,1,\ldots,n+1\}$. Put $\vec{a} = \langle a_0, a_1, \cdots, a_n\rangle$ and $\vec{z} = \langle z_0, z_1, \cdots, z_n\rangle$, and choose $\vec{b} \in H_{\vec{z}}$ such that $\vec{a} \in U_{\vec{b},\vec{z}}$. Since the mapping $s \mapsto \pi(a_{n+1} + s \cdot c_{\gamma(n+1)} \cdot z_{n+1})$ maps $I_{\vec{b},\vec{z}}$ onto $\mathbb{T}$, there exists $s \in I_{\vec{b},\vec{z}}$ such that $\pi(a_{n+1} + s \cdot c_{\gamma(n+1)} \cdot z_{n+1}) \in \pi\,[(\alpha_{n+1},\beta_{n+1})]$. This shows that (†) holds for $n+1$. Thus we can define γ inductively so that (†) holds for every $n \in \omega$. $\square$

2.2 Lemma. *Assume that $\{c_n : n < \omega\}$ is an infinite subset of $\mathbb{Q}^+$. Then there is a function $\delta : \mathbb{Q}^+ \to \omega$ such that for every $F \in \mathcal{P}_f(\mathbb{Q}^+)$ there is a function $\varphi : \mathbb{Q} \to \{0,1,2\}$ such that for all $x, y \in \mathbb{Q}$ and all $z \in F$, if $\varphi(x) = \varphi(y)$, then $x - y \neq c_{\delta(z)} \cdot z$.*

Proof. Enumerate $\mathbb{Q}^+$ as $\langle z_n\rangle_{n<\omega}$. For each $n < \omega$, let $F_n = \{z_n\}$, let $\alpha_n = \frac{1}{3}$, and let $\beta_n = \frac{2}{3}$. Pick $\gamma : \omega \to \omega$ as guaranteed by Lemma 2.1, and define $\delta(z_n) = \gamma(n)$.

Now let $F \in \mathcal{P}_f(\mathbb{Q}^+)$ be given and choose $n \in \omega$ such that $F \subseteq \{z_0, z_1, \ldots, z_n\}$. By Lemma 2.1, we may choose $s \in \mathbb{R}$ such that $\pi(s \cdot c_{\delta(z)} \cdot z) \in \pi[(\frac{1}{3}, \frac{2}{3})]$ for $z \in F$.

Define $\varphi : \mathbb{Q} \to \{0,1,2\}$ by $\varphi(x) = i \in \{0,1,2\}$ if and only if $\pi(s \cdot x) \in \pi\big[[\frac{i}{3}, \frac{i+1}{3})\big]$. If $\varphi(x) = \varphi(y)$, then $\pi\big(s \cdot (x-y)\big) \notin \pi[(\frac{1}{3}, \frac{2}{3})]$. Thus, if $z \in F$, $x - y \neq c_{\delta(z)} \cdot z$. $\square$

Now a simple compactness proof establishes that the system of equations $x_n - y_n = c_n z$ is not partition regular. In fact, for each z there is some n such that the distance $c_n z$ does not occur in any of the colors. In the proof given here, since $\mathcal{P}_f(\mathbb{Q}^+)$ is countable, we could use a sequence of functions rather than the more general net. However, we will use the same argument for $\mathbb{R}$, and don't want to write it down twice.

2.3 Theorem. *Assume that $\{c_n : n < \omega\}$ is an infinite subset of $\mathbb{Q}^+$. Then there exists $\psi : \mathbb{Q} \to \{0,1,2\}$ such that for each $z \in \mathbb{Q}^+$ there exists $n \in \omega$ so that there do not exist $x, y \in \mathbb{Q}$ with $x - y = c_n z$ and $\psi(x) = \psi(y)$.*

Proof. Pick $\delta : \mathbb{Q}^+ \to \omega$ as guaranteed by Lemma 2.2 and for each $F \in \mathcal{P}_f(\mathbb{Q}^+)$ choose a function $\varphi_F : \mathbb{Q} \to \{0,1,2\}$ such that for all $x, y \in \mathbb{Q}$ and all $z \in F$, if $\varphi_F(x) = \varphi_F(y)$, then $x - y \neq c_{\delta(z)} \cdot z$.

Now $\mathcal{P}_f(\mathbb{Q}^+)$ is directed by $\subseteq$ so $\langle \varphi_F \rangle_{\mathcal{P}_f(\mathbb{Q}^+)}$ is a net in the compact product space $\{0,1,2\}^{\mathbb{Q}}$. Pick a cluster point ψ of $\langle \varphi_F \rangle_{\mathcal{P}_f(\mathbb{Q}^+)}$. Let $z \in \mathbb{Q}^+$ and let $n = \delta(z)$. Suppose that one has $x, y \in \mathbb{Q}$ with $x - y = c_n z$ and $\psi(x) = \psi(y)$. Let $U = \{\tau \in \{0,1,2\}^{\mathbb{Q}} : \tau(x) = \psi(x)$ and $\tau(y) = \psi(y)\}$. Then U is a neighborhood of ψ so pick $F \in \mathcal{P}_f(\mathbb{Q}^+)$ such that $z \in F$ and $\varphi_F \in U$. Then $\varphi_F(x) = \varphi_F(y)$ and $x - y = c_{\delta(z)} \cdot z$, a contradiction. $\square$

We remark that the 3 colors used in the proof of Theorem 2.3 are minimal. That is, if $c_n = \dfrac{1}{2^n}$ or $c_n = \dfrac{2^n - 1}{2^n}$, then whenever $\mathbb{Q}^+$ is 2-colored there exist $z \in \mathbb{Q}^+$ and sequences $\langle x_n \rangle_{n<\omega}$ and $\langle y_n \rangle_{n<\omega}$ in $\mathbb{Q}^+$ such that $\{z\} \cup \{x_n : n < \omega\} \cup \{y_n : n < \omega\}$ is monochrome and $x_n - y_n = c_n \cdot z$ for every $n < \omega$. The proof of this assertion is a tedious and not very enlightening case analysis, so we omit it.

Before we embark upon the proof of Theorem 2.9, we would like to digress to show that the result does not remain true when we replace $\mathbb{R}$ by a rational vector space larger than $\mathbb{R}$. We urge the reader not to just skip this, however, as it helps to explain what may otherwise seem like some very unmotivated constructions in the proof of Theorem 2.9. To be precise, several of the partitions into complete multipartite hypergraphs in the proof of Theorem 2.9 can be viewed as rather natural, if one is striving to avoid the kind of situation encountered in the proof of Theorem 2.4.

In the proof of Theorem 2.4 we shall use the result of Erdős that the partition relation $\mathfrak{c}^+ \to (\omega_1)^2_\omega$ holds ([5] or see [7]). That is, if the two-element subsets $[V]^2$ of a set V larger than $\mathbb{R}$ are colored with countably many colors, then there is an uncountable set $W \subseteq V$ such that all pairs from W are monochrome.

We see in the following result that if $|V| > |\mathbb{R}|$, then not only cannot one get a 3-coloring as in Theorems 2.3 and 2.9, one cannot even get such an ω-coloring.

2.4 Theorem. *Let V be a vector space over $\mathbb{Q}$ with $|V| > |\mathbb{R}|$. Then for any $\psi : V \to \omega$, there exists $z \in V \backslash \{0\}$ such that for every $\delta \in \mathbb{Q}^+$ there exist $x \neq y$ in V such that $x - y = z \cdot \delta$ and $\psi(x) = \psi(y)$.*

Proof. Let $\psi : V \to \omega$ and suppose that for each $z \in V \backslash \{0\}$ there exists $\tau(z) \in \mathbb{Q}^+$ such that for all $x \neq y$ in V, if $\psi(x) = \psi(y)$, then $x - y \neq z \cdot \tau(z)$. Let γ be a function from $[V]^2$ to $\mathbb{Q}^+$ such that for all $x \neq y$ in V, $\gamma(\{x, y\}) \in \{\tau(x - y), \tau(y - x)\}$. Then, since $\mathfrak{c}^+ \to (\omega_1)^2_\omega$, pick an uncountable set $W \subseteq V$ and some $\delta \in \mathbb{Q}^+$ such that for all $x \neq y \in W$, $\gamma(\{x,y\}) = \delta$. Since W is uncountable, pick $x \neq y$ in W such that $\psi(x \cdot \delta) = \psi(y \cdot \delta)$. Assume without loss of generality that $\delta = \tau(x-y)$ and let $z = x-y$. Then $x \cdot \delta - y \cdot \delta = z \cdot \delta = z \cdot \tau(z)$, a contradiction. $\square$

For the proof of Theorem 2.9 we need to fix a small amount of notation.

2.5 Definition. Fix a Hamel basis $\langle e_\delta \rangle_{\delta \in \mathbb{R}}$ for $\mathbb{R}$ over $\mathbb{Q}$. For $x, \delta \in \mathbb{R}$, let $r(x,\delta) \in \mathbb{Q}$ be the e_δ-coefficient in the expansion of x. (So $x = \sum_{\delta \in \mathbb{R}} r(x,\delta) \cdot e_\delta$.) For $x \in \mathbb{R}$, let $\mathrm{supp}(x) = \{\delta \in \mathbb{R} : r(x,\delta) \neq 0\}$.

For $k \in \mathbb{N}$, let $\mathcal{I}_k = \{\vec{\delta} \in \mathbb{R}^k : \delta_1 < \delta_2 < \ldots < \delta_k\}$. For $k \in \mathbb{N}$ and $\vec{\delta} \in \mathcal{I}_k$, let $\tau(\vec{\delta}) = \big(a_0, (a_1, \ldots, a_k)\big) \in \omega \times \mathbb{Z}^k$, where $a_0 = \min\{t \in \omega : \lfloor 2^t \delta_1 \rfloor < \lfloor 2^t \delta_2 \rfloor < \ldots < \lfloor 2^t \delta_k \rfloor\}$ and for $i \in \{1, 2, \ldots, k\}$, $a_i = \lfloor 2^{a_0} \delta_i \rfloor$.

Let $\mathcal{V} = \{(\vec{z}, \vec{a}) : (\exists k \in \mathbb{N})(\vec{z} \in (\mathbb{Q} \backslash \{0\})^k \text{ and } \vec{a} \in \tau[\mathcal{I}_k])\}$. Enumerate $\mathcal{V}$ as $\langle \vec{v_n} \rangle_{n=0}^\infty$ and for each $n \in \omega$, let $k(n)$ be the member of $\mathbb{N}$ such that $\vec{v_n} = (\vec{z}, \vec{a})$ with $\vec{z} \in (\mathbb{Q} \backslash \{0\})^{k(n)}$ and $\vec{a} \in \tau[\mathcal{I}_{k(n)}]$.

For $n \in \omega$, let $\xi_n = \dfrac{1}{6 \cdot (2k(n)-1)}$, let $(\vec{z}, \vec{a}) = \vec{v_n}$, let

$$T_n = \{\textstyle\sum_{i=1}^{k(n)} z_i \cdot e_{\delta_i} : \vec{\delta} \in \mathcal{I}_{k(n)} \text{ and } \vec{a} = \tau(\vec{\delta})\}$$

and let

$$S_n = \{\textstyle\sum_{i=1}^{k(n)} z_i \cdot e_{\delta_i} : \begin{array}{l} \vec{\delta} \in \mathcal{I}_{k(n)}\,,\ \vec{a} = \tau(\vec{\delta}), \text{ and} \\ |\{i \in \{1, 2, \ldots, k(n)\} : a_i \neq 2^{a_0} \cdot \delta_i\}| \leq 1\}\,. \end{array}$$

Notice that for each $x \in \mathbb{R} \backslash \{0\}$ there is a unique $n \in \omega$ such that $x \in T_n$. Observe also that if $k(n) = 1$, then $T_n = S_n$. And observe that for $x, y, \delta \in \mathbb{R}$ and $\alpha, \beta \in \mathbb{Q}^+$ one has $r(\alpha \cdot x + \beta \cdot y, \delta) = \alpha \cdot r(x,\delta) + \beta \cdot r(y,\delta)$.

2.6 Lemma. *Let $n \in \omega$ and let $x \in T_n$. Then there exist $u_1, u_2, \ldots, u_{k(n)}, v \in S_n$ such that $x = \sum_{i=1}^{k(n)} u_i - (k(n)-1) \cdot v$.*

Proof. Assume $\vec{v_n} = (\vec{z}, \vec{a})$ and pick $\vec{\delta} \in \mathcal{I}_{k(n)}$ such that $x = \sum_{i=1}^{k(n)} z_i \cdot e_{\delta_i}$ and $\vec{a} = \tau(\vec{\delta})$. For each $i \in \{1, 2, \ldots, k(n)\}$ let $b_i = \dfrac{a_i}{2^{a_0}}$ and let $v = \sum_{i=1}^{k(n)} z_i \cdot e_{b_i}$. For each $i \in \{1, 2, \ldots, k(n)\}$, let $u_i = v - z_i \cdot e_{b_i} + z_i \cdot e_{\delta_i}$. Then one has immediately that $x = \sum_{i=1}^{k(n)} u_i - (k(n)-1) \cdot v$. To see that $\{u_1, u_2, \ldots, u_{k(n)}, v\} \subseteq S_n$ we need to show that $\vec{a} = \tau(b_1, b_2, \ldots, b_{k(n)})$ and for $i \in \{1, 2, \ldots, k(n)\}$, $\vec{a} = \tau(b_1, b_2, \ldots, b_{i-1}, \delta_i, b_{i+1}, \ldots, b_{k(n)})$. This in turn requires that $a_0 = \min\{t \in \omega : \lfloor 2^t b_1 \rfloor < \lfloor 2^t b_2 \rfloor < \ldots < \lfloor 2^t b_{k(n)} \rfloor\}$ and for $i \in \{1, 2, \ldots, k\}$, $a_0 = \min\{t \in \omega : \lfloor 2^t b_1 \rfloor < \lfloor 2^t b_2 \rfloor < \ldots \lfloor 2^t b_{i-1} \rfloor < \lfloor 2^t \delta_i \rfloor < \lfloor 2^t b_{i+1} \rfloor < \ldots < \lfloor 2^t b_{k(n)} \rfloor\}$. If $a_0 = 0$, this is immediate. Otherwise we know that some j has

$\lfloor 2^{a_0-1}\delta_j\rfloor = \lfloor 2^{a_0-1}\delta_{j+1}\rfloor$ and thus, if $l = \lfloor 2^{a_0-1}\delta_j\rfloor$, one has $2l = a_j$ and $2l+1 = a_{j+1}$ so that $l = \lfloor 2^{a_0-1}b_j\rfloor = \lfloor 2^{a_0-1}b_{j+1}\rfloor$. $\square$

2.7 Lemma. *Assume that $\{c_n : n < \omega\}$ is an infinite subset of $\mathbb{Q}^+$. Then there is a function $\gamma : \omega \to \omega$ such that whenever H is a finite nonempty subset of $\bigcup_{n=0}^{\infty} S_n$ and $D = \bigcup_{y\in H} \mathrm{supp}(y)$, there exist $\langle s_\delta\rangle_{\delta\in D}$ in $\mathbb{R}$ such that for each $y \in H$, if $y \in S_n$, then $\pi\big(\sum_{\delta\in D} s_\delta \cdot c_{\gamma(n)} \cdot r(y,\delta)\big) \in \pi[(\frac{1}{2} - \xi_n, \frac{1}{2} + \xi_n)]$.*

Proof. For each $n < \omega$, let $F_n = \{z_1, z_2, \ldots, z_{k(n)}\}$, where $\vec{v_n} = (\vec{z}, \vec{a})$. Choose $\gamma : \omega \to \omega$ as guaranteed by Lemma 2.1, where for each n, $\alpha_n = \frac{1}{2} - \xi_n$ and $\beta_n = \frac{1}{2} + \xi_n$.

Let H be a finite nonempty subset of $\bigcup_{n=0}^{\infty} S_n$ and let $D = \bigcup_{y\in H} \mathrm{supp}(y)$. Order D as $\delta_1, \delta_2, \ldots, \delta_p$ so that if $t \in \omega$, $i, j \in \{1, 2, \ldots, p\}$, $\delta_i \cdot 2^t \in \mathbb{Z}$, and $\delta_j \cdot 2^t \notin \mathbb{Z}$, then $i < j$. (That is, integers first, then odd integers over 2, etc., finishing with non dyadic reals.) We choose $s_{\delta_1}, s_{\delta_2}, \ldots, s_{\delta_p}$ in order so that if $n < \omega$, $y \in H \cap S_n$, $t \in \{1, 2, \ldots, p\}$, and $\mathrm{supp}(y) \subseteq \{\delta_1, \delta_2, \ldots, \delta_t\}$, then $\pi\big(\sum_{j=1}^{t} s_{\delta_j} \cdot c_{\gamma(n)} \cdot r(y, \delta_j)\big) \in \pi[(\frac{1}{2} - \xi_n, \frac{1}{2} + \xi_n)]$.

For $t \in \{1, 2, \ldots, p\}$, let $H_t = \{y \in H : \max\{i : \delta_i \in \mathrm{supp}(y)\} = t\}$. We claim that for each $t \in \{1, 2, \ldots, p\}$ and each $n \in \omega$, $|H_t \cap S_n| \leq 1$. If $t = 1$ and $\vec{v_n} = (\vec{z}, \vec{a})$, then $H_t \cap S_n \subseteq \{z_1 \cdot e_{\delta_1}\}$, so assume that $t > 1$. Let $n \in \omega$ and assume that $y \in H_t \cap S_n$. Let $(\vec{z}, \vec{a}) = \vec{v_n}$. Then we have some $\eta_1 < \eta_2 < \ldots < \eta_{k(n)}$ such that $y = \sum_{i=1}^{k(n)} z_i \cdot e_{\eta_i}$, $\vec{a} = \tau(\vec{\eta})$, and $|\{i \in \{1, 2, \ldots, k(n)\} : a_i \neq 2^{a_0} \cdot \eta_i\}| \leq 1$. Also, $\{\eta_1, \eta_2, \ldots, \eta_{k(n)}\} \subseteq \{\delta_1, \delta_2, \ldots, \delta_t\}$ and $\delta_t \in \{\eta_1, \eta_2, \ldots, \eta_{k(n)}\}$.

If for each $i \in \{1, 2, \ldots, k(n)\}$, $a_i = 2^{a_0} \cdot \eta_i$, then $2^{a_0} \cdot \delta_t \in \mathbb{Z}$. If there is some $j \in \{1, 2, \ldots, k(n)\}$ such that $a_j \neq 2^{a_0} \cdot \eta_j$, then $2^{a_0} \cdot \eta_j \notin \mathbb{Z}$ while for $i \neq j$, $2^{a_0} \cdot \eta_i \in \mathbb{Z}$. Because of the ordering attached to the δ_i's, one must have that this $\eta_j = \delta_t$. Thus we have, letting $b_i = \dfrac{a_i}{2^{a_0}}$,

(1) if $2^{a_0} \cdot \delta_t \in \mathbb{Z}$, then $H_t \cap S_n \subseteq \{\sum_{i=1}^{k(n)} z_i \cdot e_{b_i}\}$;

(2) if $2^{a_0} \cdot \delta_t \notin \mathbb{Z}$ and $\lfloor 2^{a_0} \cdot \delta_t\rfloor \notin \{a_1, a_2, \ldots, a_{k(n)}\}$, then $H_t \cap S_n = \emptyset$; and

(3) if $2^{a_0} \cdot \delta_t \notin \mathbb{Z}$ and $\lfloor 2^{a_0} \cdot \delta_t\rfloor = a_j$, then $H_t \cap S_n \subseteq \{\sum_{i=1}^{k(n)} z_i \cdot e_{b_i} - z_j \cdot e_{b_j} + z_j \cdot e_{\delta_t}\}$.

Therefore in any event $|H_t \cap S_n| \leq 1$ as claimed.

Let $t \in \{1, 2, \ldots, p\}$ and assume that s_{δ_i} has been chosen for each $i \in \{1, 2, \ldots, t-1\}$. Enumerate H_t as $\langle y_i\rangle_{i=1}^{l}$ where $y_i \in S_{n_i}$ and $n_1 < n_2 < \ldots < n_l$. For $i \in \{1, 2, \ldots, l\}$, let $a_{n_i} = \sum_{j=1}^{t-1} s_{\delta_j} \cdot c_{\gamma(n_i)} \cdot r(y_i, \delta_j)$. (If $t = 1$, let $a_{n,i} = 0$.) Given $i \in \{1, 2, \ldots, l\}$, we have that $r(y_i, \delta_t) \in F_{n_i}$. So by Lemma 2.1, pick s_{δ_t} such that

$$(\forall i \in \{1, 2, \ldots, l\})\big(\pi\big(a_{n_i} + s_{\delta_t} \cdot c_{\gamma(n_i)} \cdot r(y_i, \delta_t)\big) \in \pi[(\tfrac{1}{2} - \xi_{n_i} \tfrac{1}{2} + \xi_{n_i})]\big)\,. \qquad \square$$

2.8 Lemma. *Assume that $\{c_n : n < \omega\}$ is an infinite subset of $\mathbb{Q}^+$. Then there is a function $\mu : \mathbb{R}^+ \to \omega$ such that for every $F \in \mathcal{P}_f(\mathbb{R}^+)$ there is a function $\varphi : \mathbb{R} \to \{0,1,2\}$ such that for all $x, y \in \mathbb{R}$ and all $z \in F$, if $\varphi(x) = \varphi(y)$, then $x - y \neq c_{\mu(z)} \cdot z$.*

Proof. Pick $\gamma : \omega \to \omega$ as guaranteed by Lemma 2.7. For each $z \in \mathbb{R}^+$ pick the $n(z) \in \omega$ such that $z \in T_{n(z)}$ and define $\mu(z) = \gamma\big(n(z)\big)$.

Let $F \in \mathcal{P}_f(\mathbb{R}^+)$ be given. For each $z \in F$ pick by Lemma 2.6 members $u(z,1)$, $u(z,2), \ldots, u\Big(z, k\big(n(z)\big)\Big)$ and $v(z)$ of $S_{n(z)}$ such that

$$z = \textstyle\sum_{i=1}^{k(n(z))} u(z,i) - \big(k\big(n(z)\big) - 1\big) \cdot v(z)\,.$$

Let $H = \bigcup_{z \in F} \big(\{v(z)\} \cup \{u(z,i) : i \in \{1,2,\ldots,k(n(z))\}\}\big)$ and let $D = \bigcup_{y \in H} \mathrm{supp}(y)$. Pick $\langle s_\delta \rangle_{\delta \in D}$ as guaranteed by Lemma 2.7.

Define $\varphi : \mathbb{R} \to \{0,1,2\}$ by $\varphi(y) = i$ if and only if $\pi\big(\sum_{\delta \in D} s_\delta \cdot r(y,\delta)\big) \in \pi\big[[\frac{i}{3}, \frac{i+1}{3})\big]$. Let $x, y \in \mathbb{R}$, let $z \in F$, and assume that $\varphi(x) = \varphi(y) = i$. Find $m, l \in \mathbb{Z}$ such that $m + \frac{i}{3} \leq \sum_{\delta \in D} s_\delta \cdot r(x,\delta) < m + \frac{i+1}{3}$ and $l + \frac{i}{3} \leq \sum_{\delta \in D} s_\delta \cdot r(y,\delta) < l + \frac{i+1}{3}$. Then

$$(*) \qquad m - l - \tfrac{1}{3} < \textstyle\sum_{\delta \in D} s_\delta \cdot r(x - y, \delta) < m - l + \tfrac{1}{3}\,.$$

Let $n = n(z)$ and for each $i \in \{1,2,\ldots,k(n)\}$ pick $p_i \in \mathbb{Z}$ such that

$$p_i + \tfrac{1}{2} - \xi_n < \textstyle\sum_{\delta \in D} s_\delta \cdot c_{\gamma(n)} \cdot r(u(z,i),\delta) < p_i + \tfrac{1}{2} + \xi_n\,.$$

Pick $q \in \mathbb{Z}$ such that $q + \frac{1}{2} - \xi_n < \sum_{\delta \in D} s_\delta \cdot c_{\gamma(n)} \cdot r(v(z),\delta) < q + \frac{1}{2} + \xi_n$. Now $c_{\mu(z)} \cdot z = c_{\gamma(n)} \cdot \big(\sum_{i=1}^{k(n)} u(z,i) - (k(n) - 1) \cdot v(z)\big)$ so for each $\delta \in D$, $r(c_{\mu(z)} \cdot z, \delta) = \sum_{i=1}^{k(n)} c_{\gamma(n)} \cdot r(u(z,i),\delta) - (k(n) - 1) \cdot c_{\gamma(n)} \cdot r(v(z),\delta)$. Thus

$$\begin{aligned}\textstyle\sum_{i=1}^{k(n)} p_i - (k(n) - 1) \cdot q + \tfrac{1}{2} - \xi_n \cdot (2k(n) - 1) &< \textstyle\sum_{\delta \in D} s_\delta \cdot r(c_{\mu(z)} \cdot z, \delta) \\ &< \textstyle\sum_{i=1}^{k(n)} p_i - (k(n) - 1) \cdot q + \tfrac{1}{2} + \xi_n \cdot (2k(n) - 1)\,.\end{aligned}$$

Thus, letting $t = \sum_{i=1}^{k(n)} p_i - (k(n) - 1) \cdot q$ and noting that $\xi_n \cdot (2k(n) - 1) = \frac{1}{6}$, we have

$$t + \tfrac{1}{3} < \textstyle\sum_{\delta \in D} s_\delta \cdot r(c_{\mu(z)} \cdot z, \delta) < t + \tfrac{2}{3}\,.$$

Comparing these inequalities with those in $(*)$, we see that $x - y \neq c_{\mu(z)} \cdot z$. $\square$

2.9 Theorem. *Assume that $\{c_n : n < \omega\}$ is an infinite subset of $\mathbb{Q}^+$. Then there exists a function $\varphi : \mathbb{R} \to \{0,1,2\}$ such that for all $z \in \mathbb{R}^+$ there exists $n \in \omega$ so that there do not exist $x, y \in \mathbb{R}$ with $x - y = c_n z$ and $\varphi(x) = \varphi(y)$.*

Proof. The proof may be taken nearly verbatim from the proof of Theorem 2.3 using Lemma 2.8 in place of Lemma 2.2. $\square$

3. Measurable and Baire colorings

In our proof of Theorem 2.9 we used the Axiom of Choice when we chose a Hamel basis. There is by now a long history of results in Ramsey Theory showing that, while a certain system is not partition regular, if one requires that the colorings be sufficiently constructive in some sense, the system becomes partition regular with respect to such colorings. For example, it is easy to see that the extension of Ramsey's Theorem which asks that whenever $[\omega]^\omega$ is finitely colored, there must exist $D \in [\omega]^\omega$ with $[D]^\omega$ monochrome is not valid. However, Galvin and Prikry [8] showed that if each of the color classes is a Borel set in $[\omega]^\omega$ (viewed as a subspace of the product space ${}^\omega\{0,1\}$), there must exist $D \in [\omega]^\omega$ with $[D]^\omega$ monochrome. Extensions of this result were obtained by Ellentuck [4], Carlson [2], and others.

Similarly the extension of the Finite Sums Theorem [11, Corollary 5.10] which asks that whenever $\mathbb{R}^+$ is finitely colored, there must exist a sequence $\langle x_n \rangle_{n<\omega}$ in $\mathbb{R}^+$ with $AS(\langle x_n \rangle_{n<\omega}) = \{\sum_{n\in F} x_n : \emptyset \neq F \subseteq \omega\}$ monochrome is false. But Prömel and Voigt [15] showed that it holds if each of the color classes is a Baire set (meaning a member of the σ-algebra generated by the open sets and the meager sets). And Plewik and Voigt [14] showed that it holds if each of the color classes is Lebesgue measurable. These two results were simplified and given a common proof in [1].

We shall show in Theorem 3.3 that if $\{c_n : n < \omega\}$ is bounded, then the system of equations $x_n - y_n = c_n z$ is partition regular with respect to measurable colorings of $\mathbb{R}^+$ and with respect to Baire colorings of $\mathbb{R}^+$.

We denote the outer Lebesgue measure of a set $C \subseteq \mathbb{R}$ by $\mu^*(C)$. If C is Lebesgue measurable, we denote its Lebesgue measure by $\mu(C)$.

3.1 Definition. Let $C \subseteq \mathbb{R}^+$.

(a) $\overline{d}(C) = \limsup\limits_{h\downarrow 0} \dfrac{\mu^*\big(C \cap (0,h)\big)}{h}$.

(b) C is *Baire large* if and only if for every $\epsilon > 0$, $C \cap (0,\epsilon)$ is not meager.

The following simple lemma is the basis for our unified treatment of measurable colorings and Baire colorings.

3.2 Lemma. *Let X be $\mathbb{Q}^+$ or $\mathbb{R}^+$, let $C \subseteq X$ and assume that*

(1) $0 \in \overline{C}$ and

(2) There is some $\epsilon > 0$ such that $(0,\epsilon) \cap X \subseteq C - C$.

Then whenever $\{c_n : n < \omega\}$ is a bounded subset of $\mathbb{Q}^+$ there exist $z \in C$ and sequences $\langle x_n \rangle_{n<\omega}$ and $\langle y_n \rangle_{n<\omega}$ in C such that $x_n - y_n = c_n z$ for each $n < \omega$.

Proof. Pick ϵ as guaranteed by (2). Pick $d \in \mathbb{N}$ such that for all $n < \omega$, $c_n < d$. Pick $z \in C$ such that

$z < \dfrac{\epsilon}{d}$. Then for all $n < \omega$, $c_n \cdot z \in C - C$. $\square$

3.3 Theorem. *Let $\{c_n : n < \omega\}$ be a bounded subset of $\mathbb{Q}^+$, let $r \in \mathbb{N}$, and let $\mathbb{R}^+ = \bigcup_{i<r} C_i$. If for each $i \in \{0, 1, \ldots, r-1\}$, C_i is Lebesgue measurable, or for each $i \in \{0, 1, \ldots, r-1\}$, C_i is a Baire set, then there exist $i \in \{0, 1, \ldots, r-1\}$, $z \in C_i$, and sequences $\langle x_n \rangle_{n<\omega}$ and $\langle y_n \rangle_{n<\omega}$ in C_i such that $x_n - y_n = c_n z$ for each $n < \omega$.*

Proof. If each C_i is Lebesgue measurable, then for some i, $\overline{d}(C_i) > 0$. If each C_i is a Baire set, then for some i, C_i is Baire large. So pick $i \in \{0, 1, \ldots, r-1\}$ such that either C_i is measurable and $\overline{d}(C_i) > 0$ or C_i is a Baire large Baire set. In either case one has trivially that $0 \in \overline{C_i}$. We shall show that there exists $\epsilon > 0$ such that $(0, \epsilon) \subseteq C_i - C_i$, so that Lemma 3.2 applies.

Assume first that C_i is measurable and $\overline{d}(C_i) > 0$. In particular $\mu(C_i) = \mu^*(C_i) = \alpha > 0$. We may presume that $C_i \subseteq (0, 1)$. Pick sequences $\langle a_n \rangle_{n<\omega}$ and $\langle b_n \rangle_{n<\omega}$ such that $C_i \subseteq \bigcup_{n<\omega} (a_n, b_n)$, $(a_n, b_n) \cap (a_m, b_m) = \emptyset$ when $n \neq m$, and $\mu(C_i) \geq \frac{9}{10} \cdot \sum_{n<\omega} (b_n - a_n)$. Pick $n < \omega$ such that $\mu\big(C_i \cap (a_n, b_n)\big) \geq \frac{9}{10} \cdot (b_n - a_n)$. Let $\epsilon = \frac{8}{10} \cdot (b_n - a_n)$. We claim that for all $t \in (0, \epsilon)$, $C_i \cap (C_i - t) \neq \emptyset$ and thus $(0, \epsilon) \subseteq C_i - C_i$. So let $t \in (0, \epsilon)$ and suppose that $C_i \cap (C_i - t) = \emptyset$. Then $\mu\big(C_i \cap (a_n, b_n - t)\big) = \mu\big(C_i \cap (a_n, b_n)\big) - \mu\big(C_i \cap (b_n - t, b)\big) \geq \frac{9}{10} \cdot (b_n - a_n) - t$ and $\mu\big((C_i - t) \cap (a_n, b_n - t)\big) = \mu\big((C_i - t) \cap (a_n - t, b_n - t)\big) - \mu\big((C_i - t) \cap (a_n - t, a_n)\big) \geq \frac{9}{10} \cdot (b_n - a_n) - t$. Thus $b_n - t - a_n = \mu\big((a_n, b_n - t)\big) \geq \mu\Big(\big(C_i \cap (a_n, b_n - t)\big) \cup \big((C_i - t) \cap (a_n, b_n - t)\big)\Big) = \mu\big(C_i \cap (a_n, b_n - t)\big) + \mu\big((C_i - t) \cap (a_n, b_n - t)\big) \geq \frac{9}{10} \cdot (b_n - a_n) - t + \frac{9}{10} \cdot (b_n - a_n) - t$, so $t \geq \frac{8}{10} \cdot (b_n - a_n) = \epsilon$, a contradiction.

Now assume that C_i is a Baire large Baire set. Pick an open set U and a meager set M such that $C_i = U \Delta M$. Since $C_i \cap (0, 1)$ is not meager, we conclude that $U \neq \emptyset$ and therefore there is an open interval (a, b) with $(a, b) \setminus M \subseteq C_i$. Let $\epsilon = b - a$. We claim that for all $t \in (0, \epsilon)$, $C_i \cap (C_i - t) \neq \emptyset$. Suppose instead that we have $t \in (0, \epsilon)$ such that $C_i \cap (C_i - t) = \emptyset$. We shall show that $(a, b - t) \subseteq M \cup (M - t)$, which is a contradiction. So let $x \in (a, b - t)$ and assume that $x \notin M$. Then $x \in (a, b) \setminus M \subseteq C_i$ so $x \notin C_i - t$. Since $x + t \in (a, b)$ we must have that $x + t \in M$. $\square$

We shall see in Theorem 3.5 that Theorem 3.3 cannot be extended to apply to every sequence $\langle c_n \rangle_{n=0}^{\infty}$ for which $\lim_{n\to\infty} c_n = \infty$.

3.4 Lemma. *Let $s \in \mathbb{R}$ and define $\varphi : \mathbb{R} \to \{0, 1, 2\}$ by $\varphi(x) = i$ if and only if $\pi(s \cdot x) \in \pi\big[[\frac{i}{3}, \frac{i+1}{3})\big]$ If $z \in \mathbb{R}$ and $\langle x_n \rangle_{n<\omega}$ and $\langle x_n \rangle_{n<\omega}$ are sequences in $\mathbb{R}$ such that*

φ *is constant on* $\{z\} \cup \{x_n : n < \omega\} \cup \{y_n : n < \omega\}$ *and* $x_n - y_n = (n+1) \cdot z$ *for each* $n < \omega$*, then* $s \cdot z \in \mathbb{Z}$.

Proof. Pick $i \in \{0, 1, 2\}$ such that $\varphi(z) = \varphi(x_n) = \varphi(y_n) = i$ for each n. Then for each n, $\pi(s \cdot x_n) \in \pi\big[[\frac{i}{3}, \frac{i+1}{3})\big]$ and $\pi(s \cdot y_n) \in \pi\big[[\frac{i}{3}, \frac{i+1}{3})\big]$, so $\pi\big(s \cdot (x_n - y_n)\big) \in \pi\big[(-\frac{1}{3}, \frac{1}{3})\big]$ and thus $\pi(s \cdot (n+1) \cdot z) \in \pi\big[(-\frac{1}{3}, \frac{1}{3})\big]$. Since also $\pi(s \cdot z) \in \pi\big[[\frac{i}{3}, \frac{i+1}{3})\big]$, we see that $i \neq 1$.

Suppose now that $i = 0$. If $\pi(s \cdot z) = \pi(0)$ we are done, so assume that $\pi(s \cdot z) \in \pi\big[(0, \frac{1}{3})\big]$. Let $m = \lfloor s \cdot z \rfloor$. Then $m < s \cdot z < m + \frac{1}{3}$. Pick the least $n \in \mathbb{N}$ such that $(n+1) \cdot (s \cdot z - m) \geq \frac{1}{3}$. Then $n \cdot (s \cdot z - m) < \frac{1}{3}$. Thus $(n+1) \cdot m + \frac{1}{3} \leq (n+1) \cdot s \cdot z < (n+1) \cdot m + \frac{2}{3}$, contradicting the fact that $\pi(s \cdot (n+1) \cdot z) \in \pi\big[(-\frac{1}{3}, \frac{1}{3})\big]$.

Finally suppose that $i = 2$. Let $m = \lfloor s \cdot z \rfloor$. Then $m + \frac{2}{3} \leq s \cdot z < m + 1$ so $0 < 1 + m - s \cdot z \leq \frac{1}{3}$. Pick the least $n \in \mathbb{N}$ such that $(n+1) \cdot (1 + m - s \cdot z) > \frac{1}{3}$. Then $n \cdot (1 + m - s \cdot z) \leq \frac{1}{3}$ so $n + (n+1) \cdot m + \frac{1}{3} \leq s \cdot (n+1) \cdot z < n + (n+1) \cdot m + \frac{1}{3}$, again contradicting the fact that $\pi(s \cdot (n+1) \cdot z) \in \pi\big[(-\frac{1}{3}, \frac{1}{3})\big]$. $\square$

We shall call a subset of $\mathbb{R}$ or $\mathbb{Q}$ *strongly Borel* if it is a member of the Boolean algebra generated by the open sets.

3.5 Theorem. *There is a function* $\psi : \mathbb{R}^+ \to \{0, 1, \ldots, 8\}$ *such that*

(1) for each $i \in \{0, 1, \ldots, 8\}$*,* $\psi^{-1}[\{i\}]$ *is strongly Borel.*

(2) there do not exist $z \in \mathbb{R}^+$ *and sequences* $\langle x_n \rangle_{n<\omega}$ *and* $\langle y_n \rangle_{n<\omega}$ *in* $\mathbb{R}^+$ *such that* ψ *is constant on* $\{z\} \cup \{x_n : n < \omega\} \cup \{y_n : n < \omega\}$ *and* $x_n - y_n = (n+1) \cdot z$ *for each* $n < \omega$.

Proof. Let $\psi(x) = 3i + j$ where $i, j \in \{0, 1, 2\}$, $\pi(x) \in \pi\big[[\frac{i}{3}, \frac{i+1}{3})\big]$, and $\pi(\sqrt{2} \cdot x) \in \pi\big[[\frac{j}{3}, \frac{j+1}{3})\big]$. Suppose we have $z \in \mathbb{R}^+$ and sequences $\langle x_n \rangle_{n<\omega}$ and $\langle y_n \rangle_{n<\omega}$ in $\mathbb{R}^+$ such that ψ is constant on $\{z\} \cup \{x_n : n < \omega\} \cup \{y_n : n < \omega\}$ and $x_n - y_n = (n+1) \cdot z$ for each $n < \omega$. By Lemma 3.4 with $s = 1$ we have $z \in \mathbb{Z}$. By Lemma 3.4 with $s = \sqrt{2}$ we have $\sqrt{2} \cdot z \in \mathbb{Z}$. $\square$

There is a result similar to Theorem 3.3 for $\mathbb{Q}^+$.

3.6 Theorem. *Let* $\{c_n : n < \omega\}$ *be a bounded subset of* $\mathbb{Q}^+$*, let* $r \in \mathbb{N}$*, and let* $\mathbb{Q}^+ = \bigcup_{i<r} C_i$*. If for each* $i \in \{0, 1, \ldots, r-1\}$*,* C_i *is strongly Borel, then there exist* $i \in \{0, 1, \ldots, r-1\}$*,* $z \in C_i$ *and sequences* $\langle x_n \rangle_{n<\omega}$ *and* $\langle y_n \rangle_{n<\omega}$ *in* C_i *such that* $x_n - y_n = c_n z$ *for each* $n < \omega$.

Proof. Observe that any strongly Borel set is a finite union of sets of the form $F \cap G$ where F is open and G is closed. Thus we may presume that each C_i is of this form.

Further, it suffices to show that for some i,

(1) $0 \in \overline{C_i}$ and

(2) the interior in $\mathbb{Q}^+$ of C_i is nonempty.

Indeed, if $(a,b) \cap \mathbb{Q}^+ \subseteq C_i$ and $\epsilon = b - a$, then $(0,\epsilon) \cap \mathbb{Q}^+ \subseteq C_i - C_i$, so Lemma 3.2 applies.

For (1) and (2) it further suffices to show that $0 \in \overline{\mathrm{int}_{\mathbb{Q}^+}(C_i)}$ for some i. Suppose instead that $0 \notin \overline{\mathrm{int}_{\mathbb{Q}^+}(C_i)}$ for each i and pick $t > 0$ such that $(0,t) \cap \mathrm{int}_{\mathbb{Q}^+}(C_i) = \emptyset$ for each $i \in \{0, 1, \ldots, r-1\}$.

Now $\mathbb{Q}^+ \cap (0,t) \subseteq \bigcup_{i<r} \overline{C_i}$ so pick some $i < r$ such that $U = (0,t) \cap \mathrm{int}_{\mathbb{Q}^+} \overline{C_i} \neq \emptyset$. Pick a closed subset F of $\mathbb{Q}^+$ and an open subset G of $\mathbb{Q}^+$ such that $C_i = F \cap G$. Note that $U \not\subseteq \overline{G} \setminus G$. Let $V = U \setminus (\overline{G} \setminus G)$. Then V is nonempty and open. Also

$$V \subseteq U \subseteq \overline{C_i} \subseteq F \cap \overline{G} = \big(F \cap (\overline{G} \setminus G)\big) \cup (F \cap G)\,.$$

Since $V \cap (\overline{G} \setminus G) = \emptyset$, $V \subseteq F \cap G = C_i$, a contradiction. $\square$

Theorem 3.5 shows that one cannot extend Theorem 3.6 to apply to all sequences $\langle c_n \rangle_{n<\omega}$ for which $\lim\limits_{n\to\infty} c_n = \infty$. We now see that it cannot be extended to *any* such sequence.

3.7 Theorem. *Let $\{c_n : n < \omega\} \subseteq \mathbb{Q}^+$ such that $\lim\limits_{n\to\infty} c_n = \infty$. Then there is a function $\varphi : \mathbb{Q}^+ \to \{0,1,2\}$ such that*

(1) for each $i \in \{0,1,2\}$, $\varphi^{-1}[\{i\}]$ is strongly Borel.

(2) there do not exist $z \in \mathbb{Q}^+$ and sequences $\langle x_n \rangle_{n<\omega}$ and $\langle y_n \rangle_{n<\omega}$ in $\mathbb{Q}^+$ such that φ is constant on $\{x_n : n < \omega\} \cup \{y_n : n < \omega\}$ and $x_n - y_n = c_n \cdot z$ for each $n < \omega$.

Proof. Enumerate $\mathbb{Q}^+$ as $\langle z_n \rangle_{n<\omega}$. For each $n < \omega$, let $F_n = \{z_n\}$, let $\alpha_n = \frac{4}{9}$, and let $\beta_n = \frac{5}{9}$. Choose γ as guaranteed by Lemma 2.1 and for each $n < \omega$, choose $s_n \in (0,1)$ such that for all $t \in \{0,1,\ldots,n\}$, $\pi(s_n \cdot c_{\gamma(t)} \cdot z_t) \in \pi[(\frac{4}{9}, \frac{5}{9})]$. Let s be any cluster point of the sequence $\langle s_n \rangle_{n<\omega}$ in $[0,1]$. Then for each $t < \omega$, $\pi(s \cdot c_{\gamma(t)} \cdot z_t) \in \pi[(\frac{1}{3}, \frac{2}{3})]$.

Define $\varphi : \mathbb{Q}^+ \to \{0,1,2\}$ by $\varphi(x) = i$ if and only if $\pi(s \cdot x) \in \pi\big[[\frac{i}{3}, \frac{i+1}{3})\big]$. Clearly each $\varphi^{-1}[\{i\}]$ is strongly Borel. Suppose we have $z \in \mathbb{Q}^+$ and sequences $\langle x_n \rangle_{n<\omega}$ and $\langle y_n \rangle_{n<\omega}$ in $\mathbb{Q}^+$ such that φ is constant on $\{x_n : n < \omega\} \cup \{y_n : n < \omega\}$ and $x_n - y_n = c_n \cdot z$ for each $n < \omega$. Pick $n < \omega$ such that $z = z_n$. Then $\varphi(x_{\delta(n)}) = \varphi(y_{\delta(n)})$ so $\pi\big(s \cdot (x_{\delta(n)} - y_{\delta(n)})\big) \notin \pi[(\frac{1}{3}, \frac{2}{3})]$ and thus $x_{\delta(n)} - y_{\delta(n)} \neq c_{\delta(n)} \cdot z$. $\square$

References

[1] V. Bergelson, N. Hindman, and B. Weiss *All-sums sets in* $(0, 1]$ – *Category and measure*, Mathematika **44** (1997), 61-87.

[2] T. Carlson, *Some unifying principles in Ramsey Theory*, Discrete Math. **68** (1988), 117-169.

[3] W. Deuber, *Partitionen und lineare Gleichungssysteme*, Math. Zeit. **133** (1973), 109-123.

[4] E. Ellentuck, *A new proof that analytic sets are Ramsey*, J. Symbolic Logic **39** (1974), 163-165.

[5] P. Erdős, *Some set-theoretical properties of graphs*, Revista de la Univ. Nac. de Tucuman, Ser. A. Mat. y Fiz. Teor **3** (1942), 363-367.

[6] P. Erdős and S. Kakutani, *On non-denumerable graphs*, Bull. Amer. Math. Soc. **49** (1943), 457-461.

[7] P. Erdős and R. Rado, *A partition calculus in set theory*, Bull. Amer. Math. Soc. **62** (1956), 427-489.

[8] F. Galvin and K. Prikry, *Borel sets and Ramsey's Theorem*, J. Symbolic Logic **38** (1973), 193-198.

[9] R. Graham, B. Rothschild, and J. Spencer, *Ramsey Theory*, Wiley, NY, 1990.

[10] N. Hindman, I. Leader, and D. Strauss, *Open problems in partition regularity*, Comb. Prob. and Comp., to appear.

[11] N. Hindman and D. Strauss, *Algebra in the Stone-Čech compactification: theory and applications*, de Gruyter, Berlin, 1998.

[12] G. Hardy and E. Wright, An introduction to the theory of numbers, Oxford University Press, New York, 1979.

[13] K. Kunen, *Partitioning Euclidean space*, Math. Proc. Cambridge Philos. Soc. **102** (1987), 379-383.

[14] S. Plewik and B. Voigt, *Partitions of reals: measurable approach*, J. Comb. Theory (Series A) **58** (1991), 136-140.

[15] H. Prömel and B. Voigt, *A partition theorem for* $[0, 1]$, Proc. Amer. Math. Soc. **109** (1990), 281-285.

[16] R. Rado, *Studien zur Kombinatorik*, Math. Zeit. **36** (1933), 242-280.

[17] R. Rado, *Note on combinatorial analysis*, Proc. London Math. Soc. **48** (1943), 122-160.

[18] I. Schur, *Über die Kongruenz* $x^m + y^m = z^m (\text{mod } p)$, Jahresbericht der Deutschen Math.-Verein. **25** (1916), 114-117.

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