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Independent Finite Sums for $\mathbf{K_m}$-free Graphs

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Abstract. Recently, in conversation with Erdős, Hajnal asked whether or not for any triangle-free graph G on the vertex set $\mathbb{N}$, there always exists a sequence $\langle x_n\rangle_{n=1}^\infty$ so that whenever F and H are distinct finite nonempty subsets of $\mathbb{N}$, $\{\Sigma_{n\in F}\ x_n, \Sigma_{n\in H}\ x_n\}$ is not an edge of G (that is, $FS(\langle x_n\rangle_{n=1}^\infty)$ is an independent set). We answer this question in the negative. We also show that if one replaces the assumption that G is triangle-free by the assertion that for some m, G contains no complete bipartite graph $K_{m,m}$, then the conclusion does hold. If for some $m \geq 3$, G contains no K_m, we show there exists a sequence $\langle x_n\rangle_{n=1}^\infty$ so that whenever F and H are *disjoint* finite nonempty subsets of $\mathbb{N}$, $\{\Sigma_{n\in F}\ x_n, \Sigma_{n\in H}\ x_n\}$ is not an edge of G. Both of the affirmative results are in fact valid for a graph G on an arbitrary cancellative semigroup $(S,+)$.

1. Introduction.

We take $\mathbb{N}$ to be the positive integers and $\omega = \mathbb{N}\cup\{0\}$. Given a set A, we denote by $\mathcal{P}_f(A)$ the set of finite nonempty subsets of A. Given a sequence $\langle x_n\rangle_{n=1}^\infty$ in $\mathbb{N}$, we use the notation $FS(\langle x_n\rangle_{n=1}^\infty) = \{\Sigma_{n\in F}\ x_n : F \in \mathcal{P}_f(\mathbb{N})\}$. In 1972 the following theorem was proved in [6] (or see [1] or [7] for simpler proofs).

1.1 Theorem. *Let $r \in \mathbb{N}$ and let $\mathbb{N} = \bigcup_{i=1}^r A_i$. Then there exist $i \in \{1, 2, \ldots, r\}$ and a sequence $\langle x_n\rangle_{n=1}^\infty$ such that $FS(\langle x_n\rangle_{n=1}^\infty) \subseteq A_i$.*

It was already known at the time [5] that Theorem 1.1 is equivalent to the superficially weaker version which has $r = 2$. In 1995, Hajnal asked Erdős the following question. (It appears as a remark following Problem 4.4 of [4], a paper written by Erdős, Hajnal, and Pach.)

1.2 Question. *Let G be a graph on the vertex set $\mathbb{N}$ which contains no triangles. Must there exist a sequence $\langle x_n\rangle_{n=1}^\infty$ in $\mathbb{N}$ such that $FS(\langle x_n\rangle_{n=1}^\infty)$ forms an independent set?*

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Note that an affirmative answer to Question 1.2 would imply Theorem 1.1. To see this, let $\mathbb{N} = A_1 \cup A_2$ where we may presume that $A_1 \cap A_2 = \emptyset$. (We have already observed that it suffices to establish Theorem 1.1 for the case $r = 2$.) Let G be the complete bipartite graph on the sets A_1 and A_2. That is, $E(G) = \big\{\{x, y\} : x \in A_1 \text{ and } y \in A_2\big\}$. Then if $\langle x_n \rangle_{n=1}^{\infty}$ has $FS(\langle x_n \rangle_{n=1}^{\infty})$ independent one must have $FS(\langle x_n \rangle_{n=1}^{\infty}) \subseteq A_1$ or $FS(\langle x_n \rangle_{n=1}^{\infty}) \subseteq A_2$.

In fact, Theorem 1.1 follows from an affirmative answer to the weaker Question 1.3.

1.3 Question. *Let G be a graph with vertices in $\mathbb{N}$ which contains no triangles. Must there exist a sequence $\langle x_n \rangle_{n=1}^{\infty}$ in $\mathbb{N}$ such that $\{\Sigma_{n\in F}\, x_n, \Sigma_{n\in H}\, x_n\} \notin E(G)$ whenever $F, H \in \mathcal{P}_f(\mathbb{N})$ with $F \cap H = \emptyset$?*

To see that an affirmative answer to Question 1.3 implies Theorem 1.1, let the graph G be defined exactly as above and let $\langle x_n \rangle_{n=1}^{\infty}$ be as guaranteed by an affirmative answer to the question. Suppose one has $F, H \in \mathcal{P}_f(\mathbb{N})$ with $\Sigma_{n\in F}\, x_n \in A_1$ and $\Sigma_{n\in H}\, x_n \in A_2$. Pick $k > \max(F \cup H)$. Then either $\{\Sigma_{n\in F}\, x_n, x_k\} \in E(G)$ or $\{\Sigma_{n\in H}\, x_n, x_k\} \in E(G)$.

On hearing Question 1.2, Erdős "retaliated" (his word) by asking the following much weaker question.

1.4 Question. *Let G be a triangle-free graph with vertices in $\mathbb{N}$. Must there exist $x \neq y$ such that $\{x, y, x + y\}$ is an independent set?*

In [8], this question was answered in the affirmative in the following strong fashion. Here K_m is the complete graph on m vertices.

1.5 Theorem. *Let G be a graph with vertex set $\mathbb{N}$ and assume there is some $m \in \mathbb{N}$ such that G contains no K_m. Then for each $\ell \in \mathbb{N}$, there is a finite sequence $\langle x_n \rangle_{n=1}^{\ell}$ such that $FS(\langle x_n \rangle_{n=1}^{\ell})$ is an independent set.*

In [8] it was also shown that one cannot weaken the hypothesis of Theorem 1.5 to the assertion that G contains no K_ω (where K_ω is the complete graph on countably many vertices). For if $E(G) = \big\{\{x, y\} : x < y < 2x\big\}$ then G contains no K_ω, but given any $x < y$ in $\mathbb{N}$, one has $\{y, x + y\} \in E(G)$.

In Section 2 of this paper we answer Question 1.2 in the negative by exhibiting a triangle-free graph on $\mathbb{N}$ so that every $FS(\langle x_n \rangle_{n=1}^{\infty})$ induces at least one edge in the graph.

In Section 3 we provide a strong affirmative answer to Question 1.3. That is, we show that if G is a graph with vertices in $\mathbb{N}$ and there exists some $m \in \mathbb{N}\backslash\{1, 2\}$ such that G contains no K_m, then there is a sequence $\langle x_n\rangle_{n=1}^\infty$ in $\mathbb{N}$ such that whenever $F, H \in \mathcal{P}_f(\mathbb{N})$ and $F \cap H = \emptyset$, one has that $\{\Sigma_{n\in F}\, x_n, \Sigma_{n\in H}\, x_n\} \notin E(G)$.

In fact we show that the answer to Question 1.3 remains affirmative when the semigroup $(\mathbb{N},+)$ is replaced by an arbitrary semigroup $(S,+)$. We write the semigroup S additively because the origin of the questions was in $(\mathbb{N},+)$. However, we do *not* assume that the operation is commutative, so when speaking of $FS(\langle x_n\rangle_{n=1}^\infty)$ we need to specify the order of the sums, which we take to be written in increasing order of indices. (Thus, for example, $\Sigma_{t\in\{1,2,6\}}\, x_t = x_1 + x_2 + x_6$.)

The answer we give to Question 1.3 is stronger in another direction as well. That is, we show that the sequence $\langle x_n\rangle_{n=1}^\infty$ with independent finite sums can be found inside $FS(\langle y_n\rangle_{n=1}^\infty)$ for any given sequence $\langle y_n\rangle_{n=1}^\infty$, where the notion of "inside" is made precise by the following definition.

1.6 Definition. Let $(S,+)$ be a semigroup and let $\langle y_n\rangle_{n=1}^\infty$ and $\langle x_n\rangle_{n=1}^\infty$ be sequences in S. Then $\langle x_n\rangle_{n=1}^\infty$ is a *sum subsystem* of $\langle y_n\rangle_{n=1}^\infty$ if and only if there is a sequence $\langle H_n\rangle_{n=1}^\infty$ in $\mathcal{P}_f(\mathbb{N})$ such that for each $n \in \mathbb{N}$,

$$\max H_n < \min H_{n+1} \text{ and } x_n = \Sigma_{t\in H_n}\, y_t\ .$$

In Section 4 we obtain the conclusion of Question 1.2 under different (but neither weaker nor stronger) hypotheses. Using $K_{m,m}$ to denote the complete balanced bipartite graph on $2m$ vertices, we show that for every $m \in \mathbb{N}$, if G is a graph on the cancellative semigroup S which contains no $K_{m,m}$, then there is a sequence $\langle x_n\rangle_{n=1}^\infty$ in S such that no pair of finite sums (disjoint or not) form an edge of G. This result has been obtained independently in [8] for the case $S = \mathbb{N}$. Again our result in fact shows that the sequence $\langle x_n\rangle_{n=1}^\infty$ can be chosen to be a sum subsystem of any given sequence $\langle y_n\rangle_{n=1}^\infty$.

The results of Sections 3 and 4 are true, but trivial, if S is finite, so we will assume that $(S,+)$ is an infinite semigroup (and we emphasize again that we are not assuming the operation is commutative). We will utilize in these sections the algebraic structure of the semigroup $(\beta S,+)$, where $+$ denotes the extension of the operation to βS which makes $(\beta S,+)$ a right topological semigroup with S contained in its topological center. We now briefly describe the semigroup $(\beta S,+)$. See [7] for a detailed construction of βS and derivations of some of the basic algebraic facts.

We take the points of βS to be the ultrafilters on S, the principal ultrafilters being identified with the points of S. When we say that $(\beta S,+)$ is a right topological semigroup we mean that for each $p \in \beta S$ the function $\rho_p : \beta S \longrightarrow \beta S$, defined by $\rho_p(q) = q + p$, is continuous. When we say that S is contained in the topological center of $(\beta S, +)$, we mean that for each $x \in S$, the function $\lambda_x : \beta S \longrightarrow \beta S$ defined by $\lambda_x(q) = x + q$ is continuous.

The operation $+$ on βS is characterized as follows: Given $A \subseteq S$, $A \in p + q$ if and only if $\{x \in S : -x + A \in q\} \in p$ where $-x + A = \{y \in S : x + y \in A\}$. In particular, $p = p + p$ if and only if whenever $A \in p$ one has $\{x \in S : -x + A \in p\} \in p$. Observe also that if $p, q \in \beta S$, $A \in p$, and for each $x \in A$, $B(x) \in q$, then

$$\{x + y : x \in A \text{ and } y \in B(x)\} \in p + q \ .$$

Given $A \subseteq S$, $\overline{A} = c\ell A = \{p \in \beta S : A \in p\}$. The set $\{\overline{A} : A \subseteq S\}$ is a basis for the open sets (as well as a basis for the closed sets) of βS.

A significant property of any compact Hausdorff right topological semigroup is that it contains an idempotent [3, Corollary 2.10].

We shall use Lemma 1.8 frequently in Sections 3 and 4.

1.7 Definition. Let $p + p = p \in \beta S$, and let $B \in p$. Then $B^* = \{x \in B : -x + B \in p\}$.

1.8 Lemma. *Let $p + p = p \in \beta S$ and let $B \in p$. Then $B^* \in p$. Furthermore, for every $x \in B^*$, we have $-x + B^* \in p$.*

Proof. It is immediate, as we noted above, that $B^* \in p$. We know that $-x + B \in p$ and so $(-x + B)^* \in p$. We claim that $(-x + B)^* \subseteq -x + B^*$. So let $y \in (-x + B)^*$. Then $-y + (-x + B) \in p$. That is, $-(x + y) + B \in p$. Since also $x + y \in B$ we have $x + y \in B^*$ as desired. $\square$

It is the simple property established in Lemma 1.8 which makes idempotent ultrafilters a useful tool in constructing infinite sets of the form $FS(\langle x_n \rangle_{n=1}^{\infty})$. In Sections 3 and 4 of the present paper, we shall frequently be using sums of the form $x_1 + x_2 + \ldots + x_n$, where x_1 has to be chosen in some assigned member of p and each x_i has to be chosen in a member of p which depends on $x_1, x_2, \ldots, x_{i-1}$. We shall use the fact that, in the light of Lemma 1.8, a sum of this kind can be found in any given member of p. An illustration follows in the proof of Theorem 1.10.

The method of proof in Sections 3 and 4 is to take an arbitrary idempotent p in βS and an arbitrary member B of p and show that we can choose a sequence $\langle x_n \rangle_{n=1}^{\infty}$ in B as required. This allows us to obtain the sequence as a sum subsystem of another sequence because of the following old result of Fred Galvin's. By $FS(\langle y_n \rangle_{n=m}^{\infty})$ we mean of course, $\{\Sigma_{n \in F}\, y_n : F \in \mathcal{P}_f(\mathbb{N}) \text{ and } \min F \geq m\}$.

1.9 Theorem. *Let $(S,+)$ be a semigroup and let $\langle y_n\rangle_{n=1}^\infty$ be a sequence in S. There is some $p=p+p$ in βS such that for every $m\in\mathbb{N}$, $FS(\langle y_n\rangle_{n=m}^\infty)\in p$.*

Proof. See [7, Theorem 5.5]. $\square$

Working as we do with an arbitrary idempotent p in βS and an arbitrary element B of p, our results become trivial if $p\in S$. (For then $\{p\}$ is an element of the principal ultrafilter which we have identified with p, and the sequence $\langle x_n\rangle_{n=1}^\infty$ can be constantly equal to p.) Accordingly, we are interested in knowing when we can guarantee that $p\in\beta S\backslash S$. The following simple observation answers that question.

1.10 Theorem. *Let $(S,+)$ be a semigroup.*

(a) Given any sequence $\langle y_n\rangle_{n=1}^\infty$ in S, if $\bigcap_{m=1}^\infty FS(\langle y_n\rangle_{n=m}^\infty)=\emptyset$, then there is some $p=p+p$ in $\beta S\backslash S$ such that for every $m\in\mathbb{N}$, $FS(\langle y_n\rangle_{n=m}^\infty)\in p$.

(b) There is some idempotent p in $\beta S\backslash S$ if and only if there is a sequence $\langle y_n\rangle_{n=1}^\infty$ in S such that $\bigcap_{m=1}^\infty FS(\langle y_n\rangle_{n=m}^\infty)=\emptyset$.

Proof. (a) Choose p as guaranteed by Theorem 1.9. Since $\{FS(\langle y_n\rangle_{n=m}^\infty):m\in\mathbb{N}\}\subseteq p$ and $\bigcap\{FS(\langle y_n\rangle_{n=m}^\infty):m\in\mathbb{N}\}=\emptyset$, p is not principal.

(b) The sufficiency is an immediate consequence of part (a). For the necessity, let $p\in\beta S\backslash S$ such that $p=p+p$. Let $B_1=S$ and pick $y_1\in(B_1)^*$ (which is just S). Inductively, let $n\in\mathbb{N}$ and assume that we have chosen $\langle y_t\rangle_{t=1}^n$ such that for each nonempty $F\subseteq\{1,2,\ldots,n\}$, $\Sigma_{t\in F}\,y_t\in(B_{\min F})^*$. Let $B_{n+1}=S\backslash FS(\langle y_t\rangle_{t=1}^n)$ and note that since p is nonprincipal, $B_{n+1}\in p$. By Lemma 1.8 we have for each nonempty $F\subseteq\{1,2,\ldots,n\}$, $-\Sigma_{t\in F}\,y_t+(B_{\min F})^*\in p$. Choose

$$y_{n+1}\in(B_{n+1})^*\cap\bigcap\big\{-\Sigma_{t\in F}\,y_t+(B_{\min F})^*:\emptyset\neq F\subseteq\{1,2,\ldots,n\}\big\}\ .$$

This completes the inductive construction of the sequence $\langle y_n\rangle_{n=1}^\infty$.

Suppose that we have some $a\in\bigcap_{m=1}^\infty FS(\langle y_n\rangle_{n=m}^\infty)$. Pick $F\in\mathcal{P}_f(\mathbb{N})$ such that $a=\Sigma_{t\in F}\,y_t$ and let $m=\max F$. Then $a\notin B_{m+1}$ while $FS(\langle y_n\rangle_{n=m}^\infty)\subseteq B_{m+1}$, a contradiction. $\square$

We wish to thank the referee of this paper for a very thoughtful report. In particular, the referee significantly simplified our original proof of Theorem 2.2.

2. A triangle-free graph without independent finite sums.

The graph we produce is described in terms of increasing sets of integers.

2.1 Definition. (a) Given $F, H \in \mathcal{P}_f(\omega)$, write $F < H$ if and only if $\max F < \min H$. Given $x, y \in \mathbb{N}$, write $x << y$ if and only if $x = \Sigma_{n\in F}\ 2^n$, $y = \Sigma_{n\in H}\ 2^n$, and $F < H$.

(b) Define $\mu : \mathbb{N} \longrightarrow \omega$ by $\mu(x) = \min F$ where $x = \Sigma_{t\in F}\ 2^t$.

2.2 Theorem. *There is a graph G with vertex set $\mathbb{N}$ such that G contains no triangle, but given any sequence $\langle u_n\rangle_{n=1}^\infty$ in $\mathbb{N}$, there exist distinct $F, H \in \mathcal{P}_f(\mathbb{N})$ such that $\{\Sigma_{n\in F}\ u_n, \Sigma_{n\in H}\ u_n\} \in E(G)$.*

Proof. Choose any $g : \mathbb{N} \longrightarrow \mathbb{N}$ such that for each $k \in \mathbb{N}$,

$$g(k+1) > g(k).$$

Define a graph G with vertex set $\mathbb{N}$ by

$$\begin{aligned} E(G) \quad = \quad & \big\{\{x_2 + x_4 + \ldots + x_{2m}, x_1 + x_2 + x_3 + \ldots + x_{2m}\} : \\ & x_1 << x_2 << \ldots << x_{2m} \text{ and } m = g\big(\mu(x_1)\big)\big\}\ . \end{aligned}$$

An alternative description of the edges of G is as follows. A pair $\{a, b\}$, where $a < b$ and the binary supports of a and b are A and B respectively, is an edge of G if and only if the following four conditions are satisfied:

(1) $A \subseteq B$;

(2) $B\backslash A$ cuts A into exactly $g(\min B)$ pieces (i.e. maximal sets whose convex hull contains no element of B);

(3) $\min B < \min A$; and

(4) $\max B = \max A$.

(As the referee pointed out, requirement (4) could be dropped without affecting the proof.)

Let a sequence $\langle u_n\rangle_{n=1}^\infty$ be given. It is well known that one can choose an increasing sequence $\langle K_n\rangle_{n=1}^\infty$ in $\mathcal{P}_f(\mathbb{N})$ such that for each $n \in \mathbb{N}$, $\Sigma_{t\in K_n}\ u_t << \Sigma_{t\in K_{n+1}}\ u_t$. (To see this, given K_n, pick $\ell \in \mathbb{N}$ such that $2^\ell > \Sigma_{t\in K_n}\ u_t$. Then choose a set K_{n+1} with $K_n < K_{n+1}$, $|K_{n+1}| = 2^\ell$ and $u_t \equiv u_s (\text{mod } 2^\ell)$ for each $t, s \in K_{n+1}$.)

Let $m = g(\min\ K_1)$, let

$$F = K_2 \cup K_4 \cup \ldots \cup K_{2m}\ ,$$

and let

$$H = K_1 \cup K_2 \cup K_3 \cup \ldots \cup K_{2m}\ .$$

Then, by the definition of G with $x_n = \Sigma_{t\in K_n} u_t$, we have $\{\Sigma_{n\in F}\ u_n, \Sigma_{n\in H}\ u_n\} \in E(G)$.

Suppose now that we have a triangle $\{a,b,c\}$ in G with $a<b<c$ and denote the binary supports of a, b, and c by A, B, and C respectively. Since $\{a,b\}$ is an edge, $B\backslash A$ cuts A into $g(\min B)$ pieces. Since $B\backslash A$ is a subset of $C\backslash A$, it follows that $C\backslash A$ cuts A into at least $g(\min B)$ pieces. Since $\{a,c\}$ is an edge, $C\backslash A$ cuts A into exactly $g(\min C)$ pieces, whence $g(\min C)\geq g(\min B)$. On the other hand, $\min C<\min B$ (since $\{b,c\}$ is an edge); as g is strictly increasing, we have $g(\min C)<g(\min B)$, a contradiction. □

The graph in Theorem 2.2 consists of certain pairs of numbers, one of whose binary support is contained in the binary support of the other, and the smallest element in the union of the supports belongs only to one support. The following theorem shows that, if the graph is changed only slightly, the conclusion changes dramatically.

2.3 Theorem. *Choose any $f:\mathbb{N}\longrightarrow\mathbb{N}\backslash\{1\}$ Define a graph G with vertices contained in $\mathbb{N}$ by*

$$\begin{array}{rcl} E(G) & = & \big\{\{x_1+x_3+\ldots+x_{2m-1}, x_1+x_2+x_3+\ldots+x_{2m}\}: \\ & & x_1<<x_2<<\ldots<<x_{2m} \text{ and } m=f(x_1)\big\}\ . \end{array}$$

Then G has triangles. In fact, given any sequence $\langle w_n\rangle_{n=1}^{\infty}$ in $\mathbb{N}$, there is a triangle of G all of whose vertices lie in $FS(\langle w_n\rangle_{n=1}^{\infty})$.

Proof. As in the proof of Theorem 2.2 above, we may presume that $w_n<<w_{n+1}$ for each $n\in\mathbb{N}$. Let $m=f(w_1)$ and for each $i\in\{1,2,\ldots,2m-1\}$, let $x_i=y_i=w_i$. Let $z_1=x_1+x_2+x_3+\ldots+x_{2m-1}$ and let $r=f(z_1)$. For each $i\in\{2,3,\ldots,2r\}$, let $z_i=w_{2m+i}$ (or if one wants to be economical, let $z_i=w_{2m+i-2}$). Let $x_{2m}=z_3+z_5+\ldots+z_{2r-1}$, let $y_{2m}=z_2+z_3+z_4+\ldots+z_{2r}$, and let

$$\begin{array}{rclcl} a & = & x_1+x_3+\ldots+x_{2m-1} & = & y_1+y_3+\ldots+y_{2m-1}\ , \\ b & = & z_1+z_3+\ldots+z_{2r-1} & = & x_1+x_2+x_3+\ldots+x_{2m}\ , \text{ and} \\ c & = & z_1+z_2+z_3+\ldots+z_{2r} & = & y_1+y_2+y_3+\ldots+y_{2m}\ . \end{array}$$

Then $\{a,b,c\}$ is a triangle in G. □

3. When G Has No $\mathbf{K_m}$.

Throughout this section we will have a fixed infinite semigroup $(S,+)$, a fixed graph G on S and a fixed idempotent $p\in\beta S$. (Don't confuse the fact that we have "fixed" p with the old notion of a "fixed ultrafilter". The results of this section are trivial if p is principal.) Further, we fix a cardinal κ such that the cofinality of κ is greater than $|S|$. Several of the notions that we introduce depend on both G and p, but the notation will not reflect this dependence.

We mention one other notational peculiarity in this section. We shall frequently use superscripts as indices and never to denote exponentiation.

3.1 Definition. Let $a \in S$.

(a) $A_0(a) = \{b \in S : \{a, b\} \notin E(G)\}$.

(b) For each ordinal $\iota < \kappa$,

$$A_{\iota+1}(a) = \{b \in S : \{u_1 \in S : \{u_2 \in S : b + u_2 \in A_\iota(a + u_1)\} \in p\} \in p\} \ .$$

(c) For each limit ordinal ι with $0 < \iota < \kappa$,

$$A_\iota(a) = \bigcap_{\gamma<\iota} A_\gamma(a) \ .$$

(d) $A(a) = \bigcap_{\iota<\kappa} \ A_\iota(a)$.

Recall that we are not assuming that the semigroup $(S, +)$ is commutative, so, in part (b) of the following definition, we need to specify the order in which a sum is taken.

3.2 Definition. (a) Let $\mathfrak{S}$ denote the set of finite sequences of elements of S, including the empty sequence. If $\sigma = u_1u_2 \ldots u_n \in \mathfrak{S}$, we put $l(\sigma) = n$ and $\sigma' = u_1u_2 \ldots u_{n-1}$ if $n > 1$, while $\sigma' = \emptyset$ if $n = 1$. (Then $l(\sigma)$ is the *length* of σ.)

(b)We use σ_{odd} to denote the sum (in increasing order of indices) of the odd terms of σ and $\sigma_{odd,>1}$ to denote the sum of the odd terms with index greater than 1. Similarly, we define σ_{even} to be the sum (in increasing order of indices) of the even terms of σ and and $\sigma_{even,>2}$ to denote the sum of the even terms with index greater than 2.

In the following definition we denote the values of the functions U and α at the sequence σ by U_σ and α_σ respectively. We also identify elements of S with the sequences of length 1.

3.3 Definition. Let $r \in \omega$. An $\mathcal{A}$-*system of level* r is a triple (U, α, D) such that

(1) $D \subseteq \mathfrak{S}$ and $\emptyset \in D$.

(2) $U : D \longrightarrow p$.

(3) $\alpha : D \longrightarrow \kappa$.

(4) For every $\sigma \in \mathfrak{S}$ and every $u \in S$, we have $\sigma u \in D$ if and only if $\sigma \in D$, $\alpha_\sigma > 0$, and $u \in U_\sigma$.

(5) For every $\sigma \in D \backslash \{\emptyset\}$ we have $\alpha_\sigma = \alpha_{\sigma'}$ if $l(\sigma)$ is even and $\alpha_\sigma < \alpha_{\sigma'}$ if $l(\sigma)$ is odd.

(6) If $\sigma \in D$, $l(\sigma)$ is odd, and $\alpha_\sigma < r$, then $\alpha_{\sigma'} = \alpha_\sigma + 1$.

(7) $\alpha_\emptyset > r$.

An $\mathcal{A}$-system of level 0 is also called simply an $\mathcal{A}$-*system*.

Now κ contains no infinite decreasing sequences. Thus, if (U, α, D) is an $\mathcal{A}$-system and if we choose $u_1 \in U_\emptyset$, then $u_2 \in U_{u_1}$, then $u_3 \in U_{u_1u_2}$ and so on, we shall eventually have a finite sequence $\sigma = u_1 u_2 \dots u_{2s-1}$ (where $s \in \mathbb{N}$) for which $\alpha_\sigma = 0$. Saying that the system is of level r means that α assumes all the values $r, r-1, \dots, 1, 0$ on final segments of every sequence of this kind.

The reader might like to know that it is only level 1 systems which are needed in the case in which G is triangle-free. Notice that an $\mathcal{A}$-system of level $r+1$ is also an $\mathcal{A}$-system of level r.

3.4 Definition. A *terminated sequence* of the $\mathcal{A}$-system (U, α, D) is a member of $\mathfrak{S}$ of the form σu, where $\sigma \in D$, $\alpha_\sigma = 0$ and $u \in U_\sigma$.

Notice that if τ is a terminated sequence of the $\mathcal{A}$-system (U, α, D), then $\tau \notin D$. Further, if $\sigma \in D$, and $\alpha_\sigma = 0$, then $l(\sigma)$ is odd, so all terminated sequences are of even length.

3.5 Lemma. *Let $a, b \in S$, let $r \in \omega$ and suppose that $b \notin A_{r+1}(a)$. Then there is an $\mathcal{A}$-system (U, α, D) of level r such that $\{a+\sigma_{odd}, b+\sigma_{even}\} \in E(G)$ for every terminated sequence σ of the system.*

Proof. We prove by induction on r the stronger conclusion that *there is an $\mathcal{A}$-system (U, α, D) of level r such that $\alpha_u = r$ for every $u \in U_\emptyset$ and $\{a + \sigma_{odd}, b + \sigma_{even}\} \in E(G)$ for every terminated sequence σ of the system.*

If $r = 0$, the assumption that $b \notin A_1(a)$ implies that

$$\{u_1 \in S : \{u_2 \in S : \{a + u_1, b + u_2\} \in E(G)\} \in p\} \in p \ .$$

We define an $\mathcal{A}$-system (U, α, D) with the required property as follows:

$$U_\emptyset = \{u_1 \in S : \{u_2 \in S : \{a + u_1, b + u_2\} \in E(G)\} \in p\}$$

and for every $u_1 \in U_\emptyset$, $\alpha_{u_1} = 0$, $U_{u_1} = \{u_2 \in S : \{a + u_1, b + u_2\} \in E(G)\}$, and $D = \{\emptyset\} \cup U_\emptyset$.

We now make the inductive assumption that $r > 0$ and that the lemma holds for $r - 1$. Put

$$U_\emptyset = \{u_1 \in S : \{u_2 \in S : b + u_2 \notin A_r(a + u_1)\} \in p\}$$

and, for each $u_1 \in U_\emptyset$, put $U_{u_1} = \{u_2 \in S : b + u_2 \notin A_r(a + u_1)\}$. Since $b \notin A_{r+1}(a)$, we know that $U_\emptyset \in p$.

By our inductive assumption, for every $u_1 \in U_\emptyset$ and every $u_2 \in U_{u_1}$, there is an $\mathcal{A}$-system $(V^{u_1 u_2}, \beta^{u_1 u_2}, E^{u_1 u_2})$ of level $r-1$ for which $\beta^{u_1 u_2}_v = r-1$ for every $v \in V^{u_1 u_2}_\emptyset$ and $\{a+u_1+\sigma_{odd}, b+u_2+\sigma_{even}\} \in E(G)$ for every terminated sequence σ of the system.

Now, let

$$D = \{\emptyset\} \cup U_\emptyset \cup \{u_1 u_2 \sigma : u_1 \in U_\emptyset \ , u_2 \in U_{u_1}, \text{ and } \sigma \in E^{u_1 u_2}\} \ .$$

For every $u_1 \in U_\emptyset$ and $u_2 \in U_{u_1}$ we put $\alpha_{u_1} = \alpha_{u_1 u_2} = r$ and for every $\sigma \in E^{u_1 u_2}$ we put $U_{u_1 u_2 \sigma} = V^{u_1 u_2}_\sigma$ and $\alpha_{u_1 u_2 \sigma} = \beta^{u_1 u_2}_\sigma$.

It is routine to check that the $\mathcal{A}$-system (U, α, D) is as required. $\square$

The next lemma provides the essential fact connecting $\mathcal{A}$-systems and graphs.

3.6 Lemma. *Let $r \in \omega$ and let $a, b \in S$ such that $b \in \left(\bigcap_{i=0}^r A_i(a)\right) \setminus A(a)$. Then there is an $\mathcal{A}$-system of level r such that $\{a + \sigma_{odd}, b + \sigma_{even}\} \in E(G)$ for every terminated sequence σ of the system.*

Proof. Let $\gamma = \min\{\delta < \kappa : b \notin A_\delta(a)\}$. We proceed by induction on γ. If $\gamma = r + 1$, its lowest possible value, the claim is true by Lemma 3.5.

Assume then that $\gamma > r + 1$ and the statement is true at all smaller ordinals. Note that γ is neither 0 nor a limit ordinal, so $\gamma - 1$ is an ordinal smaller than γ. Let

$$M = \{u_1 \in S : \{u_2 \in S : b + u_2 \notin A_{\gamma-1}(a + u_1)\} \in p\} \ .$$

Then $M \in p$. Further, for $i \in \{0, 1, \ldots, r\}$, we have that $b \in A_{i+1}(a)$ so if

$$M_i = \{u_1 \in S : \{u_2 \in S : b + u_2 \in A_i(a + u_1)\} \in p\} \ ,$$

then $M_i \in p$.

Let $U_\emptyset = M \cap \bigcap_{i=0}^r M_i$ and given $u_1 \in U_\emptyset$, let

$$U_{u_1} = \{u_2 \in S : b + u_2 \in \left(\bigcap_{i=0}^r A_i(a + u_1)\right) \setminus A_{\gamma-1}(a + u_1) \ .$$

Then each $U_{u_1} \in p$

By our inductive assumption, for every $u_1 \in U_\emptyset$ and every $u_2 \in U_{u_1}$, there is an $\mathcal{A}$-system $(V^{u_1 u_2}, \beta^{u_1 u_2}, E^{u_1 u_2})$ of level r such that $\{a + u_1 + \sigma_{odd}, b + u_2 + \sigma_{even}\} \in E(G)$ for every terminated sequence σ of the system.

Let

$$D = \{\emptyset\} \cup U_\emptyset \cup \{u_1 u_2 \sigma : U_1 \in U_\emptyset \ , u_2 \in U_{u_1}, \text{ and } \sigma \in E^{u_1 u_2}\} \ .$$

For any $u_1 \in U_\emptyset$, $u_2 \in U_{u_1}$, and $\sigma \in E^{u_1 u_2}$, let $U_{u_1 u_2 \sigma} = V^{u_1 u_2}_\sigma$ and, if $\sigma \neq \emptyset$, let $\alpha_{u_1 u_2 \sigma} = \beta^{u_1 u_2}_\sigma$.

It remains only to define α_{u_1} and $\alpha_{u_1u_2}$ for $u_1 \in U_\emptyset$ and $u_2 \in U_{u_1}$. Since cf $|S| < \kappa$, we can choose $\nu \in \kappa$ satisfying

$$\nu > \beta_v^{u_1u_2} \text{ whenever } u_1 \in U_\emptyset \,, u_2 \in U_{u_1} \text{ and } v \in E_\emptyset^{u_1u_2} \,.$$

For each $u_1 \in U_\emptyset$ and $u_2 \in U_{u_1}$, let $\alpha_{u_1} = \alpha_{u_1u_2} = \nu$.

It is easy to see that the $\mathcal{A}$-system (U, α, D) is as required. $\square$

3.7 Lemma. *Let (U, α, D) be an $\mathcal{A}$-system of level* 1 *and let $B \in p$. Suppose that $\mu \in D$ and $\mu = \emptyset$ or $\alpha_\mu \geq 1$. Then we can choose $\sigma, \tau \in \mathfrak{S}$ satisfying the following conditions:*

(1) $\mu\sigma$ is in D and $\mu\tau$ is a terminated sequence of the system;

(2) $\alpha_{\mu\sigma} = 0$;

(3) σ_{odd}, τ_{odd} and τ_{even} are all in B. Furthermore, $\sigma_{even} \in B$ if $l(\sigma) > 1$.
We can also choose $\rho \in D$ such that $l(\rho)\in 2\mathbb{N}$, $\alpha_\rho = 1$, and ρ_{odd} and ρ_{even} are in B.

Proof. We choose $v_1 \in U_\mu \cap B^*$ and observe that $\mu v_1 \in D$.

We now make the inductive assumption that, for some $t \in \mathbb{N}$, we have chosen $\nu^t = v_1v_2 \ldots v_t$ such that $\mu\nu^t \in D$ and $\nu^t_{odd} \in B^*$ and also $\nu^t_{even} \in B^*$ if $l(\nu^t) > 1$. If t is even, we choose $v_{t+1} \in U_{\mu\nu^t} \cap (-\nu^t_{odd} + B^*)$. If t is odd, we choose $v_{t+1} \in U_{\mu\nu^t} \cap (-\nu^t_{even} + B^*)$ if $l(\nu^t) > 1$ and $v_{t+1} \in U_{\mu\nu^t} \cap B^*$ if $l(\nu^t) = 1$. If $\alpha_{\mu\nu^t} = 0$, we stop and put $\sigma = \nu^t$ and $\tau = \nu^t v_{t+1}$. If $\alpha_{\mu\nu^t} > 0$, we repeat this process with $\nu^{t+1} = v_1v_2 \ldots v_{t+1}$ in place of ν^t.

We shall eventually construct sequences σ, τ satisfying the required conditions.

In the event that $\mu = \emptyset$, we observe that we then have $l(\sigma) \geq 3$ and we put $\rho = \sigma'$. $\square$

It is possible to prove Lemma 3.9 below with an induction grounded at $n = 2$ which includes the case $n = 3$. However, the proof of Lemma 3.9 is quite complicated. For the sake of the reader who is only interested in the case in which G is triangle-free, we present a separate proof of the $n = 3$ instance of Lemma 3.9. Lemma 3.9 is not needed for the triangle-free case.

3.8 Lemma. *Let $B \in p$ and for each $i \in \{1, 2, 3\}$, let $\mathcal{A}^i = (U^i, \alpha^i, D^i)$ be an $\mathcal{A}$-system of level 1. Then there exist $a^1, a^2, a^3 \in B$ and, for each $i \in \{1, 2, 3\}$, there exists a* *terminated sequence* σ^i *of* (U^i, α^i, D^i) *such that*

$$\begin{aligned} a^1 &= \sigma^1_{odd} \\ &= \sigma^2_{odd} \\ \\ a^2 &= \sigma^1_{even} \\ &= \sigma^3_{odd} \\ \\ a^3 &= \sigma^2_{even} \\ &= \sigma^3_{even}. \end{aligned}$$

Proof. We apply Lemma 3.7 to $\mathcal{A}^1$ with $\mu = \emptyset$, and choose $\tau^1 \in D^1$ satisfying the following conditions:

$l(\tau^1)$ is even;

$\alpha^1_{\tau^1} = 1$;

$\tau^1_{odd} \in B^* \cap U^2_\emptyset$ and $\tau^1_{even} \in B^* \cap U^3_\emptyset$.

We then put $u^2_1 = \tau^1_{odd}$ and $u^3_1 = \tau^1_{even}$.

We can apply Lemma 3.7 to $\mathcal{A}^2$ with $\mu = u^2_1$. (Since $u^2_1 \in U^2_\emptyset$ and $\mathcal{A}^2$ is of level 1, $\alpha^2_\mu \geq 1$.) We choose $\tau^2 \in \mathfrak{S}$ satisfying the following conditions:

$u^2_1\tau^2 \in D^2$;

$\alpha^2_{u^2_1\tau^2} = 0$;

$\tau^2_{even} \in U^1_{\tau^1} \cap (-u^2_1 + B^*)$ and $\tau^2_{odd} \in B^* \cap U^3_{u^3_1}$.

We then put $x = \tau^2_{even}$ and $u^3_2 = \tau^2_{odd}$.

We can now apply Lemma 3.7 to $\mathcal{A}^3$ with $\mu = u^3_1u^3_2$, since $\alpha^3_\mu \geq 1$. We choose $\tau^3 \in \mathfrak{S}$ satisfying the following conditions:

$u^3_1u^3_2\tau^3$ is a terminated sequence sequence of $\mathcal{A}^3$;

$\tau^3_{odd} \in U^1_{\tau^1x} \cap (-u^3_1 + B^*)$ and $\tau^3_{even} \in U^2_{u^2_1\tau^2} \cap (-u^3_2 + B^*)$.

We put $y = \tau^3_{odd}$ and $z = \tau^3_{even}$.

We now put $\sigma^1 = \tau^1xy, \sigma^2 = u^2_1\tau^2z$ and $\sigma^3 = u^3_1u^3_2\tau^3$, and observe that these are terminated sequences of $\mathcal{A}^1, \mathcal{A}^2$ and $\mathcal{A}^3$ respectively.

We put $a^1 = \sigma^1_{odd}, a^2 = \sigma^1_{even}$ and $a^3 = \sigma^2_{even}$ and observe that these are all in B. Furthermore,

$\sigma^1_{odd} = \tau^1_{odd} + x = u^2_1 + \tau^2_{even} = \sigma^2_{odd}$;

$\sigma^1_{even} = \tau^1_{even} + y = u^3_1 + \tau^3_{odd} = \sigma^3_{odd}$;

$\sigma^2_{even} = \tau^2_{odd} + z = u^3_2 + \tau^3_{even} = \sigma^3_{even}$. □

3.9 Lemma. *Let* $n \in \mathbb{N}$ *satisfy* $n \geq 3$ *and let* $B \in p$. *Suppose that, for every* i, j *in* $\{1, 2, \ldots, n\}$ *with* $i < j$, *we have an* $\mathcal{A}$*-system* $\mathcal{A}^{i,j} = (U^{i,j}, \alpha^{i,j}, D^{i,j})$ *of level* $n - 2$.

Then there are elements $a^1, a^2, \ldots, a^n$ of B and for every $i, j \in \{1, 2, \ldots, n\}$ with $i < j$, there is a terminated sequence $\sigma^{i,j}$ of $\mathcal{A}^{i,j}$ such that $a^i = \sigma^{i,j}_{odd}$ and $a^j = \sigma^{i,j}_{even}$.

Proof. We shall prove this by induction on n. We observe that the case in which $n = 3$ is true by Lemma 3.8 . Thus we shall suppose that $n > 3$ and that our lemma has been established for $n - 1$.

Now if (U, α, D) is any $\mathcal{A}$-system of level $n - 2$, we can define a reduced $\mathcal{A}$-system (V, β, E) of level $n - 3$ in the following way: We put $E = \{(\sigma')' : \sigma \in D \text{ and } l(\sigma) > 1\}$. So $E \subseteq D$ and $\tau \in E$ implies that $\alpha_\tau > 0$. If $\tau \in E$, we put $V_\tau = U_\tau$ and $\beta_\tau = \alpha_\tau - 1$ if α_τ is in $\mathbb{N}$ and $\beta_\tau = \alpha_\tau$ otherwise. We observe that the terminated sequences of our reduced system have the form $(\sigma')'$, where σ denotes a terminated sequence of the original system.

We apply our inductive hypothesis to the reduced systems obtained from the $\mathcal{A}$-systems $\mathcal{A}^{i,j}$, where $i, j \in \{1, 2, \ldots, n-1\}$ and $i < j$. We deduce that there are sequences $\tau^{i,j}$ and elements b^i of S, defined whenever $i, j \in \{1, 2, \ldots, n-1\}$ and $i < j$, satisfying the following conditions:

(a) $\tau^{i,j} \in D^{i,j}$;
(b) $\alpha^{i,j}_{\tau^{i,j}} = 1$;
(c) $l(\tau^{i,j})$ is even;
(d) $b^i = \tau^{i,j}_{odd}$ if $i < j$;
(e) $b^i = \tau^{j,i}_{even}$ if $j < i$; and
(f) $b^i \in B^* \cap \bigcap_{j=1}^{n-1} U^{j,n}_{\emptyset}$.

We show that we can choose for each $i \in \{1, 2, \ldots, n-1\}$, a terminated sequence $\sigma^{i,n}$ of $\mathcal{A}^{i,n}$ and $x^i \in S$ such that for each $i \in \{1, 2, \ldots, n-1\}$:

(1) the first term of $\sigma^{i,n}$ is b^i;
(2) $\sigma^{i,n}_{even} = \sigma^{1,n}_{even} \in B$;
(3) $x^i = \sigma^{i,n}_{odd,>1}$;
(4) $x^1 \in (-b^1 + B) \cap \bigcap_{j=2}^{n-1} U^{1,j}_{\tau^{1,j}}$;
(5) if $i \in \{2, 3, \ldots, n-2\}$, then $x^i \in (-b^i + B) \cap \bigcap_{j=1}^{i-1} U^{j,i}_{\tau^{j,i} x^j} \cap \bigcap_{j=i+1}^{n-1} U^{i,j}_{\tau^{i,j}}$; and
(6) $x^{n-1} \in (-b^{n-1} + B) \cap \bigcap_{j=1}^{n-2} U^{j,n-1}_{\tau^{j,n-1} x^j}$.

Before showing that we can do this, let us verify that this is enough. Indeed, assume we have chosen $\sigma^{i,n}$ and x^i satisfying (1) through (6). For $i, j \in \{1, 2, \ldots, n-1\}$ with $i < j$, let $\sigma^{i,j} = \tau^{i,j} x^i x^j$ and note that, since $x^i \in U^{i,j}_{\tau^{i,j}}$ and $x^j \in U^{i,j}_{\tau^{i,j} x^i}$, $\sigma^{i,j}$ is a terminated sequence of $\mathcal{A}^{i,j}$.

For $i \in \{1, 2, \ldots, n-1\}$, let $a^i = b^i + x^i$. Since $x^i \in -b^i + B$, we have $a^i \in B$. Let $a^n = \sigma^{1,n}_{even}$. If $i \in \{1, 2, \ldots, n-1\}$, we have by (2) that $\sigma^{i,n}_{even} = a^n$. Also, $\sigma^{i,n}_{odd} = b^i + \sigma^{i,n}_{odd,>1} = b^i + x^i = a^i$.

Finally, assume that $i, j \in \{1, 2, \ldots, n-1\}$ and $i < j$. Then

$$\sigma^{i,j}_{odd} = \tau^{i,j}_{odd} + x^i = b^i + x^i = a^i$$

and

$$\sigma^{i,j}_{even} = \tau^{i,j}_{even} + x^j = b^j + x^j = a^j \ .$$

Now we proceed to the construction. For $i \in \{1, 2, \ldots, n-1\}$, let $u^i_1 = b^i$. We will inductively construct u^i_2, $\nu^i = u^i_3 u^i_4 \ldots u^i_{\ell(i)-1}$, and $z^i = u^i_{\ell(i)}$, let $\sigma^{i,n} = b^i u^i_2 \nu^i z^i = u^i_1 u^i_2 \ldots u^i_{\ell(i)}$, and let $x^i = \nu^i_{odd} = \sigma^{i,n}_{odd,>1}$ as required by (3). The process of guaranteeing that (2) holds is where the complication lies. We diagram below the final assignments for the case $n = 5$. Each of the boxes spans two lines, and the sum of the items inside the box on each line is set equal to the item in the other line, proceeding down on the left, and then up on the right.

$$\begin{array}{ccccc}
u^1_2 & + & u^1_4 + \ldots + u^1_{\ell(1)-2} & + & u^1_{\ell(1)} \\
u^2_2 & + & u^2_4 + \ldots + u^2_{\ell(2)-2} & + & u^2_{\ell(2)} \\
u^3_2 & + & u^3_4 + \ldots + u^3_{\ell(3)-2} & + & u^3_{\ell(3)} \\
u^4_2 & + & u^4_4 + \ldots + u^4_{\ell(4)-2} & + & u^4_{\ell(4)}
\end{array}$$

We shall apply Lemma 3.7 several times. In order to do so, we observe that we are dealing with $\mathcal{A}$-systems of level 2, since we are assuming that $n > 3$ and that our systems have level $n-2$. Now, if (U, α, D) is any $\mathcal{A}$-system of level 2 and if we choose $u_1 \in U_\emptyset$ and $u_2 \in U_{u_1}$ and put $\mu = u_1 u_2$, then $\alpha_\mu \geq 2$. It follows that the sequence σ guaranteed by Lemma 3.7 has length at least 3.

Recall that we are given $u^1_1 = b^1$. We choose any $u^1_2 \in B^* \cap \bigcap_{j=1}^{n-1} \left(U^{j,n}_{b^j}\right)^*$.

We apply Lemma 3.7 to $\mathcal{A}^{1,n}$ with $\mu = b^1 u^1_2$, and choose a sequence $\nu^1 \in \mathfrak{S}$ such that

$\mu\nu^1 \in D^{1,n}$, $\alpha^{1,n}_{\mu\nu^1} = 0$,
$\nu^1_{odd} \in U^{1,j}_{\tau^{1,j}} \cap (-b^1 + B^*)$ for every $j \in \{2, 3, \ldots, n-1\}$ and
$\nu^1_{even} \in -u^1_2 + \left((U^{j,n}_{b^j})^* \cap B^*\right)$ for every $j \in \{2, 3, \ldots, n-1\}$.

We put $x^1 = \nu^1_{odd}$, $y^1 = \nu^1_{even}$ and $u^2_2 = u^1_2 + \nu^1_{even}$.

We now suppose that $m \in \{2, 3, \ldots, n-1\}$ and that, for each $i \in \{1, 2, \ldots, m-1\}$, we have defined $u_2^i \in U_{b^i}^{i,n}$ and $\nu^i \in \mathfrak{S}$. We put $x^i = \nu^i_{odd}$ and $y^i = {\nu^i}_{even}$ and suppose that each of the following conditions is satisfied for every $i \in \{1, 2, \ldots, m-1\}$:

1) $b^i u_2^i \nu^i \in D^{i,n}$;
2) $\alpha^{i,n}_{b^i u_2^i \nu^i} = 0$;
3) $x^i \in U^{i,j}_{\tau^{i,j}}$ for every $j \in \{i+1, i+2, \ldots, n-1\}$;
4) $x^i \in U^{k,i}_{\tau^{k,i} x^k}$ for every $k \in \{1, 2, \ldots, i-1\}$;
5) $b^i + x^i \in B$;
6) If $i > 1$, then $u_2^i = u_2^{i-1} + y^{i-1}$;
7) $u_2^i + y^i \in B^* \cap \big(U_{b^j}^{j,n}\big)^*$ for every $j \in \{i+1, i+2, \ldots, n-1\}$;
8) $y^{j+1} + y^{j+2} + \ldots y^i \in \big(U^{j,n}_{b^j u_2^j \nu^j}\big)^* \cap \big(-u_2^{j+1} + B^*\big)$ whenever $j < i \in \{1, 2, \ldots, m-1\}$.

We show how to continue the construction.

We put $u_2^m = u_2^{m-1} + y^{m-1}$ and observe that $u_2^m \in (U_{b^j}^{j,n})^* \cap B^*$ for every $j \in \{m, m+1, \ldots, n-1\}$, by condition 7).

We now apply Lemma 3.7 with $\mu = b^m u_2^m$, to find a sequence $\nu^m \in \mathfrak{S}$ satisfying each of the above eight conditions with i replaced by m, where x^m denotes ν^m_{odd} and y^m denotes ν^m_{even}. That there is a sequence ν^m satisfying conditions 1), 2), 3), 4), 5), 7) and 8) is guaranteed by Lemma 3.7 and the observation that each of the last five of these conditions states that x^m or y^m lies in a certain member of p. (If m=2, then condition 8) says that $y^2 \in (U^{1,n}_{b^1 u_2^1 \nu^1})^* \cap (-u_2^2 + B^*)$. If $m > 2$, condition 8) says that $y^m \in (U^{m-1,n}_{b^{m-1} u_2^{m-1} \nu^{m-1}})^* \cap (-u_2^m + B^*)$ and that for each $i \in \{1, 2, \ldots, m-2\}$, $y^m \in -(y^{i+1} + y^{i+2} + \ldots + y^{m-1}) + \big((U^{i,n}_{b^i u_2^i \nu^i})^* \cap (-u_2^{i+1} + B^*)\big)$; also $(U^{m-1,n}_{b^{m-1} u_2^{m-1} \nu^{m-1}})^* \cap (-u_2^m + B^*) \in p$ by conditions 1), 6), and 7) while $-(y^{i+1} + y^{i+2} + \ldots + y^{m-1}) + \big((U^{i,n}_{b^i u_2^i \nu^i})^* \cap (-u_2^{i+1} + B^*)\big) \in p$ by condition 8).) Condition 6) is true by the definition of u_2^m.

Thus we can define $\nu^i \in \mathfrak{S}$ inductively for every $i \in \{1, 2, \ldots, n-1\}$ so that properties 1)-8) are satisfied.

By property 8), we can choose $z^{n-1} \in U^{n-1,n}_{b^{n-1} u_2^{n-1} \nu^{n-1}}$ satisfying

$z^{n-1} \in -(y^{i+1} + y^{i+2} + \ldots + y^{n-1}) + \big(U^{i,n}_{b^i u_2^i \nu^i} \cap (-u_2^{i+1} + B)\big)$ for every $i \in \{1, 2, \ldots, n-2\}$.

For each $i \in \{1, 2, \ldots, n-2\}$, we put $z^i = y^{i+1} + y^{i+2} + \ldots + y^{n-1} + z^{n-1} \in U^{i,n}_{b^i u^i \nu^i} \cap \big(-u_2^{i+1} + B\big)$. Clearly, $y^i + z^i = z^{i-1}$ if $i > 1$.

We then put $\sigma^{i,n} = b^i u_2^i \nu^i z^i$ for each $i \in \{1, 2, \ldots, n-1\}$, and observe that this is a terminated sequence of $\mathcal{A}^{i,n}$ by condition 2).

We have $\sigma_{even}^{i,n} = u_2^i + \nu_{even}^i + z^i = u_2^i + y^i + z^i = u_2^i + z^{i-1} = u_2^{i-1} + y^{i-1} + z^{i-1} = \sigma_{even}^{i-1,n}$ if $i > 1$.

Hence $\sigma_{even}^{i,n} = \sigma_{even}^{1,n}$ for every $i \in \{1, 2, \ldots, n-1\}$.

We also have $\sigma_{even}^{i,n} = u_2^i + z^{i-1} \in B$ if $i > 1$.

Thus we have established our lemma. $\square$

We now embark on a sequence of lemmas establishing that certain sets must belong to p if G has no K_m.

3.10 Lemma. *Suppose that $m \in \mathbb{N}$ and that G contains no K_m. Then, for every $r \in \omega$, it is impossible to find elements $a^1, a^2, \ldots, a^m$ of S such that $a^i \notin A_r(a^j)$ whenever $i < j$ in $\{1, 2, \ldots, m\}$.*

Proof. We prove this by induction on r. The case $r = 0$ is immediate from the assumption that G contains no K_m, and so we may suppose that $r > 0$ and that the lemma holds for $r - 1$.

Assume that we do have elements $a^1, a^2, \ldots, a^m$ of S such that $a^i \notin A_r(a^j)$ whenever $i < j$ in $\{1, 2, \ldots, m\}$. Let

$$U = \bigcap_{i=1}^{m-1} \bigcap_{j=i+1}^{m} \{u \in S : \{v \in S : a^i + v \notin A_{r-1}(a^j + u)\} \in p\} \ .$$

Then $U \in p$. Choose $b^m \in U$. Inductively, let $i \in \{1, 2, \ldots, m-1\}$, assume $b^{i+1}, b^{i+2}, \ldots, b^m$ have been chosen, and choose

$$b^i \in U \cap \bigcap_{j=i+1}^{m} \{v \in S : a^i + v \notin A_{r-1}(a^j + b^j)\} \ .$$

Then $a^i + b^i \notin A_{r-1}(a^j + b^j)$ whenever $i < j$ in $\{1, 2, \ldots, m-1\}$, contradicting our induction hypothesis. $\square$

3.11 Lemma. *Let $m \in \mathbb{N}$ and assume that G contains no K_m. Then, for every $r \in \omega$, $\{a \in S : A_r(a) \in p\} \in p$.*

Proof. Suppose instead that for some $r \in \omega$ we have $B = \{a \in S : A_r(a) \notin p\} \in p$. Choose $a^m \in B$ and for $i \in \{1, 2, \ldots, m-1\}$, choose $a^i \in B \setminus \bigcup_{j=i+1}^{m} A_r(a^j)$. This contradicts Lemma 3.10. $\square$

3.12 Lemma. *Let $m \in \mathbb{N}$ and assume that G contains no K_m. Then there do not exist elements $a^1, a^2, \ldots, a^m$ of S, such that $a^j \in \bigcap_{r=0}^{m-2} A_r(a^i) \setminus A(a^i)$ whenever $i < j$ in $\{1, 2, \ldots, m\}$.*

Proof. Suppose, on the contrary, that we do have elements with this property. Then, by Lemma 3.6, there is an $\mathcal{A}$-system $\mathcal{A}^{i,j}$ of level $m-2$, defined whenever $i<j$ in $\{1,2,\ldots,m\}$, such that $\{a^i+\sigma_{odd}, a^j+\sigma_{even}\}\in E(G)$ for every terminated sequence σ of $\mathcal{A}^{ij}$. By Lemma 3.9 (or just Lemma 3.8, if $m=3$), for every $i<j$ in $\{1,2,\ldots,m\}$, there is a terminated sequence $\sigma^{i,j}$ of $\mathcal{A}^{ij}$ and there are elements $b^1,b^2,\ldots,b^m$ of S such that $b^i=\sigma^{i,j}_{odd}$ whenever $i<j$ and $b^i=\sigma^{j,i}_{even}$ whenever $j<i$. We then have $\{a^i+b^i,a^j+b^j\}\in E(G)$ whenever $i<j$, contradicting our assumption that G contains no K_m. □

3.13 Lemma. *Let $m\in\mathbb{N}$ and suppose that G contains no K_m. Then*

$$\{a\in S: A(a)\in p\}\in p\ .$$

Proof. First observe that $\{a\in S:\bigcap_{r=0}^{m-2}A_r(a)\backslash A(a)\notin p\}\in p$. (For suppose instead that $B=\{a\in S:\bigcap_{r=0}^{m-2}A_r(a)\backslash A(a)\in p\}\in p$. Choose $a^1\in B$ and for $j\in\{2,3,\ldots,m\}$, choose $a^j\in B\cap\bigcap_{i=1}^{j-1}\bigcap_{r=0}^{m-2}A_r(a^i)\backslash A(a^i)$. This contradicts Lemma 3.12.)

Also by Lemma 3.11, $\bigcap_{r=0}^{m-2}\{a\in S:A_r(a)\in p\}\in p$. Since

$$\{a\in S:\textstyle\bigcap_{r=0}^{m-2}A_r(a)\backslash A(a)\notin p\}\cap\bigcap_{r=0}^{m-2}\{a\in S:A_r(a)\in p\}\subseteq\{a\in S:A(a)\in p\}$$

we are done. □

3.14 Definition. Let $a\in S$ and let $r\in\omega$. We put $B(a)=\{b\in S:a\in A(b)\}$ and $B_r(a)=\{b\in S:a\in A_r(b)\}$.

3.15 Lemma. *Let $m\in\mathbb{N}$ and assume that G contains no K_m. Then, for every $r\in\omega$, $\{a\in S:B_r(a)\in p\}\in p$. Furthermore, $\{a\in S:B(a)\in p\}\in p$.*

Proof. For the first assertion, suppose instead that we have some $r\in\omega$ such that $C=\{a\in S:B_r(a)\notin p\}\in p$. Choose $a^1\in C$ and for $j\in\{2,3,\ldots,m\}$, choose $a^j\in C\backslash\bigcup_{i=1}^{j-1}B_r(a^i)$. Then for $i<j$ in $\{1,2,\ldots,m\}$ one has $a^i\notin A_r(a^j)$, contradicting Lemma 3.10.

We claim also that $\{a\in S:\bigcap_{r=0}^{m-2}B_r(a)\backslash B(a)\notin p\}\in p$. Suppose instead that $D=\{a\in S:\bigcap_{r=0}^{m-2}B_r(a)\backslash B(a)\in p\}\in p$. Pick $a^m\in D$ and for $i\in\{1,2,\ldots,m-1\}$, pick $a^i\in D\cap\bigcap_{j=i+1}^{m}\bigcap_{r=0}^{m-2}B_r(a^j)\backslash B(a^j)$. Then for $i<j$ in $\{1,2,\ldots,m\}$, $a^j\in\bigcap_{r=0}^{m-2}A_r(a^i)\backslash A(a^i)$, contradicting Lemma 3.12.

Since

$$\{a\in S:\textstyle\bigcap_{r=0}^{m-2}B_r(a)\backslash B(a)\notin p\}\cap\bigcap_{r=0}^{m-2}\{a\in S:B_r(a)\in p\}\subseteq\{a\in S:B(a)\in p\}$$

we are done. □

3.16 Lemma. *Let $a\in S$ and let $b\in A(a)$. Then $\{u\in S:b\notin A_0(a+u)\setminus A(a+u)\}\in p$.*

Proof. Suppose, on the contrary, that $U = \{u \in S : b \in A_0(a+u) \setminus A(a+u)\} \in p$. For each $u \in U$, let δ_u be the first ordinal for which $b \notin A_{\delta_u}(a+u)$. We observe that δ_u is neither 0 nor a limit ordinal. Let $V(u) = \{u' : \{v : b+v \notin A_{\delta_u - 1}(a+u+u')\} \in p\}$ and note that $V(u) \in p$.

Pick a limit ordinal $\lambda < \kappa$ such that $\delta_u < \lambda$ for all $u \in U$. Now $b \in A_{\lambda+1}(a)$ so if

$$W = \{w : \{v : b+v \in A_\lambda(a+w)\} \in p\} ,$$

then $W \in p = p+p$ so $\{u : -u+W \in p\} \in p$. Pick $u \in U$ such that $-u+W \in p$ and pick $u' \in V(u) \cap (-u+W)$. Then $u+u' \in W$ so $\{v : b+v \in A_\lambda(a+u+u')\} \in p$ and $u' \in V(u)$ so $\{v : b+v \notin A_{\delta_u-1}(a+u+u')\} \in p$. Since $\lambda > \delta_u - 1$, this is a contradiction. $\square$

3.17 Lemma. *Let $a \in S$ and let $b \in A(a)$. Then $\{u \in S : -b + A(a+u) \in p\} \in p$.*

Proof. Suppose, on the contrary, that $U = \{u \in S : -b + A(a+u) \notin p\} \in p$. For each $u \in U$ let $V(u) = \{v \in S : b+v \notin A(a+u)\}$ and note that $V(u) \in p$. Thus, for each $u \in U$ and $v \in V(u)$, there exists an ordinal $\delta_{u,v} \in \kappa$ such that $b+v \notin A_{\delta_{u,v}}(a+u)$. Let $\lambda < \kappa$ be a limit ordinal satisfying $\lambda > \delta_{u,v}$ whenever $u \in U$ and $v \in V(u)$. Then $b+v \notin A_\lambda(a+u)$ whenever $u \in U$ and $v \in V(u)$. This implies that $b \notin A_{\lambda+1}(a)$, contradicting our assumption that $b \in A(a)$. $\square$

3.18 Lemma. *Suppose that $m \in \mathbb{N}$ and that G contains no K_m. Then*

$$\{a \in S : \{b \in S : a \in A(b)^*\} \in p\} \in p .$$

Proof. Suppose on the contrary that $\{a \in S : \{b \in S : a \notin A(b)^*\} \in p\} \in p$. Let

$$U = \{a \in S : \{b \in S : a \notin A(b)^*\} \in p\} \cap \{a \in S : B(a) \in p\} .$$

By Lemma 3.15, $U \in p$. Pick $a \in U$ and let $V = \{b \in S : a \notin A(b)^*\}$ and pick $b \in V^* \cap B(a)^*$. Then $b \in B(a)$ so $a \in A(b)$ so by Lemma 3.17 with a and b interchanged we have $W = \{w \in S : -a + A(b+w) \in p\} \in p$. Choose $w \in W \cap (-b+V) \cap \big(-b+B(a)\big)$. Since $b+w \in B(a)$, we have $a \in A(b+w)$. Since $w \in W$, we have $-a + A(b+w) \in p$. Thus, $a \in A(b+w)^*$, contradicting the fact that $b+w \in V$. $\square$

We are finally in a position to prove the main theorem of this section.

3.19 Theorem. *Let $m \in \mathbb{N}$ and suppose that G contains no K_m. Let $P \in p$. Then there is a sequence $\langle x_n \rangle_{n=1}^\infty$ in S such that $FS(\langle x_n \rangle_{n=1}^\infty) \subseteq p$ and whenever $F, H \in \mathcal{P}_f(\mathbb{N})$ with $F \cap H = \emptyset$, one has $\{\Sigma_{n\in F}\, x_n, \Sigma_{n\in H}\, x_n\} \notin E(G)$.*

Proof. By Lemmas 3.13 and 3.18, we may presume that

$$P \subseteq \{a \in S : A(a) \in p\} \cap \{a \in S : \{b \in S : a \in A(b)^*\} \in p\} \ .$$

Given a finite sequence $\langle x_t \rangle_{t=1}^n$ in S and $a, b \in S$, we write $a \perp b$ if and only if there exist disjoint sets F and H in $\mathcal{P}_f(\{1, 2, \ldots, n\})$ such that $a = \Sigma_{t \in F}\, x_t$ and $b = \Sigma_{t \in H}\, x_t$. (The notation depends on the choice of the sequence $\langle x_t \rangle_{t=1}^n$, but the particular sequence that we have in mind will be clear from the context.)

Choose $x_1 \in P^*$. Let $n \in \mathbb{N}$, and assume that we have chosen $x_1, x_2, \ldots, x_n \in S$ such that

(a) $FS(\langle x_t \rangle_{t=1}^n) \subseteq P^*$ and

(b) whenever $a, b \in FS(\langle x_t \rangle_{t=1}^n)$ with $a \perp b$, one has $a \in A(b)^*$.

Let $E = FS(\langle x_t \rangle_{t=1}^n)$. By Lemma 1.8, we have $P^* \in p$ and for each $a \in E$, $-a + P^* \in p$. Further, for each $a \in E$, $a \in P$ so

$$A(a) \in p \ , A(a)^* \in p \ , \text{ and } \{x \in S : a \in A(x)^*\} \in p \ .$$

Also, given $a, b \in E$ with $a \perp b$, we have by assumption that $a \in A(b)^*$ so by Lemma 1.8, $-a + A(b)^* \in p$.

Now we claim that, given $a, b \in E$ with $a \perp b$, we have $\{x \in S : b \in A(a+x)^*\} \in p$. First, by Lemma 3.16, $\{x \in S : b \notin A_0(a+x) \backslash A(a+x)\} \in p$ so either $\{x \in S : b \notin A_0(a+x)\} \in p$ or $\{x \in S : b \in A(a+x)\} \in p$. But

$$\begin{aligned} \{x \in S : b \notin A_0(a+x)\} &= \{x \in S : \{b, a+x\} \in E(G)\} \\ &= \{x \in S : a + x \notin A_0(b)\} \\ &= S \backslash \big(-a + A_0(b)\big) \ . \end{aligned}$$

Since $a \in A(b)^* \subseteq A_0(b)^*$ we have that $-a + A_0(b) \in p$. Thus $\{x \in S : b \in A(a+x)\} \in p$. Also $b \in A(a)$ so by Lemma 3.17, $\{x \in S : -b + A(a+x) \in p\} \in p$. Thus, $\{x \in S : b \in A(a+x)^*\} \in p$.

Now choose

$$\begin{aligned} x_{n+1} \in\ & P^* \cap \textstyle\bigcap_{a \in E} \big((-a + P^*) \cap \{x \in S : a \in A(x)^*\} \cap A(a)^*\big) \\ & \cap \textstyle\bigcap \{(-a + A(b)^*) \cap \{x \in S : b \in A(a+x)^*\} : a, b \in E \text{ and } a \perp b\} \ . \end{aligned}$$

Since $x_{n+1} \in P^* \cap \bigcap_{a \in E}(-a + P^*)$, we have that $FS(\langle x_t \rangle_{t=1}^{n+1}) \subseteq P^*$.

Let $a, b \in FS(\langle x_t \rangle_{t=1}^{n+1})$ with $a \perp b$. If $a, b \in FS(\langle x_t \rangle_{t=1}^n)$, there is nothing to show so assume without loss of generality that either $a = x_{n+1}$ or $a = a' + x_{n+1}$ for some $a' \in E$. Since $a \perp b$ we have $b \in E$. If $a = x_{n+1}$ we have directly that $x_{n+1} \in A(b)^*$ and

$b \in A(x_{n+1})^*$. If $a = a' + x_{n+1}$, then directly $b \in A(a' + x_{n+1})$ and $x_{n+1} \in -a' + A(b)^*$ so that $a' + x_{n+1} \in A(b)^*$. $\square$

As we promised earlier, we see that a sequence with independent finite sums can be found "inside" any given sequence. For this corollary, we need to drop our standing assumption about having fixed an idempotent $p \in \beta S$. (We choose an idempotent in the proof.) Also, strictly speaking Corollary 3.20 is not a corollary to Theorem 3.19, but is rather a corollary to its proof.

3.20 Corollary. *Let $m \in \mathbb{N}$ and suppose that G contains no K_m and let $\langle y_n \rangle_{n=1}^{\infty}$ be a sequence in S. There is a sum subsystem $\langle x_n \rangle_{n=1}^{\infty}$ of $\langle y_n \rangle_{n=1}^{\infty}$ such that whenever $F, H \in \mathcal{P}_f(\mathbb{N})$ with $F \cap H = \emptyset$, one has $\{\Sigma_{n \in F}\, x_n, \Sigma_{n \in H}\, x_n\} \notin E(G)$.*

Proof. By Theorem 1.9, pick an idempotent $p \in \beta S$ such that for every $n \in \mathbb{N}$, $FS(\langle y_k \rangle_{k=n}^{\infty}) \in p$.

We show how to modify the proof of Theorem 3.19. To start, let $k(1) = 1$ and pick $x_1 \in FS(\langle y_t \rangle_{t=1}^{\infty}) \cap P^*$. Pick $H_1 \in \mathcal{P}_f(\mathbb{N})$ such that $x_1 = \Sigma_{t \in H_1}\, y_t$ and let $k(2) = \max H_1 + 1$. At stage n in the construction require that $x_n \in FS(\langle y_t \rangle_{t=k(n)}^{\infty})$ (in addition to all of the other sets specified in that proof). Pick $H_n \in \mathcal{P}_f(\mathbb{N})$ such that $x_n = \Sigma_{t \in H_n}\, y_t$ and let $k(n+1) = \max H_n + 1$. $\square$

4. When G contains no $\mathbf{K_{m,m}}$.

We continue to assume in this section that we have an infinite (not necessarily commutative) semigroup $(S, +)$, that we have a fixed graph G with vertices in S, and a fixed idempotent $p \in \beta S \backslash S$. We add the assumption that S is cancellative. (We do not know whether this assumption is needed for the main result, Theorem 4.14, but it is required for our proof.)

4.1 Definition. For $k, l \in \mathbb{N}$, let $K_{k,l}$ denote the complete bipartite graph on sets of size k and l. That is, the vertex set of $K_{k,l}$ can be partitioned into disjoint sets C and D, with $|C| = k$ and $|D| = l$, so that the edge set of $K_{k,l}$ is $\big\{\{c, d\} : c \in C, d \in D\big\}$.

One or two of the lemmas in this section could be stated without proof, since they follow from results in Section 3, since a graph which contains no $K_{m,m}$ also contains no K_{2m}. However, all the proofs in this section are relatively simple compared to some of those in Section 3. We have therefore written Section 4 so that it can be read independently of Section 3.

The following definition extends to S the notation used in Section 2 with the semigroup $(\mathbb{N}, +)$ and the sequence $\langle 2^{t-1} \rangle_{t=1}^{\infty}$.

4.2 Definition. Let $\langle x_n \rangle_{n=1}^{\infty}$ be a sequence in S. Given $a, b \in FS(\langle x_n \rangle_{n=1}^{\infty})$, we shall write $a << b$ if and only if there exist $F, H \in \mathcal{P}_f(\mathbb{N})$ with $\max F < \min H$ such that $a = \sum_{i \in F} x_i$ and $b = \sum_{i \in H} x_i$.

Whenever we use the following lemma, we will only need finitely many terms from the sequence $\langle x_n \rangle_{n=1}^{\infty}$, but it costs us nothing to prove the stronger form.

4.3 Lemma. *Suppose that $U \in p$ and that, for every $u \in U$, $V(u) \in p$. Then there is a one to one sequence $\langle x_n \rangle_{n=1}^{\infty}$ in S such that $FS(\langle x_n \rangle_{n=1}^{\infty}) \subseteq U$ and $b \in V(a)$ whenever $a, b \in FS(\langle x_n \rangle_{n=1}^{\infty})$ and $a << b$.*

Proof. We construct our sequence inductively, first choosing x_1 to be any element of U^*. We then suppose that we have chosen $x_1, x_2, \ldots, x_n$ in S satisfying $FS(\langle x_i \rangle_{i=1}^{n}) \subseteq U^*$ and $b \in V(a)^*$ whenever $a, b \in FS(\langle x_i \rangle_{i=1}^{n})$ and $a << b$.

Let $E = FS(\langle x_i \rangle_{i=1}^{n})$. For any $a \in E$, $V(a)^* \in p$ and $-a + U^* \in p$ by Lemma 1.8. Further, given $a, b \in E$ with $a << b$, we have $b \in V(a)^*$ so $-b + V(a)^* \in p$ by Lemma 1.8. Thus we may choose

$$\begin{aligned} x_{n+1} \quad \in \quad & (U^* \backslash \{x_1, x_2, \ldots, x_n\}) \cap \textstyle\bigcap_{a \in E} \big((-a + U^*) \cap V(a)^*\big) \cap \\ & \textstyle\bigcap \{-b + V(a)^* : a, b \in E \text{ and } a << b\} \ . \end{aligned}$$

Since $x_{n+1} \in U^* \cap \bigcap_{a \in E} (-a + U^*)$, we have that $FS(\langle x_i \rangle_{i=1}^{n+1}) \subseteq U^*$.

Now, let $a, b \in FS(\langle x_i \rangle_{i=1}^{n+1})$ with $a << b$. Pick $F, H \subseteq \{1, 2, \ldots, n+1\}$ such that $\max F < \min H$ and $a = \sum_{i \in F} x_i$ and $b = \sum_{i \in H} x_i$. If $\max H < n+1$, then $b \in V(a)^*$ by the induction hypothesis. So assume $n+1 \in H$. If $H = \{n+1\}$, then $b = x_{n+1} \in V(a)^*$ by the construction. Otherwise, $b = b' + x_{n+1}$ where $b' \in E$ and $a << b'$ so that $b \in V(a)^*$ because $x_{n+1} \in (-b' + V(a)^*)$. $\square$

4.4 Definition. Let $a \in S$. We define subsets of S as follows.

$I(a) = \{b \in S : \{a, b\} \notin E(G)\}$.

$Q(a) = \{b \in S : \{x \in S : b + x \in I(a)\} \in p\}$.

$R(a) = \{b \in S : \{x \in S : b \in I(a + x)\} \in p\}$.

$T(a) = \{b \in S : \{x \in S : b + x \in I(a + x)\} \in p\}$.

Notice that $Q(a)$ and $R(a)$ can be written more simply as

$$Q(a) = \{b \in S : -b + I(a) \in p\}$$

and, since $b \in I(a + x)$ if and only if $a + x \in I(b)$,

$$R(a) = \{b \in S : -a + I(b) \in p\} \ .$$

They were written out in the longer fashion in Definition 4.4 to contrast with $T(a)$ which has no such short description.

If the semigroup S has an identity, we denote it by 0, in which case of course $S \cup \{0\} = S$. If not, $S \cup \{0\}$ denotes S with a two sided identity adjoined.

4.5 Definition. Let $a \in S \cup \{0\}$. We define subsets of $S \cup \{0\}$ as follows.
$W(a) = \{b \in S \cup \{0\} : \{x \in S : \{y \in S : b + x + y \in I(a + y)\} \in p\} \in p\}$.
$X(a) = \{b \in S \cup \{0\} : \{x \in S : \{y \in S : b + y \in I(a + x)\} \in p\} \in p\}$.
$Y(a) = \{b \in S \cup \{0\} : \{x \in S : \{y \in S : b + x \in I(a + x + y)\} \in p\} \in p\}$.
$Z(a) = \{b \in S \cup \{0\} : \{x \in S : \{y \in S : b + x + y \in I(a + x)\} \in p\} \in p\}$.

Again, each of $X(a)$, $Y(a)$, and $Z(a)$ $\big($but not $W(a)\big)$ has a simpler representation:

$$\begin{array}{rcl} X(a) & = & \{b \in S \cup \{0\} : \{x \in S : -b + I(a + x) \in p\} \in p\}\ , \\ Y(a) & = & \{b \in S \cup \{0\} : \{x \in S : -(a + x) + I(b + x) \in p\} \in p\}\ ,\ \text{and} \\ Z(a) & = & \{b \in S \cup \{0\} : \{x \in S : -(b + x) + I(a + x) \in p\} \in p\}\ . \end{array}$$

(In the case of $Y(a)$ one needed to notice that $b + x \in I(a + x + y)$ if and only if $a + x + y \in I(b + x)$.)

4.6 Lemma. *Suppose that $m \in \mathbb{N}$ and that G contains no $K_{m,m}$. Then, for every $a \in S \cup \{0\}$, $W(a) = X(a) = Y(a) = Z(a) = S \cup \{0\}$.*

Proof. (1) Suppose that $b \in (S \cup \{0\})\backslash W(a)$. Let

$$U = \{x \in S : \{y \in S : b + x + y \notin I(a + y)\} \in p\}$$

and for $x \in U$, let $V(x) = \{y \in S : b+x+y \notin I(a+y)\}$. Choose by Lemma 4.3 a sequence $\langle x_n\rangle_{n=1}^\infty$ in S such that $FS(\langle x_n\rangle_{n=1}^\infty) \subseteq U$ and $z \in V(y)$ whenever $y, z \in FS(\langle x_n\rangle_{n=1}^\infty)$ and $y << z$. For $i \in \{1, 2, \ldots, m\}$, let $c_i = \Sigma_{t=i}^{2m+1} x_t$ and $d_i = \Sigma_{t=m+i}^{2m+1} x_t$. Since S is right cancellative we have that $c_i \neq c_j$ and $d_i \neq d_j$ whenever $i \neq j$. Now, given $i, j \in \{1, 2, \ldots, m\}$, we have $\Sigma_{t=i}^{m+j-1} x_t << d_j$ so $d_j \in V(\Sigma_{t=i}^{m+j-1} x_t)$. That is, $\{b + c_i, a + d_j\} \in E(G)$, a contradiction. (Since S is left cancellative we have $b + c_i \neq b + c_j$ and $a + d_i \neq a + d_j$ whenever $i \neq j$.)

(2) Suppose that $b \in (S \cup \{0\})\backslash X(a)$. Let $U = \{x \in S : -b + I(a + x) \notin p\}$. Choose distinct $x_1, x_2, \ldots, x_m$ in U and choose distinct $y_1, y_2, \ldots, y_m$ in $S\backslash \bigcup_{i=1}^m \big(-b+I(a+x_i)\big)$. Then for any $i, j \in \{1, 2, \ldots, m\}$, $\{b + y_j, a + x_i\} \in E(G)$, a contradiction.

(3) Suppose that $b \in (S \cup \{0\})\backslash Y(a)$. Let $U = \{x \in S : -(a + x) + I(b + x) \notin p\}$ and for $x \in U$, let $V(x) = S\backslash\big(-(a + x) + I(b + x)\big)$. Choose by Lemma 4.3 a sequence $\langle x_n\rangle_{n=1}^\infty$ in S such that $FS(\langle x_n\rangle_{n=1}^\infty) \subseteq U$ and $z \in V(y)$ whenever $y, z \in FS(\langle x_n\rangle_{n=1}^\infty)$

and $y << z$. For $i \in \{1, 2, \ldots, m\}$, let $c_i = \Sigma_{t=1}^{i+1} x_t$ and $d_i = \Sigma_{t=1}^{m+i+1} x_t$. Now, given $i, j \in \{1, 2, \ldots, m\}$, we have $c_i << \Sigma_{t=i+2}^{m+j+1} x_t$ so $\Sigma_{t=i+2}^{m+j+1} x_t \in V(c_i)$. That is, $\{b + c_i, a + d_j\} \in E(G)$, a contradiction. (As in the proof of (1), we see that these are all distinct.)

(4) Notice that for any $a, b \in S \cup \{0\}$, $b \in Z(a)$ if and only if $a \in Y(b)$ so the fact that $Z(a) = S \cup \{0\}$ follows from part (3). $\square$

4.7 Lemma. *Suppose that $m \in \mathbb{N}$ and that G contains no $K_{m,m}$. We then have:*

(1) $\{a \in S : I(a) \in p\} \in p$;

(2) $\{a \in S : Q(a) \in p\} \in p$;

(3) $\{a \in S : R(a) \in p\} \in p$; *and*

(4) $\{a \in S : T(a) \in p\} \in p$.

Proof. (1) Suppose not and let $U = \{a \in S : I(a) \notin p\}$. Pick distinct $a_1, a_2, \ldots, a_m$ in U and pick distinct $b_1, b_2, \ldots, b_m$ in $\bigcap_{i=1}^m \big(S \backslash I(a_i)\big)$. Then for each $i, j \in \{1, 2, \ldots, m\}$, $\{a_i, b_j\} \in E(G)$, a contradiction.

(2) This follows from (1) and Lemma 1.8, since $I(a)^* \subseteq Q(a)$.

(3) Suppose not and let $U = \{a \in S : R(a) \notin p\}$. Choose distinct $a_1, a_2, \ldots, a_m$ in U and choose distinct $b_1, b_2, \ldots, b_m$ in $\bigcap_{i=1}^m \big(S \backslash R(a_i)\big)$. Then for each $i, j \in \{1, 2, \ldots, m\}$, $-a_i + I(b_j) \notin p$ so pick $x \in S \backslash \bigcup_{i=1}^m \bigcup_{j=1}^m \big(-a_i + I(b_j)\big)$. Then for each $i, j \in \{1, 2, \ldots, m\}$, $\{a_i + x, b_j\} \in E(G)$, a contradiction.

(4) This is nearly identical to the proof of (3). One ends up with $a_1, a_2, \ldots, a_m$, $b_1, b_2, \ldots, b_m$, and x such that $\{a_i + x, b_j + x\} \in E(G)$ whenever $i, j \in \{1, 2, \ldots, m\}$. $\square$

4.8 Definition. For every $a \in S$, $C(a) = I(a) \cap Q(a) \cap R(a) \cap T(a)$.

4.9 Lemma. *For every $a, b \in S$, $a \in C(b)$ if and only if $b \in C(a)$.*

Proof. From the definitions we have that

$$\begin{array}{rcl} a \in I(b) & \Leftrightarrow & b \in I(a)\ , \\ a \in T(b) & \Leftrightarrow & b \in T(a)\ , \text{ and} \\ a \in Q(b) & \Leftrightarrow & b \in R(a)\ . \end{array}$$

$\square$

4.10 Lemma. *Suppose that $m \in \mathbb{N}$ and that G contains no $K_{m,m}$. Then, for every $a \in S$, $C(a) = C(a)^*$.*

Proof. Suppose that $b \in C(a)$. We need to show that $-b + C(a) \in p$. Since $b \in Q(a)$, we have directly that $-b + I(a) \in p$.

Now suppose that $-b + Q(a) \notin p$ and let

$$U = S \backslash \big(-b + Q(a)\big) = \{y \in S : \{x \in S : b + y + x \notin I(a)\} \in p\}\ .$$

For $y \in U$, let $V(y) = \{x \in S : b + y + x \notin I(a)\}$. Then, since $p + p = p$, we have that $\{y + x : y \in U \text{ and } x \in V(y)\} \in p$. Since $b \in Q(a)$, we have that $\{w \in S : b + w \in I(a)\} \in p$. This is a contradiction since

$$\{y + x : y \in U \text{ and } x \in V(y)\} \cap \{w \in S : b + w \in I(a)\} = \emptyset \ .$$

By Lemma 4.6 $a \in X(b)$ so $\{x \in S : -a + I(b + x) \in p\} \in p$. Since $\{x \in S : -a + I(b + x) \in p\} \subseteq -b + R(a)$, we have that $-b + R(a) \in p$.

By Lemma 4.6 $b \in W(a)$, so $\{x \in S : \{y \in S : b + x + y \in I(a + y)\} \in p\} \in p$. Since $\{x \in S : \{y \in S : b + x + y \in I(a + y)\} \in p\} \subseteq -b + T(a)$, we know that $-b + T(a) \in p$.□

4.11 Lemma. *Suppose that $m \in \mathbb{N}$ and that G contains no $K_{m,m}$. Then, for every $a \in S$ and every $b \in C(a)$, $\{x \in S : b + x \in C(a + x)\} \in p$.*

Proof. Let $b \in C(a)$. Then $\{x \in S : b + x \in I(a + x)\} \in p$ because $b \in T(a)$. Since, by Lemma 4.6, $b \in Z(a)$, we have $\{x \in S : -(b + x) + I(a + x) \in p\} \in p$. That is, $\{x \in S : b + x \in Q(a + x)\} \in p$.

Also, by Lemma 4.6, $b \in Y(a)$ so $\{x \in S : \{y \in S : b + x \in I(a + x + y)\} \in p\} \in p$. That is, $\{x \in S : b + x \in R(a + x)\} \in p$.

Thus it remains only to show that $\{x \in S : b + x \in T(a + x)\} \in p$. Let $A = \{x \in S : b + x \in I(a + x)\}$. Since $b \in T(a)$, $A \in p$, so $\{x \in S : -x + A \in p\} \in p$. And

$$\begin{aligned} \{x \in S : -x + A \in p\} &= \{x \in S : \{y \in S : b + x + y \in I(a + x + y)\} \in p\} \\ &= \{x \in S : b + x \in T(a + x)\} \ . \end{aligned}$$

□

4.12 Lemma. *Suppose that $m \in \mathbb{N}$ and that G contains no $K_{m,m}$. Then*

$$\{a \in S : a \in C(a)\} \in p \ .$$

Proof. It is trivial that $a \in I(a)$ and $a \in T(a)$ for every $a \in S$. It is also trivial that $a \in Q(a)$ if and only if $a \in R(a)$. By Lemma 4.6, $0 \in Y(0)$ so

$$\{a \in S : \{x \in S : a \in I(a + x)\} \in p\} \in p \ .$$

That is, $\{a \in S : a \in R(a)\} \in p$. □

4.13 Lemma. *Suppose that $m \in \mathbb{N}$ and that G contains no $K_{m,m}$. Then*

$$\{a \in S : \{x \in S : x \in C(a + x)\} \in p\} \in p \ .$$

Proof. Firstly, by Lemma 4.6, $0 \in W(0)$ so $\{a \in S : \{x \in S : a + x \in I(x)\} \in p\} \in p$ so $\{a \in S : \{x \in S : x \in I(a + x)\} \in p\} \in p$.

Secondly, by Lemma 4.6, for any $a \in S$, $0 \in Z(a)$, so $\{x \in S : -x+I(a+x) \in p\} \in p$. That is, $\{x \in S : x \in Q(a+x)\} \in p$.

Thirdly, given any $a \in S$, $0 \in Y(a)$ so $\{x \in S : \{y \in S : x \in I(a+x+y)\} \in p\} \in p$. That is, $\{x \in S : x \in R(a+x)\} \in p$.

Finally, by Lemma 4.6, $0 \in W(0)$ so $\{a \in S : \{w \in S : a+w \in I(w)\} \in p\} \in p$. We claim that

$$\{a \in S : \{w \in S : a+w \in I(w)\} \in p\} \subseteq \{a \in S : \{x \in S : x \in T(a+x)\} \in p\}$$

so let $a \in S$ be given such that $\{w \in S : a+w \in I(w)\} \in p$.

Let $A = \{w \in S : w \in I(a+w)\}$. Then $A \in p$ so $\{x \in S : -x+A \in p\} \in p$. And

$$\begin{aligned} \{x \in S : -x+A \in p\} &= \{x \in S : \{y \in S : x+y \in I(a+x+y)\} \in p\} \\ &= \{x \in S : x \in T(a+x)\}\ . \end{aligned} \qquad \square$$

The following theorem is the main theorem of this section.

4.14 Theorem. *Suppose that $m \in \mathbb{N}$ and that G contains no $K_{m,m}$. Then, given any $P \in p$, there exists an infinite sequence $\langle x_n\rangle_{n=1}^{\infty}$ in S such that $FS(\langle x_n\rangle_{n=1}^{\infty}) \subseteq P$ and $\{a,b\} \notin E(G)$ whenever $a,b \in FS(\langle x_n\rangle_{n=1}^{\infty})$.*

Proof. In the light of Lemmas 4.7, 4.12, and 4.13, we may suppose that

$$P \subseteq \{a \in S : C(a) \in p\} \cap \{a \in S : a \in C(a)\} \cap \{a \in S : \{x \in S : x \in C(a+x)\} \in p\}\ .$$

We choose $\langle x_n\rangle_{n=1}^{\infty}$ inductively. Let $x_1 \in P^*$. Let $n \in \mathbb{N}$ and assume we have chosen $x_1, x_2, \ldots, x_n$ with the property that $FS(\langle x_t\rangle_{t=1}^{n}) \subseteq P^*$ and $b \in C(a)$ whenever $a,b \in FS(\langle x_t\rangle_{t=1}^{n})$.

Let $E = FS(\langle x_t\rangle_{t=1}^{n})$. By Lemma 1.8 $P^* \in p$ and for all $a \in P^*$, $-a+P^* \in p$. Given $a \in P^*$, $C(a) \in p$ and $\{x \in S : x \in C(a+x)\} \in p$. Given $a,b \in E$, we have that $a \in C(b)$ and $b \in C(a)$ and hence by Lemma 4.10 $-a+C(b) \in p$ and by Lemma 4.11 $\{x \in S : b+x \in C(a+x)\} \in p$.

Choose

$$\begin{aligned} x_{n+1} \in\ & P^* \cap \textstyle\bigcap_{a\in E}\big((-a+P^*) \cap C(a) \cap \{x \in S : x \in C(a+x)\}\big) \\ & \cap \textstyle\bigcap_{a\in E}\bigcap_{b\in E}\Big(\big(-a+C(b)\big) \cap \{x \in S : b+x \in C(a+x)\}\Big)\ . \end{aligned}$$

Then $x_{n+1} \in P^*$ and for each $a \in E$, $a+x_{n+1} \in P^*$ so $FS(\langle x_t\rangle_{t=1}^{n+1}) \subseteq P^*$.

Now let $a,b \in FS(\langle x_t\rangle_{t=1}^{n+1})$. Then without loss of generality (since $a \in C(b)$ if and only if $b \in C(a)$ by Lemma 4.9) one of the following cases holds:

(1) $a, b \in E$;

(2) $a = b = x_{n+1}$;

(3) $a = a' + x_{n+1}$ for some $a' \in E$ and $b = x_{n+1}$;

(4) $a \in E$ and $b = x_{n+1}$;

(5) $a = a' + x_{n+1}$ for some $a' \in E$ and $b = b' + x_{n+1}$ for some $b' \in E$; or

(6) $a \in E$ and $b = b' + x_{n+1}$ for some $b' \in E$.

In case (1) $b \in C(a)$ by the induction hypothesis. In case (2), $b \in C(a)$ because $P \subseteq \{x \in S : x \in C(x)\}$. In case (3), $b \in C(a)$ because $x_{n+1} \in C(a' + x_{n+1})$. In case (4), we have directly that b was chosen in $C(a)$. In case (5) we use the fact that $x_{n+1} \in \{x \in S : b' + x \in C(a' + x)\}$. And in case (6) we use the fact that $x_{n+1} \in -b' + C(a)$. $\square$

As was the case with Corollary 3.20, the following result is not a corollary to Theorem 4.14 but rather to its proof. Also as there we need to drop our standing assumption that we have fixed an idempotent in βS, because one is chosen in the proof.

4.15 Corollary. *Let $m \in \mathbb{N}$ and suppose that G contains no $K_{m,m}$ and let $\langle y_n \rangle_{n=1}^{\infty}$ be a sequence in S. There is a sum subsystem $\langle x_n \rangle_{n=1}^{\infty}$ of $\langle y_n \rangle_{n=1}^{\infty}$ such that whenever $a, b \in FS(\langle x_n \rangle_{n=1}^{\infty})$, $\{a, b\} \notin E(G)$.*

Proof. By Theorem 1.9, pick an idempotent $p \in \beta S$ such that for every $n \in \mathbb{N}$, $FS(\langle y_k \rangle_{k=n}^{\infty}) \in p$.

We show how to modify the proof of Theorem 4.14. To start, let $k(1) = 1$ and pick $x_1 \in FS(\langle y_t \rangle_{t=1}^{\infty}) \cap P^*$. Pick $H_1 \in \mathcal{P}_f(\mathbb{N})$ such that $x_1 = \Sigma_{t \in H_1} y_t$ and let $k(2) = \max H_1 + 1$. At stage n in the construction require that $x_n \in FS(\langle y_t \rangle_{t=k(n)}^{\infty})$ (in addition to all of the other sets specified in that proof). Pick $H_n \in \mathcal{P}_f(\mathbb{N})$ such that $x_n = \Sigma_{t \in H_n} y_t$ and let $k(n+1) = \max H_n + 1$. $\square$

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