TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 00, Number 0, Pages 000–000
S 0002-9947(XX)0000-0

This paper was published in *Trans. Amer. Math. Soc.* **358** (2006), 3239-3262. To the best of my knowledge, this is the final version as it was submitted to the publisher. –NH

AN INFINITARY EXTENSION OF THE GRAHAM–ROTHSCHILD PARAMETER SETS THEOREM

TIMOTHY J. CARLSON, NEIL HINDMAN, AND DONA STRAUSS

ABSTRACT. The *Graham-Rothschild Parameter Sets Theorem* is one of the most powerful results of Ramsey Theory. (The Hales-Jewett Theorem is its most trivial instance.) Using the algebra of βS, the Stone-Čech compactification of a discrete semigroup, we derive an infinitary extension of the Graham-Rothschild Parameter Sets Theorem. Even the simplest finite instance of this extension is a significant extension of the original. The original theorem says that whenever $k < m$ in $\mathbb{N}$ and the k-parameter words are colored with finitely many colors, there exist a color and an m-parameter word w with the property that whenever a k-parameter word of length m is substituted in w, the result is in the specified color. The "simplest finite instance" referred to above is that, given finite colorings of the k-parameter words for each $k < m$, there is one m-parameter word which works for each k. Some additional Ramsey Theoretic consequences are derived.

We also observe that, unlike any other Ramsey Theoretic result of which we are aware, central sets are not necessarily good enough for even the $k = 1$ and $m = 2$ version of the Graham-Rothschild Parameter Sets Theorem.

1. Introduction

Throught this paper A will denote a nonempty set and D will denote a set with a binary operation mapping $(f, g) \in D \times D$ to $fg \in D$. We assume that D has a nonempty set E of right identities for this operation. We also assume that, for each $f \in D$, we have defined a mapping $T_f : A \to A$. We shall call $(A, D, E, \langle T_f \rangle_{f \in D})$ a *parameter system.*

We write ω for the set $\{0, 1, 2, \ldots\}$ of finite ordinals and $\mathbb{N} = \omega \setminus \{0\}$. We choose a set $V = \{\nu_n : n \in \omega\}$ such that $A \cap (D \times V) = \emptyset$ and define W to be the semigroup of words over the alphabet $A \cup (D \times V)$, with concatenation as the semigroup operation. (Formally a *word* w is a function from an initial segment $\{0, 1, \ldots, k - 1\}$ of ω to the alphabet and the length $\ell(w)$ of w is k. We shall occasionally need to resort to this formal meeting, so that if $i \in \{0, 1, \ldots, \ell(w) - 1\}$, then $w(i)$ denotes the $(i + 1)^{\text{st}}$ letter of w.)

For each $n \in \mathbb{N}$, we define W_n to be the set of words over the alphabet $A \cup (D \times \{\nu_0, \nu_1, \ldots, \nu_{n-1}\})$ and we define W_0 to be the set of words over A. We note that each W_n is a subsemigroup of W.

2000 *Mathematics Subject Classification.* 05D10.
The second author acknowledges support received from the National Science Foundation (USA) via grant DMS-0070593.

Definition 1.1. Let $n \in \mathbb{N}$ and $k \in \omega$ with $k \leq n$. Then $S\binom{n}{k}$ is the set of all words $w \in W_k$ of length n such that
(1) for each $i \in \{0, 1, \ldots, k-1\}$, if any, some member of $E \times \{\nu_i\}$ occurs in w;
(2) for each $i \in \{0, 1, \ldots, k-1\}$, if any, the first occurrence of a member (s, ν_i) of $D \times \{\nu_i\}$ has $s \in E$;
(3) for each $i \in \{0, 1, \ldots, k-2\}$, if any, the first occurrence of a member of $D \times \{\nu_i\}$ in w precedes the first occurrence of of a member of $D \times \{\nu_{i+1}\}$

Definition 1.2. Let $k \in \mathbb{N}$. Then the set of *k-parameter words* is $S_k = \bigcup_{n=k}^{\infty} S\binom{n}{k}$.

Of course, $S\binom{n}{k}$ and S_k depend on A, D and E, as well as n and k. Since we consider only one parameter system at a time throughout most of the paper, we shall not normally indicate this dependence in the notation. In a context where more than one parameter system is used, we shall use $S\binom{n}{k}(\Gamma)$ and $S_k(\Gamma)$ for the sets defined above by the parameter system Γ. If $k = 0$, when we write S_k we mean simply W_0.

For each $i \in \omega$, we choose a member v_i of $E \times \{\nu_i\}$. If $D = \{e\}$ and $T_e : A \to A$ is the identity, then the k-parameter words are known as the *k-variable words*, where each v_i is a "variable".

Given $w \in S_n$ and $u \in W$ with $\ell(u) = n$, we define $w\langle u\rangle$ to be the word with length $\ell(w)$ such that for $i \in \{0, 1, \ldots, \ell(w) - 1\}$

$$w\langle u\rangle(i) = \begin{cases} w(i) & \text{if } w(i) \in A \\ T_s\big(u(j)\big) & \text{if } w(i) = (s, \nu_j) \text{ and } u(j) \in A \\ (st, \nu_l) & \text{if } w(i) = (s, \nu_j) \text{ and } u(j) = (t, \nu_l)\,. \end{cases}$$

For example, suppose that $A = \{a, b, c\}$ and $D = \{e, f, g\}$ is a group, with e the identity and $g = f^2$. Suppose also that $T_f : \begin{array}{c} a \mapsto b \\ b \mapsto c \\ c \mapsto a \end{array}$, $T_g = {T_f}^2$, and T_e is the identity. If $w = v_0 a v_1 (f, \nu_1) b (g, \nu_0) = (e, \nu_0) a (e, \nu_1)(f, \nu_1) b (g, \nu_0)$, and $u = (f, \nu_2) a$, then $w\langle u\rangle = (ef, \nu_2) a T_e(a) T_f(a) b (gf, \nu_2) = (f, \nu_2) aabb (e, \nu_2)$.

If $D = \{e\}$ and T_e is the identity map, then $w\langle u\rangle$ is simply the result of replacing each occurrence of v_i in w by $u(i)$.

The following theorem is the *Graham-Rothschild parameter sets theorem*. We use the standard "chromatic" terminology for Ramsey Theoretic results. When we say that a set is finitely colored, we mean that there is a function from that set to a finite set. A set is *monochrome* provided the given coloring function is constant on it. If D is a group, then the statement that $\langle T_f\rangle_{f \in D}$ is an action of D on A is the assertion that $T_f \circ T_g = T_{fg}$ for all $f, g \in D$ and that T_e is the identity map.

Theorem 1.3 (Graham-Rothschild)**.** *Assume that the alphabet A is finite, that D is a finite group, and that $\langle T_f\rangle_{f \in D}$ is an action of D on A. Let $m, k \in \omega$ with $m > k$ and let S_k be finitely colored. There exists $w \in S_m$ such that $\{w\langle u\rangle : u \in S\binom{m}{k}\}$ is monochrome.*

Proof. [6], or see [14] for a shorter proof. □

The case $m = 1$, $k = 0$, and $D = \{e\}$ of Theorem 1.3 is the Hales-Jewett Theorem [8]. The version of Theorem 1.3 which has $D = \{e\}$ is commonly cited in the literature as the Graham-Rothschild Parameter Sets Theorem and most of the standard consequences of the Graham-Rothschild Theorem are consequences of this special case. We shall show in Theorem 5.1 that Theorem 1.3 is derivable from this special case. This is true even if the assumptions of Theorem 1.3 are significantly weakened.

The restriction on the order of first appearances of members of $D \times \{\nu_i\}$ in the definition of n-parameter word may seem unnatural. Note however, that without that restriction, the $m = 3$, $k = 2$, and $D = \{e\}$ case of Theorem 1.3 is false. (Simply color the two variable words according to whether the first occurrence of v_0 precedes or follows the first occurrence of v_1.)

The Graham-Rothschild Parameter Sets Theorem has been recognized for its power from the time of its appearance. Section 9 of [6] contains 13 corollaries. Included among these are four results that were known at the time (namely the Hales-Jewett Theorem, van der Waerden's Theorem, Ramsey's Theorem, and the finite version of the Finite Sums Theorem). We believe that the other nine were new at the time. (These include the finite version of the Finite Unions Theorem. While the infinite version of the Finite Unions Theorem is obviously derivable from the infinite version of the Finite Sums Theorem, the finite version of the Finite Unions Theorem is not obviously derivable from the finite version of the Finite Sums Theorem.) In introducing their article about the Graham-Rothschild Parameter Sets Theorem, Prömel and Voigt [14] wrote:

> This is a complete analogue to Ramsey's theorem carried over to the structures of parameter sets and, as it turns out, Ramsey's theorem itself is an immediate consequence of the Graham-Rothschild theorem. But the concept of parameter sets does not only glue arithmetic progressions and finite sets together. Also, it provides a natural framework for seemingly different structures like Boolean lattices, partition lattices, hypergraphs and Deuber's (m, p, c)-sets, just to mention a few. So, the Graham-Rothschild theorem can be viewed as a starting point of *Ramsey Theory*.

Other strong consequences of infinitary results such as those established here are analogues of the Paris-Harrington Theorem [13]: the statement

($\star$) For any positive integers c and e there is a positive integer N such that if $[N]^e$ is colored with c colors there is a large homogeneous set of size at least $e + 1$ (a set of integers is large if its cardinality is at least as large as its least element).

is true and cannot be proved in the formal theory of Peano Arithmetic. The statement ($\star$) can be proved from the infinite version of Ramsey's Theorem by the same kind of compactness argument used to derive the finite version of Ramsey's Theorem. Perhaps the most important feature of the Paris-Harrington theorem is that it was the first example of a striking combinatorial fact which cannot be proved in the theory of formal Peano Arithmetic. Gödel's incompleteness theorem showed that there are finitary truths not provable from Peano Arithmetic, but the examples given before the Paris-Harrington Theorem were not very satisfying mathematically. By similar sorts of compactness arguments, principles similar to ($\star$) can be derived from infinitary theorems on variable words. For example, Theorem 5.4 implies the following principle:

($\star\star$) For any finite alphabet A and any positive integers c and e there is a positive integer N such that whenever the e-variable words of length N are colored with c colors there is an integer $m \geq e+1$ and a large m-variable word of length N all of whose e-variable reductions have the same color (an m-variable word is *large* if $k \leq m$ where the first occurence of a variable is in position k).

While we don't have tight bounds on what is required to prove ($\star\star$), it cannot be proved in Peano Arithmetic since it easily implies ($\star$). (Given a coloring of $[N]^e$, color an e-variable word of length N according to the color of the set of those k such that the first occurence of some variable occurs in position k.)

The following extension of the $D = \{e\}$ version of Theorem 1.3 is not new, being a direct consequence of [2, Theorem 10]. However, it is certainly not well known, even among the experts, and we shall present its derivation in Section 5. Given a set B, we write $\mathcal{P}_f(B)$ for the set of finite nonempty subsets of B.

Theorem 1.4. *Assume that A is finite, $D = \{e\}$, and, for each $n \in \omega$, S_n has been finitely colored. Then, there exists a sequence $\langle w_n\rangle_{n<\omega}$ with each $w_n \in S_n$ such that for every $m \in \omega$,*

$$S_m \cap \left\{ \textstyle\prod_{n\in F} w_n\langle u_n\rangle : F \in \mathcal{P}_f(\omega) \text{ and for all } n \in F\,,\ u_n \in \bigcup_{i=0}^{\min F} S\binom{n}{i} \right\}$$

is monochrome. (That is, the color of $\prod_{n\in F} w_n\langle u_n\rangle$ is determined solely by the number of variables in $\prod_{n\in F} w_n\langle u_n\rangle$.)

We shall derive in Section 3 the following extension of Theorems 1.3 and 1.4.

Corollary 1.5. *Assume that, for each $n \in \omega$, S_n has been finitely colored and that, for each $n \in \omega$ and each $i \in \{0, 1, \ldots, n\}$, $H_{n,i}$ is a finite subset of $S\binom{n}{i}$. Then, there exists a sequence $\langle w_n\rangle_{n<\omega}$ with each $w_n \in S_n$ such that for every $m \in \omega$,*

$$S_m \cap \{\textstyle\prod_{n\in F} w_n\langle u_n\rangle : F \in \mathcal{P}_f(\omega) \text{ and for all } n \in F\,,\ u_n \in \bigcup_{i=0}^{\min F} H_{n,i}\}$$

is monochrome. (That is, the color of $\prod_{n\in F} w_n\langle u_n\rangle$ is determined solely by the number of parameters in $\prod_{n\in F} w_n\langle u_n\rangle$.)

The special case $D = \{e\}$ of Corollary 1.5 can be derived from [2, Theorem 15] by an argument similar to the one we shall use to establish Theorem 1.4.

Notice that if A and D are finite, one may take $H_{n,i} = S\binom{n}{i}$ in Corollary 1.5.

Perhaps somewhat easier to absorb is the following corollary to Corollary 1.5.

Corollary 1.6. *Let $m \in \mathbb{N}$. Suppose that we have a finite coloring of $\bigcup_{i=0}^{m} S_i$ and a finite subset H_i of $S\binom{m}{i}$ for each $i \in \{0, 1, \ldots, m\}$. Then there exists a sequence $\langle w_n\rangle_{n=0}^{\infty}$ in S_m such that for every $l \in \{0, 1, \ldots, m\}$,*

$$S_l \cap \{\textstyle\prod_{n\in F} w_n\langle u_n\rangle : F \in \mathcal{P}_f(\omega) \text{ and for all } n \in F\,,\ u_n \in \bigcup_{i=0}^{m} H_i\}$$

is monochrome.

In particular, one immediately has the following extension of Theorem 1.3. (We shall describe in Section 3 how this extension can be derived from Theorem 1.3 without using the results of Section 2.)

Corollary 1.7. *Assume that the alphabet A is finite, that D is a finite group, and that $\langle T_f\rangle_{f\in D}$ is an action of D on A. Let $m,k \in \omega$ with $m > k$ and let S_k and S_m be finitely colored. There exists a sequence $\langle w_t\rangle_{t=0}^{\infty}$ in S_m such that $\{\prod_{t\in F} w_t : F \in \mathcal{P}_f(\omega)\}$ is monochrome and $\{\prod_{t\in F} w_t\langle u_t\rangle : F \in \mathcal{P}_f(\omega)$ and for each $n \in F$, $u_n \in S\binom{m}{k}\}$ is monochrome.*

We derive Corollary 1.5 as a straightforward consequence of the existence of a particular chain of idempotents $\langle p_n\rangle_{n=0}^{\infty}$, where each $p_n \in \beta S_n$. Section 2 is devoted to the proof of the existence of this special chain of idempotents. (More precisely, Corollary 1.5 is a special case of Theorem 3.2, the statement of which requires the introduction of additional terminology.)

Let us briefly review some facts about the Stone-Čech compactification βT of a (discrete) semigroup $(T,\cdot)$. We take the points of βT to be the ultrafilters on T, the principal ultrafilters being identified with the points of T. Given a set $A \subseteq T$, $\overline{A} = \{p \in \beta T : A \in p\}$. The set $\{\overline{A} : A \subseteq T\}$ is a basis for the open sets (as well as a basis for the closed sets) of βT. If $R \subseteq T$ we shall identify an ultrafilter p on R with the ultrafilter $\{A \subseteq T : A \cap R \in p\}$ and thereby pretend that $\beta R \subseteq \beta T$.

There is a natural extension of the operation $\cdot$ of T to βT making βT a compact right topological semigroup with T contained in its topological center. This says that for each $p \in \beta T$ the function $\rho_p : \beta T \to \beta T$ is continuous and for each $x \in T$, the function $\lambda_x : \beta T \to \beta T$ is continuous, where $\rho_p(q) = q \cdot p$ and $\lambda_x(q) = x \cdot q$. Given $B \subseteq T$ and $x \in T$, let $x^{-1}B = \{y \in T : x \cdot y \in B\}$. Then for any $p, q \in \beta T$ and any $B \subseteq T$, one has that $B \in p \cdot q$ if and only if $\{x \in T : x^{-1}B \in q\} \in p$. See [10] for an elementary introduction to the semigroup βT and for any unfamiliar algebraic facts encountered in this paper.

A subset V of a semigroup T is called a left ideal if is nonempty and $TV \subseteq V$. It is called a right ideal if it is nonempty and $VT \subseteq V$. It is called a two-sided ideal, or simply an ideal, if it is both a left ideal and a right ideal. Any compact Hausdorff right topological semigroup T has a smallest two sided ideal $K(T)$ which is the union of all of the minimal left ideals of T and is also the union of all of the minimal right ideals of T. If $x \in K(T)$, then xT is the minimal right ideal with x as a member and Tx is the minimal left ideal with x as a member. The intersection of any minimal left ideal and any minimal right ideal is a group. In particular there are idempotents in the smallest ideal. There is a partial ordering of the idempotents of T determined by $p \le q$ if and only if $p = p \cdot q = q \cdot p$. An idempotent p is minimal with respect to this order if and only if $p \in K(T)$ [10, Theorem 1.59]. Such an idempotent is called simply "minimal".

A subset B of a discrete semigroup T is *central* if and only if it is a member of a minimal idempotent of βT. Central sets are known to have remarkably strong combinatorial properties. For example [5, Theorem 8.22] any central subset of $\mathbb{N}$ contains solutions to all partition regular systems of homogeneous linear equations. See [10] and [9] for numerous other combinatorial conclusions about central sets.

Loosely speaking, Theorem 3.2 says that when one is constructing the sequence $\langle w_n\rangle_{n=0}^{\infty}$ in the statement of Corollary 1.5, one can take w_n to be any member of some central subset of S_n.

It was shown in the proof of [1, Theorem 4.1] that the $k = 0$, $m = 1$, and $D = \{e\}$ case of Corollary 1.7 (and in particular the Hales-Jewett Theorem) holds where $\varphi^{-1}[\{i\}]$ is any central subset of W_0. In fact all previous Ramsey Theoretic results of which we are aware that could be stated in terms of a finite partition of a semigroup had a conclusion valid for arbitrary central sets. We shall show at the end of Section 3 that there is a central subset B of S_1 for which the conclusion of Theorem 1.3 fails with $m = 2$ and $D = \{e\}$. (That is, there is no $w \in S_2$ such that $w\langle u\rangle \in B$ for every $u \in S\binom{2}{1}$.)

In Section 4 we shall derive some Ramsey Theoretic consequences of the results of Section 3. Additional consequences will appear in [3].

2. A Chain of Idempotents

Recall that we are assuming that we have a nonempty alphabet A, a set D with a binary operation, a nonempty set E of right identities for D, and a mapping $T_f : A \to A$ defined for every $f \in D$. Recall also that we have chosen a member v_i of $E \times \{\nu_i\}$ for each $i \in \omega$.

Suppose that $u \in W$ has length n. We shall define a homomorphism $h_u : W \to W$ by first defining h_u on all elements of $A \cup (D \times V)$. So let $w \in A \cup (D \times V)$. Then

$$h_u(w) = \begin{cases} w & \text{if } w \in A \\ T_s\big(u(j)\big) & \text{if } w = (s, \nu_j)\,,\ j < n, \text{ and } u(j) \in A \\ (st, \nu_l) & \text{if } w = (s, \nu_j)\,,\ j < n, \text{ and } u(j) = (t, \nu_l) \\ w & \text{if } w = (s, \nu_j) \text{ and } j \ge n\,. \end{cases}$$

Since W is the free semigroup on $A \cup (D \times V)$, h_u extends to a unique homomorphism defined on W, which we also denote by h_u. Thus, if $w \in W$ and $\ell(w) = k$ one has that $\ell\big(h_u(w)\big) = k$, and for $i \in \{0, 1, \ldots, k-1\}$,

$$h_u(w)(i) = \begin{cases} w(i) & \text{if } w(i) \in A \\ T_s\big(u(j)\big) & \text{if } w(i) = (s, \nu_j)\,,\ j < n, \text{ and } u(j) \in A \\ (st, \nu_l) & \text{if } w(i) = (s, \nu_j)\,,\ j < n, \text{ and } u(j) = (t, \nu_l) \\ w(i) & \text{if } w(i) = (s, \nu_j) \text{ and } j \ge n\,. \end{cases}$$

Observe that, if $n \in \mathbb{N}$, $w \in S_n$, $u \in W$, and $\ell(u) = n$, then $h_u(w) = w\langle u\rangle$. Observe also that if $u \in S\binom{n}{k}$, then $h_u : W_n \to W_k$ and $h_u : S_n \to S_k$.

We shall also use h_u to denote the continuous extension of h_u from βW to itself.

Lemma 2.1. *Let $u \in W$. Then h_u is a homomorphism from βW to βW.*

Proof. This is [10, Corollary 4.22] due originally to P. Milnes in [12]. □

In the following definition, and throughout the rest of this paper, when we write an expression such as $v_i \cdots v_j$, we assume that all intervening values of the subscript occur in order.

Definition 2.2. Let $n \in \mathbb{N}$ with $n \ge 2$.
(a) For $i \in \{0, 1, \ldots, n-1\}$, $w_{n,i}$ is the word obtained from $v_0 \cdots v_{n-1}$ by deleting v_i.

(b) For $i \in \{0,1,\ldots,n-1\}$,

$$U_{n,i} = \big\{w \in W : \quad \ell(w) = n\,,\ w(i) \in A \cup \{(s,\nu_l) : s \in D \text{ and } 0 \le l < i\}\,,\ \text{and for all } j \in \{0,1,\ldots,n-1\}, \text{ if } j < i, \text{ then } w(j) \in E \times \{\nu_j\} \text{ and if } j > i, \text{ then } w(j) \in E \times \{\nu_{j-1}\}\big\}\,.$$

Thus, if $0 < i < n-1$, a member of $U_{n,i}$ is of the form $w_0 \cdots w_{i-1} t w_i \cdots w_{n-2}$ where $t \in A \cup (D \times \{\nu_0,\nu_1,\ldots,\nu_{i-1}\})$ and each $w_i \in E \times \{\nu_i\}$.

Notice that for any $n \in \mathbb{N}$ with $n \ge 2$, $S\binom{n}{n-1} = \bigcup_{i=0}^{n-1} U_{n,i}$.

Lemma 2.3. *Let $n \in \mathbb{N}$. Then $h_{v_0 \cdots v_{n-1}}$ is the identity on W_n.*

Proof. Since $h_{v_0 \cdots v_{n-1}}$ is a homomorphism, it suffices to show that $h_{v_0 \cdots v_{n-1}}(x) = x$ for any $x \in A \cup (D \times \{\nu_0,\nu_1,\ldots,\nu_{n-1}\})$. If $x \in A$, this is immediate. If $x = (s,\nu_i)$, then $h_{v_0 \cdots v_{n-1}}(x) = (se,\nu_i)$ for some $e \in E$ and $se = s$ because e is a right identity for D. □

Lemma 2.4. *Let $n \in \mathbb{N}$ with $n \ge 2$.*
(a) If $i \in \{0,1,\ldots,n-1\}$, and $u \in U_{n,i}$, then $h_u \circ h_{w_{n,i}}$ is equal to the identity on W_{n-1}.
(b) If $i,k \in \{0,1,\ldots,n-1\}$, $i < k$, and $u \in U_{n+1,i}$, then for all $x \in W_n$, $h_u\big(h_{w_{n+1,k}}(x)\big) = h_{w_{n,k-1}}\big(h_u(x)\big)$.

Proof. (a) Since $h_u \circ h_{w_{n,i}}$ is a homomorphism, it suffices to let $x \in A \cup (D \times \{\nu_0,\nu_1,\ldots,\nu_{n-2}\})$ and show that $h_u \circ h_{w_{n,i}}(x) = x$. If $x \in A$, this is immediate. So assume that $x = (s,\nu_j)$ for some $j \in \{0,1,\ldots,n-2\}$. If $j < i$, then $w_{n,i}(j) = v_j$ and $u(j) \in E \times \{\nu_j\}$ so $h_u \circ h_{w_{n,i}}(x) = x$. If $j \ge i$, then $w_{n,i}(j) = v_{j+1}$ and $u(j+1) \in E \times \{\nu_j\}$ and so $h_u \circ h_{w_{n,i}}(x) = x$.

(b) Since $h_u \circ h_{w_{n+1,k}}$ and $h_{w_{n,k-1}} \circ h_u$ are homomorphisms, it suffices to establish the conclusion for $x \in A \cup (D \times \{\nu_0,\nu_1,\ldots,\nu_{n-1}\})$. The case in which $x \in A$ is trivial. So assume that $x = (s,\nu_j)$ for some $j \in \{0,1,\ldots,n-1\}$.

Case 1. $j < i$. Then $w_{n+1,k}(j) = w_{n,k-1}(j) = v_j$ and $u(j) = (e,\nu_j)$ for some $e \in E$. Therefore, $h_{w_{n+1,k}}(x) = h_u(x) = h_{w_{n,k-1}}(x) = (se,\nu_j) = x$. Consequently $h_u\big(h_{w_{n+1,k}}(x)\big) = h_u(x) = x$ and $h_{w_{n,k-1}}\big(h_u(x)\big) = h_{w_{n,k-1}}(x) = x$.

Case 2. $j = i$. Then $w_{n+1,k}(j) = v_j$ so $h_{w_{n+1,k}}(x) = (s,\nu_j)$.

Case 2a. $u(j) \in A$. Then $h_u\big(h_{w_{n+1,k}}(x)\big) = T_s\big(u(j)\big)$. Also, $h_u(x) = T_s\big(u(j)\big) \in A$ and so $h_{w_{n,k-1}}\big(h_u(x)\big) = T_s\big(u(j)\big)$.

Case 2b. $u(j) = (t,\nu_l)$ for some $t \in D$ and some $l < i$. Then $h_u\big(h_{w_{n+1,k}}(x)\big) = (st,\nu_l)$. Also, $h_u(x) = (st,\nu_l)$ and $w_{n,k-1}(l) = v_l$ so $h_{w_{n,k-1}}\big(h_u(x)\big) = (st,\nu_l)$.

Case 3. $i < j < k$. Then $w_{n+1,k}(j) = v_j$, $u(j) = (e,\nu_{j-1})$ for some $e \in E$, and $w_{n,k-1}(j-1) = v_{j-1}$. Therefore $h_{w_{n+1,k}}(x) = (s,\nu_j)$ so $h_u\big(h_{w_{n+1,k}}(x)\big) = (s,\nu_{j-1})$. Also, $h_u(x) = (s,\nu_{j-1})$ so $h_{w_{n,k-1}}\big(h_u(x)\big) = (s,\nu_{j-1})$.

Case 4. $k \le j$. Then $w_{n+1,k}(j) = v_{j+1}$, $u(j) = (e,\nu_{j-1})$ and $u(j+1) = (f,\nu_j)$ for some $e,f \in E$, and $w_{n,k-1}(j-1) = v_j$. Therefore $h_{w_{n+1,k}}(x) = (s,\nu_{j+1})$ so $h_u\big(h_{w_{n+1,k}}(x)\big) = (s,\nu_j)$. Also, $h_u(x) = (s,\nu_{j-1})$ and so $h_{w_{n,k-1}}\big(h_u(x)\big) = (s,\nu_j)$. □

It is standard to define partial orders of idempotents of a semigroup T by $p \le_R q$ if and only if $p \in qT$ and $p \le_L q$ if and only if $p \in Tq$. We observe that these are equivalent respectively to $p = qp$ and $p = pq$. (If $p = qx$, then $qp = qqx = qx = p$.) We extend these definitions to all of βW.

Definition 2.5. Let $p, q \in \beta W$. Then $p \leq_L q$ if and only if $p \in \beta W q$ and $p \leq_R q$ if and only if $p \in q\beta W$.

We shall use the obvious fact that, for any homomorphism $h : \beta W \to \beta W$, $p \leq_L q$ implies that $h(p) \leq_L h(q)$, and $p \leq_R q$ implies that $h(p) \leq_R h(q)$. We observe that, if p and q are idempotent, then $p \leq q$ if and only if $p \leq_R q$ and $p \leq_L q$.

Notice that $\leq_L$ and $\leq_R$ are transitive but are not, in general, reflexive on βW.

We now state several simple algebraic facts which will be needed in the proof of Theorem 2.12.

Lemma 2.6. *Let $n \in \omega$.*
(a) If $p \in \beta W_n$, $r \in \beta W$, and $p \leq_R r$, then $p = ry$ for some $y \in \beta W_n$.
(b) If $p \in \beta W_n$, $r \in \beta W$, and $p \leq_L r$, then $p = yr$ for some $y \in \beta W_n$.
(c) If $p, q \in \beta W_n$, $r \in K(\beta W_n)$, $p \leq_R r$, and $q \leq_R r$, then $p \leq_R q$.
(d) If $p, q \in \beta W_n$, $r \in K(\beta W_n)$, $p \leq_L r$, and $q \leq_L r$, then $p \leq_L q$.

Proof. We establish (a) and (c). For (a), we have that $p = ry$ for some $y \in \beta W$. Since $W \setminus W_n$ is an ideal of W, $\beta W \setminus \beta W_n$ is an ideal of βW by [10, Corollary 4.18] and so $y \in \beta W_n$.

(c) Since $r \in K(\beta W_n)$, pick a minimal right ideal R of βW_n such that $r \in R$. Then by (a) $p \in R$ and similarly $q \in R$. Thus $p \in R = q\beta W_n \subseteq q\beta W$. □

Lemma 2.7. *Let T be a compact right topological semigroup. If L is a left ideal of T and R is a right ideal of T, then there is an idempotent $p \in R \cap L$ which is minimal in T.*

Proof. By [10, Corollary 2.6 and Theorem 2.7] we may pick a minimal left ideal $L' \subseteq L$ of T and a minimal right ideal $R' \subseteq R$ of T and one has that $L' \cap R'$ is a group. □

Definition 2.8. Let $n \in \mathbb{N}$. Then $Q_n = \big\{w \in W_n :$ some member of $E \times \{\nu_{n-1}\}$ occurs in w and occurs before any other member of $D \times \{\nu_{n-1}\}\big\}$.

Lemma 2.9. *Let $n \in \mathbb{N}$, let p be an idempotent of βS_n and let q be a minimal idempotent of βW_{n+1} such that $q \leq p$. If $Q_{n+1} \in q$, then q is a minimal idempotent of βS_{n+1}.*

Proof. Given $w \in S_n$ and $u \in Q_{n+1}$, $wu \in S_{n+1}$ and thus $S_{n+1} \in pq = q$. Therefore $q \in \beta S_{n+1}$ and is thus minimal with respect to $\leq$ in βS_{n+1}. □

Lemma 2.10. *Let $n \in \mathbb{N}$, let $p \in \beta W_n$, and let $r \in \beta W_{n+1}$. If $Q_{n+1} \in r$ and $q \in pr\beta W_{n+1}$, then $Q_{n+1} \in q$.*

Proof. We have that $W_n \in p$ and $Q_{n+1} \in r$ so $W_n Q_{n+1} W_{n+1} \in q$ and $W_n Q_{n+1} W_{n+1} \subseteq Q_{n+1}$. □

Lemma 2.11. *Let $n \in \mathbb{N}$, let $p \in \beta W_{n-1}$, let $r \in \beta S_n$, and let $q \in K(\beta W_n)$. If $pq \leq_R r$, then $Q_n \in q$.*

Proof. Let $T = \{w \in W_n : \text{some member of } E \times \{\nu_{n-1}\} \text{ occurs in } w\}$. Then T is an ideal of W_n so βT is an ideal of βW_n and therefore $K(\beta W_n) \subseteq \beta T$ and thus $T \in q$. Suppose that $T \setminus Q_n \in q$. We have that $pq = rx$ for some $x \in \beta W$. Also $W_{n-1}(T \setminus Q_n) \in pq$ and $S_n W \in rx$. This is a contradiction because $W_{n-1}(T \setminus Q_n) \cap S_n W = \emptyset$. □

Theorem 10 of [2] and its consequence Theorem 5.4 follow from [2, Lemma 7.1] which establishes, in the case $D = \{e\}$ and T_e is the identity on A, the existence of a sequence of idempotents p_n in βS_n for $n \in \omega$ such that for any $n \in \omega$

- $p_{n+1} \leq p_n$
- for any $m < n$ and each $u \in S\binom{m}{n}$, $h_u(p_n) = p_m$.

Our main algebraic result, Theorem 2.12 below, shows that there is such a sequence for general D and, moreover, one may choose each of the idempotents p_n to be minimal in βS_n.

Recall that W_0 is the free semigroup (i.e., the set of words) on the alphabet A. Recall also that we do not need to assume that either A or D is finite for this result.

Theorem 2.12. *Let p be a minimal idempotent in βW_0. There is a sequence $\langle p_n \rangle_{n=0}^{\infty}$ such that*
(1) $p_0 = p$;
(2) for each $n \in \mathbb{N}$, p_n is a minimal idempotent of βS_n;
(3) for each $n \in \mathbb{N}$, $p_n \leq p_{n-1}$;
(4) for each $n \in \mathbb{N}$ and each $u \in S\binom{n}{n-1}$, $h_u(p_n) = p_{n-1}$.
Further, p_1 can be any minimal idempotent of βS_1 such that $p_1 \leq p_0$.

Proof. We first show how p_0, p_1 and p_2 can be defined. Let $p_0 = p$ and let p_1 be any minimal idempotent of βS_1 such that $p_1 \leq p_0$. Such exist because we may pick by Lemma 2.7 an idempotent

$$p_1 \in \big(p_0 \cdot \beta S_1\big) \cap \big(\beta S_1 \cdot p_0\big)$$

which is minimal in βS_1. Then $p_1 \leq p_0$. (We have $p_1 = p_0 x$ for some $x \in \beta S_1$ and so $p_0 p_1 = p_0 p_0 x = p_0 x = p_1$. Similarly, $p_1 p_0 = p_1$.) Now $S\binom{1}{0}$ consists of all words of length 1 from the alphabet A. Thus if $u \in S\binom{1}{0}$, then $h_u[S_1] \subseteq W_0$ and h_u is the identity on W_0. Therefore $h_u(p_1) \in \beta W_0$ and, since h_u is a homomorphism, $h_u(p_1) \leq h_u(p_0) = p_0$. Since p_0 is minimal in βW_0, $h_u(p_1) = p_0$. (The argument in this paragraph is due to Andreas Blass, and first appeared in [1].)

Let $\alpha = h_{v_1}(p_1)$. Then $\alpha \in \beta W_2$ so we may pick by Lemma 2.7 an idempotent

$$p_2 \in (p_1 \alpha \beta W_2) \cap (\beta W_2 \alpha p_1)$$

which is minimal in βW_2. Then as in the previous paragraph $p_2 \leq p_1$. Since $S_1 \in p_1$ and $h_{v_1}[S_1] \subseteq Q_2$, $Q_2 \in \alpha$. Thus, by Lemma 2.10 $Q_2 \in p_2$ so by Lemma 2.9 p_2 is minimal in S_2. Given any $u \in S\binom{2}{1}$, $h_u[S_2] \subseteq S_1$ and so $h_u(p_2) \in \beta S_1$. It thus suffices to show that $h_u(p_2) \leq p_1$. If $u \in U_{2,1}$, then h_u is the identity on S_1, so $h_u(p_2) \leq h_u(p_1) = p_1$. Now assume that $u \in U_{2,0}$ and pick $t \in A$ and $e \in E$ such that $u = t(e, \nu_0)$. For $w \in S_1$, $h_u(w) = h_t(w)$, and so $h_u(p_1) = h_t(p_1) = p_0$. Also,

u	$x:\ p_{n-1}$	δ_1	δ_2	δ_3	$\dots$	δ_{n-2}	δ_{n-1}
$u_{n,n-1}$	p_{n-1}						
$u_{n,n-2}$	p_{n-2}	η_1					
$u_{n,n-3}$	p_{n-2}	γ_1	η_2				
$u_{n,n-4}$	p_{n-2}	γ_1	γ_2	η_3			
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$		
$u_{n,1}$	p_{n-2}	γ_1	γ_2	γ_3	$\dots$	η_{n-2}	
$u_{n,0}$	p_{n-2}	γ_1	γ_2	γ_3	$\dots$	γ_{n-2}	η_{n-1}

TABLE 1

by Lemma 2.4(a) $h_{tv_0} \circ h_{v_1}$ is the identity on W_1. So $h_u(\alpha) = h_{t(e,\nu_0)}\big(h_{v_1}(p_1)\big) = p_1$. Therefore $h_u(p_1\alpha) = p_0p_1 = p_1$ and $h_u(\alpha p_1) = p_1p_0 = p_1$. Since $p_2 \leq_R p_1\alpha$, $h_u(p_2) \leq_R h_u(p_1\alpha) = p_1$. Since $p_2 \leq_L \alpha p_1$, $h_u(p_2) \leq_L h_u(\alpha p_1) = p_1$.

We now proceed to an inductive construction. Let $n \in \mathbb{N}$ with $n \geq 2$.

We shall introduce elements, (such as η_i or γ_i) which depend on n as well as on i. However, in an effort to reduce the number of subscripts used, we shall not indicate the dependence on n in the notation.

We make the inductive assumption that we have chosen p_i for $i \in \{0, 1, 2, \dots n\}$, η_i, η'_i, δ_i, and δ'_i for $i \in \{1, 2, 3, \dots, n-1\}$, and γ_i and γ'_i for $i \in \{2, 3, \dots, n-2\}$, if any, so that the following hypotheses are satisfied.

(a) For each $i \in \{0, 1, \dots, n\}$, p_i is a minimal idempotent of βS_i.
(b) For each $i \in \{1, 2, \dots, n\}$, $p_i \leq p_{i-1}$ and $h_u(p_i) = p_{i-1}$ for every $u \in S\binom{i}{i-1}$.
(c) For every $i \in \{1, 2, \dots, n-1\}$, η_i and η'_i are minimal idempotents in βW_{n-1}.
(d) For every $i \in \{1, 2, \dots, n-1\}$, $\eta_i \leq_L p_{n-1}$ and $\eta'_i \leq_R p_{n-1}$.
(e) For $i \in \{1, 2, \dots, n-1\}$, $\delta_i = h_{w_{n,n-i-1}}(\eta_i)$, $\delta'_i = h_{w_{n,n-i-1}}(\eta'_i)$,

$$\begin{aligned} p_n &\leq_R p_{n-1}\delta_1 \cdots \delta_{n-1}, \text{ and} \\ p_n &\leq_L \delta'_{n-1} \cdots \delta'_1 p_{n-1} . \end{aligned}$$

(f) For every $i \in \{1, 2, \dots, n-2\}$,

$$\begin{aligned} \eta_i &\leq_R \gamma_i \cdots \gamma_{n-2}\eta_{n-1} \text{ and} \\ \eta'_i &\leq_L \eta'_{n-1}\gamma'_{n-2} \cdots \gamma'_i . \end{aligned}$$

(g) For every choice of $u_{n,i} \in U_{n,i}$ for $i \in \{0, 1, \dots, n-1\}$, the entry in the row labeled by u and the column labeled by x in Tables 1 and 2 is $h_u(x)$.

We observe that these assumptions do hold if $n = 2$, with $\eta_1 = \eta'_1 = p_1$. To verify hypothesis (c) we need to show that p_1 is minimal in βW_1. Since S_1 is a right ideal of W_1, we have that βS_1 is a right ideal of βW_1 and so contains a minimal right ideal of βW_1. Therefore by [10, Theorem 1.65] $K(\beta S_1) = \beta S_1 \cap K(\beta W_1)$. For hypothesis (e), note that $\delta_1 = \delta'_1 = \alpha$. Hypothesis (f) is vacuous, and we have already verified the table entries of hypothesis (g).

Notice that since $h_{w_{n,n-i-1}}[W_{n-1}] \subseteq W_n$ one has that each $\delta_i \in \beta W_n$. Also, since $h_u[W_n] \subseteq W_{n-1}$ for each $u \in S\binom{n}{n-1}$, we have that each $\gamma_i \in \beta W_{n-1}$.

u	$x: \ \delta'_{n-1}$	δ'_{n-2}	$\ldots$	δ'_3	δ'_2	δ'_1	p_{n-1}
$u_{n,n-1}$							p_{n-1}
$u_{n,n-2}$						η'_1	p_{n-2}
$u_{n,n-3}$					η'_2	γ'_1	p_{n-2}
$u_{n,n-4}$				η'_3	γ'_2	γ'_1	p_{n-2}
$\vdots$			$\iddots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$u_{n,1}$		η'_{n-2}	$\ldots$	γ'_3	γ'_2	γ'_1	p_{n-2}
$u_{n,0}$	η'_{n-1}	γ'_{n-2}	$\ldots$	γ'_3	γ'_2	γ'_1	p_{n-2}

TABLE 2

By assumption (e), $p_n \leq_R p_{n-1}\delta_1 \cdots \delta_{n-1}$. So there is some $x \in \beta W_n$ such that $p_{n-1}\delta_1 \cdots \delta_{n-1} x = p_n = p_n p_n \in p_n \beta W$. So

$$\{x \in \beta W_n : p_{n-1}\delta_1 \cdots \delta_{n-1} x \leq_R p_n\}$$

is nonempty and is therefore a right ideal of βW_n. By Lemma 2.7, we can choose a minimal idempotent μ_n of βW_n which is in this right ideal and in the left ideal $\beta W_n p_n$ of βW_n.

Now let $i \in \{2, 3, \ldots, n-1\}$. Note that $\delta_i \cdots \delta_{n-1}\mu_n = \delta_i \cdots \delta_{n-1}\mu_n\mu_n$, so

$$\{x \in \beta W_n : p_{n-1}\delta_1\delta_2 \cdots \delta_{i-1} x \leq_R p_n \text{ and } x \leq_R \delta_i \cdots \delta_{n-1}\mu_n\}$$

is nonempty, because it contains $\delta_i \cdots \delta_{n-1}\mu_n$. It is therefore a right ideal of βW_n, and we can choose a minimal idempotent μ_i of βW_n which is in this right ideal and is also in the left ideal $\beta W_n p_n$ of βW_n.

Similarly, $\{x \in \beta W_n : p_{n-1} x \leq_R p_n \text{ and } x \leq_R \delta_1 \cdots \delta_{n-1}\mu_n\}$ is nonempty because $\delta_1 \cdots \delta_{n-1}\mu_n$ is a member, and thus we may choose a minimal idempotent μ_1 of βW_n which is in this right ideal of βW_n and also in the left ideal $\beta W_n p_n$.

Thus we have chosen minimal idempotents $\mu_1, \mu_2, \ldots, \mu_n$ in βW_n which satisfy the following conditions:

$$(*) \qquad \begin{aligned} \mu_i &\leq_L p_n \text{ for all } i \in \{1, 2, \ldots, n\}; \\ p_{n-1}\delta_1 \cdots \delta_{i-1}\mu_i &\leq_R p_n \text{ for all } i \in \{2, 3, \ldots, n\}; \\ p_{n-1}\mu_1 &\leq_R p_n; \text{ and} \\ \mu_i &\leq_R \delta_i \cdots \delta_{n-1}\mu_n \text{ for all } i \in \{1, 2, 3, \ldots, n-1\}. \end{aligned}$$

By a left-right switch of these arguments, we can chose minimal idempotents $\mu'_1, \mu'_2, \ldots, \mu'_n$ in βW_n which satisfy the following conditions:

$$(**) \qquad \begin{aligned} \mu'_i &\leq_R p_n \text{ for all } i \in \{1, 2, \ldots, n\}; \\ \mu'_i\delta'_{i-1} \cdots \delta'_1 p_{n-1} &\leq_L p_n \text{ for all } i \in \{2, 3, \ldots, n\}; \\ \mu'_1 p_{n-1} &\leq_L p_n; \text{ and} \\ \mu'_i &\leq_L \mu'_n\delta'_{n-1} \cdots \delta'_i \text{ for all } i \in \{1, 2, 3, \ldots, n-1\}. \end{aligned}$$

(While βW is right topological and not left topological, all of the algebraic facts that we are using in this proof are valid from both sides.)

For $i \in \{1, 2, \ldots, n\}$, let $\epsilon_i = h_{w_{n+1,n-i}}(\mu_i)$, let $\epsilon'_i = h_{w_{n+1,n-i}}(\mu'_i)$, and note that $\epsilon_i, \epsilon'_i \in W_{n+1}$. Then $p_n\epsilon_1 \cdots \epsilon_n \beta W_{n+1}$ and $\beta W_{n+1}\epsilon'_n \cdots \epsilon'_1 p_n$ are respectively right and left ideals of βW_{n+1}. Pick by Lemma 2.7 a minimal idempotent p_{n+1} of βW_{n+1} such that

$$p_{n+1} \in p_n\epsilon_1 \cdots \epsilon_n \beta W_{n+1} \cap \beta W_{n+1}\epsilon'_n \cdots \epsilon'_1 p_n\,.$$

u	$x:\ p_n$	ϵ_1	ϵ_2	ϵ_3	$\cdots$	ϵ_{n-1}	ϵ_n
$u_{n+1,n}$	p_n						
$u_{n+1,n-1}$	p_{n-1}	μ_1					
$u_{n+1,n-2}$	p_{n-1}	δ_1	μ_2				
$u_{n+1,n-3}$	p_{n-1}	δ_1	δ_2	μ_3			
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$		
$u_{n+1,1}$	p_{n-1}	δ_1	δ_2	δ_3	$\cdots$	μ_{n-1}	
$u_{n+1,0}$	p_{n-1}	δ_1	δ_2	δ_3	$\cdots$	δ_{n-1}	μ_n

TABLE 3

u	$x:\ \epsilon'_n$	ϵ'_{n-1}	$\cdots$	ϵ'_3	ϵ'_2	ϵ'_1	p_n
$u_{n+1,n}$							p_n
$u_{n+1,n-1}$						μ'_1	p_{n-1}
$u_{n+1,n-2}$					μ'_2	δ'_1	p_{n-1}
$u_{n+1,n-3}$				μ'_3	δ'_2	δ'_1	p_{n-1}
$\vdots$			$\iddots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$u_{n+1,1}$		μ'_{n-1}	$\cdots$	δ'_3	δ'_2	δ'_1	p_{n-1}
$u_{n+1,0}$	μ'_n	δ'_{n-1}	$\cdots$	δ'_3	δ'_2	δ'_1	p_{n-1}

TABLE 4

We claim that p_{n+1} is a minimal idempotent of βS_{n+1}. By $(*)$, $p_{n-1}\mu_1 \leq_R p_n$ so by Lemma 2.11, $Q_n \in \mu_1$. Since $h_{w_{n+1,n-1}}[Q_n] \subseteq Q_{n+1}$, $Q_{n+1} \in \epsilon_1$ and so, by Lemma 2.10, $Q_{n+1} \in p_{n+1}$. Consequently, by Lemma 2.9 p_{n+1} is minimal in βS_{n+1}.

We now claim that the induction hypotheses are satisfied for $n+1$ with η_i, η'_i, δ_i, δ'_i, γ_i, and γ'_i replaced by μ_i, μ'_i, ϵ_i, ϵ'_i, δ_i, and δ'_i respectively. That is, we claim that

(a) For each $i \in \{0,1,\ldots,n+1\}$, p_i is a minimal idempotent of βS_i.

(b) For each $i \in \{1,2,\ldots,n+1\}$, $p_i \leq p_{i-1}$ and $h_u(p_i) = p_{i-1}$ for every $u \in S\binom{i}{i-1}$.

(c) For every $i \in \{1,2,\ldots,n\}$, μ_i and μ'_i are minimal idempotents in βW_n.

(d) For every $i \in \{1,2,\ldots,n\}$, $\mu_i \leq_L p_n$ and $\mu'_i \leq_R p_n$.

(e) For $i \in \{1,2,\ldots,n\}$, $\epsilon_i = h_{w_{n+1,n-i}}(\mu_i)$, $\epsilon'_i = h_{w_{n+1,n-i}}(\mu'_i)$,

$$\begin{aligned} p_{n+1} &\leq_R p_n\epsilon_1\cdots\epsilon_n, \text{ and} \\ p_{n+1} &\leq_L \epsilon'_n\cdots\epsilon'_1 p_n\,. \end{aligned}$$

(f) For every $i \in \{1,2,\ldots,n-1\}$,

$$\begin{aligned} \mu_i &\leq_R \delta_i\cdots\delta_{n-1}\mu_n \text{ and} \\ \mu'_i &\leq_L \mu'_n\delta'_{n-1}\cdots\delta'_i\,. \end{aligned}$$

(g) For every choice of $u_{n+1,i} \in U_{n+1,i}$ for $i \in \{0,1,\ldots,n\}$, the entry in the row labeled by u and the column labeled by x in Tables 3 and 4 is $h_u(x)$.

All of these conclusions can be easily verified except (g) and the assertion in (b) that $h_u(p_{n+1}) = h_u(p_n)$ for all $u \in S\binom{n}{n-1}$. We show first that this latter assertion follows from statement (g).

For any $i \in \{0, 1, \ldots, n\}$, $h_{u_{n+1,i}}(p_{n+1}) \in \beta S_n$ and p_n is minimal in βS_n, so it suffices to show that $h_{u_{n+1,i}}(p_{n+1}) \leq p_n$. Since $p_{n+1} \leq p_n$ and $h_{u_{n+1,n}}$ is the identity on W_n, we have that $h_{u_{n+1,n}}(p_{n+1}) \leq h_{u_{n+1,n}}(p_n) = p_n$.

Now let $i \in \{0, 1, \ldots, n-1\}$ and let $u = u_{n+1,i}$. We have $p_{n+1} \leq_R p_n\epsilon_1 \cdots \epsilon_{n-i}$ and so $h_u(p_{n+1}) \leq_R h_u(p_n\epsilon_1 \cdots \epsilon_{n-i})$ and by $(*)$ and Table 3, $h_u(p_n\epsilon_1 \cdots \epsilon_{n-i}) \leq_R p_n$. Also $p_{n+1} \leq_L \epsilon'_{n-i} \cdots \epsilon'_1 p_n$ so $h_u(p_{n+1}) \leq_L h_u(\epsilon'_{n-i} \cdots \epsilon'_1 p_n)$ and by $(**)$ and Table 4, $h_u(\epsilon'_{n-i} \cdots \epsilon'_1 p_n) \leq_L p_n$.

It thus suffices to verify the entries of Table 3 and Table 4. We shall write out the verification for Table 3. The verification for Table 4 follows by a left-right switch of the arguments. To this end, let a choice of $u_{n+1,i} \in U_{n+1,i}$ for $i \in \{0, 1, \ldots, n\}$ be given.

We have that $h_{u_{n+1,n}}$ is the identity on S_n so $h_{u_{n+1,n}}(p_n) = p_n$. For $i \in \{0, 1, \ldots, n-1\}$, $h_{u_{n+1,i}} = h_{u_{n,i}}$ on S_n so $h_{u_{n+1,i}}(p_n) = h_{u_{n,i}}(p_n) = p_{n-1}$ by hypothesis (b).

The diagonal entries are correct because $\epsilon_i = h_{w_{n+1,n-i}}(\mu_i)$ for $i \in \{1, 2, \ldots, n\}$ and $h_{u_{n+1,n-i}} \circ h_{w_{n+1,n-i}}$ is the identity on W_n by Lemma 2.4(a).

Let $k \in \{1, 2, \ldots, n-1\}$, let $i \in \{0, 1, \ldots, n-k-1\}$, and let $u \in U_{n+1,i}$. To finish the proof we need to show that $h_u(\epsilon_k) = \delta_k$. Now $\epsilon_k = h_{w_{n+1,n-k}}(\mu_k)$ so we are showing that $h_u\big(h_{w_{n+1,n-k}}(\mu_k)\big) = \delta_k$. Since $i < n-k$, we have by Lemma 2.4(b) that $h_u\big(h_{w_{n+1,n-k}}(\mu_k)\big) = h_{w_{n,n-k-1}}\big(h_u(\mu_k)\big)$. So it suffices to show that $h_{w_{n,n-k-1}}\big(h_u(\mu_k)\big) = \delta_k$. Now $h_{w_{n,n-k-1}}(\eta_k) = \delta_k$ by hypothesis (e), so it suffices to show that $h_u(\mu_k) = \eta_k$. And since $h_u(\mu_k)$ and η_k are idempotents in βW_{n-1} and η_k is minimal in βW_{n-1} it suffices to show that $h_u(\mu_k) \leq \eta_k$.

Now $\mu_k \leq_L p_n$ by $(*)$ so $h_u(\mu_k) \leq_L h_u(p_n) = p_{n-1}$, the equality holding by hypothesis (b). Since $\eta_k \leq_L p_{n-1}$ by hypothesis (d), we have by Lemma 2.6(d) that $h_u(\mu_k) \leq_L \eta_k$.

It remains to show that $h_u(\mu_k) \leq_R \eta_k$. We have by $(*)$ that $\mu_k \leq_R \delta_k \cdots \delta_{n-1}\mu_n$. If $i = n-k-1$, we have that $h_u(\mu_k) \leq_R h_u(\delta_k) = \eta_k$ by hypothesis (g), so assume that $i < n-k-1$. Then $h_u(\mu_k) \leq_R h_u(\delta_k) \cdots h_u(\delta_{n-i-1}) = \gamma_k \cdots \gamma_{n-i-2}\eta_{n-i-1}$, the equality holding by hypothesis (g). If $i = 0$, we have directly that $h_u(\mu_k) \leq_R \gamma_k \cdots \gamma_{n-2}\eta_{n-1}$. Otherwise $\eta_{n-i-1} \leq_R \gamma_{n-i-1} \cdots \gamma_{n-2}\eta_{n-1}$ by hypothesis (f) so again $h_u(\mu_k) \leq_R \gamma_k \cdots \gamma_{n-2}\eta_{n-1}$. Also $\eta_k \leq_R \gamma_k \cdots \gamma_{n-2}\eta_{n-1}$ by hypothesis (f). Now $\eta_{n-1} \in K\big(\beta W_{n-1}\big)$ and $\gamma_k \cdots \gamma_{n-2} \in \beta W_{n-1}$ so $\gamma_k \cdots \gamma_{n-2}\eta_{n-1} \in K\big(\beta W_{n-1}\big)$ and thus by Lemma 2.6(c), $h_u(\mu_k) \leq_R \eta_k$. □

One might expect to be able to omit p_0 and start in Theorem 2.12 with p_1 as any minimal idempotent in βS_1. It is a consequence of Theorem 3.6 below that one cannot.

The following lemmas will be useful in the next section.

Lemma 2.13. *Let $k, m \in \omega$ with $k < m$ and let $u \in S\binom{m}{k}$. There exist $r \in S\binom{m}{m-1}$ and $s \in S\binom{m-1}{k}$ such that for all $w \in W_m$, $h_s\big(h_r(w)\big) = h_u(w)$.*

Proof. Either $u(j) \in A$ for some $j \in \{0,1,\dots,m-1\}$ or else there exists $t \in \{0,1,\dots,k-1\}$ such that elements of $D \times \{\nu_t\}$ occur more than once in u. In the second case, let t be the smallest index for which this happens. Then $u(t) \in E \times \{\nu_t\}$ and there exists $j > t$ such that $u(j) \in D \times \{\nu_t\}$. In either case, we define r and s as follows for $i \in \{0,1,\dots,m-1\}$ and $l \in \{0,1,\dots,m-2\}$:

$$r(i) = \begin{cases} v_i & \text{if } i < j \\ u(j) & \text{if } i = j \\ v_{i-1} & \text{if } j < i \end{cases} \quad \text{and } s(l) = \begin{cases} u(l) & \text{if } l < j \\ u(l+1) & \text{if } j \le l\,. \end{cases}$$

□

Lemma 2.14. *Let* $n \in \mathbb{N}$ *and let* $p_0 \in \beta W_0$ *and for each* $i \in \{1,2,\dots,n\}$, *let* $p_i \in \beta S_i$. *Assume that for each* $i \in \{0,1,\dots,n-1\}$ *and each* $u \in S\binom{i+1}{i}$, $h_u(p_{i+1}) = p_i$. *Then for each* $k < m$ *in* $\{0,1,\dots,n\}$ *and each* $u \in S\binom{m}{k}$, $h_u(p_m) = p_k$.

Proof. We proceed by induction on n, the case $n = 1$ being trivial. So let $n \in \mathbb{N}$ and assume the lemma is true for n. Let $p_0, p_1, \dots, p_{n+1}$ be as in the statement of the lemma, let $k < m$ in $\{0,1,\dots,n+1\}$, and let $u \in S\binom{m}{k}$. Pick by Lemma 2.13 $r \in S\binom{m}{m-1}$ and $s \in S\binom{m-1}{k}$ such that for all $w \in S_m$, $h_s\big(h_r(w)\big) = h_u(w)$. Since $p_m \in \beta S_m$, $h_u(p_m) = h_s\big(h_r(p_m)\big)$. Then $h_r(p_m) = p_{m-1}$ and by the induction hypothesis, $h_s(p_{m-1}) = p_k$. □

3. Extending the Graham-Rothschild Parameter Sets Theorem

Theorem 3.2 is the main Ramsey Theoretic result of this paper. In order to state it precisely, we need to formalize the notion of "tree". Recall that an ordinal is the set of its predecessors, so that, if $n \in \mathbb{N}$, then $n = \{0,1,\dots,n-1\}$.

Definition 3.1. Let X be a set.

(1) T is a *tree in* X if and only if

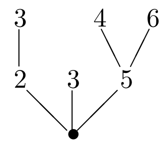
- (a) T is a nonempty set of functions,
- (b) for each $f \in T$, $\mathrm{dom}(f) \in \omega$ and $\mathrm{ran}(f) \subseteq X$, and
- (c) for each $f \in T$, if $\mathrm{dom}(f) = n > 0$, then $f_{|n-1} \in T$.

(2) If T is a tree in X and $n \in \omega$, then $T_n = \{f \in T : \mathrm{dom}(f) = n\}$.

(3) If T is a tree in X and $f \in T$, then $B_f = \{x \in X : f \cup \{(n,x)\} \in T\}$.

(4) The sequence $\langle w_n\rangle_{n=0}^{\infty}$ is a *path through* T if and only if for each $n \in \omega$, $\big\{(k,w_k) : k \in \{0,1,\dots,n\}\big\} \in T$.

The empty function is a "root" for the tree, and B_f is the set of successors to the "node" f. Consider, for example, the following diagram of a tree T.



Then formally

$$T = \big\{\emptyset, \{(0,2)\}, \{(0,3)\}, \{(0,5)\}, \{(0,2),(1,3)\}, \{(0,5),(1,4)\}, \{(0,5),(1,6)\}\big\}\,,$$

$B_\emptyset = \{2,3,5\}$ and $B_{\{(0,5)\}} = \{4,6\}$.

Notice that if $w \in S_m$ and $u \in S\binom{m}{m}$, then $w\langle u\rangle = w$.

Condition (2) of the following theorem says that each path through T satisfies the conclusion of Corollary 3.3 below and that the monochrome colors are invariant from path to path.

Theorem 3.2. *Assume that, for each $n \in \omega$, S_n has been finitely colored and that, for each $n \in \omega$ and each $i \in \{0,1,\ldots,n\}$, $H_{n,i}$ is a finite subset of $S\binom{n}{i}$. Then there is a tree T in W such that*
(1) for each $n \in \omega$ and each $f \in T_n$, B_f is a monochrome central subset of S_n and
(2) for any $m \in \omega$, the intersection of S_m with

$$\{\textstyle\prod_{n\in F} f(n)\langle u_n\rangle : f \in T\,,\ \emptyset \neq F \subseteq \mathrm{dom}(f),\ \text{and for all } n \in F\,,\ u_n \in \bigcup_{i=0}^{\min F} H_{n,i}\}$$

is monochrome.

Proof. Pick a sequence of idempotents $\langle p_n\rangle_{n=0}^\infty$ as guaranteed by Theorem 2.12. For $n \in \omega$, choose a monochrome set $C_n \subseteq S_n$ such that $C_n \in p_n$.

Let $T_0 = \{\emptyset\}$ and let $V_{\emptyset,0} = C_0$. Inductively, let $k \in \omega$ and assume that for $l \in \{0,1,\ldots,k\}$ we have defined T_l and for each $f \in T_l$ and each $i \in \{0,1,\ldots,l\}$ we have defined $V_{f,i}$ such that

(1) $\bigcup_{l=0}^k T_l$ is a tree in W;
(2) if $l \in \{0,1,\ldots,k-1\}$ and $f \in T_l$, then $B_f \subseteq C_l$ and $B_f \in p_l$;
(3) if $l \in \{0,1,\ldots,k\}$, $f \in T_l$, and $j \in \{0,1,\ldots,l\}$, then $V_{f,j} \subseteq C_j$ and $V_{f,j} \in p_j$;
(4) if $l \in \{0,1,\ldots,k-1\}$, $i \in \{0,1,\ldots,l\}$, $f \in T_l$, and $u \in H_{l,i}$, then $B_f \subseteq {h_u}^{-1}[V_{f,i}]$;
(5) if $l \in \{1,2,\ldots,k\}$, $m \in \{0,1,\ldots,l-1\}$, $j \in \{0,1,\ldots,m\}$, $g \in T_l$, and $f = g_{|m}$, then $V_{g,j} \subseteq V_{f,j}$;
(6) if $l \in \{1,2,\ldots,k\}$, $i \in \{0,1,\ldots,l\}$, $j \in \{0,1,\ldots,i\}$, $g \in T_l$, $f = g_{|l-1}$, and $u \in H_{l-1,i}$, then $h_u\big(g(l-1)\big)V_{g,j} \subseteq V_{f,i}$; and
(7) if $l \in \{1,2,\ldots,k\}$, $j \in \{0,1,\ldots,l\}$, $i \in \{0,1,\ldots,j\}$, $g \in T_l$, $f = g_{|l-1}$, and $u \in H_{l-1,i}$, then $h_u\big(g(l-1)\big)V_{g,j} \subseteq V_{f,j}$.

These hypotheses are valid for $k = 0$, all except (1) and (3) vacuously.

Now for $f \in T_k$ and $i \in \{0,1,\ldots,k\}$, let

$$U_{f,i} = V_{f,i} \cap \textstyle\bigcap_{j=i}^k \{w \in W : w^{-1}V_{f,j} \in p_j\} \cap \bigcap_{j=0}^i \{w \in W : w^{-1}V_{f,i} \in p_j\}\,.$$

(We include the $j = i$ term in both intersections to avoid worrying about $i = 0$ or $i = k$.) Given $j \in \{i, i+1, \ldots, k\}$ we have that $p_j = p_i p_j$ and $V_{f,j} \in p_j$ so $\{w \in W : w^{-1}V_{f,j} \in p_j\} \in p_i$. Given $j \in \{0,1,\ldots,i\}$, we have that $p_i = p_i p_j$ and $V_{f,i} \in p_i$ so $\{w \in W : w^{-1}V_{f,i} \in p_j\} \in p_i$. Consequently $U_{f,i} \in p_i$.

Given $i \in \{0,1,\ldots,k\}$, $f \in T_k$, and $u \in S\binom{k}{i}$, we have by Lemma 2.14 that $h_u(p_k) = p_i$ and so ${h_u}^{-1}[U_{f,i}] \in p_k$. For $f \in T_k$, let

$$D_f = C_k \cap \textstyle\bigcap\{{h_u}^{-1}[U_{f,i}] : i \in \{0,1,\ldots,k\} \text{ and } u \in H_{k,i}\}$$

and note that $D_f \in p_k$ because $H_{k,i}$ is finite. (In this and all similar expressions we take $C \cap \bigcap \emptyset = C$. Thus, if $\bigcup_{i=0}^{k} H_{k,i} = \emptyset$, then $D_f = C_k$.)

Now let $T_{k+1} = \{f \cup \{(k,x) : f \in T_k \text{ and } x \in D_f\}$. For $g \in T_{k+1}$, let $f = g_{|k}$, let $V_{g,k+1} = C_{k+1}$, and for $j \in \{0,1,\ldots,k\}$, let

$$\begin{aligned} V_{g,j} = V_{f,j} \quad & \cap \bigcap \Big\{ \big(h_u(g(k)) \big)^{-1} V_{f,i} : i \in \{j, j+1, \ldots, k\} \text{ and } u \in H_{k,i} \Big\} \\ & \cap \bigcap \Big\{ \big(h_u(g(k)) \big)^{-1} V_{f,j} : i \in \{0,1,\ldots,j\} \text{ and } u \in H_{k,i} \Big\} . \end{aligned}$$

Then hypothesis (1) is satisfied directly and given $f \in T_l$ we have that $B_f = D_f$ so hypothesis (2) holds.

To verify hypothesis (3), let $g \in T_{k+1}$ and let $j \in \{0,1,\ldots,k+1\}$. If $j = k+1$ we have that $V_{g,j} = C_j \in p_j$ so assume that $j \leq k$ and let $f = g_{|k}$. Then $f \in T_k$ and $V_{g,j} \subseteq V_{f,j} \subseteq C_j$ and $V_{f,j} \in p_j$. Note that $g(k) \in B_f = D_f$, so for $i \in \{0,1,\ldots,k\}$ and $u \in H_{k,i}$, $h_u(g(k)) \in U_{f,i}$. Thus if $i \leq j$, $\big(h_u(g(k))\big)^{-1} V_{f,j} \in p_j$ and if $i \geq j$, $\big(h_u(g(k))\big)^{-1} V_{f,i} \in p_j$, so $V_{g,j} \in p_j$.

Hypothesis (4) follows directly from the definition of D_f for $f \in T_k$. Hypothesis (5) follows from the definition of $V_{g,j}$ for $g \in T_{k+1}$ and $j \in \{0,1,\ldots,k\}$ and the fact that hypothesis (5) holds at earlier stages. Hypotheses (6) and (7) follow directly from the definition of $V_{g,j}$ for $g \in T_{k+1}$ and $j \in \{0,1,\ldots,k\}$.

The induction being complete, we have from hypothesis (2) that the first conclusion of the theorem holds. To verify the second conclusion, we show that $\prod_{n\in F} f(n)\langle u_n\rangle \in C_m$ whenever $f \in T$, $\emptyset \neq F \subseteq \text{dom}(f)$, $\prod_{n\in F} f(n)\langle u_n\rangle \in S_m$, and for all $n \in F$, $u_n \in \bigcup_{i=0}^{\min F} H_{n,i}$. To do this, let $f \in T$. We show by induction on $|F|$ that if $\emptyset \neq F \subseteq \text{dom}(f)$, $k = \min F$, $g = f_{|\{0,1,\ldots,k-1\}}$, for each $n \in F$, $u_n \in \bigcup_{i=0}^{k} H_{n,i}$, and $\prod_{n\in F} f(n)\langle u_n\rangle \in S_m$, then $\prod_{n\in F} f(n)\langle u_n\rangle \in V_{g,m}$. (Since $V_{g,m} \subseteq C_m$ by hypothesis (3), this will suffice.)

Assume first that $F = \{k\}$. Then $h_{u_k}\big(f(k)\big) = f(k)\langle u_k\rangle \in S_m$ so $u_k \in H_{k,m}$. And $w_k \in B_g \subseteq {h_{u_k}}^{-1}[V_{g,m}]$ by hypothesis (4), so $h_{u_k}(w_k) \in V_{g,m}$ as required.

So assume that $|F| > 1$. Let $r = \min(F \setminus \{k\})$, let $\hat{g} = f_{|\{0,1,\ldots,r-1\}}$, and let $g^* = f_{|\{0,1,\ldots,k\}}$. Pick j such that $\prod_{n\in F\setminus\{k\}} f(n)\langle u_n\rangle \in S_j$. Note that $j \leq m \leq k < r$. Then by the induction hypothesis $\prod_{n\in F\setminus\{k\}} f(n)\langle u_n\rangle \in V_{\hat{g},j}$ and by hypothesis (5), $V_{\hat{g},j} \subseteq V_{g^*,j}$. Pick $i \in \{0,1,\ldots,k\}$ such that $u_k \in S\binom{k}{i}$. If $j \leq i$, then $h_{u_k}\big(f(k)\big)V_{g^*,j} \subseteq V_{g,i}$ by hypothesis (6) and so $\prod_{n\in F} f(n)\langle u_n\rangle \in V_{g,i} = V_{g,m}$. If $j \geq i$, then $h_{u_k}\big(f(k)\big)V_{g^*,j} \subseteq V_{g,j}$ by hypothesis (7) and so $\prod_{n\in F} f(n)\langle u_n\rangle \in V_{g,j} = V_{g,m}$. □

The fact that the set of successors to each node of the tree constructed in Theorem 3.2 is central, means that that set of successors itself has very rich combinatorial structure as given by the Noncommutative Central Sets Theorem [10, Theorem 14.15]. Some of the more simply stated consequences of the centrality of B_f hold because $\ell : S_m \to \mathbb{N}$ is a homomorphism and thus so is its continuous extension $\ell : \beta S_m \to \beta\mathbb{N}$. Therefore, by [10, Exercise 1.7.3 and Lemma 3.30], if $f \in T$, then $\ell[B_f]$ is central in $\mathbb{N}$. Thus, for example, there will be members of B_f whose lengths form arbitrarily long arithmetic progressions.

Corollary 3.3. *Assume that, for each $n \in \omega$, S_n has been finitely colored and that, for each $n \in \omega$ and each $i \in \{0,1,\dots,n\}$, $H_{n,i}$ is a finite subset of $S\binom{n}{i}$. Then, there exists a sequence $\langle w_n\rangle_{n<\omega}$ with each $w_n \in S_n$ such that for every $m \in \omega$,*

$$S_m \cap \{\textstyle\prod_{n\in F} w_n\langle u_n\rangle : F \in \mathcal{P}_f(\omega) \text{ and for all } n \in F\,,\ u_n \in \bigcup_{i=0}^{\min F} H_{n,i}\}$$

is monochrome. (That is, the color of $\prod_{n\in F} w_n\langle u_n\rangle$ is determined solely by the number of parameters in $\prod_{n\in F} w_n\langle u_n\rangle$.)

Proof. Let T be a tree as guaranteed by Theorem 3.2 and let $\langle w_n\rangle_{n=0}^{\infty}$ be any path through T. □

Note that in the statement of Corollary 3.3, the requirement $u_n \in \bigcup_{i=0}^{\min F} H_{i,n}$ cannot be replaced by the requirement that $u_n \in \bigcup_{i=0}^{n} H_{i,n}$. To see this, for each $n \in \omega$, let $r_n = n+1$ and define $\varphi_n : S_n \to \{1,2,\dots,r_n\}$ by $\varphi_n(w) \equiv \ell(w) \bmod n$. Then given w_1 and n greater than $\ell(w_1)$, one cannot have $\varphi_n(w_1 w_n) = \varphi_n(w_n)$. (Recall that $w_1 = w_1\langle v_1\rangle$ and $w_n = w_n\langle v_1 \cdots v_n\rangle$.)

If one is only interested in the following corollary, one may prove it in a fashion similar to the proof of Theorem 3.2.

Corollary 3.4. *Let $m \in \mathbb{N}$. Suppose that we have a finite coloring of $\bigcup_{i=0}^{m} S_i$ and a finite subset H of $\bigcup_{i=0}^{m} S\binom{m}{i}$. Then there exists a sequence $\langle w_n\rangle_{n=0}^{\infty}$ in S_m such that for every $l \in \{0,1,\dots,m\}$,*

$$S_l \cap \{\textstyle\prod_{n\in F} w_n\langle u_n\rangle : F \in \mathcal{P}_f(\omega) \text{ and for all } n \in F\,,\ u_n \in H\}$$

is monochrome.

Proof. For each $n \in \omega$, we define a finite coloring of S_n by stating that it coincides with the given coloring if $n \le m$ and that it is the constant coloring if $n > m$. Choose any $a \in A$ and, for $n \in \mathbb{N}$, put $a^n = a \cdots a \in S\binom{n}{0}$ and let $a^0 = \emptyset$.

For each $n \in \omega$ and $i \in \{0,1,\dots,m\}$, let $H_{m+n,i} = \{ua^n : u \in H \cap S\binom{m}{i}\}$. For all other values of r and i, put $H_{r,i} = \emptyset$. Pick a sequence $\langle w'_n\rangle_{n=0}^{\infty}$ as guaranteed by Corollary 3.3. For $n \in \omega$, let $s_n = v_0 \cdots v_{m-1} a^n$ and let $w_n = h_{s_n}(w'_{m+n})$. To see that the sequence $\langle w_n\rangle_{n=0}^{\infty}$ is as required, let $l \in \{0,1,\dots,m\}$. We shall show that

$$S_l \cap \{\textstyle\prod_{n\in F} w_n\langle u_n\rangle : F \in \mathcal{P}_f(\omega) \text{ and for all } n \in F\,,\ u_n \in H\}$$

is a subset of

$$S_l \cap \{\textstyle\prod_{k\in F'} w'_k\langle u_k\rangle : F' \in \mathcal{P}_f(\omega) \text{ and for all } k \in F'\,,\ u_k \in \bigcup_{i=0}^{\min F'} H_{k,i}\}\,.$$

Let $F \in \mathcal{P}_f(\omega)$ and for each $n \in F$, let $u_n \in H$. Let $F' = m + F$ and for $n \in F$, let $u'_{m+n} = h_{u_n}(s_n) = u_n a^n$. Then $\min F' \ge m$. We claim that for each $k \in F'$, $u'_k \in \bigcup_{i=0}^{\min F'} H_{k,i}$. To see this, let $k \in F'$ and let $n = k - m$, so that $u'_k = h_{u_n}(s_n)$. Pick $i \in \{0,1,\dots,m\}$ such that $u_n \in S\binom{m}{i}$. Then $u'_k = u_n a^n \in H_{k,i}$. To complete the proof we show that for $n \in F$, $w'_{m+n}\langle u'_{m+n}\rangle = w_n\langle u_n\rangle$. Note that $h_{u_n} \circ h_{s_n} = h_{u'_{m+n}}$. Thus

$$w_n\langle u_n\rangle = h_{u_n}(w_n) = h_{u_n}\big(h_{s_n}(w'_{m+n})\big) = h_{u'_{m+n}}(w'_{m+n}) = w'_{m+n}\langle u'_{m+n}\rangle\,.$$

□

The following corollary is then immediate.

Corollary 3.5. *Assume that the alphabet A is finite, that D is a finite group, and that $\langle T_f\rangle_{f\in D}$ is an action of D on A. Let $m,k\in\omega$ with $m>k$ and let S_k and S_m be finitely colored. There exists a sequence $\langle w_t\rangle_{t=0}^{\infty}$ in S_m such that $\{\prod_{t\in F} w_t : F\in\mathcal{P}_f(\omega)\}$ is monochrome and $\{\prod_{t\in F} w_t\langle u_t\rangle : F\in\mathcal{P}_f(\omega)$ and for each $n\in F$, $u_n\in S\binom{m}{k}\}$ is monochrome.*

Corollary 3.5 can be derived from Theorem 1.3 without using the results of Section 2 as follows. Let $k,m\in\omega$ with $m>k$. Using Theorem 1.3 show that there exist idempotents $p\in\beta S_k$ and $q\in\beta S_m$ such that $h_u(q)=p$ for every $u\in S\binom{m}{k}$. Then derive the corollary in a fashion similar to the proof of Theorem 3.2.

Based on previous experience with many Ramsey Theoretic problems we would expect that in the statement of Theorem 1.3 one could take any color class which is central in S_k. We see now that this is not the case.

Theorem 3.6. *There is a central subset M of S_1 such that there is no $w\in S_2$ with the property that $w\langle u\rangle\in M$ for every $u\in S\binom{2}{1}$.*

Proof. Recall that we are assuming that $A\neq\emptyset$, so pick $a\in A$. For each $k\in\mathbb{N}$, let $L_k=\{w\in S_1 : |\{i : w(i)=v_0\}|\geq|\{i : w(i)=a\}|+k\}$ and let $L=\bigcap_{k=1}^{\infty}\overline{L_k}$. Trivially $L\neq\emptyset$. Given any $w\in S_1$, if $m=\ell(w)$, then for each $z\in L_{m+k}$, one has $wz\in L_k$. Consequently L is a left ideal of βS_1. By [10, Corollary 2.6] pick a minimal idempotent $p\in L$. Let $M=L_1$. Then $M\in p$ so M is central.

Now let $w\in S_2$ and suppose that $w\langle u\rangle\in M$ for every $u\in S\binom{2}{1}$. Let $u_1=av_0$ and let $u_2=v_0a$. Let

$$\begin{aligned} b &= |\{i : w(i)=a\}|\,, \\ c &= |\{i : w(i)=v_0\}|\ , \text{ and} \\ d &= |\{i : w(i)=v_1\}|\,. \end{aligned}$$

Then $d=|\{i : w\langle u_1\rangle(i)=v_0\}|\geq|\{i : w\langle u_1\rangle(i)=a\}|+1=b+c+1$ and $c=|\{i : w\langle u_2\rangle(i)=v_0\}|\geq|\{i : w\langle u_2\rangle(i)=a\}|+1=b+d+1$ and so $d\geq 2b+d+2$, a contradiction. $\square$

4. Some Ramsey Theoretic Consequences

We present in this section a new and simpler derivation of a known result and a new extension of a recent result of Gunderson, Leader, Prömel, and Rödl. Both deal with the notion of a *first entries matrix*. Given a matrix which is denoted by an upper case letter, we shall follow the custom of denoting the entries by the lower case of the same letter.

Definition 4.1. Let $a,b\in\mathbb{N}$ and let M be an $a\times b$ matrix with entries from $\mathbb{Z}$. Then M satisfies the *first entries condition* if and only if no row of M is $\vec{0}$ and there exist $c_1,c_2,\ldots,c_b\in\mathbb{N}$ such that, for any row i of M, if the first nonzero entry of row i occurs in column j, then $m_{i,j}=c_j$. Each c_j is called a *first entry* of M. A *first entries matrix* is a matrix with entries from $\mathbb{Z}$ which satisfies the first entries condition.

First entries matrices are based on Deuber's (m,p,c)-sets which were used [4] to prove Rado's conjecture about sets containing solutions to all partition regular systems of homogeneous linear equations. First entries matrices provide characterizations of all image partition regular matrices. See [10, Chapter 15].

Definition 4.2. For each $j \in \mathbb{N}$ and each $w \in W$, let $\alpha_j(w)$ be the number of occurrences of v_{j-1} in W.

The following lemma will be used in both of the featured results of this section. In it, no special assumptions about the alphabet A are needed.

Lemma 4.3. *Let $a,b \in \mathbb{N}$, let M be an $a \times b$ first entries matrix which has all first entries equal, and let C be a central subset of $\mathbb{N}$. Assume that $D = E = \{e\}$. For each $i \in \{1,2,\ldots,a\}$, define $f_i : W \to \mathbb{Z}$ by $f_i(w) = \sum_{j=1}^{b} m_{i,j}\alpha_j(w)$. Then there is a minimal idempotent s of S_b such that $\bigcap_{i=1}^{a} {f_i}^{-1}[C] \in s$.*

Proof. Pick a minimal idempotent q' of $\beta\mathbb{N}$ such that $C \in q'$. Pick $c \in \mathbb{N}$ such that all first entries of M are equal to c. We claim that we may choose a minimal idempotent q of $\beta\mathbb{N}$ such that $cq = q'$, where the product cq is computed in the semigroup $(\beta\mathbb{N},\cdot)$. (When we refer to the semigroup $\beta\mathbb{N}$ without an operation mentioned, we are speaking of $(\beta\mathbb{N},+)$.) To see this, define $\gamma : \beta\mathbb{N} \to c\beta\mathbb{N}$ by $\gamma(p) = cp$. Then γ is a surjective homomorphism from $\beta\mathbb{N}$ onto $c\beta\mathbb{N}$ so by [10, Exercise 1.7.3], $\gamma[K(\beta\mathbb{N})] = K(c\beta\mathbb{N})$. Further, by [10, Lemma 6.6], $q' \in c\beta\mathbb{N}$ and by [10, Theorem 1.65] $K(c\beta\mathbb{N}) = c\beta\mathbb{N} \cap K(\beta\mathbb{N})$ so $q' \in K(c\beta\mathbb{N})$. Pick $r \in K(\beta\mathbb{N})$ such that $\gamma(r) = q'$ and pick a minimal left ideal L of $\beta\mathbb{N}$ such that $r \in L$. Then $L \cap \gamma^{-1}[\{q'\}]$ is a compact right topological semigroup, so has an idempotent q.

Note that for each $i \in \{1,2,\ldots,a\}$, f_i is a homomorphism so its continuous extension, also denoted by f_i, from βW to $\beta\mathbb{Z}$ is a homomorphism. Now $\alpha_1[S_1] = \omega$ so ${\alpha_1}^{-1}[\{q\}]$ is a compact subsemigroup of βS_1 and so we may pick an idempotent $r \in \beta S_1$ such that $\alpha_1(r) = q$. Let p_0 be any minimal idempotent of βW_0 and pick by Lemma 2.7 an idempotent $p_1 \in p_0 r \beta S_1 \cap \beta S_1 r p_0$ which is minimal in βS_1. Since $p_1 \leq p_0$ we may pick a sequence $\langle p_n \rangle_{n=0}^{\infty}$ as guaranteed by Theorem 2.12.

Now $p_1 = p_0 r x$ for some $x \in \beta S_1$ and so $\alpha_1(p_1) = \alpha_1(p_0) + \alpha_1(r) + \alpha_1(x) = 0 + \alpha_1(r) + \alpha_1(x)$ and thus $\alpha_1(p_1) \leq_R \alpha_1(r) = q$. Similarly $\alpha_1(p) \leq_L q$. Since q is minimal in $\beta\mathbb{N}$, we have $\alpha_1(p) = q$.

Next we observe that for any $j \in \{1,2,\ldots,b\}$, $\alpha_j(p_j) = q$. To see this, pick $d \in A$ and define $u \in S\binom{j}{1}$ by agreeing that for $t \in \{0,1,\ldots,j-1\}$,

$$u(t) = \begin{cases} d & \text{if } t \neq j-1 \\ v_0 & \text{if } t = j-1\,. \end{cases}$$

Then $\alpha_j = \alpha_1 \circ h_u$ on S_j and $h_u(p_j) = p_1$ by Lemma 2.14 so $\alpha_j(p_j) = \alpha_1\big(h_u(p_j)\big) = \alpha_1(p_1) = q$.

Now we claim that for each $i \in \{1,2,\ldots,a\}$, $f_i(p_b) = q'$. This will complete the proof because we may take $s = p_b$. Let $i \in \{1,2,\ldots,a\}$ and let j be the column with the first nonzero entry of row i of M. Then for $w \in W$, $f_i(w) = c\alpha_j(w) + \sum_{t=j+1}^{b} m_{i,t}\alpha_t(w)$. For $w \in S_j$ and $t > j$, $\alpha_t(w) = 0$ and so on S_j, $f_i(w) = c\alpha_j(w)$ and consequently $f_i(p_j) = c\alpha_j(p_j) = cq = q'$. Since $p_b \leq p_j$ we have that $f_i(p_b) \leq f_i(p_j) = q'$ and so, since q' is minimal, $f_i(p_b) = q'$. □

Definition 4.4. If M is an $a \times b$ matrix and $\vec{x} \in \mathbb{Z}^b$, then $\eta(M\vec{x})$ is the set of entries of $M\vec{x}$.

The following result is not new. (See for example [10, Theorem 15.5]1.) The proof here is shorter (given the earlier development).

Theorem 4.5. *Let $a, b \in \mathbb{N}$ and let M be an $a \times b$ first entries matrix. Let C be a central subset of $\mathbb{N}$. Then there exists $\vec{x} \in \mathbb{N}^b$ such that $\eta(M\vec{x}) \subseteq C$.*

Proof. Let $c_1, c_2, \ldots, c_b$ be the first entries of M, let $c = \prod_{j=1}^b c_j$, and for $j \in \{1, 2, \ldots, b\}$, let $d_j = \dfrac{c}{c_j}$. Define the $a \times b$ matrix N by $n_{i,j} = d_j m_{i,j}$ for $i \in \{1, 2, \ldots, a\}$ and $j \in \{1, 2, \ldots, b\}$. Then N is a first entries matrix with all first entries equal to c. Let $D = E = \{e\}$. For $i \in \{1, 2, \ldots, a\}$ define $f_i : W \to \mathbb{Z}$ by $f_i(w) = \sum_{j=1}^b n_{i,j}\alpha_j(w)$. Pick by Lemma 4.3 a minimal idempotent s of βS_b such that $\bigcap_{i=1}^a {f_i}^{-1}[C] \in s$. Pick $w \in S_b \cap \bigcap_{i=1}^a {f_i}^{-1}[C]$. For $j \in \{1, 2, \ldots, b\}$ let $x_j = d_j\alpha_j(w)$. Then each $x_j \in \mathbb{N}$ and for $i \in \{1, 2, \ldots, a\}$, $\sum_{j=1}^b m_{i,j}x_j = \sum_{j=1}^b d_j m_{i,j}\alpha_j(w) = \sum_{j=1}^b n_{i,j}\alpha_j(w) = f_i(w) \in C$. □

Recently Gunderson, Leader, Prömel, and Rödl proved the following theorem. (By a K_k we mean a complete graph on k vertices. A set is independent with respect to a graph if there are no edges between members of the set.)

Theorem 4.6. *Let $a, b, k \in \mathbb{N}$ and let M be an $a \times b$ first entries matrix. Then there exist $c, d \in \mathbb{N}$ and a $c \times d$ first entries matrix P such that all first entries of P are equal and whenever $\vec{x} \in \mathbb{N}^d$ and G is a K_k-free graph on $\eta(P\vec{x})$, there exists $\vec{y} \in \mathbb{N}^b$ such that $\eta(M\vec{y})$ is an independent subset of $\eta(P\vec{x})$.*

Proof. [7]. □

The following corollary is an immediate consequence of Theorem 4.6.

Corollary 4.7. *Let $a, b, k \in \mathbb{N}$ and let M be an $a \times b$ first entries matrix. Let G be a K_k-free graph on $\mathbb{N}$. Then there exists $\vec{y} \in \mathbb{N}^b$ such that $\eta(M\vec{y})$ is independent.*

We establish now the following simultaneous extension of Theorem 4.5 and Corollary 4.7.

Theorem 4.8. *Let $a, b, k \in \mathbb{N}$, let M be an $a \times b$ first entries matrix, and let G be a K_k-free graph on $\mathbb{N}$. There exists a sequence $\langle \vec{x_n} \rangle_{n=1}^\infty$ in $\mathbb{N}^b$ such that for every $F \in \mathcal{P}_f(\mathbb{N})$, $\eta\big(M(\sum_{n\in F} \vec{x_n})\big)$ is an independent subset of C.*

Proof. Pick by Theorem 4.6 some $c, d \in \mathbb{N}$ and a $c \times d$ first entries matrix P with all first entries equal such that for each $\vec{z} \in \mathbb{N}^d$, there exists $\vec{y} \in \mathbb{N}^b$ such that $\eta(M\vec{y})$ is an independent subset of $\eta(P\vec{z})$. Let $D = E = \{e\}$. For $i \in \{1, 2, \ldots, c\}$ define $f_i : W \to \mathbb{Z}$ by $f_i(w) = \sum_{j=1}^d p_{i,j}\alpha_j(w)$. Pick by Lemma 4.3 a minimal idempotent s of βS_d such that $\bigcap_{i=1}^c {f_i}^{-1}[C] \in s$.

For each $w \in S_d$, let $\vec{z_w} = \begin{pmatrix} \alpha_1(w) \\ \vdots \\ \alpha_d(w) \end{pmatrix}$, pick $\vec{y_w} \in \mathbb{N}^b$ such that $\eta(M\vec{y_w})$ is an independent subset of $\eta(P\vec{z_w})$, and choose $\gamma_w : \{1, 2, \ldots, a\} \to \{1, 2, \ldots, c\}$ such that for each $i \in \{1, 2, \ldots, a\}$, $\sum_{j=1}^b m_{i,j}y_{w,j} = \sum_{j=1}^d p_{\gamma_w(i),j}\alpha_j(w)$.

For each $\mu : \{1,2,\ldots,a\} \to \{1,2,\ldots,c\}$, let $H_\mu = \{w \in S_d : \gamma_w = \mu\}$ and pick $\mu : \{1,2,\ldots,a\} \to \{1,2,\ldots,c\}$ such that $H_\mu \in s$. Choose by [10, Theorem 5.8] a sequence $\langle w_n\rangle_{n=1}^\infty$ in S_d such that $FP(\langle w_n\rangle_{n=1}^\infty) \subseteq H_\mu \cap \bigcap_{i=1}^c {f_i}^{-1}[C]$, where $FP(\langle w_n\rangle_{n=1}^\infty) = \{\prod_{n\in F} w_n : F \in \mathcal{P}_f(\mathbb{N})\}$ and $\prod_{n\in F} w_n$ is computed in increasing order of indices.

For each $n \in \mathbb{N}$, let $\vec{x_n} = \vec{y_{w_n}}$. To complete the proof, let $F \in \mathcal{P}_f(\mathbb{N})$ and let $u = \prod_{n\in F} w_n$. It suffices to show that $M(\sum_{n\in F}\vec{x_n}) = M\vec{y_u}$. To this end, let $i \in \{1,2,\ldots,a\}$. The entry in row i of $M(\sum_{n\in F}\vec{x_n})$ is

$$\begin{aligned}\textstyle\sum_{j=1}^b m_{i,j}(\sum_{n\in F} y_{w_n,j}) &= \textstyle\sum_{n\in F}\sum_{j=1}^b m_{i,j}y_{w_n,j}\\ &= \textstyle\sum_{n\in F}\sum_{j=1}^d p_{\mu(i),j}\alpha_j(w_n)\\ &= \textstyle\sum_{j=1}^d p_{\mu(i),j}\big(\sum_{n\in F}\alpha_j(w_n)\big)\\ &= \textstyle\sum_{j=1}^d p_{\mu(i),j}\alpha_j(u)\\ &= \textstyle\sum_{j=1}^b m_{i,j}y_{u,j}\,.\end{aligned}$$

□

We remark that Theorem 4.8 can be proved without using Theorem 2.12, using instead methods such as those in the proof of [11, Theorem 3.16].

5. Some Derivations from Known Results

In this section we show that the commonly quoted version of the Graham-Rothschild Parameter Sets Theorem in which $D = \{e\}$ implies the full version as stated in Theorem 1.3 – in fact a strengthening of that full version, because in Theorem 5.1 it is not required that D be a group, or even a semigroup. We also present a derivation of Corollary 3.3 for the case in which A is finite and $D = \{e\}$ from [2, Theorem 10].

The following theorem may be known but is certainly not well known, even though its proof is very simple. (Before obtaining the proof we inquired of several experts whether such a derivation was possible, and none of them knew.)

Theorem 5.1. *Assume the Graham Rothschild Theorem as stated in Theorem 1.3 for the case in which $D = \{e\}$. Let $\Gamma = (A, D, \{e\}, \langle T_f\rangle_{f\in D})$ be a parameter system for which A and D are finite and T_e is the identity. Let $m, k \in \omega$ with $k < m$. Then, whenever $S_k(\Gamma)$ is finitely colored, there exists $w \in S_m(\Gamma)$ such that $\{w\langle u\rangle : u \in S\binom{m}{k}(\Gamma)\}$ is monochrome.*

Proof. Let $n = |D|$ and let $D = \{g_0, g_1, \ldots, g_{n-1}\}$, with $g_0 = e$. We define $\theta : A \cup (D\times V) \to A\cup(D\times V)$ by $\theta(a) = a$ if $a \in A$ and $\theta\big((g_i,\nu_j)\big) = (e, \nu_{nj+i})$. As usual, let W denote the semigroup of words over $A\cup(D\times V)$. Then θ extends to a homomorphism from W to W, which we shall also denote by θ. We observe that θ is a bijection from W onto W', the semigroup of words over $A\cup(\{e\}\times V)$. Let Γ' denote the parameter system $(A, \{e\}, \{e\}, \langle T_e\rangle)$.

We define $\lambda : D \to \{0,1,\ldots,n-1\}$ by $\lambda(g_i) = i$.

Since $W' \subseteq W$, $h_w : W \to W$ is defined for every $w \in W'$. Further, if $w \in W'$, then $h_w[W'] \subseteq W'$. We shall show that, for every $u \in S\binom{m}{k}(\Gamma)$, there exists $u' \in S\binom{nm}{nk}(\Gamma')$ such that $\theta^{-1}\circ h_{u'}\circ\theta = h_u$. To this end, let $u \in S\binom{m}{k}(\Gamma)$ be given and let $t \in \{0,1,\ldots,m-1\}$. If $u(t) \in A$, we let $u'(nt+i) = T_{g_i}\big(u(t)\big)$ for each $i \in \{0,1,\ldots,n-1\}$. If $u(t) = (f,\nu_j) \in D \times V$, we put: $u'(nt+i) = (e,\nu_{nj+\lambda(g_i f)})$ for each $i \in \{0,1,\ldots,n-1\}$.

We claim that $\theta^{-1} \circ h_{u'} \circ \theta = h_u$. To this end, let $x \in A \cup (D \times V)$. If $x \in A$, then $\theta^{-1} \circ h_{u'} \circ \theta(x) = x = h_u(x)$, so assume that $x = (g_i, \nu_j)$ for some $i \in \{0,1,\ldots,n-1\}$ and some $j \in \omega$. If $j \geq m$, then $h_u(x) = x$ and $\theta(x) = (e,\nu_{nj+i})$ where $nj+i \geq nm$ so that $\theta^{-1}\big(h_{u'}(e,\nu_{nj+i})\big) = \theta^{-1}(e,\nu_{nj+i}) = x$. So assume that $j < m$ and assume first that $u(j) \in A$. Then $h_u(x) = T_{g_i}\big(u(j)\big)$. Also $u'(nj+i) = T_{g_i}\big(u(j)\big)$ so $h_{u'}\big(\theta(x)\big) = T_e\Big(T_{g_i}\big(u(j)\big)\Big) = T_{g_i}\big(u(j)\big)$ and thus $\theta^{-1} \circ h_{u'} \circ \theta(x) = T_{g_i}\big(u(j)\big)$. Finally assume that $j < m$ and $u(j) = (f,\nu_t)$ for some $f \in D$ and some $t \in \omega$. Then $h_u(x) = (g_i f, \nu_t)$. Also $\theta(x) = (e,\nu_{nj+i})$ and $u'(nj+i) = (e,\nu_{nt+\lambda(g_i f)})$ so $h_{u'}\big(\theta(x)\big) = (e,\nu_{nt+\lambda(g_i f)})$. Also $\theta(g_i f,\nu_t) = (e,\nu_{nt+\lambda(g_i f)})$ so $\theta^{-1} \circ h_{u'} \circ \theta(x) = (g_i f, \nu_t)$.

We must check that $u' \in S\binom{nm}{nk}(\Gamma')$. Clearly, u' has length nm. Since (e,ν_j) occurs in u if and only if $j \in \{0,1,\ldots,k-1\}$ and since $\lambda(g_i e) = i$, (e,ν_{nj+i}) occurs in u' if and only if $j \in \{0,1,\ldots,k-1\}$ and $i \in \{0,1,\ldots,n-1\}$. So (e,ν_s) occurs in u' if and only if $s \in \{0,1,\ldots,nk-1\}$.

Finally, let $j \in \{0,1,\ldots,k-1\}$ and let $i \in \{0,1,\ldots,n-1\}$. If t is the first index for which (e,ν_j) occurs in u, then $nt+i$ is the first index for which (e,ν_{nj+i}) occurs in u'. This establishes that $u' \in S\binom{nm}{nk}(\Gamma')$.

Now it is easy to verify that $\theta^{-1}[S_{nk}(\Gamma')] \subseteq S_k(\Gamma)$ and $\theta^{-1}[S_{nm}(\Gamma')] \subseteq S_m(\Gamma)$. Let $r \in \mathbb{N}$ and let $\varphi : S_k(\Gamma) \to \{1,2,\ldots,r\}$ be a finite coloring of $S_k(\Gamma)$. Then $\varphi \circ \theta^{-1}$ is a finite coloring of $S_{nk}(\Gamma')$ so pick by Theorem 1.3 $w' \in S_{nm}(\Gamma')$ and $i \in \{1,2,\ldots,r\}$ such that $\varphi \circ \theta^{-1}\big(h_s(w')\big) = i$ for every $s \in S\binom{nm}{nk}(\Gamma')$. Let $w = \theta^{-1}(w')$. Then $w \in S_m(\Gamma)$. To see that w is as required, let $u \in S\binom{m}{k}$. Then

$$\varphi(w\langle u\rangle) = \varphi\big(h_u(w)\big) = \varphi\big(\theta^{-1} \circ h_{u'} \circ \theta(w)\big) = \varphi \circ \theta^{-1}\big(h_{u'}(w')\big) = i\,.$$

□

We now introduce some notation adapted from [2]. Let

$$\mathcal{S} = \{\langle s_n\rangle_{n=0}^{\infty} : \text{ for all } n \in \omega\,,\ s_n \text{ is an } n\text{-variable word over } A\}\,.$$

(This is what is denoted in [2] by $\mathcal{S}^+(A,\vec{e})$ where $e(n) = n$ for all $n \in \omega$.)

Given $\vec{s},\vec{t} \in \mathcal{S}$, $\vec{s} \leq \vec{t}$ if and only if there exists an increasing sequence $\langle H_n\rangle_{n=0}^{\infty}$ in $\mathcal{P}_f(\omega)$ (meaning that $\max H_n < \min H_{n+1}$ for each n) and for each $n \in \omega$ and each $k \in H_n$ there exists $u_k \in \bigcup_{l=0}^{n} S\binom{k}{l}$ such that $s_n = \prod_{k\in H_n} t_k\langle u_k\rangle$. (Since $s_n \in S_n$, one has that for some $k \in H_n$, $u_k \in S\binom{k}{n}$.) Note that $\leq$ is transitive. (This is the relation that is denoted in [2] by $\leq^+$.)

Give $\mathcal{S}$ the topology with basis $\{B(n,\vec{t}) : n \in \omega \text{ and } \vec{t} \in \mathcal{S}\}$, where

$$B(n,\vec{t}) = \{\vec{s} \in \mathcal{S} : \vec{s} \leq \vec{t} \text{ and for all } i \in \{0,1,\ldots,n-1\}\,,\ s_i = t_i\}\,.$$

Notice that if $\vec{s} \in B(n,\vec{t})$, then $B(n,\vec{s}) \subseteq B(n,\vec{t})$.

Theorem 5.2. *Let X be open in $\mathcal{S}$, let $\vec{s} \in \mathcal{S}$, and let $n \in \omega$. Then there exists $\vec{t} \in B(n, \vec{s})$ such that either $B(n, \vec{t}) \subseteq X$ or $B(n, \vec{t}) \cap X = \emptyset$.*

Proof. This is an immediate consequence of [2, Theorem 10] because open sets are Baire. □

Lemma 5.3. *Let $n \in \omega$, let $k \in \mathbb{N}$, let $\varphi : S_n \to \{1, 2, \ldots, k\}$, and let $\vec{s} \in \mathcal{S}$. There exists $\vec{t} \in B(n, \vec{s})$ such that φ is constant on $\{r_n : \vec{r} \in B(n, \vec{t})\}$.*

Proof. We proceed by induction on k, the case $k = 1$ being trivial. Let $k \in \mathbb{N}$ and assume the lemma is true for k. Let $\varphi : S_n \to \{1, 2, \ldots, k+1\}$ and let $X = \{\vec{t} \in \mathcal{S} : \varphi(t_n) = k+1\}$. If $\vec{t} \in X$, then $B(n+1, \vec{t}) \subseteq X$ so X is open. Let $\vec{s} \in \mathcal{S}$ and pick $\vec{t} \in B(n, \vec{s})$ such that either $B(n, \vec{t}) \subseteq X$ or $B(n, \vec{t}) \cap X = \emptyset$.

If $B(n, \vec{t}) \subseteq X$, then φ is constantly equal to $k + 1$ on $\{r_n : \vec{r} \in B(n, \vec{t})\}$, so assume that $B(n, \vec{t}) \cap X = \emptyset$. Define $\tau : S_n \to \{1, 2, \ldots, k\}$ by

$$\tau(w) = \begin{cases} \varphi(w) & \text{if } \varphi(w) \le k \\ 1 & \text{if } \varphi(w) = k+1\,. \end{cases}$$

Pick $\vec{u} \in B(n, \vec{t})$ such that τ is constant on $\{r_n : \vec{r} \in B(n, \vec{u})\}$. Then $\vec{u} \in B(n, \vec{s})$. We claim that φ is constant on $\{r_n : \vec{r} \in B(n, \vec{u})\}$. Indeed, given $\vec{r} \in B(n, \vec{u})$ one has $\vec{r} \in B(n, \vec{t})$ so $\varphi(r_n) \ne k+1$ and thus $\varphi(r_n) = \tau(r_n)$. □

Theorem 5.4. *Assume that A is finite, $D = \{e\}$, and, for each $n \in \omega$, S_n has been finitely colored. Then, there exists a sequence $\langle w_n \rangle_{n<\omega}$ with each $w_n \in S_n$ such that for every $m \in \omega$,*

$$S_m \cap \Big\{ \textstyle\prod_{n\in F} w_n\langle u_n\rangle : F \in \mathcal{P}_f(\omega) \text{ and for all } n \in F\,,\ u_n \in \bigcup_{i=0}^{\min F} S\binom{n}{i} \Big\}$$

is monochrome. (That is, the color of $\prod_{n\in F} w_n\langle u_n\rangle$ is determined solely by the number of variables in $\prod_{n\in F} w_n\langle u_n\rangle$.)

Proof. We may assume that T_e is the identity since the general case then follows easily. To see this, notice that whenever $w \in S\binom{n}{m}$ and $u \in S\binom{m}{k}$, there is a $u' \in S\binom{m}{k}$ such that $w\langle u\rangle$ is the same as $w\langle u'\rangle$, where $w\langle u'\rangle$ is computed after reinterpreting T_e to be the identity on A.

For each $n \in \omega$, let φ_n be a finite coloring of S_n. Choose by Lemma 5.3 $\vec{s_0} \in \mathcal{S}$ such that φ_0 is constant on $\{r_0 : \vec{r} \in B(0, \vec{s_0})\}$.

Let $n \in \mathbb{N}$ and assume that we have chosen $\vec{s_{n-1}}$. Choose $\vec{s_n} \in B(n, \vec{s_{n-1}})$ such that φ_n is constant on $\{r_n : \vec{r} \in B(n, \vec{s_n})\}$.

For $n \in \omega$, let $w_n = s_{n,n}$, i.e. entry n of $\vec{s_n}$. We claim that the sequence $\langle w_n\rangle_{n=0}^{\infty}$ is as required. So let $m \in \omega$, let $F \in \mathcal{P}_f(\omega)$, for each $n \in F$, let $u_n \in \bigcup_{i=0}^{\min F} S\binom{n}{i}$, and assume that $\prod_{n\in F} w_n\langle u_n\rangle \in S_m$.

We shall show that $\prod_{n\in F} w_n\langle u_n\rangle \in \{r_m : \vec{r} \in B(m, \vec{s_m})\}$. To this end, let $k = \max F$. Then for each $n \in F$, $w_n = s_{n,n} = s_{n+1,n} = \ldots = s_{k,n}$. Notice that $m \le \min F$ because $\prod_{n\in F} w_n\langle u_n\rangle \in S_m$ so for some n, $u_n \in S\binom{n}{m}$. For this n, $u_n \in S\binom{n}{i}$ for some $i \le \min F$ and so $m = i$.

For $n < m$, let $r_n = w_n = s_{k,n}$. Let $r_m = \prod_{n \in F} w_n\langle u_n \rangle$. And for $n > m$, let $r_n = s_{k,k+n-m}\langle z_n \rangle$ where z_n is any member of $S\binom{k+n-m}{n}$. Then $\vec{r} \in B(m, \vec{s_k}) \subseteq B(m, \vec{s_m})$. $\square$

References

1. V. Bergelson, A. Blass, and N. Hindman, *Partition theorems for spaces of variable words*, Proc. London Math. Soc. **68** (1994), 449-476.
2. T. Carlson, *Some unifying principles in Ramsey Theory*, Discrete Math. **68** (1988), 117-169.
3. T. Carlson, N. Hindman, and D. Strauss, *Ramsey theoretic consequences of some new results about algebra in the Stone-Čech compactification*, manuscript. (Currently available at `http://members.aol.com/nhindman/`.)
4. W. Deuber, *Partitionen und lineare Gleichungssysteme*, Math. Zeit. **133** (1973), 109-123.
5. H. Furstenberg, *Recurrence in ergodic theory and combinatorical number theory*, Princeton University Press, Princeton, 1981.
6. R. Graham and B. Rothschild, *Ramsey's Theorem for n-parameter sets*, Trans. Amer. Math. Soc. **159** (1971), 257-292.
7. D. Gunderson, I. Leader, H. Prömel and V, Rödl, *Independent Deuber sets in graphs on the natural numbers*, J. Comb. Theory (Series A) **103** (2003), 305-322.
8. A. Hales and R. Jewett, *Regularity and positional games*, Trans. Amer. Math. Soc. **106** (1963), 222-229.
9. N. Hindman, I. Leader, and D. Strauss, *Infinite partition regular matrices - solutions in central sets*, Trans. Amer. Math. Soc. **355** (2003), 1213-1235.
10. N. Hindman and D. Strauss, *Algebra in the Stone-Čech compactification: theory and applications*, de Gruyter, Berlin, 1998.
11. N. Hindman and D. Strauss, *Independent sums of arithmetic progressions in K_m-free graphs*, Ars Combinatoria **70** (2004), 221-243.
12. P. Milnes, *Compactifications of topological semigroups*, J. Australian Math. Soc. **15** (1973), 488-503.
13. J. Paris and L. Harrington *A mathematical incompleteness in Peano arithmetic*, in Handbook of mathematical logic, J. Barwise, ed., North Holland, Amsterdam, 1977, 1133-1142.
14. H. Prömel and B. Voigt, *Graham-Rothschild parameter sets*, in Mathematics of Ramsey Theory, J. Nešetřil and V. Rödl, eds., Springer-Verlag, Berlin, 1990, 113-149.

Department of Mathematics, Ohio State University, Columbus, OH 43210
E-mail address: `carlson@math.ohio-state.edu`

Department of Mathematics, Howard University, Washington, DC 20059
E-mail address: `nhindman@aol.com`
URL: `http://members.aol.com/nhindman/`

Department of Pure Mathematics, University of Hull, Hull HU6 7RX, UK
E-mail address: `d.strauss@maths.hull.ac.uk`