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The First Nontrivial Hales-Jewett Number is Four

Neil Hindman * Eric Tressler †

Abstract

We show that whenever the length four words over a three letter alphabet are two-colored, there must exist a monochromatic combinatorial line. We also provide some computer generated lower bounds for some other Hales-Jewett numbers.

1 Introduction

Several of the classic results in Ramsey Theory assert, given a certain size k and number of colors r, the existence of a positive integer n such that whenever an appropriate object of size n is r-colored, there must exist a monochromatic object of size k. (By an r-coloring of a set X we mean, of course, a function taking X to some r-element set. To say that $Y \subseteq X$ is monochromatic is to say that the given function is constant on Y.)

Consider the following examples, given here in historical order. Schur's Theorem [6] says that for any r there is some n such that whenever the set $[n] := \{1, ..., n\}$ is r-colored, there must exist some x, y such that $\{x, y, x + y\}$ is monochromatic.

Also, van der Waerden's Theorem [8] says that if k and r are positive integers, there is some n such that whenever the $[n]$ is r-colored, there must exist a monochromatic length k arithmetic progression.

The finite version of Ramsey's Theorem [5] says that given positive integers k, m, and r, there is some n such that whenever the k-element subsets of an

*Department of Mathematics, Howard University, Washington, DC 20059, nhindman@aol.com. This author acknowledges support received from the National Science Foundation via Grant DMS-0554803.

†Department of Mathematics, University of California, San Diego, La Jolla, CA 92093-0112, etressle@math.ucsd.edu

n-element set X are r-colored, there must be an m-element subset Y of X, all of whose k-element subsets are the same color.

And the Hales-Jewett Theorem [2] says that for any positive integers k and r there exists $n = HJ(k,r)$ such that whenever the set of length n words over a k-letter alphabet are r-colored, there must exist a monochromatic line. Here perhaps a bit of explanation is in order. A *word* over an *alphabet* (= set) A is a finite sequence in A, and the *length* of the word is the number of terms in the sequence. For our purposes, the informal view of a sequence as terms listed in order will do, so that, for example, 1323 is a length 4 word over the alphabet $\{1,2,3\}$ (and also over the alphabet $\{1,2,3,4,5\}$ for that matter). A *variable word* over A is a word over $A \cup \{v\}$ in which v occurs, where v is a "variable" which is not in A. If $w = w(v)$ is a variable word over A and $a \in A$, then $w(a)$ is the word in which all occurrences of v are replaced by a. Thus, for example, if $w(v) = 1v3v$, then $w(1) = 1131$ and $w(2) = 1232$. A *combinatorial line* over A is $\{w(a) : a \in A\}$ where $w(v)$ is a variable word over A. Again, if $A = \{1,2,3\}$, then $\{1131, 1232, 1333\}$ is the combinatorial line determined by $w(v) = 1v3v$.

A substantial amount of effort has been invested in finding the value of the smallest n which "works" for particular instances of Schur's Theorem, van der Waerden's Theorem, and Ramsey's Theorem. For example, the smallest n guaranteeing a monochromatic length k arithmetic progression when $[n]$ is 2-colored are respectively 9, 35, and 178 for $k = 3$, $k = 4$, and $k = 5$. See [1, Chapter 4] and [4] for substantial information about known specific values of van der Waerden numbers, Schur numbers, and Ramsey numbers.

The original proofs of these theorems produced exceedingly large upper bounds for n (except for Schur's Theorem, where the original proof shows that $n = \lfloor r!e \rfloor$ will do). The easiest way to prove Ramsey's Theorem and the Hales-Jewett theorem is to prove the infinite versions. One then deduces the finite versions, but this method yields no upper bounds at all. Twenty years ago there was a great deal of excitement when Shelah showed [7] that there are upper bounds for the van der Waerden and Hales-Jewett numbers that are primitive recursive. See [1] for a detailed discussion of the Hales-Jewett theorem and also of the proof by Shelah.

Uniquely among the classical theorems mentioned above, no nontrivial values of $HJ(k,r)$ had been known. It's clear that $HJ(k,1) = 1$ for any k, and that $HJ(2,r) = r$ is not hard to prove. (If w is a word of length l over the alphabet $\{1,2\}$ and $\varphi(w)$ is the number of 1's occurring in w, then there is no monochromatic combinatorial line and so $HJ(2,r) \geq r$. If $w_i = a_{i,1}a_{i,2}\cdots a_{i,l}$ where $a_{i,t} = 2$ if $t < i$ and $a_{i,t} = 1$ if $t \geq i$, then whenever $i \neq j$, $\{w_i, w_j\}$ is a combinatorial line, and so $HJ(2,r) \leq r$.) The first nontrivial value of HJ, then, is $HJ(3,2)$, which we show here, in Section 2, to be 4. In Section 3 we present an algorithm which we used to determine that HJ(3,2)=4 before the detailed proof of Section 2 was found and present some lower bounds for other Hales-Jewett numbers obtained using that algorithm.

2 HJ(3,2)=4

This section is devoted entirely to a proof of the following theorem.

Theorem 1. *Let the length four words on the alphabet* $\{1,2,3\}$ *be two colored. Then there exists a monochromatic combinatorial line.*

Proof. Suppose instead that we have a 2-coloring of the 4-letter words over $\{1,2,3\}$ with respect to which there is no monochromatic combinatorial line. Let A be the set of words with the first color and let B be the set of words with the second color. Now $\{1111, 2222, 3333\}$ is a combinatorial line, so we may assume without loss of generality that $1111 \in A$, $2222 \in A$, and $3333 \in B$.

The proof now proceeds through four lemmas. In the proofs of these lemmas, we shall follow the customary abuse of notation wherein we substitute "$P \Rightarrow Q$" for the instance of modus ponens which should say "$(P \Rightarrow Q)$ and P, therefore Q".

Lemma 2. *If* $\{2111, 1211\} \subseteq A$*, then* $2211 \in B$*.*

Proof. Suppose instead that $\{2111, 1211, 2211\} \subseteq A$.

$$\begin{aligned}
1111 \in A \text{ and } 2211 \in A &\quad\Rightarrow\quad 3311 \in B.\\
1211 \in A \text{ and } 2211 \in A &\quad\Rightarrow\quad 3211 \in B.\\
1111 \in A \text{ and } 2111 \in A &\quad\Rightarrow\quad 3111 \in B.
\end{aligned}$$

But $\{3311, 3211, 3111\}$ is a combinatorial line. □

Lemma 3. *It is not the case that* $\{1112, 1121, 1211, 2111\} \subseteq A$*.*

Proof. Suppose that $\{1112, 1121, 1211, 2111\} \subseteq A$.

$$\begin{aligned}
1112 \in A \text{ and } 2222 \in A &\quad\Rightarrow\quad 3332 \in B.\\
3332 \in B \text{ and } 3333 \in B &\quad\Rightarrow\quad 3331 \in A.\\
3331 \in A \text{ and } 1111 \in A &\quad\Rightarrow\quad 2221 \in B.\\
1111 \in A \text{ and } 1121 \in A &\quad\Rightarrow\quad 1131 \in B.\\
1131 \in B \text{ and } 3333 \in B &\quad\Rightarrow\quad 2232 \in A.\\
\text{Lemma 2} &\quad\Rightarrow\quad 2211 \in B.\\
2221 \in B \text{ and } 2211 \in B &\quad\Rightarrow\quad 2231 \in A.\\
2111 \in A \text{ and } 2222 \in A &\quad\Rightarrow\quad 2333 \in B.\\
2232 \in A \text{ and } 2231 \in A &\quad\Rightarrow\quad 2233 \in B.\\
1211 \in A \text{ and } 2222 \in A &\quad\Rightarrow\quad 3233 \in B.\\
3233 \in B \text{ and } 3333 \in B &\quad\Rightarrow\quad 3133 \in A.\\
2333 \in B \text{ and } 2233 \in B &\quad\Rightarrow\quad 2133 \in A.\\
2233 \in B \text{ and } 3333 \in B &\quad\Rightarrow\quad 1133 \in A.
\end{aligned}$$

But $\{1133, 2133, 3133\}$ is a combinatorial line. $\square$

Lemma 4. *It is not the case that some two of* 1112*,* 1121*,* 1211*, and* 2111 *are in* A*.*

Proof. Suppose instead without loss of generality that $\{1211, 2111\} \subseteq A$. By Lemma 3 we can assume without loss of generality that $1112 \in B$.

$$\begin{array}{rcl}
2222 \in A \text{ and } 1211 \in A & \Rightarrow & 3233 \in B. \\
3333 \in B \text{ and } 3233 \in B & \Rightarrow & 3133 \in A. \\
1111 \in A \text{ and } 3133 \in A & \Rightarrow & 2122 \in B. \\
2122 \in B \text{ and } 1112 \in B & \Rightarrow & 3132 \in A. \\
3132 \in A \text{ and } 3133 \in A & \Rightarrow & 3131 \in B. \\
3131 \in B \text{ and } 3333 \in B & \Rightarrow & 3232 \in A. \\
1111 \in A \text{ and } 2111 \in A & \Rightarrow & 3111 \in B. \\
3232 \in A \text{ and } 2222 \in A & \Rightarrow & 1212 \in B. \\
3111 \in B \text{ and } 3333 \in B & \Rightarrow & 3222 \in A. \\
3232 \in A \text{ and } 3222 \in A & \Rightarrow & 3212 \in B. \\
3111 \in B \text{ and } 3212 \in B & \Rightarrow & 3313 \in A. \\
1212 \in B \text{ and } 3212 \in B & \Rightarrow & 2212 \in A.
\end{array}$$

But $\{1111, 2212, 3313\}$ is a combinatorial line. $\square$

Lemma 5. $\{1112, 1121, 1211, 2111, 2221, 2212, 2122, 1222\} \subseteq B$*.*

Proof. Suppose not. We have not distinguished between 2 and 1 so we may assume without loss of generality that $2111 \in A$. We have that $\{1211, 1121, 1112\} \subseteq B$ by Lemma 4.

$$\begin{aligned}
1111 \in A \text{ and } 2111 \in A &\Rightarrow 3111 \in B.\\
3111 \in B \text{ and } 3333 \in B &\Rightarrow 3222 \in A.\\
2111 \in A \text{ and } 2222 \in A &\Rightarrow 2333 \in B.\\
3222 \in A \text{ and } 2222 \in A &\Rightarrow 1222 \in B.\\
2333 \in B \text{ and } 3333 \in B &\Rightarrow 1333 \in A.\\
1222 \in B \text{ and } 1211 \in B &\Rightarrow 1233 \in A.\\
1233 \in A \text{ and } 1333 \in A &\Rightarrow 1133 \in B.\\
1133 \in B \text{ and } 3333 \in B &\Rightarrow 2233 \in A.\\
1222 \in B \text{ and } 1112 \in B &\Rightarrow 1332 \in A.\\
1332 \in A \text{ and } 1333 \in A &\Rightarrow 1331 \in B.\\
1331 \in B \text{ and } 3333 \in B &\Rightarrow 2332 \in A.\\
2332 \in A \text{ and } 1332 \in A &\Rightarrow 3332 \in B.\\
2233 \in A \text{ and } 1233 \in A &\Rightarrow 3233 \in B.\\
2332 \in A \text{ and } 2222 \in A &\Rightarrow 2112 \in B.\\
3233 \in B \text{ and } 3333 \in B &\Rightarrow 3133 \in A.\\
3133 \in A \text{ and } 1111 \in A &\Rightarrow 2122 \in B.\\
2233 \in A \text{ and } 2222 \in A &\Rightarrow 2211 \in B.\\
3332 \in B \text{ and } 3333 \in B &\Rightarrow 3331 \in A.\\
3331 \in A \text{ and } 1111 \in A &\Rightarrow 2221 \in B.\\
2122 \in B \text{ and } 2112 \in B &\Rightarrow 2132 \in A.\\
2221 \in B \text{ and } 2211 \in B &\Rightarrow 2231 \in A.\\
2221 \in B \text{ and } 1211 \in B &\Rightarrow 3231 \in A.\\
2132 \in A \text{ and } 3133 \in A &\Rightarrow 1131 \in B.\\
3231 \in A \text{ and } 2231 \in A &\Rightarrow 1231 \in B.
\end{aligned}$$

But $\{1131, 1231, 1331\}$ is a combinatorial line. □

We are now ready to conclude the proof of Theorem 1.

We have by Lemma 5 that $\{1112, 1121, 1211, 2111, 2221, 2212, 2122, 1222\} \subseteq B$ and we have not distinguished between 1 and 2. (We distinguished between 1 and 2 in the proof of Lemma 5, but that distinction has disappeared.) Since $\{3331, 3332, 3333\}$ is a combinatorial line, we may assume without loss of generality that $3331 \in A$.

We have that all words with three 1's and one 2 are in B and all words with three 2's and one 1 are in B, so all words with two 3's, one 1, and one 2 are in A. (To see for example that $3132 \in A$, use the fact that $2122 \in B$ and $1112 \in B$.)

$$
\begin{aligned}
3331 \in A \text{ and } 3321 \in A &\Rightarrow 3311 \in B. \\
3331 \in A \text{ and } 2331 \in A &\Rightarrow 1331 \in B. \\
1331 \in B \text{ and } 3333 \in B &\Rightarrow 2332 \in A. \\
3311 \in B \text{ and } 3333 \in B &\Rightarrow 3322 \in A. \\
2332 \in A \text{ and } 2222 \in A &\Rightarrow 2112 \in B. \\
3322 \in A \text{ and } 2222 \in A &\Rightarrow 1122 \in B. \\
2112 \in B \text{ and } 2122 \in B &\Rightarrow 2132 \in A. \\
1112 \in B \text{ and } 1122 \in B &\Rightarrow 1132 \in A.
\end{aligned}
$$

But $\{1132, 2132, 3132\}$ is a combinatorial line. □

3 An Algorithm

Another method of proving that $HJ(3,2) = 4$ requires a computer (or some months of free time), but is very elementary, and gives a reasonable idea for obtaining constructive lower bounds on other Hales-Jewett numbers. Owing to the extremely large upper bound, of course, it is possible that any constructive lower bound is still well short of the mark.

The algorithm is quite simple (and can easily be generalized, but we will use $k = 3$ and $r = 2$ here for clarity). First, one enumerates and stores the 2-colorings of the length 1 words (here and below, over the alphabet $\{1, 2, 3\}$) that avoid a monochromatic line (the "good" colorings); these are the 6 nonconstant colorings.

Now we make the simple observation that in any good 2-coloring of the length-2 words, each set of the form $\{1x, 2x, 3x\}$ with $x \in [3]$ must correspond to one of the 6 good colorings of $[3]^1$, or else that set comprises a monochromatic line. Using this fact, we can examine all of the possibly good colorings of the length 2 words by considering 6^3 possibilities instead of all $2^9 = 8^3$ colorings. The good colorings are stored – it turns out that there are 66 of them.

In any possible good 2-coloring of the words of length 3, each set of the form $\{11x$, $12x$, $13x$, $21x$, $22x$, $23x$, $31x$, $32x$, $33x\}$ with $x \in [3]$ must have one of the 66 colorings mentioned above. In searching the colorings of the length 3 words, this lets us examine just 66^3 possibilities instead of $2^{27} = 512^3$. Of the 66^3 we examine, we find 1644 good colorings, which are stored as before.

Repeating this process, in the 1644^3 possible good colorings of the length 4 words, we find in each case a monochromatic line. Thus, $HJ(3,2) = 4$. Note that in this last step, we have a search space of $1644^3 \approx 2^{32}$ instead of one with size 2^{81}.

Using this algorithm together with a simple simulated annealing algorithm (see [3] for a description of simulated annealing) we have easily obtained the bounds $HJ(4,2) > 5$, $HJ(5,2) > 6$, and $HJ(3,3) > 6$. Note that even if, for instance, $HJ(3,3) = 7$, the search space in this case has order 3^{3^7}, too large for the methods of this section to approach.

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