This paper was published in *African Diaspora J. Math.* 8 (2009), No. 2, 125-136. To the best of my knowledge this is the final version as it was submitted to the publisher.–NH

HOMOMORPHISMS ON COMPACT SUBSETS OF βS

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Abstract

Let S and T be infinite discrete semigroups, let $\mathcal{A} \subseteq \mathcal{P}(S)$, and assume that $\mathcal{A}$ has the finite intersection property. Let $f : S \to T$ and let $\widetilde{f} : \beta S \to \beta T$ be its continuous extension. We obtain necessary and sufficient conditions for the restriction of $\widetilde{f}$ to $\bigcap_{A\in\mathcal{A}} c\ell_{\beta S}(A)$ to be a homomorphism and to be injective. We also investigate certain simpler conditions that are known to be sufficient for this restriction to be a homomorphism.

AMS Subject Classification: 54D35; 22A15.

Keywords: Stone-Čech compactification, discrete semigroup, homomorphism, injective.

1 Introduction

We are concerned in this paper with the Stone-Čech compactification βS of a discrete semigroup S and the extension of the operation on S to βS. So we shall begin with a brief introduction to this structure. For a more complete introduction see the book [3].

Given a discrete semigroup $(S,\cdot)$, we take the points of the Stone-Čech compactification βS of S to be the ultrafilters on S, identifying the principal ultrafilters with the points of S and thereby pretending that $S \subseteq \beta S$. A basis for the open sets of βS (as well as for the closed sets) is $\{\overline{A} : A \subseteq S\}$ where $\overline{A} = \{p \in \beta S : A \in p\}$. Then $\overline{A} = c\ell_{\beta S}(A)$. The operation extends to βS making $(\beta S,\cdot)$ into a compact right topological semigroup (meaning that for each $p \in \beta S$, the function $\rho_p : \beta S \to \beta S$ defined by $\rho_p(q) = q \cdot p$ is continuous) with S contained in its topological center (meaning that for each $x \in S$, the function $\lambda_x : \beta S \to \beta S$ defined by $\lambda_x(q) = x \cdot q$ is continuous). Given $p, q \in \beta S$ and $A \subseteq S$, one has that $A \in p \cdot q$ if and only if

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$\{x \in S : x^{-1}A \in q\} \in p$, where $x^{-1}A = \{y \in S : x \cdot y \in A\}$. (If $(S,\cdot)$ is a group, this agrees with the customary definition $x^{-1}A = \{x^{-1} \cdot z : z \in A\}$.) We shall use the fact that $p \cdot q$ has a basis of sets of the form $\bigcup_{a\in A} aB_a$, where $A \in p$ and, for each $a \in A$, $B_a \in q$.

We shall use S^* to denote the remainder space $\beta S \setminus S$ and, for a subset A of S, we shall use A^* to denote $cl_{\beta S}(A) \cap S^*$.

The algebraic structure of βS can be very rich, and has had significant combinatorial applications. It is a simple fact that a nonempty subset X of βS is closed if and only if there is a family $\mathcal{A} \subseteq \mathcal{P}(S)$ such that $\mathcal{A}$ has the finite intersection property and $X = \bigcap_{A\in\mathcal{A}} \overline{A}$. It is an old result of Paul Milnes [4] that if S is a discrete semigroup, T is a compact Hausdorff right topological semigroup, f is a homomorphism from S to T such that for each $x \in S$, $\lambda_{f(x)}$ is continuous, then the continuous extension $\widetilde{f} : \beta S \to T$ is a homomorphism. As a consequence, if S and T are discrete semigroups and $f : S \to T$ is a homomorphism, then the continuous extension $\widetilde{f} : \beta S \to \beta T$ is also a homomorphism.

For some of the combinatorial applications, as well as for a significant amount of the knowledge of the algebraic structure of βT, it has been important that for $\widetilde{f}$ to be a homomorphism on certain compact subsemigroups of βS, f need not be a homomorphism on all of S.

Theorem 1.1. *Let $(S,\cdot)$ be a semigroup, let $\mathcal{A} \subseteq \mathcal{P}(S)$ have the finite intersection property, and let $H = \bigcap_{A\in\mathcal{A}} \overline{A}$. Let $(T,\cdot)$ be a compact right topological semigroup and let $f : S \longrightarrow T$ have the property that for all $x \in S$, $\lambda_{f(x)}$ is continuous. Assume that there is some $A \in \mathcal{A}$ such that for each $x \in A$, there exists $B \in \mathcal{A}$ for which $f(x \cdot y) = f(x) \cdot f(y)$ for every $y \in B$. Then for all $p,q \in H$, $\widetilde{f}(p \cdot q) = \widetilde{f}(p) \cdot \widetilde{f}(q)$.*

Proof. [3, Theorem 4.21]. □

One is primarily interested in $\bigcap_{A\in\mathcal{A}} \overline{A}$ when it is a subsemigroup, though that assumption is not used in Theorem 1.1. In [3] we provided a simple sufficient condition for this to hold.

Theorem 1.2. *Let $(S,\cdot)$ be a semigroup, let $\mathcal{A} \subseteq \mathcal{P}(S)$ have the finite intersection property, and let $H = \bigcap_{A\in\mathcal{A}} \overline{A}$. If for each $A \in \mathcal{A}$ and each $x \in A$, there exists $B \in \mathcal{A}$ such that $x \cdot B \subseteq A$, then H is a subsemigroup of βS.*

Proof. [3, Theorem 4.20]. □

In fact, when [3] was written, a necessary and sufficient condition for the conclusion of Theorem 1.2 was known. (Why we didn't at least mention it is known only to God, and She is not telling. We did mention that Theorem 1.2 was a special case of a result from [1].) For a set X, we write $\mathcal{P}_f(X)$ for the set of finite nonempty subsets of X.

Theorem 1.3. *Let $(S,\cdot)$ be a semigroup, let $\mathcal{A} \subseteq \mathcal{P}(S)$ have the finite intersection property, let $H = \bigcap_{A\in\mathcal{A}} \overline{A}$, and let $\mathcal{R} = \{B \subseteq S : (\forall A \in \mathcal{A})(B \cap A \neq \emptyset)\}$. Then H is a subsemigroup of βS if and only if $(\forall A \in \mathcal{A})(\forall B \in \mathcal{R})\big(\exists F \in \mathcal{P}_f(B)\big)(\exists C \in \mathcal{A})(C \subseteq \bigcup_{x\in F} x^{-1}A)$.*

Proof. [1, Theorem 2.6]. □

It is essentially trivial that the condition of Theorem 1.1 is not necessary, as we shall see in Theorem 1.4. In that theorem we assume that $S^* = \beta S \setminus S$ is a subsemigroup of βS. A necessary and sufficient condition that S^* be a subsemigroup of βS, which is an immediate consequence of Theorem 1.3 with $\mathcal{A}$ as the set of all cofinite subsets of S, is given in [3, Theorem 4.28]. In particular, if S is either right or left cancellative, then S^* is a subsemigroup of βS.

Theorem 1.4. *Let $T = \mathbb{N} \cup \{\infty\}$ where topologically T is the one point compactification of $\mathbb{N}$ and algebraically the operation on T is ordinary addition on $\mathbb{N}$ with $\infty + x = x + \infty = \infty$ for all $x \in T$. Let S be any infinite discrete semigroup and let $f : S \to T$. If $f^{-1}[\{a\}]$ is finite for all $a \in \mathbb{N}$, and $\widetilde{f} : \beta S \to T$ is the continuous extension of f, then the restriction of $\widetilde{f}$ to S^* is a homomorphism.*

Proof. It suffices to show that for all $p \in S^*$, $\widetilde{f}(p) = \infty$. Suppose instead that we have $p \in S^*$ such that $\widetilde{f}(p) = a \in \mathbb{N}$. Then $f^{-1}[\{a\}]$ is in p, and is therefore infinite, a contradiction. □

Thus, it is possible for the restriction of $\widetilde{f}$ to S^* to be a homomorphism while for all x and y in S, $f(x \cdot y) \neq f(x) \cdot f(y)$. For example, let $S = (\mathbb{N}, +)$ and $f(x) = x + 1$.

The situation is not nearly so simple if we consider $f : S \to T$ where S and T are both discrete semigroups. In this case, one lets $\mathcal{A}$ be a collection of subsets of S with the finite intersection property, lets $H = \bigcap_{A \in \mathcal{A}} \overline{A}$, and asks whether the restriction of $\widetilde{f}$ to H is a homomorphism, where $\widetilde{f} : \beta S \to \beta T$ is the continuous extension of f. (When we say the restriction is a homomorphism we are not demanding that H be a subsemigroup of βS, though it will be in the more interesting situations. We only ask that whenever $p, q \in H$ one has $\widetilde{f}(p \cdot q) = \widetilde{f}(p) \cdot \widetilde{f}(q)$.) We shall obtain necessary and sufficient conditions for the restriction of $\widetilde{f}$ to H to be a homomorphism in Section 2. We shall show, in contrast with Theorem 1.4, that if $\widetilde{f}(p \cdot q) = \widetilde{f}(p) \cdot \widetilde{f}(q)$ for all p and q in a nonempty open subset of S^*, then one must have many a and b in S for which $f(a \cdot b) = f(a) \cdot f(b)$.

In Section 3 we restrict ourselves to the situation where $H = S^*$, that is where $\mathcal{A} = \{F \subseteq S : S \setminus F \text{ is finite}\}$. In Section 4 we go to the other extreme and only consider whether $\widetilde{f}(p)$ is an idempotent when p is an idempotent. On the one hand, this arises when $\mathcal{A} = p$. On the other hand, if H is a subsemigroup of βS, it will contain an idempotent – usually many, and if the restriction of $\widetilde{f}$ to H is a homomorphism, then the image of an idempotent will be an idempotent.

2 Restrictions to H

In this section we derive some necessary and sufficient conditions for the restriction of $\widetilde{f}$ to H to be injective and for it to be a homomorphism, where S and T are infinite discrete semigroups, $f : S \to T$, $\mathcal{A}$ is a set of subsets of S with the finite intersection property, and $H = \bigcap_{A \in \mathcal{A}} \overline{A}$.

We begin with the following characterization of when the restriction of $\widetilde{f}$ to H is injective. This simple fact seems not to have been noted before.

Theorem 2.1. *Let S and T be infinite discrete semigroups, let $\mathcal{A}$ be a set of subsets of S with the finite intersection property, let $H = \bigcap_{A \in \mathcal{A}} \overline{A}$, let $f : S \to T$, and let $\widetilde{f} : \beta S \to \beta T$ be* *the continuous extension of f. The restriction of $\widetilde{f}$ to H is injective if and only if for every $A \subseteq S$, there exists $B \in \mathcal{A}$ such that $f[A\cap B] \cap f[B\setminus A] = \emptyset$.*

Proof. Sufficiency. Let $p, q \in H$ and assume that $p \neq q$. Pick $A \in p \setminus q$. Pick $B \in \mathcal{A}$ such that $f[A\cap B] \cap f[B\setminus A] = \emptyset$. Then $A\cap B \in p$ and $B\setminus A \in q$ so $f[A\cap B] \in \widetilde{f}(p)$ and $f[B\setminus A] \in \widetilde{f}(q)$.

Necessity. Assume that the restriction of $\widetilde{f}$ to H is injective and suppose that we have $A \subseteq S$ such that for all $B \in \mathcal{A}$, $f[A\cap B] \cap f[B\setminus A] \neq \emptyset$. Let $\mathcal{R} = \{f[A\cap B] \cap f[B\setminus A] : B \in \mathcal{A}\}$. Then $\mathcal{R}$ has the finite intersection property so pick $r \in \beta T$ such that $\mathcal{R} \subseteq r$.

Let $\mathcal{B} = \{A\cap B\cap f^{-1}[C] : B \in \mathcal{A} \text{ and } C \in R\}$ and let $\mathcal{C} = \{(B\setminus A)\cap f^{-1}[C] : B \in \mathcal{A} \text{ and } C \in R\}$. Then $\mathcal{B}$ and $\mathcal{C}$ have the finite intersection property, so pick $p, q \in \beta S$ such that $\mathcal{B} \subseteq p$ and $\mathcal{C} \subseteq q$. Then $p, q \in H$ and $\widetilde{f}(p) = \widetilde{f}(q) = r$. □

Given a set $\mathcal{A}$ of subsets of S and a cardinal $\kappa \leq |S|$, we say that $\mathcal{A}$ has the κ-*uniform finite intersection property* if and only if whenever $\mathcal{F} \in \mathcal{P}_f(\mathcal{A})$, one has $|\bigcap \mathcal{F}| \geq \kappa$. And we let $U_\kappa(S) = \{p \in \beta S : (\forall A \in p)(|A| \geq \kappa)\}$.

In our characterizations of when the restriction of $\widetilde{f}$ to H is a homomorphism, it is more convenient to take a negative point of view. That is, whe characterize when it is *not* a homomorphism.

Theorem 2.2. *Let S and T be infinite discrete semigroups, let $\mathcal{A}$ be a set of subsets of S with the finite intersection property, let $H = \bigcap_{A\in\mathcal{A}} \overline{A}$, let $f : S \to T$, and let $\widetilde{f} : \beta S \to \beta T$ be the continuous extension of f. Statements (a), (b), and (c) are equivalent. If in addition $\omega \leq \kappa \leq |S|$, $H \subseteq U_\kappa(S)$, and $|\mathcal{A}| = \kappa$, then all five statements are equivalent.*

(a) There exist $p, q \in H$ such that $\widetilde{f}(p \cdot q) \neq \widetilde{f}(p) \cdot \widetilde{f}(q)$.

(b) There exist $B \subseteq S$ and $D : S \to \mathcal{P}(S)$ such that $\mathcal{A} \cup \{B\}$ and $\mathcal{A} \cup \{D(x) : x \in S\}$ have the finite intersection property and $\left(\bigcup_{x\in B} f\left[x \cdot D(x)\right]\right) \cap \left(\bigcup_{x\in B} f(x) \cdot f\left[D(x)\right]\right) = \emptyset$.

(c) There exist indexed families $\langle x_\mathcal{F} \rangle_{\mathcal{F}\in\mathcal{P}_f(\mathcal{A})}$ and $\langle y_\mathcal{F} \rangle_{\mathcal{F}\in\mathcal{P}_f(\mathcal{A})}$ such that

  (i) for all $\mathcal{F} \in \mathcal{P}_f(\mathcal{A})$, $x_\mathcal{F} \in \bigcap \mathcal{F}$ and $y_\mathcal{F} \in \bigcap \mathcal{F}$; and

  *(ii) $\{f(x_\mathcal{F} \cdot y_\mathcal{G}) : \mathcal{F}, \mathcal{G} \in \mathcal{P}_f(\mathcal{A}) \text{ and } \mathcal{F} \subseteq \mathcal{G}\} \cap$
  $\{f(x_\mathcal{F}) \cdot f(y_\mathcal{G}) : \mathcal{F}, \mathcal{G} \in \mathcal{P}_f(\mathcal{A}) \text{ and } \mathcal{F} \subseteq \mathcal{G}\} = \emptyset$.*

(d) There exist $B \subseteq S$ and $D : S \to \mathcal{P}(S)$ such that $\mathcal{A} \cup \{B\}$ and $\mathcal{A} \cup \{D(x) : x \in S\}$ have the κ-uniform finite intersection property and
$\left(\bigcup_{x\in B} f\left[x \cdot D(x)\right]\right) \cap \left(\bigcup_{x\in B} f(x) \cdot f\left[D(x)\right]\right) = \emptyset$.

(e) There exist indexed families $\langle x_\mathcal{F} \rangle_{\mathcal{F}\in\mathcal{P}_f(\mathcal{A})}$ and $\langle y_\mathcal{F} \rangle_{\mathcal{F}\in\mathcal{P}_f(\mathcal{A})}$ such that

  (i) for all $\mathcal{F}, \mathcal{G} \in \mathcal{P}_f(\mathcal{A})$, if $\mathcal{F} \neq \mathcal{G}$, then $x_\mathcal{F} \neq x_\mathcal{G}$ and $y_\mathcal{F} \neq y_\mathcal{G}$;

  (ii) for all $\mathcal{F} \in \mathcal{P}_f(\mathcal{A})$, $x_\mathcal{F} \in \bigcap \mathcal{F}$ and $y_\mathcal{F} \in \bigcap \mathcal{F}$; and

  *(iii) $\{f(x_\mathcal{F} \cdot y_\mathcal{G}) : \mathcal{F}, \mathcal{G} \in \mathcal{P}_f(\mathcal{A}) \text{ and } \mathcal{F} \subseteq \mathcal{G}\} \cap$
  $\{f(x_\mathcal{F}) \cdot f(y_\mathcal{G}) : \mathcal{F}, \mathcal{G} \in \mathcal{P}_f(\mathcal{A}) \text{ and } \mathcal{F} \subseteq \mathcal{G}\} = \emptyset$.*

Proof. (a) $\Rightarrow$ (b). Pick p and q in H such that $\widetilde{f}(p\cdot q)\neq\widetilde{f}(p)\cdot\widetilde{f}(q)$ and pick

$$A\in\widetilde{f}(p\cdot q)\setminus\widetilde{f}(p)\cdot\widetilde{f}(q)\,.$$

Since $\widetilde{f}\circ\rho_q$ and $\rho_{\widetilde{f}(q)}\circ\widetilde{f}$ are continuous, pick $B\in p$ such that $\widetilde{f}\circ\rho_q[\overline{B}]\subseteq\overline{A}$ and $\rho_{\widetilde{f}(q)}\circ\widetilde{f}[\overline{B}]\subseteq\overline{T\setminus A}$.

For each $x\in B$, since $\widetilde{f}\circ\lambda_x$ and $\lambda_{f(x)}\circ\widetilde{f}$ are continuous, pick $D(x)\in q$ such that $\widetilde{f}\circ\lambda_x[\overline{D(x)}]\subseteq\overline{A}$ and $\lambda_{f(x)}\circ\widetilde{f}[\overline{D(x)}]\subseteq\overline{T\setminus A}$. For $x\in S\setminus B$, let $D(x)=S$.

Since $\mathcal{A}\cup\{B\}\subseteq p$ and $\mathcal{A}\cup\{D(x):x\in S\}\subseteq q$, they each have the finite intersection property. And $\left(\bigcup_{x\in B}f\left[x\cdot D(x)\right]\right)\subseteq A$ and $\left(\bigcup_{x\in B}f(x)\cdot f\left[D(x)\right]\right)\subseteq T\setminus A$.

(b) $\Rightarrow$ (c). Pick B and D as guaranteed by statement (b). For $\mathcal{F}\in\mathcal{P}_f(\mathcal{A})$, pick $x_{\mathcal{F}}\in B\cap\bigcap\mathcal{F}$. For $\mathcal{G}\in\mathcal{P}_f(\mathcal{A})$, pick $y_{\mathcal{G}}\in\bigcap\mathcal{G}\cap\bigcap\{D(x_{\mathcal{F}}):\emptyset\neq\mathcal{F}\subseteq\mathcal{G}\}$.

Then $\{f(x_{\mathcal{F}}\cdot y_{\mathcal{G}}):\mathcal{F},\mathcal{G}\in\mathcal{P}_f(\mathcal{A})$ and $\mathcal{F}\subseteq\mathcal{G}\}\subseteq\bigcup_{x\in B}f[x\cdot D(x)]$ and $\{f(x_{\mathcal{F}})\cdot f(y_{\mathcal{G}}):\mathcal{F},\mathcal{G}\in\mathcal{P}_f(\mathcal{A})$ and $\mathcal{F}\subseteq\mathcal{G}\}\subseteq\bigcup_{x\in B}f(x)\cdot f[D(x)]$.

(c) $\Rightarrow$ (a). Pick $\langle x_{\mathcal{F}}\rangle_{\mathcal{F}\in\mathcal{P}_f(\mathcal{A})}$ and $\langle y_{\mathcal{F}}\rangle_{\mathcal{F}\in\mathcal{P}_f(\mathcal{A})}$ as guaranteed by statement (c). For $\mathcal{F}\in\mathcal{P}_f(\mathcal{A})$, let $B_{\mathcal{F}}=\{x_{\mathcal{G}}:\mathcal{G}\in\mathcal{P}_f(\mathcal{A})$ and $\mathcal{F}\subseteq\mathcal{G}\}$ and let $C_{\mathcal{F}}=\{y_{\mathcal{G}}:\mathcal{G}\in\mathcal{P}_f(\mathcal{A})$ and $\mathcal{F}\subseteq\mathcal{G}\}$. Then $\{B_{\mathcal{F}}:\mathcal{F}\in\mathcal{P}_f(\mathcal{A})\}$ and $\{C_{\mathcal{F}}:\mathcal{F}\in\mathcal{P}_f(\mathcal{A})\}$ have the finite intersection property so pick p and q in βS such that $\{B_{\mathcal{F}}:\mathcal{F}\in\mathcal{P}_f(\mathcal{A})\}\subseteq p$ and $\{C_{\mathcal{F}}:\mathcal{F}\in\mathcal{P}_f(\mathcal{A})\}\subseteq q$. Given $A\in\mathcal{A}$, $B_{\{A\}}\subseteq A$ and $C_{\{A\}}\subseteq A$ so $A\in p$ and $A\in q$. Thus $p\in H$ and $q\in H$.

Let $A=\{f(x_{\mathcal{F}}\cdot y_{\mathcal{G}}):\mathcal{F},\mathcal{G}\in\mathcal{P}_f(\mathcal{A})$ and $\mathcal{F}\subseteq\mathcal{G}\}$ and let $E=\{f(x_{\mathcal{F}})\cdot f(y_{\mathcal{G}}):\mathcal{F},\mathcal{G}\in\mathcal{P}_f(\mathcal{A})$ and $\mathcal{F}\subseteq\mathcal{G}\}$. Then $A\cap E=\emptyset$ so it suffices to show that $A\in\widetilde{f}(p\cdot q)$ and $E\in\widetilde{f}(p)\cdot\widetilde{f}(q)$.

Pick any $D\in\mathcal{A}$. Then $B_{\{D\}}\in p$. We claim that $B_{\{D\}}\subseteq\{z\in S:z^{-1}f^{-1}[A]\in q\}$ so that $f^{-1}[A]\in p\cdot q$ and consequently $A\in\widetilde{f}(p\cdot q)$. So let $\mathcal{F}\in\mathcal{P}_f(\mathcal{A})$ with $D\in\mathcal{F}$. We claim that $C_{\mathcal{F}}\subseteq x_{\mathcal{F}}^{-1}f^{-1}[A]$ and thus $x_{\mathcal{F}}^{-1}f^{-1}[A]\in q$. Let $\mathcal{G}\in\mathcal{P}_f(\mathcal{A})$ with $\mathcal{F}\subseteq\mathcal{G}$. Then $f(x_{\mathcal{F}}\cdot y_{\mathcal{G}})\in A$ so $y_{\mathcal{G}}\in x_{\mathcal{F}}^{-1}f^{-1}[A]$.

Now we claim that $f[B_{\{D\}}]\subseteq\{z\in T:z^{-1}E\in\widetilde{f}(q)\}$ so that $E\in\widetilde{f}(p)\cdot\widetilde{f}(q)$ as required. So let $\mathcal{F}\in\mathcal{P}_f(\mathcal{A})$ with $D\in\mathcal{F}$. Given $\mathcal{G}\in\mathcal{P}_f(\mathcal{A})$ with $\mathcal{F}\subseteq\mathcal{G}$, one has $f(x_{\mathcal{F}})\cdot f(y_{\mathcal{G}})\in E$ so $f[C_{\mathcal{F}}]\subseteq f(x_{\mathcal{F}})^{-1}E$.

We have shown that statements (a), (b), and (c) are equivalent. Now assume that $\omega\leq\kappa\leq|S|$, $H\subseteq U_\kappa(S)$, and $|\mathcal{A}|=\kappa$. Trivially statement (e) implies statement (c). Further, the proof that statement (a) implies statement (d) may be taken verbatim from the proof that statement (a) implies statement (b) except that where, in that proof, we noted that since "$\mathcal{A}\cup\{B\}\subseteq p$ and $\mathcal{A}\cup\{D(x):x\in S\}\subseteq q$, they each have the finite intersection property" we now note that since $p,q\in H\subseteq U_\kappa(S)$, it follows that $\mathcal{A}\cup\{B\}$ and $\mathcal{A}\cup\{D(x):x\in S\}$ each have the κ-uniform finite intersection property.

(d) $\Rightarrow$ (e). Pick B and D as guaranteed by statement (d). Since $|\mathcal{A}|=\kappa$, we also have that $|\mathcal{P}_f(\mathcal{A})|=\kappa$. Enumerate $\mathcal{P}_f(\mathcal{A})$ as $\langle\mathcal{F}_\alpha\rangle_{\alpha<\kappa}$. Inductively choose for $\alpha<\kappa$, $x_{\mathcal{F}_\alpha}\in(B\cap\bigcap\mathcal{F}_\alpha)\setminus\{x_{\mathcal{F}_\delta}:\delta<\alpha\}$. When that induction is complete so that $x_{\mathcal{F}}$ has been chosen for each $\mathcal{F}\in\mathcal{P}_f(\mathcal{A})$, inductively choose for $\alpha<\kappa$,

$$y_{\mathcal{F}_\alpha}\in\bigcap\mathcal{F}_\alpha)\cap\bigcap\{D(x_{\mathcal{F}}):\emptyset\neq\mathcal{F}\subseteq\mathcal{F}_\alpha\}\setminus\{y_{\mathcal{F}_\delta}:\delta<\alpha\}\,.$$

Then $\{f(x_{\mathcal{F}}\cdot y_{\mathcal{G}}):\mathcal{F},\mathcal{G}\in\mathcal{P}_f(\mathcal{A})$ and $\mathcal{F}\subseteq\mathcal{G}\}\subseteq\bigcup_{x\in B}f[x\cdot D(x)]$ and $\{f(x_{\mathcal{F}})\cdot f(y_{\mathcal{G}}):\mathcal{F},\mathcal{G}\in\mathcal{P}_f(\mathcal{A})$ and $\mathcal{F}\subseteq\mathcal{G}\}\subseteq\bigcup_{x\in B}f(x)\cdot f[D(x)]$. □

We next obtain another necessary and sufficient condition which is valid for semigroups satisfying a weak cancellation assumption and functions for which the preimages of points are not too large.

Definition 2.3. Let $(S,\cdot)$ be an infinite semigroup with $|S| \geq \kappa$. A subset A of S is a *left solution set of* S if and only if there exist $w,z \in S$ such that $A = \{x \in S : w = zx\}$. The semigroup S is *weakly left cancellative* if and only if every left solution set of S is finite. The semigroup S is κ-*weakly left cancellative* if and only if the union of fewer than κ left solution sets of S must have cardinality less than κ.

Notice that if κ is regular, A is κ-weakly left cancellative if and only if each left solution set has cardinality less than κ. In particular, S is weakly left cancecellative if and only if S is ω-weakly left cancellative. If κ is singular, then A is κ-weakly left cancellative if and only if there is a cardinal $\gamma < \kappa$ such that each left solution set has cardinality less than or equal to γ.

Theorem 2.4. *Let S and T be infinite discrete semigroups, let $\mathcal{A}$ be a set of subsets of S with the finite intersection property, let $H = \bigcap_{A\in\mathcal{A}} \overline{A}$, let $f : S \to T$, and let $\widetilde{f} : \beta S \to \beta T$ be the continuous extension of f. Assume that $|S| \geq \kappa \geq \omega$, $H \subseteq U_\kappa(S)$, and $|\mathcal{A}| = \kappa$. For $x,z \in S$, let $E(x,z) = \{y \in S : f(x\cdot y) \neq f(z)\cdot f(y)\}$. Then statement (a) implies statement (b). If in addition S is κ-weakly left cancellative and either κ is regular and for each $z \in T$, $|f^{-1}[\{z\}]| < \kappa$, or there exists a cardinal $\delta < \kappa$ such that for each $z \in T$, $|f^{-1}[\{z\}]| \leq \delta$, then both statements are equivalent.*

(a) There exist $p,q \in H$ such that $\widetilde{f}(p\cdot q) \neq \widetilde{f}(p)\cdot\widetilde{f}(q)$.

(b) There exists $B \subseteq S$ such that $\mathcal{A}\cup\{B\}$ and $\mathcal{A}\cup\{E(x,z) : x,z \in B\}$ have the κ-uniform finite intersection property.

Proof. (a) $\Rightarrow$ (b). Pick indexed families $\langle x_\mathcal{F}\rangle_{\mathcal{F}\in\mathcal{P}_f(\mathcal{A})}$ and $\langle y_\mathcal{F}\rangle_{\mathcal{F}\in\mathcal{P}_f(\mathcal{A})}$ as guaranteed by Theorem 2.2(e). Let $B = \{x_\mathcal{F} : \mathcal{F} \in \mathcal{P}_f(\mathcal{A})\}$. Given $\mathcal{F} \in \mathcal{P}_f(\mathcal{A})$,

$$\{x_\mathcal{G} : \mathcal{G} \in \mathcal{P}_f(\mathcal{A}) \text{ and } \mathcal{F} \subseteq \mathcal{G}\} \subseteq B \cap \bigcap \mathcal{F}\,,$$

so $\mathcal{A}\cup\{B\}$ has the κ-uniform finite intersection property.

To see that $\mathcal{A}\cup\{E(x,z) : x,z \in B\}$ has the κ-uniform finite intersection property, let $\mathcal{F} \in \mathcal{P}_f(\mathcal{A})$, let $l,m \in \mathbb{N}$, and let $\mathcal{H}_1,\mathcal{H}_2,\ldots,\mathcal{H}_l$ and $\mathcal{J}_1,\mathcal{J}_2,\ldots,\mathcal{J}_m$ be members of $\mathcal{P}_f(\mathcal{A})$. Let $\mathcal{K} = \mathcal{F}\cup\bigcup_{i=1}^{l}\mathcal{H}_i\cup\bigcup_{j=1}^{m}\mathcal{J}_j$. Then $\{x_\mathcal{G} : \mathcal{G} \in \mathcal{P}_f(\mathcal{A}) \text{ and } \mathcal{K} \subseteq \mathcal{G}\} \subseteq \bigcap\mathcal{F}\cap\bigcap_{i=1}^{l}\bigcap_{j=1}^{m} E(x_{\mathcal{H}_i}, x_{\mathcal{J}_j})$.

Now assume that S is κ-weakly left cancellative and either κ is regular and for each $z \in T$, $|f^{-1}[\{z\}]| < \kappa$, or there exists a cardinal $\delta < \kappa$ such that for each $z \in T$, $|f^{-1}[\{z\}]| \leq \delta$.

(b) $\Rightarrow$ (a). Pick B as guaranteed by statement (b). We shall show that statement (c) of Theorem 2.2 holds. For $\mathcal{F} \in \mathcal{P}_f(\mathcal{A})$, choose $x_\mathcal{F} \in B\cap\bigcap\mathcal{F}$. Now well order $\mathcal{P}_f(\mathcal{A})$ as $\langle\mathcal{G}_\alpha\rangle_{\alpha<\kappa}$, choosing $\mathcal{G}_0 = \{A\}$ for some $A \in \mathcal{A}$. Pick $y_{\mathcal{G}_0} \in A\cap E(x_{\mathcal{G}_0}, x_{\mathcal{G}_0})$.

Let $0 < \alpha < \kappa$ and assume we have chosen $y_{\mathcal{G}_\sigma} \in \bigcap\mathcal{G}_\sigma$ for each $\sigma < \alpha$ such that

$$\{f(x_\mathcal{F}\cdot y_{\mathcal{G}_\sigma}) : \sigma < \alpha \text{ and } \emptyset \neq \mathcal{F} \subseteq \mathcal{G}_\sigma\} \cap \{f(x_\mathcal{F})\cdot f(y_{\mathcal{G}_\sigma}) : \sigma < \alpha \text{ and } \emptyset \neq \mathcal{F} \subseteq \mathcal{G}_\sigma\} = \emptyset\,.$$

Observe that the hypothesis holds for $\alpha = 1$.

Let

$$H=\{z\in T:\ (\exists\sigma<\alpha)(\exists\tau\le\alpha)(\exists\mathcal{F})(\exists\mathcal{H})(\emptyset\neq\mathcal{F}\subseteq\mathcal{G}_\sigma,\ \emptyset\neq\mathcal{H}\subseteq\mathcal{G}_\tau,\text{ and } f(x_{\mathcal{H}})\cdot z=f(x_{\mathcal{F}}\cdot y_{\mathcal{G}_\sigma})\}$$

and let

$$L=\{y\in S:\ (\exists\sigma<\alpha)(\exists\tau\le\alpha)(\exists\mathcal{F})(\exists\mathcal{H})(\emptyset\neq\mathcal{F}\subseteq\mathcal{G}_\sigma,\ \emptyset\neq\mathcal{H}\subseteq\mathcal{G}_\tau,\text{ and } f(x_{\mathcal{H}}\cdot y)=f(x_{\mathcal{F}})\cdot f(y_{\mathcal{G}_\sigma})\}.$$

Now H is the union of fewer than κ left solution sets so $|H|<\kappa$ and thus $|f^{-1}[H]|<\kappa$.

Let $A=\{f(x_{\mathcal{F}})\cdot f(y_{\mathcal{G}_\sigma}):\sigma<\alpha$ and $\emptyset\neq\mathcal{F}\subseteq\mathcal{G}_\sigma\}$. Then $|A|<\kappa$ so $|f^{-1}[A]|<\kappa$. Now $L=\{y\in S:(\exists\tau\le\alpha)(\exists\mathcal{H})(\emptyset\neq\mathcal{H}\subseteq\mathcal{G}_\tau$, and $x_{\mathcal{H}}\cdot y\in f^{-1}[A])\}$ so L is the union of fewer than κ left solution sets and so $|L|<\kappa$.

Pick $y_{\mathcal{G}_\alpha}\in(\bigcap\mathcal{G}_\alpha\cap\bigcap\{E(x_{\mathcal{F}},x_{\mathcal{H}}):\emptyset\neq\mathcal{F}\subseteq\mathcal{G}_\alpha$ and $\emptyset\neq\mathcal{H}\subseteq\mathcal{G}_\alpha\})\setminus(f^{-1}[H]\cup L)$.

Now suppose that we have $\sigma\le\alpha$, $\delta\le\alpha$, $\emptyset\neq\mathcal{F}\subseteq\mathcal{G}_\sigma$, and $\emptyset\neq\mathcal{H}\subseteq\mathcal{G}_\delta$ such that $f(x_{\mathcal{H}}\cdot y_{\mathcal{G}_\delta})=f(x_{\mathcal{F}})\cdot f(y_{\mathcal{G}_\sigma})$. If $\sigma<\alpha$ and $\delta<\alpha$ we contradict the induction hypothesis. If $\sigma=\delta=\alpha$, then $y_{\mathcal{G}_\alpha}\notin E(x_{\mathcal{H}},x_{\mathcal{F}})$. If $\delta<\alpha=\sigma$, then $y_{\mathcal{G}_\alpha}\in f^{-1}[H]$. If $\sigma<\alpha=\delta$, then $y_{\mathcal{G}_\alpha}\in L$. In any event we get a contradiction. □

In the event that the family $\mathcal{A}$ is countably infinite, we may presume that $\mathcal{A}=\{A_n:n\in\mathbb{N}\}$ and for each n, $A_{n+1}\subseteq A_n$. In this case the condition of Theorem 2.2(e) becomes simpler.

Theorem 2.5. *Let S and T be infinite semigroups and let $\mathcal{A}=\{A_n:n\in\mathbb{N}\}$ be a set of subsets of S such that for each $n\in\mathbb{N}$, A_n is infinite and $A_{n+1}\subseteq A_n$ and $\bigcap_{n=1}^\infty A_n=\emptyset$. Let $f:S\to T$, let $\widetilde{f}:\beta S\to\beta T$ be the continuous extension of f, and let $H=\bigcap_{n=1}^\infty\overline{A_n}$. There exist p and q in H such that $\widetilde{f}(p\cdot q)\neq\widetilde{f}(p)\cdot\widetilde{f}(q)$ if and only if there exist injective sequences $\langle x_n\rangle_{n=1}^\infty$ and $\langle y_n\rangle_{n=1}^\infty$ in S such that*

$$\{f(x_n\cdot y_m):n,m\in\mathbb{N}\text{ and }n\le m\}\cap\{f(x_n)\cdot f(y_m):n,m\in\mathbb{N}\text{ and }n\le m\}=\emptyset.$$

Proof. Necessity. Pick $\langle x_{\mathcal{F}}\rangle_{\mathcal{F}\in\mathcal{P}_f(\mathcal{A})}$ and $\langle y_{\mathcal{F}}\rangle_{\mathcal{F}\in\mathcal{P}_f(\mathcal{A})}$ as guaranteed by Theorem 2.2(e). For $n\in\mathbb{N}$ let $x_n=x_{\{A_n\}}$ and $y_n=x_{\{A_n\}}$.

Sufficiency. Pick $\langle x_n\rangle_{n=1}^\infty$ and $\langle y_n\rangle_{n=1}^\infty$ as guaranteed. For $\mathcal{F}\in\mathcal{P}_f(\mathcal{A})$ let $n=\max\{t:A_t\in\mathcal{F}\}$ and let $x_{\mathcal{F}}=x_n$ and $y_{\mathcal{F}}=y_n$. Then $\langle x_{\mathcal{F}}\rangle_{\mathcal{F}\in\mathcal{P}_f(\mathcal{A})}$ and $\langle y_{\mathcal{F}}\rangle_{\mathcal{F}\in\mathcal{P}_f(\mathcal{A})}$ satisfy Theorem 2.2(c). □

We shall show in Theorem 2.8 that an injective map $f:S\to T$ between left cancellative semigroups can only have the property that $\widetilde{f}$ is a homomorphism on a compact G_δ subset H of βS if, for every $p,q\in H$, $\{s\in S:\{t\in S:f(st)=f(s)f(t)\}\in q\}\in p$.

We have need of two preliminary lemmas.

Lemma 2.6. *Let S and T be infinite semigroups, let $\mathcal{A}=\{A_n:n\in\mathbb{N}\}$ be a set of subsets of S such that for each $n\in\mathbb{N}$, A_n is infinite and $A_{n+1}\subseteq A_n$, let $H=\bigcap_{n=1}^\infty\overline{A_n}$, let $f:S\to T$, let $\widetilde{f}:\beta S\to\beta T$ be the continuous extension of f, and assume that for all $p,q\in H$, $\widetilde{f}(p\cdot q)=\widetilde{f}(p)\cdot\widetilde{f}(q)$. For each $q\in H$ there exists $B\subseteq S$ such that $H\subseteq\overline{B}$ and for each $a\in B$, $\widetilde{f}(a\cdot q)=f(a)\cdot\widetilde{f}(q)$.*

Proof. Let $q\in H$. The continuous functions $\widetilde{f}\circ\rho_q$ and $\rho_{\widetilde{f}(q)}\circ\widetilde{f}$ agree on H so by [5, Corollary 2.3(i)] we can choose $B\subseteq S$ such that $H\subseteq\overline{B}$ and $\widetilde{f}\circ\rho_q$ and $\rho_{\widetilde{f}(q)}\circ\widetilde{f}$ agree on B. □

We need to extend the theorem that, for any set S, any $f : S \to S$ and any $p \in \beta S$, $\widetilde{f}(p) = p$ if and only if $\{s \in S : f(s) = s\} \in p$ ([3, Theorem 3.35]). The extension is certainly well known. We give the simple proof, however, because we do not have a reference.

Lemma 2.7. *Let S and T be any nonempty discrete spaces and let f and g be mappings from S to T. Let $\widetilde{f}$ and $\widetilde{g}$ be their continuous extensions from βS to βT. Suppose that $\widetilde{f}(p) = \widetilde{g}(p)$ for some $p \in \beta S$. If f is injective on a member of p, then $\{s \in S : f(s) = g(s)\} \in p$.*

Proof. Suppose that f is injective on $B \in p$. Pick $a \in S$ and define $h : T \to S$ by, for $t \in T$,

$$h(t) = \begin{cases} s & \text{if } t = f(s) \text{ for some } s \in B \\ a & \text{if } t \notin f[B] \end{cases}$$

Then $\widetilde{h \circ g}(p) = \widetilde{h}\big(\widetilde{g}(p)\big) = \widetilde{h}\big(\widetilde{f}(p)\big) = \widetilde{h \circ f}(p) = p$, the latter equality holding because $h \circ f$ is the identity on B. Thus by [3, Theorem 3.35] we have that $\{s \in S : h\big(g(s)\big) = s\} \in p$. Also, $g^{-1}\big[f[B]\big] \in p$ so $\{s \in S : h\big(g(s)\big) = s\} \cap g^{-1}\big[f[B]\big] \in p$ and for s in this intersection, $g(s) = f(s)$. □

Theorem 2.8. *Let S and T be infinite semigroups, let $\mathcal{A} = \{A_n : n \in \mathbb{N}\}$ be a set of subsets of S such that for each $n \in \mathbb{N}$, A_n is infinite and $A_{n+1} \subseteq A_n$, let $H = \bigcap_{n=1}^{\infty} \overline{A_n}$, let $f : S \xrightarrow{1\text{-}1} T$, and let $\widetilde{f} : \beta S \to \beta T$ be the continuous extension of f. Assume that either S or T is left cancellative. The following statements are equivalent.*

(a) For all $p, q \in H$, $\widetilde{f}(p \cdot q) = \widetilde{f}(p) \cdot \widetilde{f}(q)$.

(b) For each $q \in H$, there exists $B \subseteq S$ such that $H \subseteq \overline{B}$ and for all $a \in B$, $\{b \in S : f(a \cdot b) = f(a) \cdot f(b)\} \in q$.

Proof. $(a) \Rightarrow (b)$. Pick B as guaranteed by Lemma 2.6 and let $a \in B$. Then either $f \circ \lambda_a$ or $\lambda_{f(a)} \circ f$ is injective (depending on whether S or T is left cancellative) so by Lemma 2.7, $\{b \in S : f(a \cdot b) = f(a) \cdot f(b)\} \in q$.

$(b) \Rightarrow (a)$. Let $q \in H$ and pick B as guaranteed. For any $a \in B$, $\widetilde{f} \circ \lambda_a$ and $\lambda_{f(a)} \circ \widetilde{f}$ agree on a member of q, so agree at q. Therefore $\widetilde{f} \circ \rho_q$ and $\rho_q \circ \widetilde{f}$ agree on B and therefore on H. □

We shall show now that if $\widetilde{f}(p \cdot q) = \widetilde{f}(p) \cdot \widetilde{f}(q)$ for all $p, q \in A^* = \overline{A} \setminus A$ for some infinite subset of S (in particular if $\widetilde{f}$ is a homomorphism on a compact G_δ subset of S^*), then there must be an infinite number of $a \in S$ for which there are an infinite number of $b \in S$ satisfying $f(a \cdot b) = f(a) \cdot f(b)$.

Theorem 2.9. *Let S and T be infinite semigroups, let A be an infinite subset of S, let $f : S \to T$, and let $\widetilde{f}$ be the continuous extension of f. If $\widetilde{f}(p \cdot q) = \widetilde{f}(p) \cdot \widetilde{f}(q)$ for all $p, q \in A^*$, then for infinitely many $a \in A$, $\{b \in A : f(a \cdot b) = f(a) \cdot f(b)\}$ is infinite.*

Proof. By passing to a subset, we may assume that A is countably infinite. For each $a \in A$, let $C_a = \{q \in A^* : \widetilde{f}(a \cdot q) = f(a) \cdot \widetilde{f}(q)\}$. Let $D = \{a \in A : C_a$ has nonempty interior in $S^*\}$. We shall show that D is infinite and for all $a \in D$, $\{b \in A : f(a \cdot b) = f(a) \cdot f(b)\}$ is infinite.

Suppose first that D is finite. We claim that $A^* = \bigcup_{a\in A\setminus D} C_a$. So let $q \in A^*$. Then the functions $\widetilde{f}\circ\rho_q$ and $\rho_{\widetilde{f}(q)}\circ\widetilde{f}$ agree on A^* so by [5, Corollary 2.3(ii)] they agree for all but finitely many members of A and therefore agree at some $a \in A\setminus D$.

Note that each C_a is closed. For each $a\in A\setminus D$, $S^*\setminus C_a$ is an open dense subset of S^* so by the Baire Category Theorem, $S^*\setminus\bigcup_{a\in A\setminus D} C_a = S^*\setminus A^*$ is dense in S^*, a contradiction. Thus D is infinite as claimed.

Let $a \in D$ and pick an infinite subset E of A such that $E^*\subseteq C_a$. We shall show that $\{b\in E : f(a\cdot b) = f(a)\cdot f(b)\}$ is infinite. Consider the continuous functions $\widetilde{f}\circ\lambda_a$ and $\lambda_{f(a)}\circ\widetilde{f}$. Assume first that there is an infinite subset K of E on which both of these functions are constant, and pick $q\in K^*$. Let c and d be the constant values of $\widetilde{f}\circ\lambda_a$ and $\lambda_{f(a)}\circ\widetilde{f}$ on K respectively. Then $c=\widetilde{f}(a\cdot q) = f(a)\cdot\widetilde{f}(q) = d$ so $K\subseteq\{b\in E : f(a\cdot b) = f(a)\cdot f(b)\}$.

Now assume that there is no infinite subset of E on which $\widetilde{f}\circ\lambda_a$ and $\lambda_{f(a)}\circ\widetilde{f}$ are both constant. We may thus choose a sequence $\langle x_n\rangle_{n=1}^\infty$ in E on which one of these functions is injective. Pick $q\in E^*$ such that $\{x_n : n\in\mathbb{N}\}\in q$. By Lemma 2.7, we have that $\{b\in S : f(a\cdot b) = f(a)\cdot f(b)\}\in q$. □

3 Restrictions to S^*

If S is an infinite semigroup and $\mathcal{A} = \{S\setminus F : F$ is a finite subset of $S\}$, then $S^* = \bigcap_{A\in\mathcal{A}}\overline{A}$ so the results of Section 2 apply to S^*. In this section we give some characterizations of when the restriction of $\widetilde{f}$ to S^* is injective or a homomorphism which do not seem to follow from the results of Section 2.

Theorem 3.1. *Let S and T be infinite semigroups, let $f : S\to T$, and let $\widetilde{f} : \beta S\to\beta T$ be the continuous extension of f.*

(1) The restriction of $\widetilde{f}$ to S^* is injective if and only if there is a finite subset F of S such that the restriction of $\widetilde{f}$ to $S\setminus F$ is injective.

(2) If the restriction of $\widetilde{f}$ to S^* is not injective, then $\big|\{p\in S^* : (\exists q\in S^*\setminus\{p\})(\widetilde{f}(q) = \widetilde{f}(p))\}\big| \geq 2^{\mathfrak{c}}$, where $\mathfrak{c}$ is the cardinality of the continuum.

Proof. (1). Sufficiency. Let $\mathcal{A} = \{S\setminus F : F$ is a finite subset of $S\}$ and pick a finite subset F of S such that the restriction of f to $S\setminus F$ is injective. Let $B = S\setminus F$. Then for any $A\subseteq S$, $f[A\cap B]\cap f[B\setminus A] = \emptyset$, so Theorem 2.1 applies.

(1). Necessity. Suppose the conclusion fails. Pick $x_1\neq y_1$ in S such that $f(x_1) = f(y_1)$. Inductively, let $n\in\mathbb{N}$ and assume we have chosen $x_1, x_2, \ldots, x_n$ and $y_1, y_2, \ldots, y_n$. Let $F = \{x_1, x_2, \ldots, x_n\}\cup\{y_1, y_2, \ldots, y_n\}$ and choose $x_{n+1}\neq y_{n+1}$ in $S\setminus F$ such that $f(x_{n+1}) = f(y_{n+1})$. Pick any $p\in S^*$ such that $\{x_n : n\in\mathbb{N}\}\in p$. Define $g : \{x_n : n\in\mathbb{N}\}\to\{y_n : n\in\mathbb{N}\}$ by $g(x_n) = y_n$. Let $\mathcal{B} = \{g[A\cap\{x_n : n\in\mathbb{N}\}] : A\in p\}$. Then $\mathcal{B}$ has the ω-uniform finite intersection property, so pick $q\in S^*$ such that $\mathcal{B}\subseteq q$. (In fact, there is a unique choice for q.) Then $\widetilde{f}(p) = \widetilde{f}(q)$.

(2). In the proof of the necessity of statement (1), p could be any member of S^* such that $\{x_n : n\in\mathbb{N}\}\in p$. There are $2^{\mathfrak{c}}$ such choices. (See, for example, [3, Theorem 3.62].) □

In the event that S is countable (in which case $\{S\setminus F : F$ is a finite subset of $S\}$ is countable), the following theorem follows from Theorem 2.5.

Theorem 3.2. *Let S and T be infinite semigroups, let $f : S \to T$, and let $\widetilde{f} : \beta S \to \beta T$ be the continuous extension of f. There exist $p,q \in S^*$ such that $\widetilde{f}(p\cdot q) \neq \widetilde{f}(p)\cdot\widetilde{f}(q)$ if and only if there exist injective sequences $\langle x_n\rangle_{n=1}^{\infty}$ and $\langle y_n\rangle_{n=1}^{\infty}$ in S such that $\{f(x_n\cdot y_m) : n,m \in \mathbb{N}$ and $n \leq m\} \cap \{f(x_n)\cdot f(y_m) : n,m \in \mathbb{N}$ and $n \leq m\} = \emptyset$.*

Proof. Necessity. Let $\mathcal{A} = \{S\setminus F : F$ is a finite subset of $S\}$. Pick by Theorem 2.2(b), $B \subseteq S$ and $D : S \to \mathcal{P}(S)$ such that $\mathcal{A}\cup\{B\}$ and $\mathcal{A}\cup\{D(x) : x\in S\}$ have the finite intersection property and $\left(\bigcup_{x\in B} f\left[x\cdot D(x)\right]\right)\cap\left(\bigcup_{x\in B} f(x)\cdot f\left[D(x)\right]\right) = \emptyset$. Then B is infinite so pick an injective sequence $\langle x_n\rangle_{n=1}^{\infty}$ in B. Pick $y_1 \in D(x_1)$. For $m\in\mathbb{N}$, $\bigcap_{n=1}^{m+1} D(x_n)$ is infinite so pick $y_{m+1} \in \bigcap_{n=1}^{m+1} D(x_n)\setminus\{y_1,y_2,\ldots,y_m\}$.

Sufficiency. Pick p and q in S^* such that $\{x_n : n\in\mathbb{N}\}\in p$ and $\{y_n : n\in\mathbb{N}\}\in q$. Given $n\in\mathbb{N}$, $\{y_m : m\geq n\}\in q$ so $\{x_n\cdot y_m : n,m\in\mathbb{N}$ and $n\leq m\}\in p\cdot q$ and thus

$$\{f(x_n\cdot y_m) : n,m\in\mathbb{N} \text{ and } n\leq m\}\in\widetilde{f}(p\cdot q)\,.$$

Also, given $n\in\mathbb{N}$, $\{f(y_m) : m\geq n\}\in\widetilde{f}(q)$ so

$$\{f(x_n)\cdot f(y_m) : n,m\in\mathbb{N} \text{ and } n\leq m\}\in\widetilde{f}(p)\cdot\widetilde{f}(q)\,.$$

□

4 Image of Idempotents

We are interested in the question of whether $\widetilde{f}(p)$ is an idempotent, given that p is an idempotent, for two reasons. On the one hand, we could have that $\mathcal{A} = p$, in which case this is exactly the question of whether the restriction of $\widetilde{f}$ to $H = \bigcap_{A\in\mathcal{A}}\overline{A}$ is an isomorphism. On the other hand, if H is a subsemigroup, it is guaranteed to have an idempotent by [2, Corollary 2.10].

For the first result of this section, we do not presume that p is an idempotent.

Theorem 4.1. *Let S and T be infinite semigroups, let $f : S\to T$, let $\widetilde{f} : \beta S\to\beta T$ be the continuous extension of f, and let $p\in\beta S$. Then $\widetilde{f}(p)$ is an idempotent if and only if $(\forall A\in p)(\exists B\in p)(\forall x\in B)(\exists C\in p)(f(x)\cdot f[C]\subseteq f[A])$.*

Proof. Necessity. Let $A\in p$. Then $f[A]\in\widetilde{f}(p)$ so $E = \{z\in T : z^{-1}f[A]\in\widetilde{f}(p)\}\in\widetilde{f}(p)$. Let $B = f^{-1}[E]$ and let $x\in B$. Then $f(x)^{-1}f[A]\in\widetilde{f}(p)$ Let $C = f^{-1}\left[f(x)^{-1}f[A]\right]$.

Sufficiency. We show that $\widetilde{f}(p)\subseteq\widetilde{f}(p)\cdot\widetilde{f}(p)$, so let $D\in\widetilde{f}(p)$ and let $A = f^{-1}[D]$. Pick $B\in p$ as guaranteed for A. We claim that $f[B]\subseteq\{z\in T : z^{-1}D\in\widetilde{f}(p)\}$. Let $z\in f[B]$ and pick $x\in B$ such that $z = f(x)$. Pick $C\in p$ such that $f(x)\cdot f[C]\subseteq f[A]$. Then $f[C]\subseteq z^{-1}D$. □

Theorem 4.2. *Let S and T be semigroups, let $f : S\to T$, let $\widetilde{f} : \beta S\to\beta T$ be the continuous extension of f, and let $p,q\in\beta S$. Then $\widetilde{f}(p\cdot q) = \widetilde{f}(p)\cdot\widetilde{f}(q)$ if and only if*

$$(\forall B\in p)(\forall D : S\to q)(\exists u,v\in B)\big(\exists x\in D(u)\big)\big(\exists y\in D(v)\big)\big(f(u\cdot x) = f(v)\cdot f(y)\big)\,.$$

Proof. We observe that $\{\bigcup_{u\in B} f[uD(u)] : B \in p \text{ and } D : S \to q\}$ is a basis for the ultrafilter $\widetilde{f}(p\cdot q)$ and $\{\bigcup_{u\in B} f(u)f[D(u)] : B \in p \text{ and } D : S \to q\}$ is a basis for the ultrafilter $\widetilde{f}(p)\cdot\widetilde{f}(q)$. If $\widetilde{f}(p\cdot q) = \widetilde{f}(p)\cdot\widetilde{f}(q)$, $B \in p$, and $D : S \to q$, then $\bigcup_{u\in B} f[uD(u)] \in \widetilde{f}(p\cdot q)$ and $\bigcup_{u\in B} f(u)f[D(u)] \in \widetilde{f}(p)\cdot\widetilde{f}(q)$ and so $\bigcup_{u\in B} f[uD(u)] \cap \bigcup_{u\in B} f(u)f[D(u)] \neq \emptyset$.

Conversely, assume that $\widetilde{f}(p\cdot q) \neq \widetilde{f}(p)\cdot\widetilde{f}(q)$ and pick $A \in \widetilde{f}(p\cdot q)\setminus\widetilde{f}(p)\cdot\widetilde{f}(q)$. Pick $B_1 \in p$ and $D_1 : S \to q$ such that $\bigcup_{u\in B_1} f[uD_1(u)] \subseteq A$ and pick $B_2 \in p$ and $D_2 : S \to q$ such that $\bigcup_{u\in B_2} f(u)f[D_2(u)] \cap A = \emptyset$. Let $B = B_1 \cap B_2$ and define $B : S \to q$ by, for $u \in S$, $D(u) = D_1(u) \cap D_2(u)$. Then for all $u, v \in B$, all $x \in D(u)$, and all $y \in D(v)$, $f(u\cdot x) \neq f(v)\cdot f(y)$. □

Corollary 4.3. *Let S and T be infinite semigroups, let $f : S \to T$, let $\widetilde{f} : \beta S \to \beta T$ be the continuous extension of f, and let p be an idempotent in βS. Then $\widetilde{f}(p)$ is an idempotent if and only if* $(\forall B\in p)(\forall D: S\to p)(\exists u,v\in B)\big(\exists x\in D(u)\big)\big(\exists y\in D(v)\big)\big(f(u\cdot x) = f(v)\cdot f(y)\big)$.

The following theorem gives an attractive sufficient condition.

Theorem 4.4. *Let S and T be infinite semigroups, let $f : S \to T$, let $\widetilde{f} : \beta S \to \beta T$ be the continuous extension of f, and let p be an idempotent in βS. If*

$$\{x \in S : \{y \in S : f(x\cdot y) = f(x)\cdot f(y)\} \in p\} \in p\,,$$

then $\widetilde{f}(p)$ is an idempotent.

Proof. We verify the condition of Theorem 4.1. Let $A \in p$ and let

$$B = \{x \in S : x^{-1}A \in p\} \cap \{x \in S : \{y \in S : f(x\cdot y) = f(x)\cdot f(y)\} \in p\}\,.$$

Since p is an idempotent, $B \in p$. Let $x \in B$ and let $C = x^{-1}A \cap \{y \in S : f(x\cdot y) = f(x)\cdot f(y)\}$. Then $C \in p$. If $y \in C$, then $f(x)\cdot f(y) = f(x\cdot y) \in f[A]$. □

We see that the condition of Theorem 4.4 far from being necessary. Note that, in the following example, f is a bijection, S is cancellative and T is left cancellative. So this example should be contrasted with Theorem 2.8 above.

Example 4.5. Let $S = (\mathbb{N},+)$ and T denote $\mathbb{N}$ with the right zero semigroup operation; i.e. $x\cdot y = y$ for every $x, y \in \mathbb{N}$. Then βT is also a right zero semigroup. Let $f : S \to T$ be the identity map. Then $\widetilde{f}$ maps every idempotent of βN to an idempotent, because every element of βT is idempotent. However, the equation $f(x+y) = f(y)$ does not hold for any $x, y \in S$.

In the following example, S and T are both cancellative.

Example 4.6. Let S denote the subsemigroup $\{n \in \mathbb{N} : n \geq 4\}$ of $(\mathbb{N},+)$. We give an example of a function $f : S \to \mathbb{N}$ with the property that $\widetilde{f}(p)$ is idempotent for some idempotent $p \in \beta S$, but such that there are no elements $s, t \in S$ for which $f(s+t) = f(s) + f(t)$.

For $s \in \mathbb{N}$, $\text{supp}(s)$ will denote the binary support of s, defined to be the set in $\mathcal{P}_f(\omega)$ for which $s = \sum_{i\in\text{supp}(s)} 2^i$. We define $f : S \to \mathbb{N}$ by $f(s) = \max\big(\text{supp}(s)\big)$. Since f maps $\{2^{n+1} : n \in \mathbb{N}\}$ onto $\mathbb{N}\setminus\{1\}$, $\widetilde{f}$ maps $cl_{\beta S}(\{2^{n+1} : n \in \mathbb{N}\}) \sim \beta(\{2^{n+1} : n \in \mathbb{N}\})$ onto $\beta\mathbb{N}\setminus\{1\}$

by [3, Exercise 3.3.3] and [3, Exercise 3.4.1]. So we can choose $x \in cl_{\beta S}(\{2^{n+1} : n \in \mathbb{N}\})$ for which $\widetilde{f}(x)$ is an idempotent q.

Now if $s,t \in S$ satisfy $f(s) < \min(\text{supp}(t))$, $f(s+t) = f(t)$. Hence, for any $y \in \beta S$, $\widetilde{f}(y+x) = \widetilde{f}(x) = q$, as can be seen by allowing t to tend to x and then allowing s to tend to y in the equation $f(s+t) = f(t)$. So $\widetilde{f}^{-1}[\{q\}]$ contains the left ideal $\beta S + x$ of βS and therefore contains an idempotent p (see [3, Corollary 2.6]).

Let $s,t \in S$. Then $f(s+t) \in \{f(s), f(t), f(s)+1, f(t)+1\}$. Since $f(s), f(t) \geq 2$, $f(s+t) \neq f(s) + f(t)$.

In the above example, f is badly not injective.

Question 4.7. *Let S and T be cancellative semigroups, let $f : S \xrightarrow{1\text{-}1} T$, and let $\widetilde{f} : \beta S \to \beta T$ be its continuous extension. If $\widetilde{f}$ takes some idempotent of βS to an idempotent of βT, does it follow that there exist $x, y \in S$ such that $f(x \cdot y) = f(x) \cdot f(y)$?*

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