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SUBSEMIGROUPS OF βS CONTAINING THE IDEMPOTENTS

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Abstract. Let S be a discrete semigroup and let $P(S)$ be the set of points p in the Stone-Čech compactification, βS, of S with the property that every neighborhood of p contains arbitrarily large finite sum sets or finite product sets, (depending on whether the operation in S is denoted by $+$ or $\cdot$). Then $P(S)$ contains all of the idempotents of βS, where the operation on βS extends that on S making βS into a right topological semigroup with S contained in its topological center. If S is commutative, then $P(S)$ is a compact subsemigroup of βS. Responding to a question of Vitaly Bergelson, we show that if S is any semigroup which can be embedded in a compact topological group, then $P(S)$ is not the smallest closed semigroup containing the idempotents of βS and the closure of the semigroup generated by the idempotents of βS is not a semigroup.

1. Introduction

In 1933 Richard Rado published [8] his remarkable theorem characterizing those finite matrices with rational coefficients which are kernel partition regular over the set $\mathbb{N}$ of positive integers. (A $u \times v$ matrix A is *kernel partition regular over* $\mathbb{N}$ if and only if whenever $\mathbb{N}$ is partitioned into finitely many classes, there will exist $\vec{x} \in \mathbb{N}^v$ with all of its entries in one class such that $A\vec{x} = \vec{0}$.) As an easy consequence, one sees that the matrix $\begin{pmatrix} 1 & 1 & -1 \end{pmatrix}$ is

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kernel partition regular over $\mathbb{N}$. That is, whenever $\mathbb{N}$ is partitioned into finitely many cells, there will be in one cell some x, y, and z with $x + y = z$. This result is Schur's Theorem [10]. More generally, it is an easy consequence of Rado's Theorem that whenever $r \in \mathbb{N}$ and $\mathbb{N}$ is divided into finitely many cells, there will exist a finite sequence $\langle x_t \rangle_{t=1}^{r}$ in $\mathbb{N}$ such that $FS(\langle x_t \rangle_{t=1}^{r})$ is contained in one cell, where $FS(\langle x_t \rangle_{t=1}^{r}) = \big\{ \sum_{t\in F} x_t : \emptyset \neq F \subseteq \{1, 2, \ldots, r\} \big\}$. (See [4, Corollary 2.4] for the details of how this follows easily from Rado's Theorem.) Much later Jon Sanders [9] and Jon Folkman (unpublished) independently derived this same result.

In [3] an infinite version of this result was obtained. That is, whenever $\mathbb{N}$ is divided into finitely many cells, there will exist a sequence $\langle x_t \rangle_{t=1}^{\infty}$ in $\mathbb{N}$ such that $FS(\langle x_t \rangle_{t=1}^{\infty})$ is contained in one cell, where $FS(\langle x_t \rangle_{t=1}^{\infty}) = \{\sum_{t\in F} x_t : F \in \mathcal{P}_f(\mathbb{N})\}$. (Here $\mathcal{P}_f(X)$ is the set of finite nonempty subsets of X.) The proof given in [3] was excruciatingly complicated. There is a much simpler proof due to Fred Galvin and Steven Glazer, not published by either of them. See the Notes to [6, Chapter 5] for the history of the discovery of this proof.

Theorem 1.1. *Let $A \subseteq \mathbb{N}$. There exists a sequence $\langle x_t \rangle_{t=1}^{\infty}$ in $\mathbb{N}$ with $FS(\langle x_t \rangle_{t=1}^{\infty}) \subseteq A$ if and only if there exists an idempotent p in $(\beta\mathbb{N}, +)$ such that $A \in p$.*

Proof. [6, Theorem 5.12]. □

In fact Theorem 1.1 holds more generally. Given any semigroup $(S, \cdot)$, not necessarily commutative, and given a sequence $\langle x_t \rangle_{t=1}^{\infty}$ in S, one defines $FP(\langle x_t \rangle_{t=1}^{\infty}) = \{\prod_{t\in F} x_t : F \in \mathcal{P}_f(\mathbb{N})\}$, where the product $\prod_{t\in F} x_t$ is taken in increasing order of indices. One then has that for any $A \subseteq S$ there exists a sequence $\langle x_t \rangle_{t=1}^{\infty}$ in S with $FP(\langle x_t \rangle_{t=1}^{\infty}) \subseteq A$ if and only if there exists an idempotent p in $(\beta S, \cdot)$ such that $A \in p$.

Given a discrete space X, we are taking the points of βX to be the ultrafilters on X, identifying the principal ultrafilters with the points of X and thereby pretending that $X \subseteq \beta X$. We let $X^* = \beta X \setminus X$. Given $A \subseteq X$, $\overline{A} = c\ell_{\beta X} A = \{p \in \beta S : A \in p\}$. If $(S, \cdot)$ is a discrete semigroup, the operation extends to βS making $(\beta S, \cdot)$ a right topological semigroup (meaning that for each $p \in \beta S$, the function $\rho_p : \beta S \to \beta S$ defined by $\rho_p(q) = q \cdot p$ is continuous) with

S contained in its topological center (meaning that for each $x \in S$, the function $\lambda_x : \beta S \to \beta S$ defined by $\lambda_x(q) = x \cdot q$ is continuous). Given $p, q \in \beta S$ and $A \subseteq S$, one has that $A \in p \cdot q$ if and only if $\{x \in S : x^{-1}A \in q\} \in p$, where $x^{-1}A = \{y \in S : x \cdot y \in A\}$. If the operation in S is denoted by $+$, we have that $A \in p + q$ if and only if $\{x \in S : -x + A \in q\} \in p$, where $-x + A = \{y \in S : x + y \in A\}$. It is a fundamental fact, due originally to R. Ellis [1], that any compact Hausdorff right topological semigroup has an idempotent. See [6] for an elementary introduction to the structure of βS.

Recently Vitaly Bergelson asked whether there is some nice algebraic description of the set of ultrafilters on $\mathbb{N}$ every member of which contains arbitrarily large finite sums sets. This would be the set $P(\mathbb{N})$ defined below. (We state the definition multiplicatively because we will be dealing with these sets in a quite general context.)

Definition 1.2. Let $(S, \cdot)$ be a semigroup.

(a) For each $r \in \mathbb{N}$,
$P_r(S) = \{p \in S^* : (\forall A \in p)(\exists \langle x_t \rangle_{t=1}^r)(FP(\langle x_t \rangle_{t=1}^r) \subseteq A)\}$.
(b) $P(S) = \bigcap_{r=1}^\infty P_r(S)$.

If S is commutative, it is easy to see that $P_r(S)$ is a compact subsemigroup of βS. (Given $r \in \mathbb{N}$, $p, q \in P_r(S)$, and $A \in p \cdot q$, one has that $B = \{x \in S : x^{-1}A \in q\} \in p$ so pick $\langle x_t \rangle_{t=1}^r$ with $FP(\langle x_t \rangle_{t=1}^r) \subseteq B$. Then $C = \bigcap\{z^{-1}A : z \in FP(\langle x_t \rangle_{t=1}^r)\} \in q$ so pick $\langle y_t \rangle_{t=1}^r$ with $FP(\langle y_t \rangle_{t=1}^r) \subseteq C$. Then $FP(\langle x_t \cdot y_t \rangle_{t=1}^r) \subseteq A$.) By [5, Theorem 3.9] (using a result of Nešetřil and Rödl [7]), for each $r > 1$, $P_{r+1}(\mathbb{N}, +)$ is a proper subset of $P_r(\mathbb{N}, +)$. Further, it is an immediate consequence of Theorem 1.1 that all idempotents of $\beta\mathbb{N}$ are in $P(\mathbb{N}, +)$. Thus, a tempting answer to Bergelson's question would be that $P(\mathbb{N}, +)$ is the smallest compact subsemigroup of $(\beta\mathbb{N}, +)$ containing the idempotents.

However, it was shown in [5] that the closure of the semigroup generated by the idempotents of $(\beta\mathbb{N}, +)$ is not a semigroup and that there is a compact subsemigroup of $(\beta\mathbb{N}, +)$ (denoted there by M) which lies strictly between the smallest subsemigroup of $(\beta\mathbb{N}, +)$ containing the idempotents and $P(\mathbb{N}, +)$. In Section 2 of this paper we extend these results to semigroups which can be algebraically embedded in compact topological groups.

In Section 3 we restrict our attention to $P(\mathbb{N},+)$, noting that as a consequence of the following result, $P(\mathbb{N},+)$ is an ideal of $(\beta\mathbb{N},\cdot)$.

Theorem 1.3. *Let $r \in \mathbb{N}$. Then $P_r(\mathbb{N},+)$ is an ideal of $(\beta\mathbb{N},\cdot)$.*

Proof. Let $p \in P_r(\mathbb{N},+)$ and let $q \in \beta\mathbb{N}$. To see that $p \cdot q \in P_r(\mathbb{N},+)$, let $A \in p \cdot q$. Pick $\langle x_t \rangle_{t=1}^r$ such that $FS(\langle x_t \rangle_{t=1}^r) \subseteq \{y \in \mathbb{N} : y^{-1}A \in q$. Then $B = \bigcap\{y^{-1}A : y \in FS(\langle x_t \rangle_{t=1}^r)\} \in q$ so pick $a \in B$. Then $FS(\langle x_t \cdot a \rangle_{t=1}^r) \subseteq A$.

To see that $q \cdot p \in P_r(\mathbb{N},+)$, let $A \in q \cdot p$. Pick $a \in \mathbb{N}$ such that $a^{-1}A \in p$ and pick $\langle x_t \rangle_{t=1}^r$ such that $FS(\langle x_t \rangle_{t=1}^r) \subseteq a^{-1}A$. Then $FS(\langle a \cdot x_t \rangle_{t=1}^r) \subseteq A$. □

Another tempting answer to Bergelson's question then becomes that $P(\mathbb{N},+)$ is the smallest compact subset of $\beta\mathbb{N}$ which is both a subsemigroup of $(\beta\mathbb{N},+)$ and an ideal of $(\beta\mathbb{N},\cdot)$. We show in Section 3 that this is not the case.

All hypothesized topological spaces are Hausdorff.

2. Semigroups embeddable in compact topological groups

We show in this section that if S is any semigroup which can be embedded in a compact topological group, then the closure of the semigroup generated by the idempotents of S^* is not a semigroup. (As is well known, such semigroups include all free semigroups and all commutative cancellative semigroups.) We also show, under the same assumption on S, that there is an element of $P(S)$ which is not a member of the smallest compact subsemigroup of βS containing the idempotents of S^*. (This result is less interesting in the case that S is not commutative, since then it is unlikely that $P(S)$ will be a semigroup.)

The following lemma is, as we are fond of saying, well known by those who know it well.

Lemma 2.1. *Let $(S,\cdot)$ be a countably infinite semigroup. If S can be algebraically embedded in a compact topological group, then S can be algebraically embedded in a compact metrizable topological group.*

Proof. Let G be a compact topological group with identity 1 and let $\varphi : S \to G$ be an injective homomorphism. Let

$$H = \{\varphi(s)\varphi(t)^{-1} : s, t \in S \text{ and } s \neq t\}\,.$$

For $a \in H$ pick by [2, Theorem 22.14] a compact metrizable topological group C_a and a continuous homomorphism $h_a : G \to C_a$ such that $h_a(a) \neq h_a(1)$. Let $C = \times_{a \in H} C_a$ and define $\psi : S \to C$ by $\psi(s)(a) = h_a\big(\varphi(s)\big)$ for each $a \in H$. Given $s \neq t$ in S, if $a = \varphi(s)\varphi(t)^{-1}$, then $\psi(s)(a) \neq \psi(t)(a)$ so ψ is injective. □

The Lemma 2.3 will be used in the proofs of both of the theorems of this section. If $n \in \mathbb{N}$, $\mathrm{supp}(n)$ is the subset of ω determined by $n = \sum_{t \in \mathrm{supp}(n)} 2^t$, where $\omega = \mathbb{N} \cup \{0\}$.

Definition 2.2. (a) $\mathbb{H} = \bigcap_{n=1}^{\infty} c\ell_{\beta\mathbb{N}}(\mathbb{N}2^n)$.

(b) Let X be a subset of a semigroup. A function $\psi : \omega \to X$ will be called an $\mathbb{H}$-*map* if it is bijective and if $\psi(m+n) = \psi(m)\psi(n)$ whenever $m, n \in \mathbb{N}$ satisfy $\max \mathrm{supp}(m) + 1 < \min \mathrm{supp}(n)$.

Note that by [6, Lemma 6.6], $\mathbb{H}$ contains all of the idempotents of $(\beta\mathbb{N}, +)$.

Lemma 2.3. *Let S be a countable semigroup which can be embedded in a compact topological group. Then there exist a countable group G containg S, an $\mathbb{H}$-map $\psi : \omega \to G$, and a subsemigroup V of G^* which contains all of the idempotents of G^* such that $\widetilde{\psi}_{|\mathbb{H}}$ is an isomorphism from $\mathbb{H}$ onto V. Further, there is a sequence $\langle s_n \rangle_{n=1}^{\infty}$ in S such that for each n,* $\max \mathrm{supp}\ \psi^{-1}(s_n) + 1 < \min \mathrm{supp}\ \psi^{-1}(s_{n+1})$.

Proof. By Lemma 2.1 there exist a compact metrizable topological group C with identity 1 and an injective homomorphism $\varphi : S \to C$. Let G be the subgroup of C generated by $\varphi[S]$ and let βG_d be the Stone-Čech compactification of G with the discrete topology. We may assume in fact that $S \subseteq G$. Let $\iota : G \to C$ be the inclusion map and let $\widetilde{\iota} : \beta G_d \to C$ be its continuous extension. Let $V = G^* \cap \widetilde{\iota}^{-1}[\{1\}]$. By [6, Theorem 7.28] V is a subsemigroup of G^* which contains all of the idempotents of G^* and there is an $\mathbb{H}$-map $\psi : \omega \to G$ such that $\widetilde{\psi}_{|\mathbb{H}}$ is an isomorphism from $\mathbb{H}$ onto V.

Now pick an idempotent $q \in S^*$. (By [6, Theorem 4.36] S^* is a subsemigroup of βS so has an idempotent.) We choose the sequence $\langle s_n\rangle_{n=1}^\infty$ inductively, letting s_1 be any element of S. Let $n\in\mathbb{N}$ and assume that $s_1,s_2,\ldots,s_n$ have been chosen. Let $k=\max\operatorname{supp}\psi^{-1}(s_n)+2$. Now $q\in V$ so $\widetilde{\psi}^{-1}(q)$ is an idempotent in $\mathbb{H}$ and thus $\mathbb{N}2^k\in\widetilde{\psi}^{-1}(q)$. By [6, Lemma 3.30] $\psi[\mathbb{N}2^k]\in q$ so pick $s_{n+1}\in\psi[\mathbb{N}2^k]$. $\square$

Note that the idempotents p_n hypothesized in the next lemma exist by [6, Lemma 5.11].

Lemma 2.4. *Let $\langle x_t\rangle_{t=1}^\infty$ be a sequence in $\mathbb{N}$ such that for all t, $\max\operatorname{supp}(x_t)<\min\operatorname{supp}(x_{t+1})$. Let $\{E_n:n\in\mathbb{N}\}$ be a partition of $\mathbb{N}$ into infinite sets and for each n let p_n be an idempotent in $\beta\mathbb{N}$ such that for each $m\in\mathbb{N}$,*

$$\{\textstyle\sum_{t\in F}x_t : F\in\mathcal{P}_f(E_n) \text{ and } \min F>m\}\in p_n\,.$$

Let p be a cluster point in $\beta\mathbb{N}$ of the sequence $\langle p_n\rangle_{n=1}^\infty$ and let $A=\big\{\sum_{t\in F}x_t+\sum_{t\in G}x_t : F\in\mathcal{P}_f(E_1)$ and $(\exists n)\big(\max F<n<\min G$ and $G\in\mathcal{P}_f(E_n)\big)\big\}$. Then $A\in p_1+p$ and there do not exist $r\in\mathbb{H}$ and an idempotent q such that $A\in r+q$.

Proof. To see that $A\in p_1+p$ we show that

$$FS(\langle x_t\rangle_{t\in E_1})\subseteq\{a\in\mathbb{N}:-a+A\in p\}\,.$$

So let $F\in\mathcal{P}_f(E_1)$, let $a=\sum_{t\in F}x_t$, and let

$$B=\{\textstyle\sum_{t\in G}x_t:(\exists n)(\max F<n<\min G \text{ and } G\in\mathcal{P}_f(E_n)\}\,.$$

Then $B\subseteq -a+A$ so it suffices to show that $B\in p$. Suppose instead $B\notin p$ and pick $n>\max F$ such that $p_n\in\overline{\mathbb{N}\setminus B}$. Then $\{\sum_{t\in G}x_t:G\in\mathcal{P}_f(E_n)$ and $\min G>n\}$ is an element of p_n which is contained in B, a contradiction.

Now suppose that we have $r\in\mathbb{H}$ and an idempotent q such that $A\in r+q$. Let $X=FS(\langle x_t\rangle_{t=1}^\infty)$. We claim first that $X\in q$, so suppose instead that $X\notin q$. Pick $a\in\mathbb{N}$ such that $-a+A\in q$ and pick $k\in\mathbb{N}$ such that $\max\operatorname{supp}(a)<\min\operatorname{supp}(x_k)$ and let $m=\max\operatorname{supp}(x_k)+1$. Pick $b\in(-a+A)\cap\mathbb{N}2^m\setminus X$. Then $a+b=\sum_{t\in F}x_t+\sum_{t\in G}x_t$, where $F\in\mathcal{P}_f(E_1)$ and there is some n with $\max F<n<\min G$ and $G\in\mathcal{P}_f(E_n)$. Let $H=\{t\in F\cup G:t<k\}$ and let $K=\{t\in F\cup G:t>k\}$. Then since $\operatorname{supp}(x_k)\cap\operatorname{supp}(a+b)=\emptyset$, we have $H\cup K=F\cup G$. Also, $\max\operatorname{supp}(a)<\min\operatorname{supp}(\sum_{t\in K}x_t)$ and $\max\operatorname{supp}(\sum_{t\in H}x_t)<\min\operatorname{supp}(b)$ so $a=\sum_{t\in H}x_t$ and $b=\sum_{t\in K}x_t$, so $b\in X$, a contradiction.

Define $g : X \to \mathbb{N}$ by $g(\sum_{t\in F} x_t) = n$ if and only if $\max F \in E_n$. We next claim that there is some $n \in \mathbb{N}$ such that

$$\{y \in X : g(y) \leq n\} \in q\,.$$

So suppose instead that for all $n \in \mathbb{N}$, $\{y \in X : g(y) > n\} \in q$. Pick $a \in \mathbb{N}$ such that $-a + A \in q$. Let $l = \max \operatorname{supp}(a)$. Now

$$\{b \in (-a + A) : -b + (-a + A) \in q\} \in q\,,$$

so pick $b \in (-a + A) \cap X \cap \mathbb{N}2^{l+1}$ such that $-b + (-a + A) \in q$. Pick $F \in \mathcal{P}_f(E_1)$ and $n \in \mathbb{N}$ such that $\max F < n < \min G$, $G \in \mathcal{P}_f(E_n)$, and $a + b = \sum_{t\in F} x_t + \sum_{t\in G} x_t$. Then $g(b) = n$. Let $k = \max \operatorname{supp}(b)$ and pick

$$b' \in \big(-b + (-a + A)\big) \cap X \cap \mathbb{N}2^{k+1} \cap \{y \in X : g(y) > n\}\,.$$

Then $a + b + b' \in A$ so $a + b + b' = \sum_{t\in F'} x_t + \sum_{t\in G'} x_t$ for some $F' \in \mathcal{P}_f(E_1)$ and some $G' \in \mathcal{P}_f(E_m)$ where $\max F' < m < \min G'$. then $m = g(b')$ so $m > n$. But then $a + b + b' = \sum_{t\in H} x_t$ where $H \cap E_1 \neq \emptyset$, $H \cap E_n \neq \emptyset$, and $H \cap E_m \neq \emptyset$, a contradiction. Thus we do have some $n \in \mathbb{N}$ such that $\{y \in X : g(y) \leq n\} \in q$.

Now let $k = \max \operatorname{supp}(x_n)$ and pick $a \in \mathbb{N}2^{k+1}$ such that $-a + A \in q$ (using the fact here that $r \in \mathbb{H}$). Let $l = \max \operatorname{supp}(a)$ and pick $b \in (-a + A) \cap \mathbb{N}2^{l+1} \cap \{y \in X : g(y) \leq n\}$. Pick $F \in \mathcal{P}_f(E_1)$ and m and G such that $G \in \mathcal{P}_f(E_m)$ and $\max F < m < \min G$. Then $g(b) = m$ so $m \leq n$. But $\min \operatorname{supp}(a + b) > k$ so $\max F \geq \min F > n$ so $m > n$, a contradiction. □

Recall that a semigroup $(S, \cdot)$ is *weakly cancellative* provided that for all $x, y \in S$, $\{s \in S : x \cdot s = y \text{ or } s \cdot x = y\}$ is finite.

Lemma 2.5. *Let $(S, \cdot)$ be an infinite weakly cancellative semigroup. Then there is a countable subsemigroup T of S such that if $q, r \in \beta S$, $q = q \cdot q$, and $r \cdot q \in \overline{T}$, then $r \in \overline{T}$ and $q \in \overline{T}$. Furthermore, if A is the subsemigroup of $\overline{T}$ generated by the idempotents of T^* and B is the subsemigroup of βS generated by the idempotents of S^*, then $c\ell A = \overline{T} \cap c\ell B$.*

Proof. Let C_1 be an arbitrary countable subsemigroup of S. Given $n \in \mathbb{N}$ and C_n, let

$$D_n = C_n \cup \{s \in S : (\exists x \in C_n)(x \cdot s \in C_n \text{ or } s \cdot x \in C_n)\}$$

and let C_{n+1} be the semigroup generated by D_n. Let $T = \bigcup_{n=1}^{\infty} C_n$. Trivially T is a countable subsemigroup of S. Notice also that if $x \in T$, $s \in S$, and either $xs \in T$ or $sx \in T$, then $s \in T$.

Now assume that $q, r \in \beta S$, $q = q \cdot q$, and $r \cdot q \in \overline{T}$. Then $T \in r \cdot q = r \cdot q \cdot q$ so $\{x \in S : x^{-1}T \in q\} \in r \cdot q$. Pick $x \in T$ such that $x^{-1}T \in q$. Then $x^{-1}T \subseteq T$ so $T \in q$.

Now $\{s \in S : s^{-1}T \in q\} \in r$. We claim that

$$\{s \in S : s^{-1}T \in q\} \subseteq T$$

so that $T \in r$. Let $s \in S$ such that $s^{-1}T \in q$. Pick $x \in s^{-1}T \cap T$. Then $sx \in T$ so $s \in T$.

One easily shows by induction on k that if $k \in \mathbb{N}$ and $r_1, r_2, \ldots, r_k$ are idempotents in βS and $r_1 \cdot r_2 \cdots r_k \in \overline{T}$ then $\{r_1, r_2, \ldots, r_k\} \subseteq \overline{T}$.

Trivially $c\ell A \subseteq \overline{T} \cap c\ell B$. For the reverse inclusion, let $p \in \overline{T} \cap c\ell B$ and let $C \in p$. Now $C \cap T \in p$ so $\overline{C \cap T} \cap B \neq \emptyset$ so pick $k \in \mathbb{N}$ and idempotents $r_1, r_2, \ldots, r_k$ in S^* such that $r_1 \cdot r_2 \cdots r_k \in \overline{C \cap T}$. Then $\{r_1, r_2, \ldots, r_k\} \subseteq \overline{T}$ so $r_1 \cdot r_2 \cdots r_k \in \overline{C} \cap A$. □

We are now ready to fulfill the first of our objectives of this section.

Theorem 2.6. *Let S be a semigroup which is embeddable in a compact topological group and let B be the subsemigroup of βS generated by the idempotents of S^*. Then $c\ell B$ is not a semigroup. In fact, there exist an idempotent q_1 of S^* and a point q in the closure of the set of idempotents of S^* such that $q_1 \cdot q \notin c\ell B$.*

Proof. By Lemma 2.5 we may assume that S is countable. Pick by Lemma 2.3 a countable group G containg S, an $\mathbb{H}$-map $\psi : \omega \to G$, and a subsemigroup V of G^* which contains all of the idempotents of G^* such that $\widetilde{\psi}_{|\mathbb{H}}$ is an isomorphism from $\mathbb{H}$ onto V. Also pick a sequence $\langle s_n \rangle_{n=1}^{\infty}$ in S such that for each n, $\max \operatorname{supp} \psi^{-1}(s_n) + 1 < \min \operatorname{supp} \psi^{-1}(s_{n+1})$. For each n, let $x_n = \psi^{-1}(s_n)$.

Let $\{E_n : n \in \mathbb{N}\}$ be a partition of $\mathbb{N}$ into infinite sets and for each n let p_n be an idempotent in $\beta\mathbb{N}$ such that for each $m \in \mathbb{N}$,

$$\{\textstyle\sum_{t \in F} x_t : F \in \mathcal{P}_f(E_n) \text{ and } \min F > m\} \in p_n\,.$$

Let p be a cluster point in $\beta\mathbb{N}$ of the sequence $\langle p_n \rangle_{n=1}^{\infty}$ and pick by Lemma 2.4 some $A \in p_1 + p$ such that there do not exist $r \in \mathbb{H}$ and an idempotent $q \in \mathbb{N}^*$ such that $A \in r + q$.

Let $q_1 = \widetilde{\psi}(p_1)$ and let $q = \widetilde{\psi}(p)$. Then q_1 is an idempotent of G^* and since $p_1 \in c\ell\{x_n : n \in \mathbb{N}\}$, $q_1 \in c\ell\{s_n : n \in \mathbb{N}\}$ so q_1 is an idempotent of S^*. Similarly, each $\widetilde{\psi}(p_n) \in S^*$ so $q \in S^*$ and $q \in c\ell\{\widetilde{\psi}(p_n) : n \in \mathbb{N}\}$.

Now $A \in p_1 + p$ so $\psi[A] \in \widetilde{\psi}(p_1 + p) = q_1 \cdot q$. Suppose $\overline{\psi[A]} \cap B \neq \emptyset$ and pick $k \in \mathbb{N}$ and idempotents $r_1, r_2, \ldots, r_k$ in S^* such that $\psi[A] \in r_1 \cdot r_2 \cdots r_k$. (We may presume that $k \geq 2$, since $r_1 = r_1 \cdot r_1$.) Then $\widetilde{\psi}^{-1}(r_1 \cdot r_2 \cdots r_{k-1}) \in \mathbb{H}$ and $\widetilde{\psi}^{-1}(r_k)$ is an idempotent of $\mathbb{N}^*$ and $A \in \widetilde{\psi}^{-1}(r_1 \cdot r_2 \cdots r_{k-1}) + \widetilde{\psi}^{-1}(r_k)$, a contradiction. $\square$

We now turn our attention to showing that under the same hypotheses $P(S)$ is not the smallest compact subsemigroup of βS containing the idempotents of S^*.

Lemma 2.7. *Let S and T be discrete semigroups, let $h : S \to \beta T$ be a homomorphism and let $\widetilde{h} : \beta S \to \beta T$ denote the continuous extension of h. Then $\widetilde{h}[P(S)] \subseteq P(T)$.*

Proof. Let $x \in P(S)$, let $B \in \widetilde{h}(x)$ and let $n \in \mathbb{N}$ with $n \geq 3$. Pick $C \in x$ such that $\widetilde{h}[\overline{C}\,] \subseteq \overline{B}$ and pick $\langle a_t \rangle_{t=1}^{n}$ such that $FP(\langle a_t \rangle_{t=1}^{n}) \subseteq C$. We shall construct inductively $\langle b_t \rangle_{t=1}^{n}$ such that $FP(\langle b_t \rangle_{t=1}^{n}) \subseteq B$.

Given $z \in FP(\langle h(a_t) \rangle_{t=2}^{n})$, we have $h(a_1)z \in \overline{B}$ so pick $D_z \in h(a_1)$ such that $D_z z \subseteq \overline{B}$. Also $B \in h(a_1)$ so pick

$$b_1 \in B \cap \bigcap\{D_z : z \in FP(\langle h(a_t) \rangle_{t=2}^{n})\}\,.$$

Now let $m \in \{1, 2, \ldots, n-2\}$ and assume we have chosen $\langle b_t \rangle_{t=1}^{m}$ such that for each $c \in FP(\langle b_t \rangle_{t=1}^{m})$ and $z \in FP(\langle h(a_t) \rangle_{t=m+1}^{n})$, $cz \in \overline{B}$. Given $c \in FP(\langle b_t \rangle_{t=1}^{m})$ and $z \in FP(\langle h(a_t) \rangle_{t=m+2}^{n})$ one has $h(a_{m+1})z \in \overline{B}$, $ch(a_{m+1}) \in \overline{B}$, and $ch(a_{m+1})z \in \overline{B}$. Since λ_c and ρ_z are continuous, we may pick D_z, E_c, and $F_{c,z}$ in $h(a_{m+1})$ such that $D_z z \subseteq \overline{B}$, $cE_c \subseteq \overline{B}$, and $cF_{c,z} z \subseteq \overline{B}$. Pick

$$\begin{aligned} b_{m+1} \in \quad & B \cap \bigcap\{D_z : z \in FP(\langle h(a_t) \rangle_{t=m+1}^{n})\} \\ & \cap \bigcap\{E_c : c \in FP(\langle b_t \rangle_{t=1}^{m})\} \\ & \cap \bigcap\{F_{c,z} : c \in FP(\langle b_t \rangle_{t=1}^{m}) \text{ and } z \in FP(\langle h(a_t) \rangle_{t=m+1}^{n})\} \end{aligned}$$

Having chosen $\langle b_t \rangle_{t=1}^{n-1}$, pick for each $c \in FP(\langle b_t \rangle_{t=1}^{n-1})$, $E_c \in h(a_n)$ such that $cE_c \subseteq \overline{B}$ and pick $b_n \in B \bigcap\{E_c : c \in FP(\langle b_t \rangle_{t=1}^{n-1})\}$. $\square$

Recall that if $q \in \beta\mathbb{N}$, $\langle x_n \rangle_{n=1}^{\infty}$ is a sequence in a Hausdorff topological space X, and $y \in X$, then $y = q\text{-}\lim_{n\in\mathbb{N}} x_n$ if and only if whenever U is a neighborhood of y in X, $\{n \in \mathbb{N} : x_n \in U\} \in q$.

Lemma 2.8. *Let $(S, \cdot)$ be a semigroup, let $p \in \beta S$, let $q \in P(\mathbb{N}, +)$, and let $r = q\text{-}\lim_{n\in\mathbb{N}} p^n$. Then $r \in P(S)$.*

Proof. This follows immediately from Lemma 2.7 and the observation that the map $n \mapsto p^n$ is a homomorphism from $(\mathbb{N}, +)$ into βS. $\square$

Given $v \in \mathbb{N}^*$ and $n \in \mathbb{N}$, we write $n * v$ for the sum of v with itself n times. (The notation $n \cdot v$ represents the operation in the semigroup $(\beta\mathbb{N}, \cdot)$, and $n * v$ need not equal $n \cdot v$. For example, if v is an idempotent, then $2 * v = v$ and $2 \cdot v \neq v$.)

Lemma 2.9. *Let $\langle x_n \rangle_{n=1}^{\infty}$ be a sequence in $\mathbb{N}$ such that for each $n \in \mathbb{N}$, $\max \operatorname{supp}(x_n) < \min \operatorname{supp}(x_{n+1})$, let $q \in P(\mathbb{N})$, let*

$$v \in \{x_n : n \in \mathbb{N}\}^*,$$

and let $p = q\text{-}\lim_{n\in\mathbb{N}} n * v$. Then

$$p \in \mathbb{H} \cap P(\mathbb{N}) \setminus c\ell \bigcup\{\beta\mathbb{N} + e : e \in \mathbb{N}^* \text{ and } e + e = e\}.$$

Proof. By Lemma 2.8 $p \in P(\mathbb{N})$. One shows easily by induction on n that for $n, k \in \mathbb{N}$, $\{\sum_{t\in F} x_t : |F| = n \text{ and } \min F > k\} \in n * v$. In particular, each $n * v \in \mathbb{H}$ and so $p \in \mathbb{H}$. Let

$$A = \{\textstyle\sum_{t\in F} x_t : F \in \mathcal{P}_f(\mathbb{N}) \text{ and } \min F > |F|\}.$$

Then given any $n \in \mathbb{N}$, $\{\sum_{t\in F} x_t : |F| = n \text{ and } \min F > n\} \subseteq A$ so $A \in p$.

We claim that $\overline{A} \cap \bigcup\{\beta\mathbb{N} + e : e \in \mathbb{N}^* \text{ and } e + e = e\} = \emptyset$. Suppose instead we have some $e = e + e$ such that $\overline{A} \cap (\beta\mathbb{N} + e) \neq \emptyset$ and pick $y \in \mathbb{N}$ such that $-y + A \in e$. Pick $l \in \mathbb{N}$ such that $\min \operatorname{supp}(x_l) > \max \operatorname{supp}(y)$. We claim that for each $m \geq l$, if $k = \max \operatorname{supp}(x_m)$, then $y \in FS(\langle x_t \rangle_{t=1}^{m})$ and $(-y + A) \cap \mathbb{N}2^{k+1} \subseteq FS(\langle x_t \rangle_{t=m+1}^{\infty})$. So let $z \in (-y + A) \cap \mathbb{N}2^{k+1}$. Then $y + z \in A$ so pick $F \in \mathcal{P}_f(\mathbb{N})$ such that $y + z = \sum_{t\in F} x_t$. Let $H = \{t \in F : t \leq m\}$ and let $K = \{t \in F : t > m\}$. Now $\operatorname{supp}(y) \cup \operatorname{supp}(z) = \operatorname{supp}(y + z) = \bigcup_{t\in F} \operatorname{supp}(x_t) = \bigcup_{t\in H} \operatorname{supp}(x_t) \cup \bigcup_{t\in K} \operatorname{supp}(x_t)$. Also $\max \bigcup_{t\in H} \operatorname{supp}(x_t) \leq \max \operatorname{supp}(x_m) = k < \min \operatorname{supp}(z)$ and

$\max \operatorname{supp}(y) < \min \operatorname{supp}(x_l) < \min \bigcup_{t \in K} \operatorname{supp}(x_t)$ so $\operatorname{supp}(y) = \bigcup_{t \in H} \operatorname{supp}(x_t)$ and $\operatorname{supp}(z) = \bigcup_{t \in K} \operatorname{supp}(x_t)$ and so $y = \sum_{t \in H} x_t$ and $z = \sum_{t \in K} x_t$. Taking $l = m$, we have $y \in FS(\langle x_t \rangle_{t=1}^{l})$. Also we have that for all $m \geq l$, $FS(\langle r_t \rangle_{t=m+1}^{\infty}) \in e$.

Given any $B \in e$, we let $B^\star = \{z \in B : -z + B \in e\}$. Then by [6, Lemma 4.14], whenever $z \in B^\star$, $-z + B^\star \in e$. Now we choose inductively $\langle F_i \rangle_{i=1}^{l}$ with $\min F_1 > l$ and for each $i \in \{1, 2, \ldots, l-1\}$, $\max F_i < \min F_{i+1}$, with $\sum_{j=1}^{i} \sum_{t \in F_j} x_t \in (-y + A)^\star$. Since $(-y + A)^\star \in e$, pick F_1 with $\min F_1 > l$ such that $\sum_{t \in F_1} x_t \in (-y + A)^\star$. Having chosen $\langle F_j \rangle_{j=1}^{i}$, let $m = \max F_i$ and pick $z \in -(\sum_{j=1}^{i} \sum_{t \in F_j} x_t) + (-y + A)^\star \cap FS(\langle x_t \rangle_{t=m+1}^{\infty})$ and pick F_{i+1} with $\min F_{i+1} \geq m + 1$ such that $z = \sum_{t \in F_{i+1}} x_t$. Now $y + \sum_{j=1}^{l} \sum_{t \in F_j} x_t \in A$ and $y = \sum_{t \in H} x_t$ for some H with $\max H \leq l$ so $\min(H \cup \bigcup_{j=1}^{l} F_j) \leq l$ while $|H \cup \bigcup_{j=1}^{l} F_j| \geq l + 1$, a contradiction. $\square$

Theorem 2.10. *Let S be a semigroup which can be embedded in a compact topological group. Let*

$$L = c\ell \bigcup \{\beta S \cdot e : e \in S^* \text{ and } e = e \cdot e\}.$$

Then L is a left ideal of βS and there exists $r \in P(S) \setminus L$. (So if S is commutative, $L \cap P(S)$ is a compact subsemigroup of βS containing the idempotents of S^* and properly contained in $P(S)$.)

Proof. By [6, Theorem 2.17], L is a left ideal of βS. We first show that it suffices to assume that S is countable. To see this, pick by Lemma 2.5 a countable subsemigroup T of S such that if $q, r \in \beta S$, $q = q \cdot q$, and $r \cdot q \in \overline{T}$, then $r \in \overline{T}$ and $q \in \overline{T}$. Assume that we have some $r \in P(T) \setminus c\ell \bigcup \{\overline{T} \cdot e : e \in T^* \text{ and } e = e \cdot e\}$. Then $r \in P(S)$. If $A \in r$ such that $\overline{A} \cap \bigcup \{\overline{T} \cdot e : e \in T^* \text{ and } e = e \cdot e\} = \emptyset$, then $\overline{A} \cap \bigcup \{\beta S \cdot e : e \in S^* \text{ and } e = e \cdot e\} = \emptyset$. So we shall assume that S is countable.

Pick by Lemma 2.3 a countable group G containg S, an $\mathbb{H}$-map $\psi : \omega \to G$, and a subsemigroup V of G^* which contains all of the idempotents of G^* such that $\widetilde{\psi}_{|\mathbb{H}}$ is an isomorphism from $\mathbb{H}$ onto V. Also pick a sequence $\langle s_n \rangle_{n=1}^{\infty}$ in S such that for each n, $\max \operatorname{supp} \psi^{-1}(s_n) + 1 < \min \operatorname{supp} \psi^{-1}(s_{n+1})$. For each n, let $x_n = \psi^{-1}(s_n)$. Let $q \in P(\mathbb{N})$, let $v \in \{x_n : n \in \mathbb{N}\}^*$, and let

$p = q\text{-}\lim_{n\in\mathbb{N}} n * v$. Then by Lemma 2.9

$$p \in \mathbb{H} \cap P(\mathbb{N}) \setminus c\ell \bigcup \{\beta\mathbb{N} + e : e \in \mathbb{N}^* \text{ and } e + e = e\}\,.$$

Let $r = \widetilde{\psi}(p)$ and let $w = \widetilde{\psi}(v)$. Then it is routine to establish that $r = q\text{-}\lim_{n\in\mathbb{N}} w^n$ and so, by Lemma 2.8 $r \in P(S)$.

Now we claim that $r \notin L$. To see this, pick $A \in p$ such that $\overline{A} \cap \bigcup\{\beta\mathbb{N} + e : e \in \mathbb{N}^* \text{ and } e + e = e\} = \emptyset$. Then $\psi[A] \in r$. We claim that $\overline{\psi}[A] \cap \bigcup\{\beta S \cdot e : e \in S^* \text{ and } e = e \cdot e\} = \emptyset$. Suppose instead that we have some $e = e \cdot e \in S^*$ and $y \in S$ such that $\psi[A] \in y \cdot e$. Now $\widetilde{\psi}^{-1}(e)$ is an idempotent in $\mathbb{N}^*$, so it suffices to show that $A \in \psi^{-1}(y) + \widetilde{\psi}^{-1}(e)$. Let $u = \psi^{-1}(y)$ and let $k = \max \operatorname{supp}(u)$. Since $y^{-1}\psi[A] \in e$ and consequently $\psi^{-1}\big[y^{-1}\psi[A]\big] \in \psi^{-1}(e)$, it suffices to show that $\mathbb{N}2^{k+2} \cap \psi^{-1}\big[y^{-1}\psi[A]\big] \subseteq -\psi^{-1}(y) + A$. So let $z \in \mathbb{N}2^{k+2} \cap \psi^{-1}\big[y^{-1}\psi[A]\big]$. Then $y\psi(z) \in \psi[A]$ so $\psi(u + z) = \psi(u)\psi(z) = y\psi(z) \in \psi[A]$ so $u + z \in A$ and thus $z \in -\psi^{-1}(y) + A$. $\square$

3. $\beta\mathbb{N}$

Recall that we have seen that $P(\mathbb{N}, +)$, in addition to being a compact subsemigroup of $(\beta\mathbb{N}, +)$ containing the idempotents, is also a two sided ideal of $(\beta\mathbb{N}, \cdot)$. We see now that it is not the smallest such.

Theorem 3.1. *There is a compact subsemigroup of* $(\beta\mathbb{N}, +)$ *which contains the idempotents of* $(\beta\mathbb{N}, +)$*, is a two sided ideal of* $(\beta\mathbb{N}, \cdot)$*, and is properly contained in* $P(\mathbb{N}, +)$*.*

Proof. Choose $v \in \{2^{2^n} : n \in \mathbb{N}\}^*$ and $q \in P(\mathbb{N})$. Let $p = q\text{-}\lim_{n\in\mathbb{N}} n * v$. Then by Lemma 2.8, $p \in P(\mathbb{N})$. Let

$$L = c\ell \bigcup \{\beta\mathbb{N} + e : e \in \mathbb{N}^* \text{ and } e + e = e\}\,.$$

Then by Lemma 2.9, $p \notin L$.

Define $f : \mathbb{N} \to \mathbb{R}$ by putting $f(n) = \log_2(n)$, and let $\widetilde{f} : \beta\mathbb{N} \to u\mathbb{R}$ denote the continuous extension of f, where $u\mathbb{R}$ denotes the uniform compactification of $\mathbb{R}$. We observe that $\mathbb{R}$ can be regarded as a subspace of $u\mathbb{R}$, because $\mathbb{R}$ can be embedded in $u\mathbb{R}$ by a topological isomorphism. Then by [11, Lemma 2.1], $\widetilde{f}$ has the following properties:

(i) $\widetilde{f}(x \cdot y) = \widetilde{f}(x) + \widetilde{f}(y)$ for every $x, y \in \beta\mathbb{N}$ and
(ii) $\widetilde{f}(x + y) = \widetilde{f}(y)$ for every $x \in \beta\mathbb{N}$ and every $y \in \mathbb{N}^*$.

For a subset S of $\mathbb{R}$, $\rho(S)$ will denote $c\ell_{u\mathbb{R}}(S) \setminus \mathbb{R}$. Let $X = \rho(\{2^n : n \in \mathbb{N}\})$. We claim that $X \subseteq \mathrm{int}_{\rho(\mathbb{R})}(\rho(\mathbb{R}) \setminus (\rho(\mathbb{R}) + \rho(\mathbb{R}))$. To see this, put $E = \{2^n : n \in \mathbb{N}\}$ and put $F = \mathbb{R} \setminus \big((-1,1) + E\big)$. Using the uniform structure on $\mathbb{R}$ defined by the usual metric, it follows from [6, Exercise 21.5.3] that there is a uniformly continuous function $\phi : \mathbb{R} \to [0,1]$ such that $\phi[E] = \{0\}$ and $\phi[F] = \{1\}$. Let $\widetilde{\phi} : u\mathbb{R} \to [0,1]$ denote the continuous extension of ϕ. If $W = \widetilde{\phi}^{-1}\big[[0, \frac{1}{2})\big]$, then W is an open neighbourhood of X in $u\mathbb{R}$ and $W \subseteq cl_{u\mathbb{R}}\big((-1,1)+E\big)$. We shall show that $W \cap \big(\rho(\mathbb{R})+\rho(\mathbb{R})\big) = \emptyset$. To see this, assume that $\xi, \eta \in \rho(\mathbb{R})$ and that $\xi + \eta \in W$. We can choose $s, t \in \mathbb{R}$ with $|s-t| > 2$ such that $s+\eta$ and $t+\eta$ are both in W, because $\{\zeta \in u\mathbb{R} : \zeta+\eta \in W\}$ is a neighbourhood of ξ in $u\mathbb{R}$, and so its intersection with $\mathbb{R}$ is unbounded. (If $B \subseteq \mathbb{R}$ is bounded, then $c\ell_{u\mathbb{R}}(B) \subseteq \mathbb{R}$.) Then $-s+W$ and $-t+W$ are both neighbourhoods of η. However, we claim that $(-s+W) \cap (-t+W) \cap \mathbb{R}$ is bounded. To see this, note that for any $x \in (-s+W) \cap (-t+W) \cap \mathbb{R}$ we may pick $n, m \in \mathbb{N}$ such that $|s+x-2^n| < 1$ and $|t+x-2^m| < 1$. Since $|s-t| > 2$ we have that $n \neq m$. On the other hand, $|2^n - 2^m| = |2^n - x - s + t + x - 2^m + s - t| < 2 + |s-t|$. Thus there are only finitely many pairs (n,m) for which there is some x with $|s+x-2^n| < 1$ and $|t+x-2^m| < 1$. Given any such (n,m) and x, $|x| \leq |x+s-2^n| + |s-2^n| < 1 + |s-2^n|$ so $(-s+W) \cap (-t+W) \cap \mathbb{R}$ is bounded as claimed. But this contradicts the assumption that $\eta \in \rho(\mathbb{R})$.

It follows from (ii) above that $J = \widetilde{f}^{-1}[\rho(\mathbb{R}) \setminus W]$ is a closed subset of $\mathbb{N}^*$ which is a left ideal of $(\beta\mathbb{N}, +)$. Furthermore, it follows from (i) above that $\mathbb{N}^* \cdot \mathbb{N}^* \subseteq J$. and in particular J is a two sided ideal of $(\mathbb{N}^*, \cdot)$. Let V denote the smallest compact subset of $\beta\mathbb{N}$ which is both a left ideal of $(\beta\mathbb{N}, +)$ and satisfies $\mathbb{N}^* \cdot \mathbb{N}^* \subseteq V$. Then $V \subseteq J$. We claim that V is an ideal of $(\beta\mathbb{N}, \cdot)$. To see this, let $n \in \mathbb{N}$. Then by [6, Theorem 6.54] $nV = Vn$ so it suffices to show that $nV \subseteq V$. To see this, let $W = \{p \in \beta\mathbb{N} : n \cdot p \in V\}$. Then it is easy to verify that W is a compact subset of $\beta\mathbb{N}$ which is both a left ideal of $(\beta\mathbb{N}, +)$ and satisfies $\mathbb{N}^* \cdot \mathbb{N}^* \subseteq W$, and consequenty $V \subseteq W$ and therefore $nv \subseteq V$ as required.

We claim that $L \cup V$ is a closed left ideal of $(\beta\mathbb{N}, +)$ and an ideal of $(\beta\mathbb{N}, \cdot)$. It is obviously a closed left ideal of $(\beta\mathbb{N}, +)$. To see that it is an ideal of $(\beta\mathbb{N}, \cdot)$, it is routine to verify that for any $n \in \mathbb{N}$, $n \cdot L = L \cdot n \subseteq L$. Also, for any $x \in \mathbb{N}^*$, $(x \cdot L) \cup (L \cdot x) \subseteq \mathbb{N}^* \cdot \mathbb{N}^* \subseteq V$.

We claim that the element $p \in P(\mathbb{N})$ defined above is not in V. To see this, observe that $\widetilde{f}(v) \in c\ell_{u\mathbb{R}}(E)$ and hence, by property (ii) above, that $\beta\mathbb{N} + v \subseteq \widetilde{f}^{-1}[c\ell_{u\mathbb{R}} E] \subseteq \widetilde{f}^{-1}[W]$. So $(\beta\mathbb{N} + v) \cap J = \emptyset$ and consequently $(\beta\mathbb{N} + v) \cap V = \emptyset$. Now let $r = q\text{-}\lim_{n \in \mathbb{N}} (n-1) * v$. Then $p = r + v \in \beta\mathbb{N} + v$, so $p \notin V$. We have already noted that $p \notin L$. Thus $P(\mathbb{N}) \nsubseteq L \cup V$. $\square$

References

[1] R. Ellis, *Lectures on topological dynamics*, Benjamin, New York, 1969.

[2] E. Hewitt and K. Ross, *Abstract Harmonic Analysis, I*, Springer-Verlag, Berlin, 1963.

[3] N. Hindman, *Finite sums from sequences within cells of a partition of* $\mathbb{N}$, J. Comb. Theory (Series A) **17** (1974), 1-11.

[4] N. Hindman, *Partition regularity of matrices*, in Combinatorial Number Theory, B. Landman, M. Nathanson, J. Nesetril, R. Nowakowski, and C. Pomerance, editors, deGruyter, Berlin, 2007, 265-298.

[5] N. Hindman and D. Strauss, *Compact subsemigroups of* $(\beta\mathbb{N}, +)$ *containing the idempotents*, Proc. Edinburgh Math. Soc. **39** (1996), 291-307.

[6] N. Hindman and D. Strauss, *Algebra in the Stone-Čech compactification: theory and applications*, de Gruyter, Berlin, 1998.

[7] J. Nešetřil and V. Rödl, *Finite union theorem with restrictions*, Graphs and Comb. **2** (1986), 357-361.

[8] R. Rado, *Studien zur Kombinatorik*, Math. Zeit. **36** (1933), 242-280.

[9] Sanders, J., *A generalization of Schur's theorem*, Ph.D. Dissertation, Yale University (1968).

[10] I. Schur, *Über die Kongruenz* $x^m + y^m = z^m (\text{mod } p)$, Jahresbericht der Deutschen Math.-Verein. **25** (1916), 114-117.

[11] D. Strauss, *The smallest ideals of* $\beta\mathbb{N}$ *under addition and multiplication*, Topology Appl. **149** (2005), 289-292.

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