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Independent Sums of Arithmetic Progressions in $\mathrm{K_m}$-free Graphs

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Abstract. We establish that for any $m \in \mathbb{N}$ and any K_m-free graph G on $\mathbb{N}$ there exist large additive and multiplicative structures that are independent with respect to G. In particular, there exists for each $l \in \mathbb{N}$ an arithmetic progression A_l of length l with increment chosen from the finite sums of a prespecified sequence $\langle t_{l,n}\rangle_{n=1}^\infty$ such that $\bigcup_{l=1}^\infty A_l$ is an independent set. Moreover, if F and H are disjoint finite subsets of $\mathbb{N}$, and for each $t \in F \cup H$, $a_t \in A_t$, then $\{\Sigma_{t\in F}\, a_t, \Sigma_{t\in H}\, a_t\}$ is not an edge of G. If G is $K_{m,m}$-free, one may drop the disjointness assumption on the sets F and H. Analogous results are valid for geometric progressions.

1. Introduction

In 1995 A. Hajnal asked P. Erdős whether, given any triangle-free graph G on the set $\mathbb{N}$ of positive integers, there must exist a sequence $\langle x_n\rangle_{n=1}^\infty$ such that $FS(\langle x_n\rangle_{n=1}^\infty) = \{\sum_{n\in F} x_n : F$ is a finite nonempty subset of $\mathbb{N}\}$ is an independent set. (This appears as a remark following Problem 4.4 of [6].)

This question was answered in the negative in [3]. On the other hand it was established independently in [13] and [3] that if for some m, G contains no complete bipartite graph $K_{m,m}$, then there must exist a sequence $\langle x_n\rangle_{n=1}^\infty$ such that $FS(\langle x_n\rangle_{n=1}^\infty)$ is independent. It was shown further in [3] that if, for some m, G does not contain a complete graph on m vertices,

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then there is a sequence $\langle x_n\rangle_{n=1}^{\infty}$ such that there are no edges of G of the form $\{\sum_{n\in F} x_n, \sum_{n\in H} x_n\}$ with F and H disjoint.

A typical statement of Ramsey Theory (or at least that portion that deals with partitions or "colorings") can be described as follows. Let X be a set and let $\mathcal{G}$ be a set of "good" subsets of X. *If X is divided into finitely many classes, then there is a member B of $\mathcal{G}$ contained in one of these classes.* We shall list in Observation 2.1 some simple facts about such a set $\mathcal{G}$. These include the fact that having a member which is independent with respect to any K_m-free graph is stronger than being partition regular.

In [8], D. Gunderson, I. Leader, H. Prömel, and V. Rödl established, with an elegantly simple proof, the following extension of van der Waerden's Theorem.

1.1 Theorem. *Let $m \in \mathbb{N}$ and let G be a graph on $\mathbb{N}$ which contains no K_m. Then for every $l \in \mathbb{N}$, there is a length l arithmetic progression which is an independent set.*

Proof. [8, Theorem 2.2]. □

They also establish, with a considerably more complicated proof, that the progression can be chosen so that it together with its increment forms an independent set.

In this paper we obtain several extensions of Theorem 1.1. In Section 2 we develop some tools needed for the rest of the paper and derive generalizations of the Central Sets Theorem for both K_m-free graphs and $K_{m,m}$-free graphs.

In Section 3 we extend Theorem 1.1 in much the same way that Deuber's (m, p, c)-sets Theorem [2] was extended in [4], adding additional features. That is, we shall establish a general result (Theorem 3.10), among whose consequences is the following: If $m \in \mathbb{N}$, G is a K_m-free graph on $\mathbb{N}$, and for each $l \in \mathbb{N}$, $\langle t_{l,n}\rangle_{n=1}^{\infty}$ is a sequence in $\mathbb{N}$, then there exists for each $l \in \mathbb{N}$ an arithmetic progression A_l of length l with increment from $FS(\langle t_{l,n}\rangle_{n=1}^{\infty})$ such that $\bigcup_{l=1}^{\infty} A_l$ is independent with respect to G, and further, if F and H are disjoint finite subsets of $\mathbb{N}$, and for each $t \in F \cup H$, $a_t \in A_t$, then $\{\sum_{t\in F} a_t, \sum_{t\in H} a_t\}$ is not an edge of G. Further, if G is $K_{m,m}$-free, we show that one need not require that $F \cap H = \emptyset$.

We shall utilize extensively the algebraic structure of the Stone-Čech compactification βS of a discrete semigroup $(S, +)$. We take the points of βS to be the ultrafilters on S, the principal ultrafilters being identified with the points of S. Given a set $A \subseteq S$, $\overline{A} = \{p \in \beta S : A \in p\}$. The set $\{\overline{A} : A \subseteq S\}$ is a basis for the open sets (as well as a basis for the closed sets) of βS.

There is a natural extension of the operation $+$ of S to βS, also denoted by $+$. (We are denoting the operation by $+$ because we are primarily concerned with commutative semigroups. However, even when $(S,+)$ is commutative, $(\beta S,+)$ is almost certain not to be commutative.) This extension makes $(\beta S,+)$ a compact right topological semigroup with S contained in its topological center. This says that for each $p\in\beta S$ the function $\rho_p:\beta S\to\beta S$ is continuous and for each $x\in S$, the function $\lambda_x:\beta S\to\beta S$ is continuous, where $\rho_p(q)=q+p$ and $\lambda_x(q)=x+q$. Given $p,q\in\beta S$ and $A\subseteq S$, $A\in p+q$ if and only if $\{y\in S:-y+A\in q\}\in p$, where $-y+A=\{z\in S:y+z\in A\}$. (There is no requirement that there be such an object as $-y$. When writing multiplicatively, the notation $-y+A$ is replaced by $y^{-1}A$.) See [10] for an elementary introduction to the semigroup βS.

Any compact Hausdorff right topological semigroup $(T,\cdot)$ has a smallest two sided ideal $K(T)$ which is the union of all of the minimal left ideals of T, each of which is closed [10, Theorem 2.8], and any compact right topological semigroup contains idempotents. Since the minimal left ideals are themselves compact right topological semigroups, this says in particular that there are idempotents in the smallest ideal. There is a partial ordering of the idempotents of T determined by $p\leq q$ if and only if $p=p+q=q+p$. An idempotent p is minimal with respect to this order if and only if $p\in K(T)$ [10, Theorem 1.59]. Such an idempotent is called simply "minimal". A set $C\subseteq S$ is *central* if and only if C is a member of some minimal idempotent.

Central sets were introduced by Furstenberg [7] and defined in terms of notions of topological dynamics. These sets enjoy very strong combinatorial properties. (See [7, Proposition 8.21] or [10, Chapter 14].) See [10, Theorem 19.27] for a proof of the equivalence of the definition above with the original dynamical definition.

In our results about K_m-free graphs and $K_{m,m}$-free graphs, we obtain independent configurations that are also contained in a specified central set. Whenever S is finitely colored, one of the color classes must be central. Accordingly, we obtain configurations that are independent with respect to a specified graph and monochrome with respect to a given coloring. In the case of K_m-free graphs, this monochromicity is a corollary of the independence result alone by Observation 2.1.

Given an idempotent $p\in\beta S$ and a set $A\subseteq S$, we let

$$A^\star=\{x\in A:-x+A\in p\}\,.$$

By [10, Lemma 4.14] if $A\in p$, then $A^\star\in p$, and if $x\in A^\star$, then $-x+A^\star\in p$.

Given a finite sequence $\langle y_t\rangle_{t=1}^n$ in a semigroup $(S,+)$ we let

$$FS(\langle y_t\rangle_{t=1}^n)=\left\{\textstyle\sum_{t\in F}y_t:\emptyset\neq F\subseteq\{1,2,\ldots,n\}\right\}.$$

Similarly, if the operation is denoted by $\cdot$, then

$$FP(\langle y_t\rangle_{t=1}^{n}) = \left\{ \prod_{t\in F} y_t : \emptyset \neq F \subseteq \{1,2,\ldots,n\}\right\},$$

where the product $\prod_{t\in F} y_t$ is taken in increasing order of indices.

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2. Independent Structures Within Central Sets

We begin by recording some simple observations about the relationship between partition theorems and graphs.

2.1 Observation. *Let X be a set and let $\mathcal{G} \subseteq \mathcal{P}(X)$. Statements (a), (a′), and (a″) are equivalent, as are statements (c) and (c′). Statement (a) implies statements (b) and (c). The conjunction of statements (b) and (c) does not imply statement (a). Neither of statements (b) or (c) implies the other. In fact with $X = \mathbb{N}$, statement (b) can hold while (c) fails with $m = 2$, and statement (c) can hold while (b) fails with $r = 3$.*

- *(a) Whenever $m \in \mathbb{N}$ and G is a graph with vertex set X which contains no K_m, there exists $B \in \mathcal{G}$ such that B is independent with respect to G.*
- *(a′) Whenever $m \in \mathbb{N}$, $\mathcal{H}$ is a finite set of K_m-free graphs on X, and F is a finite set of functions from X to X, there exists $B \in \mathcal{G}$ such that $f[B]$ is independent with respect to G for every $f \in F$ and every $G \in \mathcal{H}$.*
- *(a″) Whenever $m, r \in \mathbb{N}$, G is a K_m-free graph on X, and $X = \bigcup_{i=1}^{r} A_i$, there exist $B \in \mathcal{G}$ and $i \in \{1,2,\ldots,r\}$ such that B is independent with respect to G and $B \subseteq A_i$.*
- *(b) Whenever $r \in \mathbb{N}$ and $X = \bigcup_{i=1}^{r} A_i$, there exist $i \in \{1,2,\ldots,r\}$ and $B \in \mathcal{G}$ such that $B \subseteq A_i$.*
- *(c) Whenever $m \in \mathbb{N}$ and G is a graph with vertex set X which contains no $K_{m,m}$, there exists $B \in \mathcal{G}$ such that B is independent with respect to G.*
- *(c′) Whenever $m \in \mathbb{N}$, $\mathcal{H}$ is a finite set of $K_{m,m}$-free graphs on X, and F is a finite set of functions from X to X, there exists $B \in \mathcal{G}$ such that $f[B]$ is independent with respect of G for every $f \in F$ and every $G \in \mathcal{H}$.*

Proof. If G contains no $K_{m,m}$ then G contains no K_{2m}, so (a) implies (c). Trivially (a″) implies both (a) and (b) and (c′) implies (c).

(a) implies (a′). For each $f \in F$ and each $G \in \mathcal{H}$, we can define a graph $H_{f,G}$ on X by putting $\{s,t\} \in E(H_{f,G})$ if and only if $\{f(s), f(t)\} \in E(G)$. (Notice that if $f(s) = f(t)$, then $\{s,t\} \notin E(H_{f,G})$.) Then $H_{f,G}$ is K_m-free. Let $r = |\{H_{f,G} : f \in F \text{ and } G \in \mathcal{H}\}|$. Pick by the finite version of Ramsey's

Theorem [16] (or see [10, Exercise 5.5.1]) some $n \in \mathbb{N}$ such that whenever a K_n is r-colored, there must be a monochrome K_m. Let L be the graph on X with $E(L) = \bigcup_{G \in \mathcal{H}} \bigcup_{f \in F} E(H_{f,G})$. Then L is K_n-free so pick $B \in \mathcal{G}$ such that B is independent with respect to L.

(a$'$) implies (a$''$). Let G be a K_m-free graph, let $X = \bigcup_{i=1}^{r} A_i$, and for simplicity assume that $A_i \cap A_j = \emptyset$ when $i \neq j$. Let H be the graph with vertex set X and edge set

$$E(H) = \big\{\{x, y\} : \text{there is no } i \in \{1, 2, \ldots, r\} \text{ with } \{x, y\} \subseteq A_i\big\}.$$

Then H contains no K_{r+1}. Apply (a$'$) with $\mathcal{H} = \{G, H\}$, $F = \{\text{identity}\}$, and $m' = \max\{m, r + 1\}$.

The proof that (c) implies (c$'$) is identical to the proof that (a) implies (a$'$), replacing the appeal to Ramsey's Theorem by an appeal to the fact that for each $r, m \in \mathbb{N}$ there exists $n \in \mathbb{N}$ such that whenever a $K_{n,n}$ is r-colored, there exists a monochrome $K_{m,m}$. (See [12, Problem 10.37] for the derivation of a stronger fact.)

The fact that the conjunction of statements (b) and (c) does not imply statement (a) follows from the results on independent $FS(\langle x_n \rangle_{n=1}^{\infty})$ cited above.

To see that statement (b) does not imply statement (c), even with $m = 2$, let G be a graph with vertex set $X = \mathbb{N}$ which contains no $K_{2,2}$ but has infinite chromatic number. (P. Erdős established [5] by probabilistic methods that there are graphs with arbitrarily large girth g – meaning no cycles of length less than g – and arbitarily large chromatic number, and a $K_{2,2}$ is exactly a cycle of length 4. The first explicit construction of such graphs was given by L. Lovász [11] and a greatly simplified construction was given by J. Nešetřil and V. Rödl [15].) Let $\mathcal{G} = E(G)$. By the definition of chromatic number, any finite coloring of $\mathbb{N}$ yields a monochrome edge of G.

Finally, to see that statement (c) does not imply statement (b), even with $r = 3$, let $X = \mathbb{N}$ and let $\mathcal{G} = \big\{\{x + y, 2x + 3y\} : x, y \in \mathbb{N}\big\}$. For $i \in \{1, 2, 3\}$, let $A_i = \bigcup_{n=1}^{\infty}\{x \in \mathbb{N} : 2^{3n-i} \leq x < 2^{3n-i+1}\}$. Then, given $i \in \{1, 2, 3\}$ and $x, y \in \mathbb{N}$, if $2^{3n-i} \leq x + y < 2^{3n-i+1}$, one has $2^{3n-i+1} < 2x + 3y < 2^{3n-i+3}$. Thus statement (b) fails with $r = 3$.

To see that statement (c) holds, let G be a graph with respect to which no element of $\mathcal{G}$ is independent. Let $m \in \mathbb{N}$. We show that G contains a $K_{m,m}$. For $i \in \{1, 2, \ldots, m\}$ let $a_i = 3m + i$ and let $b_i = 8m + i$. Now let $i, j \in \{1, 2, \ldots, m\}$, let $x = m + 3i - j$, and let $y = 2m + j - 2i$. Then $x + y = a_i$ and $2x + 3y = b_j$ so $\{a_i, b_j\} \in E(G)$. □

A question left open by Observation 2.1 is whether there exist X and $\mathcal{G}$ such that statement (c) holds while statement (b) fails with $r = 2$.

The remaining results of this section depend rather heavily on results from [3] which we summarize in the following lemma. The reader who wishes to verify the results that we cite from [3] in the following lemma will find that, unfortunately, the proofs of these results were much more complicated than is normal for proofs using the algebraic structure of βS.

2.2 Lemma. *Let $m \in \mathbb{N}$, let $(S,+)$ be an infinite semigroup, let G be a graph on S which contains no K_m, and let p be an idempotent in βS. Then there exists $P \in p$ such that, for each $a \in P$, there is a set $A(a) \in p$ with the following properties:*

(1) for each $a \in P$, $A(a) \subseteq \{b \in S : \{a,b\} \notin E(G)\}$;

(2) for each $a \in P$, $\{b \in P : a \in A(b)^\star\} \in p$; and

(3) for all $a, b \in P$, if $a \in A(b)^\star$ and $b \in A(a)$, then $\{x \in P : b \in A(a+x)^\star\} \in p$.

Proof. For each $a \in S$ define $A(a) \subseteq S$ as in [3, Definition 3.1]. Then from that definition we have that $A(a) \subseteq \{b \in S : (a,b) \notin G\}$. By [3, Lemma 3.13], $\{a \in S : A(a) \in p\} \in p$. By [3, Lemma 3.18],

$$\{a \in S : \{b \in S : a \in A(b)^\star\} \in p\} \in p\,.$$

Let $P = \{a \in S : A(a) \in p\} \cap \{a \in S : \{b \in S : a \in A(b)^\star\} \in p\}$. Then conclusions (1) and (2) hold immediately.

To verify statement (3), let $a, b \in S$ and assume that $a \in A(b)^\star$ and $b \in A(a)$. Then by [3, Lemma 3.16], $\{x \in S : b \notin A_0(a+x) \setminus A(a+x)\} \in p$ and so $\{x \in S : b \notin A_0(a+x)\} \in p$ or $\{x \in S : b \in A(a+x)\} \in p$. But

$$\{x \in S : b \notin A_0(a+x)\} = \{x \in S : a+x \notin A_0(b)\} = S \setminus \big(-a + A_0(b)\big)\,.$$

Since $a \in A(b)^\star \subseteq A_0(b)^\star$, $-a + A_0(b) \in p$ and so $\{x \in S : b \notin A_0(a+x)\} \notin p$. Therefore $\{x \in S : b \in A(a+x)\} \in p$.

Also, by [3, Lemma 3.17], $\{x \in S : -b + A(a+x) \in p\} \in p$. Therefore

$$\begin{aligned}&\{x \in S : b \in A(a+x)^\star\} =\\ &\{x \in S : b \in A(a+x)\} \cap \{x \in S : -b + A(a+x) \in p\} \in p\,.\end{aligned}$$ □

2.3 Definition. Let $(S,+)$ be an infinite semigroup, let $P \in p$, let $A : P \to p$, let $C \in p$ with $C \subseteq P$, let $n \in \mathbb{N}$, and let $g : \{1,2,\ldots,n\} \to S$.

(a) The sequence g has property (†) if and only if $FS(\langle g(t)\rangle_{t=1}^n) \subseteq C^\star$ and $\sum_{t\in J} g(t) \in A\big(\sum_{t\in K} g(t)\big)^\star$ for every pair of disjoint nonempty subsets J and K of $\{1,2,\ldots,n\}$.

(b) The sequence g has property (††) if and only if $FS(\langle g(t)\rangle_{t=1}^n) \subseteq C^\star$ and $\sum_{t\in J} g(t) \in A\big(\sum_{t\in K} g(t)\big)^\star$ for every pair of nonempty subsets J and K of $\{1,2,\ldots,n\}$.

The notation does not reflect the fact that properties (†) and (††) depend on the choice of p, P, A, and C.

2.4 Theorem. *Let $m \in \mathbb{N}$, let $(S,+)$ be an infinite semigroup, let G be a graph on S which contains no K_m, and let p be an idempotent in βS. There exist $P \in p$ and $A : P \to p$ such that*

(1) for each $a \in P$, $A(a) \subseteq \{b \in S : \{a,b\} \notin E(G)\}$;

(2) for each $a \in P$, $\{b \in P : a \in A(b)^\star\} \in p$;

(3) for all $a, b \in P$, if $a \in A(b)^\star$ and $b \in A(a)$, then $\{x \in P : b \in A(a+x)^\star\} \in p$; and

(4) for every $C \in p$ with $C \subseteq P$, every $n \in \mathbb{N}$, and every $g : \{1,2,\ldots,n\} \to S$, if g has property (†), then $\{x \in S : g^\frown x$ has property (†)$\} \in p$.

Proof. Pick $P \in p$ and $A : P \to p$ as guaranteed by Lemma 2.2. Then conclusions (1), (2), and (3) hold. To verify conclusion (4), let $C \in p$ with $C \subseteq P$, let $n \in \mathbb{N}$, and let $g : \{1,2,\ldots,n\} \to S$ such that g has property (†). Let $B = FS(\langle g(t)\rangle_{t=1}^n)$. For $a, b \in B$, we write $a \perp b$ if there exist disjoint nonempty subsets J and K of $\{1,2,\ldots,n\}$ such that $a = \sum_{t \in J} y_t$ and $b = \sum_{t \in K} y_t$. Let

$$\begin{aligned} D \;=\; & C^\star \cap \textstyle\bigcap_{a \in B} \big((-a + C^\star) \cap A(a)^\star \cap \{x \in P : a \in A(x)^\star\}\big) \cap \\ & \textstyle\bigcap \{(-a + A(b)^\star) \cap \{x \in P : a \in A(b+x)^\star\} : a, b \in B \text{ and } a \perp b\}\,. \end{aligned}$$

By conclusions (2) and (3) and the fact that g satisfies (†) we have that $D \in p$. It is routine to verify that if $x \in D$, then $g^\frown x$ satifies property (†). □

There is a result for $K_{m,m}$-free graphs which is similar to Lemma 2.2. (The proofs of the results cited from [3] for the case of $K_{m,m}$-free graphs are not nearly so complicated as the ones for K_m-free graphs.)

2.5 Lemma. *Let $(S,+)$ be a cancellative semigroup and let p be an idempotent in βS. Let $m \in \mathbb{N}$ and let G be a graph on S which contains no $K_{m,m}$. Then there exist $P \in p$ and for each $a \in P$, a set $A(a) \in p$, such that, for every $a \in P$*

(1) $A(a) \subseteq \{b \in S : \{a,b\} \notin E(G)\}$;

(2) for every $b \in P$, $b \in A(a) \Leftrightarrow a \in A(b)$;

(3) $a \in A(a)$;

(4) $A(a)^\star = A(a)$;

(5) $\{x \in -a + P : x \in A(a+x)\} \in p$; and

(6) $\{x \in -a + P : b + x \in A(a+x)\} \in p$ for every $b \in A(a)$.

Proof. For $a \in S$, let $C(a)$ be as defined in [3, Definitions 4.4 and 4.8] and let $Q = \{a \in S : a \in C(a)\,, C(a) \in p\,,$ and $\{x \in S : x \in C(a+x)\} \in p\}$. Then by [3, Lemmas 4.7, 4.12, and 4.13], $Q \in p$. Let $P = Q^\star$ and for $a \in P$, let $A(a) = C(a)$. By the definition of $A(a)$, statement (1) holds and by the definition of P, statements (3), and (5) hold. Statements (2), (4), and (6) hold by [3, Lemmas 4.9, 4.10, and 4.11]. □

2.6 Theorem. *Let $m \in \mathbb{N}$, let $(S,+)$ be an infinite cancellative semigroup, let G be a graph on S which contains no $K_{m,m}$, and let p be an idempotent in βS. There exist $P \in p$ and $A : P \to p$ such that*

(1) $A(a) \subseteq \{b \in S : \{a,b\} \notin E(G)\}$;
(2) for every $b \in P$, $b \in A(a) \Leftrightarrow a \in A(b)$;
(3) $a \in A(a)$;
(4) $A(a)^\star = A(a)$;
(5) $\{x \in -a + P : x \in A(a+x)\} \in p$;
(6) $\{x \in -a + P : b + x \in A(a+x)\} \in p$ for every $b \in A(a)$; and
(7) for every $C \in p$ with $C \subseteq P$, every $n \in \mathbb{N}$, and every $g : \{1,2,\ldots,n\} \to S$, if g has property (††), then $\{x \in S : g^\frown x \text{ has property (††)}\} \in p$.

Proof. Pick $P \in p$ and $A : P \to p$ as guaranteed by Lemma 2.5. Then conclusions (1) through (6) hold. To verify conclusion (7), let $C \in p$ with $C \subseteq P$, let $n \in \mathbb{N}$, and let $g : \{1,2,\ldots,n\} \to S$ such that g has property (††). Let $B = FS(\langle g(t)\rangle_{t=1}^n)$. Let $D =$

$$C^\star \cap \textstyle\bigcap_{a \in B} \big((-a + C^\star) \cap A(a) \cap \{x \in C \cap (-a+P) : x \in A(a+x)\}\big)$$
$$\cap \textstyle\bigcap \{\big(-a + A(b)\big) \cap \{x \in C \cap (-b+P) : a + x \in A(b+x)\} : a, b \in B\}\,.$$

By conclusions (2) through (6) and the fact that g satisfies (††), we have that $D \in p$. (Given $a, b \in B$, $a \in A(b) = A(b)^\star$ so $-a + A(b) \in p$.) One easily verifies that if $x \in D$, then $g^\frown x$ satifies property (††), using the fact that $x \in A(y)$ if and only if $y \in A(x)$. □

We are now ready for the major results of this section, which are generalizations of the commutative Central Sets Theorem [10, Theorem 14.11]. One can derive in the same way similar generalizations of the noncommutative Central Sets Theorem [10, Theorem 14.15]. However, those theorems are considerably more complicated to state.

Given a set X, we write $\mathcal{P}_f(X)$ for the set of finite nonempty subsets of X.

2.7 Theorem. *Let $m \in \mathbb{N}$, let $(S,+)$ be an infinite commutative cancellative semigroup, let G be a graph on S which contains no K_m, let C be a central subset of S, and for each $l \in \mathbb{N}$, let $\langle y_{l,n}\rangle_{n=1}^\infty$ be a sequence in S. There exist a sequence $\langle a_n\rangle_{n=1}^\infty$ in S and a sequence $\langle H_n\rangle_{n=1}^\infty$ of finite nonempty subsets of $\mathbb{N}$ such that $\max H_n < \min H_{n+1}$ for each $n \in \mathbb{N}$ and for each $f : \mathbb{N} \to \mathbb{N}$ for which $f(n) \le n$ for each $n \in \mathbb{N}$, one has*

(1) for every finite nonempty $F \subseteq \mathbb{N}$, $\sum_{n \in F} \big(a_n + \sum_{t \in H_n} y_{f(n),t}\big) \in C$ and

(2) for every pair of disjoint finite nonempty subsets F and K of $\mathbb{N}$,

$$\left\{ \sum_{n\in F} \left(a_n + \sum_{t\in H_n} y_{f(n),t}\right), \sum_{n\in K} \left(a_n + \sum_{t\in H_n} y_{f(n),t}\right)\right\} \notin E(G)\,.$$

Proof. Pick a minimal idempotent $p \in \beta S$ such that $C \in p$. Let P and $\{A(a) : a \in P\}$ be as guaranteed for p and G by Theorem 2.4. We may suppose that $C \subseteq P$. Let $\Phi = \{f : f : \mathbb{N} \to \mathbb{N}$ and for all $m \in \mathbb{N}$, $f(m) \leq m\}$.

By [10, Theorem 14.11], we can choose $a_1 \in S$ and $H_1 \in \mathcal{P}_f(\mathbb{N})$ such that $a_1 + \sum_{t\in H_1} y_{1,t} \in C^\star$. We then make the inductive assumption that we have chosen $a_1, a_2, \ldots, a_n \in S$ and $H_1, H_2, \ldots, H_n \in \mathcal{P}_f(\mathbb{N})$ so that

(a) $\max H_i < \min H_{i+1}$ for every $i \in \{1, 2, \ldots, n-1\}$;

(b) for every $J \in \mathcal{P}_f(\{1, 2, \ldots, n\})$ and every $f \in \Phi$, $\sum_{m\in J} \left(a_m + \sum_{t\in H_m} y_{f(m),t}\right) \in C^\star$; and

(c) whenever $J, K \in \mathcal{P}_f(\{1, 2, \ldots, n\})$ with $J \cap K = \emptyset$, $f \in \Phi$, $a = \sum_{m\in J} \left(a_m + \sum_{t\in H_m} y_{f(m),t}\right)$, and $b = \sum_{m\in K} \left(a_m + \sum_{t\in H_m} y_{f(m),t}\right)$, one has $a \in A(b)^\star$.

Notice that by Theorem 2.4(1), induction hypothesis (c) implies conclusion (2) of the theorem, so it suffices to show that we can continue the induction.

For each $f \in \Phi$, let $F_f = \langle a_m + \sum_{t\in H_m} y_{f(m),t}\rangle_{m=1}^n$. Then F_f has property (†). Let $X_f = \{x \in S : F_f{}^\frown x$ has property (†)$\}$. By Theorem 2.4, $X_f \in p$. Let $X = \bigcap_{f\in\Phi} X_f \in p$. By [10, Theorem 14.11] we can choose $a_{n+1} \in S$ and $H_{n+1} \in \mathcal{P}_f(\mathbb{N})$, with $\min(H_{n+1}) > \max(H_n)$, so that $a_{n+1} + \sum_{t\in H_{n+1}} y_{i,t} \in X$ for every $i \in \{1, 2, \ldots, n+1\}$. This shows that the induction can be continued, and so the theorem follows. $\square$

2.8 Theorem. *Let $m \in \mathbb{N}$, let $(S, +)$ be an infinite commutative semigroup, let G be a graph on S which contains no $K_{m,m}$, let C be a central subset of S, and for each $l \in \mathbb{N}$, let $\langle y_{l,n}\rangle_{n=1}^\infty$ be a sequence in S. There exist a sequence $\langle a_n\rangle_{n=1}^\infty$ in S and a sequence $\langle H_n\rangle_{n=1}^\infty$ of finite nonempty subsets of $\mathbb{N}$ such that $\max H_n < \min H_{n+1}$ for each $n \in \mathbb{N}$. Also for each $f : \mathbb{N} \to \mathbb{N}$ such that $f(n) \leq n$ for each $n \in \mathbb{N}$, one has*

(1) for every finite nonempty $F \subseteq \mathbb{N}$, $\sum_{n\in F} \left(a_n + \sum_{t\in H_n} y_{f(n),t}\right) \in C$ and

(2) for every pair of finite nonempty subsets F and K of $\mathbb{N}$,

$$\left\{ \sum_{n\in F} \left(a_n + \sum_{t\in H_n} y_{f(n),t}\right), \sum_{n\in K} \left(a_n + \sum_{t\in H_n} y_{f(n),t}\right)\right\} \notin E(G)\,.$$

Proof. Pick a minimal idempotent $p \in \beta S$ such that $C \in p$. The proof is now nearly identical to that of Theorem 2.7, replacing appeals to Theorem 2.4 by appeals to Theorem 2.6, changing induction hypothesis (c) by dropping the requirement that $J \cap K = \emptyset$, and replacing property (†) by property (††). $\square$

3. Arithmetic Progressions and Image Partition Regular Matrices

We establish in this section several generalizations of Theorem 1.1, among the simplest of which is the fact that if G is a K_m-free graph on $\mathbb{N}$ and $\langle t_n\rangle_{n=1}^{\infty}$ is any sequence in $\mathbb{N}$, then there exist arbitrarily long arithmetic progressions which are independent with respect to G and have increment chosen from $FS(\langle t_n\rangle_{n=1}^{\infty})$.

3.1 Definition. Let S be a set and let x be a "variable" not in S.

(a) Then

$$\begin{aligned} \mathcal{W}(S) &= \{w : w \text{ is a function, } \mathrm{dom}(w)\in\mathcal{P}_f(\mathbb{N}), \text{ and } \mathrm{ran}(w)\subseteq S\}\,, \\ \mathcal{W}(S,x) &= \{w\in\mathcal{W}(S\cup\{x\}) : x\in\mathrm{ran}(w)\}, \text{ and} \\ \mathcal{T}(S) &= \mathcal{W}(S)\cup\mathcal{W}(S,x)\,. \end{aligned}$$

(b) Given $w,v\in\mathcal{T}(S)$ we define $w\cdot v$ by $\mathrm{dom}(w\cdot v)=\mathrm{dom}(w)\cup\mathrm{dom}(v)$ and for $j\in\mathrm{dom}(w\cdot v)$,

$$(w\cdot v)(j) \quad = \quad \begin{cases} w(j) & \text{if } j\in\mathrm{dom}(w) \\ v(j) & \text{if } j\notin\mathrm{dom}(w)\,. \end{cases}$$

Given S, $\mathcal{W}(S)$ is the set of *located words* over S. It is easy to check that $\mathcal{W}(S)$ and $\mathcal{T}(S)$ are semigroups and that $\mathcal{W}(S,x)$ is a right ideal of $\mathcal{T}(S)$. We only really care about the product $w\cdot v$ when $\mathrm{dom}(w)\cap\mathrm{dom}(v)=\emptyset$, in fact when $\max\big(\mathrm{dom}(w)\big)<\min\big(\mathrm{dom}(v)\big)$. An alternative treatment, which was used in [1], is to leave $w\cdot v$ undefined when $\max\big(\mathrm{dom}(w)\big)\geq\min\big(\mathrm{dom}(v)\big)$. In this case $\mathcal{W}(S)$ and $\mathcal{T}(S)$ become *partial semigroups* and $\mathcal{W}(S,x)$ becomes a (partial semigroup) ideal of $\mathcal{T}(S)$.

In the following, we take $\mathcal{T}^0=\{\emptyset\}$. The proof of this lemma uses an old idea of A. Blass which was first used in [1].

3.2 Lemma. *Let S be a set, let $\mathcal{W}=\mathcal{W}(S)$, let $\mathcal{T}=\mathcal{T}(S)$, and for $n\in\mathbb{N}$, let $T_n=\{w\in\mathcal{T} : \min\big(\mathrm{dom}(w)\big)>n\}$. Let $T=\bigcap_{n=1}^{\infty} c\ell_{\beta\mathcal{T}}(T_n)$. Let Θ be a finite set of functions from $\mathcal{T}$ to $\mathcal{W}$ such that for each $\theta\in\Theta$,*

(1) θ is the identity on $\mathcal{W}$;

(2) for all $n\in\mathbb{N}$, $\theta[T_n]\subseteq T_n$; and

(3) for all $w,v\in\mathcal{T}$, if $\max\big(\mathrm{dom}(w)\big)<\min\big(\mathrm{dom}(v)\big)$, then $\theta(w\cdot v)=\theta(w)\cdot\theta(v)$.

Let q be a minimal idempotent in $T\cap\beta\mathcal{W}$. Then there is an idempotent $r\in\beta\big(\mathcal{W}(S,x)\big)\cap T$ such that $\widetilde{\theta}(r)=q$ for each $\theta\in\Theta$, where $\widetilde{\theta}:\beta\mathcal{T}\to\beta\mathcal{W}$ is the continuous extension of θ. Also, for all $Q\in q$, all $R\in r$, and all $\psi:\bigcup_{n=0}^{\infty}\mathcal{T}^n\to r$, there exists a sequence $\langle w_n\rangle_{n=1}^{\infty}$ in R such that $\prod_{n\in F}\theta_n(w_n)\in Q\cup R$ whenever $F\in\mathcal{P}_f(\mathbb{N})$ and $\{\theta_n : n\in F\}\subseteq\Theta\cup\{\iota\}$, *where $\iota : \mathcal{T} \to \mathcal{T}$ is the identity. Further, $w_1 \in \psi(\emptyset)$ and for all n, $w_{n+1} \in \psi(w_1, w_2, \ldots, w_n)$ and* $\max\big(\mathrm{dom}(w_n)\big) < \min\big(\mathrm{dom}(w_{n+1})\big)$.

Proof. For each $n \in \mathbb{N}$, $T_n \cdot T_n \subseteq T_n$ so by [10, Theorem 4.20], T is a subsemigroup of $\beta\mathcal{T}$. By [10, Corollary 4.18], $\beta\big(\mathcal{W}(S,x)\big)$ is a right ideal of $\beta\mathcal{T}$ and so $\beta\big(\mathcal{W}(S,x)\big) \cap T$ is a right ideal of T. Given $n \in \mathbb{N}$ and $w \in \mathcal{W}(S,x) \cap T_n$, let $k = \max\big(\mathrm{dom}(w)\big)$; then $T_k \cdot w \in \mathcal{W}(S,x) \cap T_n$ so by [10, Lemma 14.9] $\beta\big(\mathcal{W}(S,x)\big) \cap T$ is a left ideal of T. Since $\beta\big(\mathcal{W}(S,x)\big) \cap T$ is an ideal of T, we have $K(T) \subseteq \beta\big(\mathcal{W}(S,x)\big)$.

Choose by [10, Theorem 1.60] an idempotent r which is minimal in T with $r \cdot q = q \cdot r = r$. Then $r \in K(T) \subseteq \beta\big(\mathcal{W}(S,x)\big)$. Let any $\theta \in \Theta$ be given. We have by hypotheses (1) and (2) that $\widetilde{\theta}[T] \subseteq T \cap \beta\mathcal{W}$ and that $\widetilde{\theta}$ is the identity on $\beta\mathcal{W}$. We have by hypothesis (3) and [10, Theorem 4.21] that $\widetilde{\theta}$ is a homomorphism on T. Since $q \in \beta\mathcal{W}$, we have that $\widetilde{\theta}(q) = q$. Thus $\widetilde{\theta}(r) \cdot q = \widetilde{\theta}(r \cdot q) = \widetilde{\theta}(r) = \widetilde{\theta}(q \cdot r) = q \cdot \widetilde{\theta}(r)$. Thus, since q is minimal in $T \cap \beta\mathcal{W}$, $\widetilde{\theta}(r) = q$ and thus $\widetilde{\theta}^{-1}[Q] \in r$ for every $Q \in q$.

Let $Q \in q$, let $R \in r$, and let $\psi : \bigcup_{n=0}^{\infty} \mathcal{T}^n \to r$. Since $r \in \beta\big(\mathcal{W}(S,x)\big)$ we may presume that $R \subseteq \mathcal{W}(S,x)$. Let

$$\begin{aligned} Q^\sharp &= \{v \in Q : v^{-1}Q \in q \text{ and } v^{-1}R \in r\} \text{ and let} \\ R^\sharp &= \{w \in R : w^{-1}R \in q \text{ and } w^{-1}R \in r\}\,. \end{aligned}$$

Since $q = q \cdot q$, $\{v \in \mathcal{T} : v^{-1}Q \in q\} \in q$. Since $r = q \cdot r$,

$$\{v \in \mathcal{T} : v^{-1}R \in r\} \in q\,.$$

Thus $Q^\sharp \in q$. Likewise, since $r = r \cdot r = r \cdot q$, $R^\sharp \in r$.

Notice that if $y \in \mathcal{T}$ and $y^{-1}R \in r$, in particular if $y \in Q^\sharp \cup R^\sharp$, then $y^{-1}R^\sharp \in r$. To see this note that

$$\begin{aligned} y^{-1}R^\sharp &= \{z \in \mathcal{T} : yz \in R\,,\ (yz)^{-1}R \in q, \text{ and } (yz)^{-1}R \in r\} \\ &= y^{-1}R \cap \{z \in \mathcal{T} : z^{-1}(y^{-1}R) \in q\} \cap \{z \in \mathcal{T} : z^{-1}(y^{-1}R) \in r\}\,. \end{aligned}$$

Since $y^{-1}R \in r = r \cdot q$, $\{z \in \mathcal{T} : z^{-1}(y^{-1}R) \in q\} \in r$. Since $y^{-1}R \in r = r \cdot r$, $\{z \in \mathcal{T} : z^{-1}(y^{-1}R) \in r\} \in r$.

Also, if $w \in R^\sharp$, then $w^{-1}R \in q = q \cdot q$ so $\{z \in \mathcal{T} : z^{-1}(w^{-1}R) \in q\} \in q$ and $w^{-1}R \in r = q \cdot r$ so $\{z \in \mathcal{T} : z^{-1}(w^{-1}R) \in r\} \in q$. That is, if $w \in R^\sharp$, then $w^{-1}R^\sharp \in q$. Similarly, if $v \in Q^\sharp$, then $v^{-1}Q^\sharp \in q$.

Now we choose the sequence $\langle w_n \rangle_{n=1}^{\infty}$ inductively, choosing

$$w_1 \in R^\sharp \cap \psi(\emptyset) \cap \textstyle\bigcap_{\theta \in \Theta} \theta^{-1}[Q^\sharp]$$

which is a member of r. Now let $n \in \mathbb{N}$ and assume that we have chosen $w_1, w_2, \ldots, w_n$ such that

(i) $\prod_{i \in F} \theta_i(w_i) \in Q^\sharp \cup R^\sharp$ whenever $\emptyset \neq F \subseteq \{1, 2, \ldots, n\}$ and $\{\theta_i : i \in F\} \subseteq \Theta \cup \{\iota\}$;

(ii) $\max\big(\mathrm{dom}(w_i)\big) < \min\big(\mathrm{dom}(w_{i+1})\big)$ for all $i \in \{1,2,\ldots,n-1\}$; and

(iii) $w_{i+1} \in \psi(w_1, w_2, \ldots, w_i)$ for all $i \in \{1,2,\ldots,n-1\}$.

Let

$$B = \big\{ \textstyle\prod_{i\in F} \theta_i(w_i) : \emptyset \neq F \subseteq \{1,2,\ldots,n\} \text{ and } \{\theta_i : i \in F\} \subseteq \Theta \cup \{\iota\}\big\}\,.$$

Notice that since $B \subseteq Q^\sharp \cup R^\sharp$, we have $w^{-1}R^\sharp \in r$ and $w^{-1}(Q^\sharp \cup R^\sharp) \in q$ for every $w \in B$.

Let $k = \max\big(\mathrm{dom}(w_n)\big)$ and let

$$\begin{aligned} D \;=\;\; & R^\sharp \cap T_k \cap \psi(w_1, w_2, \ldots, w_n) \cap \\ & \textstyle\bigcap_{w\in B} w^{-1}R^\sharp \cap \bigcap_{\theta\in\Theta} \theta^{-1}\big[Q^\sharp \cap \bigcap_{w\in B} w^{-1}(Q^\sharp \cup R^\sharp)\big]\,. \end{aligned}$$

Then $D \in r$ so pick $w_{n+1} \in D$. Hypotheses (ii) and (iii) hold directly. To verify hypothesis (i), let $F \subseteq \{1,2,\ldots,n+1\}$ with $n+1 \in F$ and let $\{\theta_i : i \in F\} \subseteq \Theta \cup \{\iota\}$. If $F = \{n+1\}$, we have $w_{n+1} \in R^\sharp$ and $\theta(w_{n+1}) \in Q^\sharp$ for each $\theta \in \Theta$.

Assume that $F \neq \{n+1\}$, let $H = F\backslash\{n+1\}$, and let $w = \prod_{i\in H} \theta_i(w_i)$. Then $w \cdot w_{n+1} \in R^\sharp$ and $W \cdot \theta(w_{n+1}) \in Q^\sharp \cup R^\sharp$ for each $\theta \in \Theta$. $\square$

Given t in a commutative semigroup $(S,+)$ and $b \in \mathbb{N}$, we denote by $b \cdot t$ the sum of t with itself b times.

3.3 Definition. Let $(S,+)$ be a commutative semigroup.

(a) Define $\sigma : \mathcal{W}(S) \to S$ by $\sigma(w) = \sum_{j\in\mathrm{dom}(w)} w(j)$.

(b) Let $i, b, k \in \mathbb{N}$ and for each $i \in \{1,2,\ldots,k\}$, let $\langle t_{i,n}\rangle_{n=1}^\infty$ be a sequence in S. For each $i \in \{1,2,\ldots,k\}$ and each $b \in \mathbb{N}$ define $h_{i,b} : \mathcal{T}(S) \to \mathcal{W}(S)$ as follows. Given $w \in \mathcal{T}(S)$, $\mathrm{dom}\big(h_{i,b}(w)\big) = \mathrm{dom}(w)$ and for $j \in \mathrm{dom}(w)$,

$$h_{i,b}(w)(j) = \begin{cases} w(j) & \text{if } w(j) \neq x \\ b \cdot t_{i,j} & \text{if } w(j) = x\,. \end{cases}$$

The following lemma will be used to strengthen Theorem 1.1. It uses the ingenious proof given in [8, Lemma 4.1].

3.4 Lemma. *Let $(S,+)$ be a commutative cancellative semigroup with an identity 0, which has the property that $bs \neq cs$ whenever $s \in S\backslash\{0\}$ and b and c are distinct positive integers. Let $k, m, l \in \mathbb{N}$ and let G be a graph on S which is K_m-free. Let p be a minimal idempotent in βS and, for each $i \in \{1,2,\ldots,k\}$, let $\langle t_{i,n}\rangle_{n=1}^\infty$ be a sequence in S with the property that $\sum_{n\in F} t_{i,n} \neq 0$ whenever $F \in \mathcal{P}_f(\mathbb{N})$. Let $\mathcal{W} = \mathcal{W}(S)$, let $\mathcal{T} = \mathcal{T}(S)$, and for $n \in \mathbb{N}$, let $T_n = \{w \in \mathcal{T} : \min\big(\mathrm{dom}(w)\big) > n\}$. Let $T = \bigcap_{n=1}^\infty c\ell_{\beta\mathcal{T}}(T_n)$. There exist a minimal idempotent $q \in T \cap \beta\mathcal{W}$ and an idempotent $r \in \beta\big(\mathcal{W}(S,x)\big) \cap T$ such that*

(1) $\widetilde{\sigma}(q) = p$;

(2) for all $i \in \{1,2,\ldots,k\}$ *and all* $b \in \{1,2,\ldots,l\}$, $\widetilde{h_{i,b}}(r) = q$;

(3) for all $Q \in q$, *all* $R \in r$, *and all* $\psi : \bigcup_{n=0}^{\infty} \mathcal{T}^n \to r$, *there exists a sequence* $\langle w_n \rangle_{n=1}^{\infty}$ *in* R *such that*

(a) $\prod_{n\in F} \theta_n(w_n) \in Q \cup R$ *whenever* $F \in \mathcal{P}_f(\mathbb{N})$ *and* $\{\theta_n : n \in F\} \subseteq \big\{h_{i,b} : i \in \{1,2,\ldots,k\}$ *and* $b \in \{1,2,\ldots,l\}\big\} \cup \{\iota\}$,

(b) $w_1 \in \psi(\emptyset)$ *and for all* n, $w_{n+1} \in \psi(w_1, w_2, \ldots, w_n)$ *and* $\max\big(\mathrm{dom}(w_n)\big) < \min\big(\mathrm{dom}(w_{n+1})\big)$; *and*

(4)
$$\bigcap_{i=1}^{k} \big\{w \in \mathcal{W}(S,x) : \big\{\sigma\big(h_{i,b}(w)\big) : b \in \{1,2,\ldots,l\}\big\} \text{ is independent with respect to } G\big\} \in r\,.$$

Proof. Let $\widetilde{\sigma} : \beta\mathcal{W} \to \beta S$ be the continuous extension of σ and let γ be the restriction of $\widetilde{\sigma}$ to $T \cap \beta\mathcal{W}$. We note that for each $n \in \mathbb{N}$, $\sigma[T_n \cap \mathcal{W}] = S$ and consequently $\gamma[T \cap \beta\mathcal{W}] = \beta S$. Next observe that if $w \in \mathcal{W}$, $n = \max\big(\mathrm{dom}(w)\big)$, and $v \in T_n \cap \mathcal{W}$, then $\sigma(w \cdot v) = \sigma(w) + \sigma(v)$ and so, by [10, Theorem 4.21] γ is a homomorphism.

Since γ is a surjective homomorphism, we have by [10, Exercise 1.7.3] that $\gamma[K(T \cap \beta\mathcal{W})] = K(\beta S)$ and thus we may pick some minimal left ideal L of $T \cap \beta\mathcal{W}$ such that $L \cap \gamma^{-1}[\{p\}] \neq \emptyset$. Then $L \cap \gamma^{-1}[\{p\}]$ is a compact subsemigroup of T so pick an idempotent $q \in L \cap \gamma^{-1}[\{p\}]$, and note that q is then minimal in $T \cap \beta\mathcal{W}$.

Let $\Theta = \big\{h_{i,b} : i \in \{1,2,\ldots,k\}$ and $b \in \{1,2,\ldots,l\}\big\}$ and note that Θ satisfies hypotheses (1), (2), and (3) of Lemma 3.2. Pick an idempotent $r \in \beta\big(\mathcal{W}(S,x)\big) \cap T$ as guaranteed by Lemma 3.2. Then conclusions (1), (2), and (3) of the current lemma hold.

Suppose that conclusion (4) fails and let $M = \mathcal{W}(S,x) \setminus \bigcap_{i=1}^{k} \big\{w \in \mathcal{W}(S,x) : \big\{\sigma\big(h_{i,b}(w)\big) : b \in \{1,2,\ldots,l\}\big\}$ is independent with respect to $G\big\}$.

We define $\delta : \mathcal{W}(S,x) \to (\{1,2,\ldots,k\} \times \{1,2,\ldots,l\}^2) \cup \{0\}$ by choosing $\delta(w)$ to be any triple (i,b,c) for which $\big\{\sigma\big(h_{i,b}(w)\big), \sigma\big(h_{i,c}(w)\big)\big\} \in E(G)$ if any such triple exists and letting $\delta(w) = 0$ otherwise. Choose $R \in r$ such that δ is constant on R. We may presume that $R \subseteq M$.

Let $Q = \sigma^{-1}[C]$ and choose a sequence $\langle w_n \rangle_{n=1}^{\infty}$ in R as guaranteed by conclusion (3) for the function ψ which is constantly equal to $\mathcal{W}(S,x)$.

We claim that the constant value of δ is 0. Suppose instead that we have $(i,b,c) \in (\{1,2,\ldots,k\} \times \{1,2,\ldots,l\}^2)$ such that $\delta(w) = (i,b,c)$ for all $w \in R$. Notice that $b \neq c$. Let $w = w_1 \cdot w_2 \cdots w_m$ and, for each $n \in \{1,2,\ldots,m\}$, let

$$v_n = h_{i,b}(w_1) \cdot h_{i,b}(w_2) \cdots h_{i,b}(w_{n-1}) \cdot h_{i,c}(w_n) \cdot h_{i,c}(w_{n+1}) \cdots h_{i,c}(w_m)\,.$$

For $j \in \{1,2,\ldots,m\}$, let $F_j = {w_j}^{-1}[\{x\}]$ and let $H_j = \mathrm{dom}(w_j) \setminus F_j$. Then for each $n \in \{1,2,\ldots,m\}$,

$$\sigma(v_n) = \sum_{j=1}^m \sum_{s\in H_j} w_j(s) + \sum_{j=1}^{n-1} \sum_{s\in F_j} b\cdot t_{i,s} + \sum_{j=n}^m \sum_{s\in F_j} c\cdot t_{i,s}\,.$$

Consequently, since S is cancellative, if $n < n'$, then $\sigma(v_n) \neq \sigma(v_{n'})$. This is true because if $\sigma(v_n) = \sigma(v_{n'})$, then

$$c\cdot(\sum_{j=n}^{n'-1} \sum_{s\in F_j} t_{i,s}) = b\cdot(\sum_{j=n}^{n'-1} \sum_{s\in F_j} t_{i,s})\,,$$

which is impossible by our assumptions about S and the sequence $\langle t_{i,s}\rangle_{s=1}^\infty$.

For each pair $n, n' \in \{1,2,\ldots,m\}$ with $n < n'$ let

$$u_{n,n'} = h_{i,b}(w_1)\cdots h_{i,b}(w_{n-1})\cdot w_n \cdots w_{n'-1}\cdot h_{i,c}(w_{n'})\cdots h_{i,c}(w_m)\,.$$

Then each $u_{n,n'} \in R$ and so $\delta(u_{n,n'}) = (i,b,c)$ and thus

$$\big\{\sigma\big(h_{i,b}(u_{n,n'})\big), \sigma\big(h_{i,c}(u_{n,n'})\big)\big\} \in E(G)\,.$$

But $h_{i,b}(u_{n,n'}) = v_{n'}$ and $h_{i,c}(u_{n,n'}) = v_n$ so that $\{\sigma(v_1),\sigma(v_2),\ldots,\sigma(v_m)\}$ is a K_m in G, a contradiction.

Consequently, δ is constantly 0 on R. Thus, for each $i \in \{1,2,\ldots,k\}$, $\big\{\sigma\big(h_{i,b}(w_1)\big) : b \in \{1,2,\ldots,l\}\big\}$ is independent with respect to G. This contradicts the fact that $w_1 \in M$. $\square$

3.5 Theorem. *Let $(S,+)$ be a commutative cancellative semigroup with an identity 0, which has the property that $bs \neq cs$ whenever $s \in S\backslash\{0\}$ and b and c are distinct positive integers. Let $k,m,l \in \mathbb{N}$ and let G be a graph on S which is K_m-free. Let $C \subseteq S$ be a central set and, for each $i \in \{1,2,\ldots,k\}$, let $\langle t_{i,n}\rangle_{n=1}^\infty$ be a sequence in S with the property that $\sum_{n\in F} t_{i,n} \neq 0$ whenever $F \in \mathcal{P}_f(\mathbb{N})$. Then there exist $a \in S$ and $F \in \mathcal{P}_f(\mathbb{N})$ such that each of the arithmetic progressions*

$$A_i = \big\{a + bd_i : b \in \{1,2,\ldots,l\}\big\}\,,$$

with $d_i = \sum_{n\in F} t_{i,n}$, is contained in C and is independent with respect to G.

Proof. Pick a minimal idempotent $p \in \beta S$ such that $C \in p$. Pick q and r as guaranteed by Lemma 3.4. By conclusion (4) of Lemma 3.4, we may choose $w \in \mathcal{W}(S,x)$ so that for all $i \in \{1,2,\ldots,k\}$, $\big\{\sigma\big(h_{i,b}(w)\big) : b \in \{1,2,\ldots,l\}\big\}$ is independent with respect to G. By conclusions (1) and (3) of Lemma 3.4 we may presume that $h_{i,b}(w) \in \sigma^{-1}[C]$ for each $i \in \{1,2,\ldots,k\}$ and each $b \in \{1,2,\ldots,l\}$.

Let $F = w^{-1}[\{x\}]$, let $H = \mathrm{dom}(w)\backslash F$, and let $a = \sum_{s\in H} w(s)$. For $i \in \{1,2,\ldots,k\}$, let $d_i = \sum_{s\in F} t_{i,s}$. Then for each b, $\sigma\big(h_{i,b}(w)\big) = a + b\cdot d_i$. Further, each $h_{i,b}(w) \in \sigma^{-1}[C]$ so $\big\{a + b\cdot d_i : b \in \{1,2,\ldots,l\}\big\} \subseteq C$. $\square$

3.6 Question. *Can we choose the arithmetic progressions A_i of the preceding theorem so that the common difference of A_i is independent of each element in A_i with respect G?*

Theorem 3.5 can be applied to obtain independent arithmetic progressions in $\mathbb{N}$ with their common differences in arbitrary finite sum sets, and independent geometric progressions in $\mathbb{N}$ with their common ratios in arbitrary finite product sets. We let $\omega = \mathbb{N} \cup \{0\}$.

3.7 Corollary. *Let $k, m, l \in \mathbb{N}$ and let $C \subseteq \mathbb{N}$ be a central set. Let G be a graph on $\mathbb{N}$ which is K_m-free. For each $i \in \{1, 2, \ldots, k\}$, let $\langle t_{i,n} \rangle_{n=1}^{\infty}$ be a sequence in $\mathbb{N}$. Then there exist $F \in \mathcal{P}_f(\mathbb{N})$ and $a \in \mathbb{N}$ such that, for each $i \in \{1, 2, \ldots, k\}$, the arithmetic progression*

$$A_i = \left\{a + bd_i : b \in \{1, 2, \ldots, l\}\right\},$$

with $d_i = \sum_{n \in F} t_{i,n}$, is contained in C and is independent with respect to G.

Proof. We may presume that $l > 1$. For $j \in \{0, 1\}$, let $D_j = \{2^{2n+j}(2t + 1) : n, t \in \omega\}$. Pick a minimal idempotent $p \in \beta\mathbb{N}$ with $C \in p$ and pick $j \in \{0, 1\}$ such that $D_j \in p$. We apply Theorem 3.5 with $S = \omega$ and with C replaced by $C \cap D_j$. This gives the required arithmetic progressions A_i with $a \in \omega$. To see that $a \in \mathbb{N}$, note that for any $i \in \{1, 2, \ldots, k\}$, if $d_i \in D_j$, then $2d_i \notin D_j$ and thus one cannot have $\{0 + d_i, 0 + 2d_i\} \subseteq D_j$. $\square$

3.8 Corollary. *Let $k, m, l \in \mathbb{N}$ and let C be a central set in $(\mathbb{N}, \cdot)$. Let G be a graph on $\mathbb{N}$ which is K_m-free. For each $i \in \{1, 2, \ldots, k\}$, let $\langle t_{i,n} \rangle_{n=1}^{\infty}$ be a sequence in $\mathbb{N} \backslash \{1\}$. Then there exist $F \in \mathcal{P}_f(\mathbb{N})$ and $a \in \mathbb{N}$ such that, for each $i \in \{1, 2, \ldots, k\}$, the geometric progression*

$$B_i = \left\{a{r_i}^b : b \in \{1, 2, \ldots, l\}\right\},$$

with $r_i = \prod_{n \in F} t_{i,n}$, is contained in C and is independent with respect to G.

Proof. We apply Theorem 3.5 with $S = (\mathbb{N}, \cdot)$. This gives the required geometric progressions B_i. $\square$

We now derive a result combining aspects of Corollaries 3.7 and 3.8.

3.9 Theorem. *Let $k, l, m \in \mathbb{N}$. For each $i \in \{1, 2, \ldots, k\}$, let $\langle t_{i,n} \rangle_{n=1}^{\infty}$ be a sequence in $\mathbb{N}$ and let $\langle u_{i,n} \rangle_{n=1}^{\infty}$ be a sequence in $\mathbb{N} \backslash \{1\}$. Let G be a graph on $\mathbb{N}$ which is K_m-free. Then, given any finite coloring of $\mathbb{N}$, there exist $a, b \in \mathbb{N}$ and $F, K \in \mathcal{P}_f(\mathbb{N})$ such that, for each $i \in \{1, 2, \ldots, k\}$, $A_i = \left\{a + cd_i : c \in \{1, 2, \ldots, l\}\right\}$ and $B_i = \left\{b{r_i}^c : c \in \{1, 2, \ldots, l\}\right\}$, where $d_i = \sum_{n \in F} t_{i,n}$ and $r_i = \prod_{n \in K} u_{i,n}$, are independent with respect to G and $\bigcup_{i=1}^{k}(A_i \cup B_i)$ is monochrome. Furthermore, for each $y \in \bigcup_{i=1}^{k} A_i$ and each $z \in \bigcup_{i=1}^{k} B_i$, $\{y, z\} \notin E(G)$.*

Proof. There is an element $p \in \beta\mathbb{N}$ which is a minimal idempotent in $(\beta\mathbb{N}, \cdot)$ and is in the closure of the set of minimal idempotents in $(\beta\mathbb{N}, +)$ by [10, Lemma 17.2]. So every member of p is central in $(\mathbb{N}, +)$ and in $(\mathbb{N}, \cdot)$.

Let a finite coloring of $\mathbb{N}$ be given and pick monochrome $C \in p$. Pick by Theorem 2.4 a set $D \in p$ such that for each $y \in D$,

$$\{z \in \mathbb{N} : \{y, z\} \notin E(G)\} \in p\,.$$

By Corollary 3.7, we can choose $a \in \mathbb{N}$ and $F \in \mathcal{P}_f(\mathbb{N})$ so that, for each $i \in \{1, 2, \ldots, k\}$, A_i is independent with respect to G and $A_i \subseteq C \cap D$, where A_i is as in the statement of the theorem. We can then complete the proof of this theorem by applying Corollary 3.8 with

$$C \cap \bigcap \big\{\{z \in \mathbb{N} : \{y, z\} \notin E(G)\} : y \in \textstyle\bigcup_{i=1}^{k} A_i\big\}$$

in place of C. $\square$

As we have previously remarked, the following theorem extends Theorem 1.1 in much the same way that Deuber's (m, p, c)-sets Theorem [2] was extended in [4].

3.10 Theorem. *Let $(S, +)$ be a commutative cancellative semigroup with an identity 0, which has the property that $bs \neq cs$ whenever $s \in S\backslash\{0\}$ and b and c are distinct positive integers. Let $m \in \mathbb{N}$. For each $l \in \mathbb{N}$ and each $i \in \{1, 2, \ldots, l\}$, let $\langle t_{i,l,n}\rangle_{n=1}^{\infty}$ be a sequence in S such that $\sum_{n\in F} t_{i,l,n} \neq 0$ whenever $F \in \mathcal{P}_f(\mathbb{N})$. Let G be a graph on S which is K_m-free and let C be a central subset of S. Then, for each $l \in \mathbb{N}$, there exist $a_l \in S$ and $F_l \in \mathcal{P}_f(\mathbb{N})$ such that the arithmetic progressions defined by*

$$A_{i,l} = \big\{a_l + b \cdot (\textstyle\sum_{n\in F_l} t_{i,l,n}) : b \in \{1, 2, \ldots, l\}\big\}$$

for $i \in \{1, 2, \ldots, l\}$, have the following properties:

(1) $\sum_{n\in J} g(n) \in C$ for every $g \in \times_{l=1}^{\infty}(\bigcup_{i=1}^{l} A_{i,l})$ and every $J \in \mathcal{P}_f(\mathbb{N})$;

(2) for every $f \in \times_{l=1}^{\infty}\{1, 2, \ldots, l\}$, $\bigcup_{l=1}^{\infty} A_{f(l),l}$ is independent with respect to G; and

(3) for every $g \in \times_{l=1}^{\infty}(\bigcup_{i=1}^{l} A_{i,l})$ and every pair of disjoint sets J and K in $\mathcal{P}_f(\mathbb{N})$, $\{\sum_{n\in J} g(n), \sum_{n\in K} g(n)\} \notin E(G)$.

Proof. Let p be a minimal idempotent in βS for which $C \in p$. Pick $P \in p$ and $A : P \to p$ as guaranteed by Theorem 2.4. We may suppose that $C \subseteq P$. Pick by Theorem 3.5 $a_1 \in S$ and $F_1 \in \mathcal{P}_f(\mathbb{N})$ such that $a_1 + \sum_{n\in F_1} t_{1,1,n} \in C^\star$.

Let $l \in \mathbb{N}$ and assume that we have chosen $a_1, a_2, \ldots, a_l \in S$ and $F_1, F_2, \ldots, F_l \in \mathcal{P}_f(\mathbb{N})$ so that every $g \in \times_{j=1}^{l}(\bigcup_{i=1}^{j} A_{i,j})$ satisfies (†), where $A_{i,j}$ is as in the statement of the theorem.

For $g \in \times_{j=1}^{l}(\bigcup_{i=1}^{j} A_{i,j})$ let $X_g = \{x \in S : g^\frown x$ has property $(\dagger)\}$; then by Theorem 2.4 $X_g \in p$. Let $X = \bigcap\{X_g : g \in \times_{j=1}^{l}(\bigcup_{i=1}^{j} A_{i,j})\}$.

Choose by Theorem 3.5, $a_{l+1} \in S$ and $F_{l+1} \in \mathcal{P}_f(\mathbb{N})$ such that for each $i \in \{1, 2, \ldots, l+1\}$, $A_{i,l+1} \subseteq X$ and $A_{i,l+1}$ is independent with respect to G. Now, given $g \in \times_{j=1}^{l+1}(\bigcup_{i=1}^{j} A_{i,j})\}$, if $h = g_{|\{1,2,\ldots,l\}}$, then $g = h^\frown g(l+1)$ so g has property $(\dagger)$. $\square$

As corollaries we get sequences of independent arithmetic progressions and sequences of independent geometric progressions in $\mathbb{N}$.

3.11 Corollary. *Let $m \in \mathbb{N}$. For each $l \in \mathbb{N}$ and each $i \in \{1, 2, \ldots, l\}$, let $\langle t_{i,l,n}\rangle_{n=1}^{\infty}$ be a sequence in $\mathbb{N}$. Let G be a graph on $\mathbb{N}$ which is K_m-free and let C be a central subset of $(\mathbb{N}, +)$. Then, for each $l \in \mathbb{N}$, there exist $a_l \in \omega$ and $F_l \in \mathcal{P}_f(\mathbb{N})$ such that the arithmetic progressions defined by*

$$A_{i,l} = \big\{a_l + b \cdot (\textstyle\sum_{n \in F_l} t_{i,l,n}) : b \in \{1, 2, \ldots, l\}\big\}$$

for $i \in \{1, 2, \ldots, l\}$, have the following properties:

(1) $\sum_{n \in J} g(n) \in C$ for every $g \in \times_{l=1}^{\infty}(\bigcup_{i=1}^{l} A_{i,l})$ and every $J \in \mathcal{P}_f(\mathbb{N})$;

(2) for every $f \in \times_{l=1}^{\infty}\{1, 2, \ldots, l\}$, $\bigcup_{l=1}^{\infty} A_{f(l),l}$ is independent with respect to G; and

(3) for every $g \in \times_{l=1}^{\infty}(\bigcup_{i=1}^{l} A_{i,l})$ and every pair of disjoint sets J and K in $\mathcal{P}_f(\mathbb{N})$, $\{\sum_{n \in J} g(n), \sum_{n \in K} g(n)\} \notin E(G)$.

Proof. Apply Theorem 3.10 with $S = (\omega, +)$, noting that any set which is central in $(\mathbb{N}, +)$ is also central in $(\omega, +)$. $\square$

3.12 Corollary. *Let $m \in \mathbb{N}$. For each $l \in \mathbb{N}$ and each $i \in \{1, 2, \ldots, l\}$, let $\langle t_{i,l,n}\rangle_{n=1}^{\infty}$ be a sequence in $\mathbb{N}\backslash\{1\}$. Let G be a graph on $\mathbb{N}$ which is K_m-free and let C be a central subset of $(\mathbb{N}, \cdot)$. Then, for each $l \in \mathbb{N}$, there exist $a_l \in S$ and $F_l \in \mathcal{P}_f(\mathbb{N})$ such that the geometric progressions defined by*

$$A_{i,l} = \big\{a_l \cdot (\textstyle\prod_{n \in F_l} t_{i,l,n})^b : b \in \{1, 2, \ldots, l\}\big\}$$

for $i \in \{1, 2, \ldots, l\}$, have the following properties:

(1) $\prod_{n \in J} g(n) \in C$ for every $g \in \times_{l=1}^{\infty}(\bigcup_{i=1}^{l} A_{i,l})$ and every $J \in \mathcal{P}_f(\mathbb{N})$;

(2) for every $f \in \times_{l=1}^{\infty}\{1, 2, \ldots, l\}$, $\bigcup_{l=1}^{\infty} A_{f(l),l}$ is independent with respect to G; and

(3) for every $g \in \times_{l=1}^{\infty}(\bigcup_{i=1}^{l} A_{i,l})$ and every pair of disjoint sets J and K in $\mathcal{P}_f(\mathbb{N})$, $\{\prod_{n \in J} g(n), \prod_{n \in K} g(n)\} \notin E(G)$.

Proof. Apply Theorem 3.10 with $S = (\mathbb{N}, \cdot)$. $\square$

We leave to the reader the formulation of the similar corollaries of Theorems 3.13 and 3.14.

Notice that, because of the negative answer in [3] to Hajnal's question, if $S = (\mathbb{N}, +)$, one cannot ask that $\{\sum_{t\in J} g(t), \sum_{t\in G} g(t)\} \notin E(G)$ whenever J and K are distinct nonempty finite subsets of $\mathbb{N}$ and $g \in (\bigcup_{i=1}^{k} A_{il})^{\mathbb{N}}$. On the other hand, we have a stronger result for $K_{m,m}$-free graphs.

3.13 Theorem. *Let $(S, +)$ be a commutative cancellative semigroup with an identity 0, which has the property that $bs \neq cs$ whenever $s \in S\backslash\{0\}$ and b and c are distinct positive integers. Let $m \in \mathbb{N}$. For each $l \in \mathbb{N}$ and each $i \in \{1, 2, \ldots, l\}$, let $\langle t_{i,l,n}\rangle_{n=1}^{\infty}$ be a sequence in S such that $\sum_{n\in F} t_{i,l,n} \neq 0$ whenever $F \in \mathcal{P}_f(\mathbb{N})$. Let G be a graph on S which is $K_{m,m}$-free and let C be a central subset of S. Then, for each $l \in \mathbb{N}$, there exist $a_l \in S$ and $F_l \in \mathcal{P}_f(\mathbb{N})$ such that the arithmetic progressions defined by*

$$A_{i,l} = \big\{a_l + b \cdot (\textstyle\sum_{n\in F_l} t_{i,l,n}) : b \in \{1, 2, \ldots, l\}\big\}$$

for $i \in \{1, 2, \ldots, l\}$, have the following properties:

(1) $\sum_{n\in J} g(n) \in C$ for every $g \in \times_{l=1}^{\infty}(\bigcup_{i=1}^{l} A_{i,l})$ and every $J \in \mathcal{P}_f(\mathbb{N})$;

(2) for every $f \in \times_{l=1}^{\infty}\{1, 2, \ldots, l\}$, $\bigcup_{l=1}^{\infty} A_{f(l),l}$ is independent with respect to G; and

(3) for every $g \in \times_{l=1}^{\infty}(\bigcup_{i=1}^{l} A_{i,l})$ and every pair of sets J and K in $\mathcal{P}_f(\mathbb{N})$, $\{\sum_{n\in J} g(n), \sum_{n\in K} g(n)\} \notin E(G)$.

Proof. This is identical to the proof of Theorem 3.10, using Theorem 2.6 instead of Theorem 2.4. $\square$

In the following theorem we do not obtain arbitrarily long progressions as we did in Theorem 3.10, but we get large independent structures obtained by translating sums of progressions.

3.14 Theorem. *Let $(S, +)$ be a commutative cancellative semigroup with an identity 0, which has the property that $bs \neq cs$ whenever $s \in S\backslash\{0\}$ and b and c are distinct positive integers. Let $m, k, l \in \mathbb{N}$. For each $i \in \{1, 2, \ldots, k\}$, let $\langle t_{i,n}\rangle_{n=1}^{\infty}$ be a sequence in S such that $\sum_{n\in F} t_{i,n} \neq 0$ whenever $F \in \mathcal{P}_f(\mathbb{N})$. Let G be a graph on S which is K_m-free and let C be a central subset of S. Then, for each $n \in \mathbb{N}$, there exist $a_n \in S$ and $F_n \in \mathcal{P}_f(\mathbb{N})$ such that the arithmetic progressions defined by*

$$A_{i,n} = \big\{a_n + b \cdot (\textstyle\sum_{j\in F_n} t_{i,j}) : b \in \{1, 2, \ldots, l\}\big\}$$

for $i \in \{1, 2, \ldots, k\}$, have the following properties:

(1) $A_{i,n} \subseteq C$ for each $i \in \{1, 2, \ldots, k\}$ and each $n \in \mathbb{N}$;

(2) if $i \in \{1, 2, \ldots, k\}$, J and K are disjoint finite subsets of $\mathbb{N}$ with $K \neq \emptyset$, and $g \in \times_{n=1}^{\infty}(\bigcup_{i=1}^{k} A_{i,n})$, then

$$\textstyle\sum_{t\in J} g(t) + \big\{ \sum_{n\in K} \big(a_n + b \cdot (\sum_{j\in F_n} t_{i,j})\big) : b \in \{1, 2, \ldots, l\}\big\}$$

is independent with respect to G; and

(3) *if J and K are disjoint sets in $\mathcal{P}_f(\mathbb{N})$ and $g \in \times_{n=1}^{\infty}(\bigcup_{i=1}^{k} A_{i,n})$, then* $\{\sum_{t\in J} g(t), \sum_{t\in K} g(t)\} \notin E(G)$.

Proof. Let $\mathcal{T} = \mathcal{T}(S)$ and let $\mathcal{W} = \mathcal{W}(S)$. Let p be a minimal idempotent in βS with $C \in p$. Pick $P \in p$ and $A : P \to p$ as guaranteed by Lemma 2.4. We may presume that $C \subseteq P$. Choose idempotents q and r as guaranteed by Lemma 3.4. Let $Q = \sigma^{-1}[C]$ and let

$$R = \bigcap_{i=1}^{k} \Big(\bigcap_{b=1}^{l} {h_{i,b}}^{-1}\big[\sigma^{-1}[C^\star]\big] \cap \big\{w \in \mathcal{W}(S,x) : \big\{\sigma\big(h_{i,b}(w)\big) : b \in \{1,2,\ldots,l\}\big\} \text{ is independent with respect to } G\big\}\Big)\,.$$

Then by Lemma 3.4 $Q \in q$ and $R \in r$. We now define $\psi : \bigcup_{n=0}^{\infty} \mathcal{T}^n \to r$.

Let $\psi(\emptyset) = R^\star$. Now let $n \in \mathbb{N}$ and let $w_1, w_2, \ldots, w_n \in \mathcal{T}$ be given. Let $L = \times_{t=1}^{n}\big\{h_{i,b}(w_t) : i \in \{1,2,\ldots,k\} \text{ and } b \in \{1,2,\ldots,l\}\big\}$. For each $f \in L$, let $X_f = \{x \in S : (\sigma \circ f)^\frown x \text{ has property } (\dagger)\}$. If, for all $f \in L$, $\big\langle\sigma\big(f(t)\big)\big\rangle_{t=1}^{n}$ has property $(\dagger)$, then let $\psi(w_1, w_2, \ldots, w_n) = \bigcap_{i=1}^{k}\bigcap_{b=1}^{l} {h_{i,b}}^{-1}\big[\sigma^{-1}[\bigcap_{f\in L} X_f]\big\}$, in which case by Lemmas 2.4 and 3.4, $\psi(w_1, w_2, \ldots, w_n) \in r$. Otherwise, let $\psi(w_1, w_2, \ldots, w_n) = \mathcal{W}(S,x)$.

Choose $\langle w_n\rangle_{n=1}^{\infty}$ as guaranteed by Lemma 3.4(3). We claim that for each $n \in \mathbb{N}$ and each

$$f \in \times_{t=1}^{n}\big\{h_{i,b}(w_t) : i \in \{1,2,\ldots,k\} \text{ and } b \in \{1,2,\ldots,l\}\big\}\,,$$

$\big\langle\sigma\big(f(t)\big)\big\rangle_{t=1}^{n}$ has property $(\dagger)$. For $n = 1$ this is the assertion that $\sigma\big(h_{i,b}(w_1)\big) \in C^\star$ for each $i \in \{1,2,\ldots,k\}$ and each $b \in \{1,2,\ldots,l\}$, which is true because $w_1 \in R$. Assume then that the claim is true for n and let $f \in \times_{t=1}^{n+1}\big\{h_{i,b}(w_t) : i \in \{1,2,\ldots,k\} \text{ and } b \in \{1,2,\ldots,l\}\big\}$. Then $w_{n+1} \in \psi(w_1, w_2, \ldots, w_n)$ so if $g = f_{|\{1,2,\ldots,n\}}$ we have $\sigma\big(f(n+1)\big) \in X_g$ so $\sigma \circ f$ has property $(\dagger)$.

For each $n \in \mathbb{N}$, let $F_n = \{t \in \mathbb{N} : w_n(t) = x\}$, let $H_n = \mathrm{dom}(w_n)\backslash F_n$, and let $a_n = \sum_{s\in H_n} w_n(s)$. To verify conclusion (1) first note that for $i \in \{1,2,\ldots,k\}$, $b \in \{1,2,\ldots,l\}$, and $n \in \mathbb{N}$, $a_n + b \cdot (\sum_{j\in F_n} t_{i,j}) = \sigma\big(h_{i,b}(w_n)\big)$. Thus, conclusion (1) holds because for each $n \in \mathbb{N}$ and each $f \in \times_{t=1}^{n}\big\{h_{i,b}(w_t) : i \in \{1,2,\ldots,k\} \text{ and } b \in \{1,2,\ldots,l\}\big\}$, $\big\langle\sigma\big(f(t)\big)\big\rangle_{t=1}^{n}$ has property $(\dagger)$.

To verify conclusion (2), let $i \in \{1,2,\ldots,k\}$, let J and K be finite subsets of $\mathbb{N}$ with $K \neq \emptyset$ and let $g \in \times_{n=1}^{\infty}(\bigcup_{i=1}^{k} A_{i,n})$. For each $n \in J$, we have $g(n) = \sigma(h_{i_n,u_n}(w_n))$ for some $i_n \in \{1,2,\ldots,k\}$ and some $u_n \in \{1,2,\ldots,l\}$. Put $h_n = h_{i_n,u_n}$. For each $n \in K$, put $h_n = \iota$, the identity map from $\mathcal{T}$ to itself. Let $w = \prod_{n\in J\cup K} h_n(w_n)$. If $n \in J$, then $h_n(w_n) \in \mathcal{W}$ and so $h_{i,b}\big(h_n(w_n)\big) = h_n(w_n)$ for every $i \in \{1,2,\ldots,k\}$ and every $b \in \{1,2,\ldots,l\}$. Thus, for each $i \in \{1,2,\ldots,k\}$,

$$\begin{aligned}&\big\{\sigma\big(h_{i,b}(w)\big) : b \in \{1,2,\ldots,l\}\big\} = \\ &\textstyle\sum_{n\in J} g(n) + \{\sum_{n\in K}\big(a_n + b(\sum_{j\in F_n} t_{i,j})\big) : b \in \{1,2,\ldots,l\}\big\}\,.\end{aligned}$$

So (2) follows from the fact that

$$\begin{aligned}w \in \textstyle\bigcap_{i=1}^{k}\big\{w \in \mathcal{W}(S,x) : &\big\{\sigma\big(h_{i,b}(w)\big) : b \in \{1,2,\ldots,l\}\big\} \\ &\text{is independent with respect to } G\big\}\,.\end{aligned}$$

To verify conclusion (3), let J and K be disjoint sets in $\mathcal{P}_f(\mathbb{N})$ and let $g \in \times_{n=1}^{\infty}(\bigcup_{i=1}^{k} A_{i,n})$. Let $n = \max(J \cup K)$ and choose $f \in \times_{t=1}^{n}\big\{h_{i,b}(w_t) : i \in \{1,2,\ldots,k\}$ and $b \in \{1,2,\ldots,l\}\big\}$ such that for all $t \in \{1,2,\ldots,n\}$, $g(t) = \sigma\big(f(t)\big)$. Then conclusion (3) follows from the fact that $\langle\sigma\big(f(t)\big)\rangle_{t=1}^{n}$ has property (†). □

The following terminology is due to Walter Deuber, and the images of these matrices are intimately related with his (m,p,c)-sets, which in turn correspond to solution sets to systems of partition regular homogeneous linear equations. (See [9] or [10, Chapter 15].)

3.15 Definition. A $u \times v$ matrix A with entries from $\mathbb{Z}$ is *image partition regular if and only if* whenever $r \in \mathbb{N}$ and $\mathbb{N}$ is r-colored, there must exist $\vec{x} \in \mathbb{N}^v$ such that the entries of $A\vec{x}$ are monochrome.

Arithmetic progressions are special instances of images of image partition regular matrices. That is, a length l arithmetic progression is the set of entries of

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ \vdots & \vdots \\ 1 & l-1 \end{pmatrix}\begin{pmatrix} a \\ d \end{pmatrix}.$$

The following theorem is analogous to Theorem 2.7.

3.16 Theorem. *Let $m \in \mathbb{N}$, let G be a graph on $\mathbb{N}$ which contains no K_m, and let C be a central subset of $\mathbb{N}$. Let $\langle D_t\rangle_{t=1}^{\infty}$ enumerate the image partition regular matrices, where each D_t is a $u(t) \times v(t)$ matrix. There exists a sequence $\langle\vec{x}_t\rangle_{t=1}^{\infty}$ such that each $\vec{x}_t \in \mathbb{N}^{v(t)}$ and*

(1) for each $F \in \mathcal{P}_f(\mathbb{N})$, if for each $t \in F$, a_t is an entry of $D_t\vec{x}_t$, then $\sum_{t\in F} a_t \in C$ and

(2) for each pair of disjoint sets $F, K \in \mathcal{P}_f(\mathbb{N})$, if for each $t \in F \cup K$, a_t is an entry of $D_t\vec{x}_t$, then $\{\sum_{t\in F} a_t, \sum_{t\in K} a_t\} \notin E(G)$.

Proof. Pick a minimal idempotent $p \in \beta\mathbb{N}$ such that $C \in p$. We may assume that $C \subseteq P$, where P is the member of p guaranteed by Lemma 2.2. Let $\{A(a) : a \in P\}$ be as in Lemma 2.2.

Then $C^\star$ is central so by [10, Theorems 15.5 and 15.24], pick $\vec{x}_1 \in \mathbb{N}^{v(1)}$ such that all entries of $D_1\vec{x}_1$ are in $C^\star$.

Inductively, let $n > 1$ and assume that we have chosen $\vec{x}_t \in \mathbb{N}^{v(t)}$ for each $t \in \{1, 2, \ldots, n-1\}$. For each $t \in \{1, 2, \ldots, n-1\}$, let B_t be the set of entries of $D_t\vec{x}_t$. Let $B = \{\sum_{t\in F} f(t) : F \in \mathcal{P}_f(\{1, 2, \ldots, n-1\})$ and $f \in \times_{t=1}^{n-1} B_t\}$. We write $a \perp b$ if there exist $f \in \times_{t=1}^{n-1} B_t$ and disjoint nonempty $F, K \subseteq \{1, 2, \ldots, n-1\}$ such that $a = \sum_{t\in F} f(t)$ and $b = \sum_{t\in K} f(t)$. We assume that $B \subseteq C^\star$ and that $b \in A(a)^\star$ whenever $a, b \in B$ and $a \perp b$.

Then $B \subseteq C^\star \subseteq P$ and so for each $b \in B$, $-b + C^\star \in p$, $A(b)^\star \in p$, and $\{x \in P : b \in A(x)^\star\} \in p$. Further, given $a, b \in B$ with $a \perp b$, one has $b \in A(a)^\star$ and $a \in A(b)^\star$ so $-a + A(b)^\star \in p$ and $\{x \in P : b \in A(a+x)^\star\} \in p$. Let $E =$

$$C^\star \cap \bigcap_{b\in B}\big(A(b)^\star \cap (-b + C^\star) \cap \{x \in P : b \in A(x)^\star\}\big) \cap$$
$$\bigcap \big\{(-a + A(b)^\star) \cap \{x \in P : b \in A(a + x)^\star\} : a, b \in B \text{ and } a \perp b\big\}.$$

Then $E \in p$, so E is central and thus, again using [10, Theorems 15.5 and 15.24], pick $\vec{x}_n \in \mathbb{N}^{v(n)}$ such that all entries of $D_n\vec{x}_n$ are in E. Let B_n be the set of entries of $D_n\vec{x}_n$.

Let $H = \{\sum_{t\in F} f(t) : F \in \mathcal{P}_f(\{1, 2, \ldots, n-1\})$ and $f \in \times_{t=1}^{n} B_t\}$. Since $B_n \subseteq C^\star \cap \bigcap_{b\in B}(-b + C^\star)$, we have that $H \subseteq C^\star$. To complete the proof, let $f \in \times_{t=1}^{n} B_t$ and let F and K be disjoint nonempty subsets of $\{1, 2, \ldots, n\}$ with $n \in F$, let $a = \sum_{t\in F} f(t)$, and let $b = \sum_{t\in K} f(t)$. We show that $a \in A(b)^\star$ and $b \in A(a)^\star$.

Suppose first that $F = \{n\}$. Then $b \in B$ so $a \in A(b)^\star$ and $b \in A(a)^\star$. Next assume that $F \neq \{n\}$, let $H = F\setminus\{n\}$, and let $c = \sum_{t\in H} f(t)$. Then $b, c \in B$ and $b \perp c$ so $a \in -c + A(b)^\star$ and $b \in A(a + c)^\star$. $\square$

We would like to extend the conclusions of Theorem 3.16 to include the assertion that for each t, the set of entries of $D_t\vec{x}_t$ is an independent set. We are unable to do this. However, we note that either of two assertions (Conjectures 3.19 and 3.20), both of which seem quite likely to be true, would yield our desired conclusion.

In [8] it was announced that the following conjecture ([8, Conjecture 7.2]) has been proved, with the promise that the proof will appear in a subsequent paper.

3.17 Conjecture. *Let $u, v, m \in \mathbb{N}$, let G be a graph on $\mathbb{N}$ which contains no K_m, and let A be a $u \times v$ image partition regular matrix. There exists $\vec{x} \in \mathbb{N}^v$ such that the entries of $A\vec{x}$ are independent.*

Consider now Deuber's Theorem.

3.18 Theorem (Deuber). *Let $u, v, m \in \mathbb{N}$ and let A be a $u \times v$ image partition regular matrix. There exist $l, n \in \mathbb{N}$ and an $l \times n$ image partition regular matrix B such that whenever $\vec{y} \in \mathbb{N}^n$ with $B\vec{y} \in \mathbb{N}^l$ and the entries of $B\vec{y}$ are m-colored, there exists $\vec{x} \in \mathbb{N}^v$ such that the entries of $A\vec{x}$ are (contained in the entries of $B\vec{y}$ and) monochrome.*

Proof. [2, Satz 3.1] and [9, Theorem 3.1]. □

One is thus led naturally to the following.

3.19 Conjecture. *Let $u, v, m \in \mathbb{N}$ and let A be a $u \times v$ image partition regular matrix. There exist $l, n \in \mathbb{N}$ and an $l \times n$ image partition regular matrix B such that whenever $\vec{y} \in \mathbb{N}^n$ with $B\vec{y} \in \mathbb{N}^l$ and G is a K_m-free graph on the entries of $B\vec{y}$, there exists $\vec{x} \in \mathbb{N}^v$ such that the entries of $A\vec{x}$ are (contained in the entries of $B\vec{y}$ and) independent.*

If Conjecture 3.19 is valid, then one can get the desired strengthened version of Theorem 3.16.

As we have previously noted, central sets satisfy very strong combinatorial conclusions. Notice in particular, as a consequence of Theorem 3.5, given any $m \in \mathbb{N}$ and a K_m-free graph G on $\mathbb{N}$, every central set in $\mathbb{N}$ contains arbitrarily long independent arithmetic progressions.

3.20 Conjecture. *Let $u, v, m \in \mathbb{N}$, let G be a K_m-free graph on $\mathbb{N}$, let C be a central subset of $\mathbb{N}$, and let A be a $u \times v$ image partition regular matrix. There exists $\vec{x} \in \mathbb{N}^v$ such that $A\vec{x} \in C^u$ and the entries of $A\vec{x}$ are independent.*

If Conjecture 3.20 is correct, then the proof of Theorem 3.16 can be modified to require that the entries of $D_t\vec{x}_t$ are independent.

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