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# Recent results on partition regularity of infinite matrices

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## Abstract

We survey results obtained in the last ten years on image and kernel partition regularity of infinite matrices.

## 1 Introduction

We let  $\mathbb{N}$  be the set of positive integers and  $\omega = \mathbb{N} \cup \{0\}$ . We shall treat  $u \in \mathbb{N}$  as an ordinal, so that  $u = \{0, 1, \dots, u - 1\}$ . Also  $\omega = \{0, 1, 2, \dots\}$  is the first infinite ordinal. Thus, if  $u, v \in \mathbb{N} \cup \{\omega\}$ , and  $A$  is a  $u \times v$  matrix, the rows and columns of  $A$  will be indexed by  $u = \{i : i < u\}$  and  $v = \{i : i < v\}$ , respectively.

As is standard in Ramsey Theory, a *finite coloring* of a set  $X$  is a function whose domain is  $X$  and whose range is finite. Similarly, a  $\kappa$ -coloring has range with cardinality  $\kappa$ . Given a coloring  $f$  of  $X$ , a subset  $B$  of  $X$  is *monochromatic* if and only if  $f$  is constant on  $B$ .

**Definition 1.1.** Let  $u, v \in \mathbb{N} \cup \{\omega\}$ , let  $A$  be a  $u \times v$  matrix with rational entries and finitely many nonzero entries per row, let  $S$  be a nontrivial subsemigroup of  $(\mathbb{Q}, +)$ , and let  $G$  be the subgroup of  $\mathbb{Q}$  generated by  $S$ .

- (a) The matrix  $A$  is *kernel partition regular over  $S$*  if and only if whenever  $S \setminus \{0\}$  is finitely colored, there exists  $\vec{x} \in (S \setminus \{0\})^v$  such that  $A\vec{x} = \vec{0}$  and the entries of  $\vec{x}$  are monochromatic.
- (b) The matrix  $A$  is *image partition regular over  $S$*  if and only if whenever  $S \setminus \{0\}$  is finitely colored, there exists  $\vec{x} \in (S \setminus \{0\})^v$  such that the entries of  $A\vec{x}$  are monochromatic.
- (c) The matrix  $A$  is *weakly image partition regular over  $S$*  if and only if whenever  $S \setminus \{0\}$  is finitely colored, there exists  $\vec{x} \in G^v$  such that the entries of  $A\vec{x}$  are monochromatic.

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The definition of kernel partition regularity given is the only one that makes sense. However, different choices can be made for image partition regularity and weak image partition regularity, and the reader will find a sampling of these different choices in the references to this paper (unfortunately including some of the references with Hindman as an author). The reader will also find some places where one or the other of these notions is referred to simply as “partition regular”.

Space consideration prevents me from explaining why these notions are interesting. The reader who needs convincing on this point is referred to the introduction to [11].

In 2005 I presented a survey [10] about image and kernel partition regularity of finite and infinite matrices at the *Integers Conference 2005 in Celebration of the 70<sup>th</sup> Birthday of Ron Graham*. The situation with respect to finite matrices was largely settled at that time, and consequently, there have not been many new results dealing with partition regularity of finite matrices published since then. In particular, characterizations of both kernel and image partition regularity were known (and included in [10]). And the relations among kernel and image partition regularity over various subsemigroups of  $(\mathbb{R}, +)$  were largely settled.

In the case of infinite matrices, nothing close to a characterization of kernel or image partition regularity was known then, and that remains true today. (However, the main result of Section 3 completely characterizes kernel partition regularity in terms of image partition regularity.) And there were (and are) many open problems related to relationships among the notions. Progress on these problems is the subject of the current paper.

Throughout this paper, when I write that something “was known” without saying when, the reader may assume I mean “was known when [10] was written”.

In Section 2 we present four examples showing that different suggestions for necessary or sufficient conditions for image or kernel partition regularity do not work.

In Section 3 we present a result showing that, given a matrix  $A$  there is a matrix  $B$  such that, for each nontrivial subsemigroup  $S$  of  $\mathbb{Q}$ ,  $B$  is image partition regular over  $S$  if and only if  $A$  is kernel partition regular over  $S$ . We also present examples showing that two attempts to go in the other direction do not work.

In Section 4 we present a result establishing that if  $R$  and  $S$  are subrings of  $\mathbb{Q}$  with 1 as a member and  $R \setminus S \neq \emptyset$ , then there is a matrix which is kernel partition regular over  $R$  but not kernel partition regular over  $S$ .

As a consequence of the results presented in Sections 3 and 4 one has immediately that if  $R$  and  $S$  are subrings of  $\mathbb{Q}$  with 1 as a member and  $R \setminus S \neq \emptyset$ , then there is a matrix which is image partition regular over  $R$  but not image partition regular over  $S$ . We begin Section 5 with this observation. Section 5 also includes new results showing that an elaborate pattern of implications

among various versions of image partition regularity has no valid implications except those diagramed.

Among the earliest known infinite matrices that are image partition regular over  $\mathbb{N}$  are the Milliken-Taylor matrices. Section 6 consists of some new results about these matrices.

The final section, Section 7, presents some results establishing that certain matrices are or are not image partition regular.

Throughout the paper, we will assume that hypothesized matrices have finitely many nonzero entries in each row. We will follow the custom of denoting the entries of a matrix with a capital letter name by the lower case letter corresponding to that name. Given a semigroup  $S$  we will abbreviate “kernel partition regular over  $S$ ”, “image partition regular over  $S$ ”, and “weakly image partition regular over  $S$ ” by  $\text{KPR}/S$ ,  $\text{IPR}/S$ , and  $\text{WIPR}/S$  respectively.

## 2 In search of necessary or sufficient conditions

Rado in [21] and [22] characterized kernel partition regularity of a finite matrix  $A$  with rational entries over  $\mathbb{N}$ ,  $\mathbb{Z}$ ,  $\mathbb{Q}$ , or  $\mathbb{R}$  via the *columns property*. (That is, if  $S$  is any one of  $\mathbb{N}$ ,  $\mathbb{Z}$ ,  $\mathbb{Q}$ , or  $\mathbb{R}$ , then  $A$  is  $\text{KPR}/S$  if and only if  $S$  satisfies the columns property.)

**Definition 2.1.** Let  $u, v \in \mathbb{N}$  and let  $A$  be a  $u \times v$  matrix with entries from  $\mathbb{Q}$ . Denote the columns of  $A$  by  $\langle \vec{c}_i \rangle_{i=0}^{v-1}$ . The matrix  $A$  satisfies the *columns property* if and only if there exist  $m \in \{1, 2, \dots, v\}$  and a partition  $\langle I_t \rangle_{t=0}^{m-1}$  of  $\{0, 1, \dots, v-1\}$  such that

- (1)  $\sum_{i \in I_0} \vec{c}_i = \vec{0}$  and
- (2) for each  $t \in \{1, 2, \dots, m-1\}$ ,  $\sum_{i \in I_t} \vec{c}_i$  is a linear combination with coefficients from  $\mathbb{Q}$  of  $\{\vec{c}_i : i \in \bigcup_{j=0}^{t-1} I_j\}$ .

The columns property has an obvious extension to infinite matrices.

**Definition 2.2.** Let  $A$  be a countably infinite matrix with entries from  $\mathbb{Q}$  and columns indexed by a set  $J$ . Denote the columns of  $A$  by  $\langle \vec{c}_i \rangle_{i \in J}$ . The matrix  $A$  satisfies the *columns property* if and only if there exists a partition  $\langle I_\sigma \rangle_{\sigma < \mu}$  of  $J$ , where  $\mu \in \mathbb{N} \cup \{\omega\}$ , such that

- (1)  $\sum_{i \in I_0} \vec{c}_i = \vec{0}$  and
- (2) for each  $t \in \mu \setminus \{0\}$ ,  $\sum_{i \in I_t} \vec{c}_i$  is a linear combination with coefficients from  $\mathbb{Q}$  of  $\{\vec{c}_i : i \in \bigcup_{j < t} I_j\}$ .

Note that the sums make sense even if  $I_j$  is infinite, since each row has only finitely many nonzero entries. It has been known that there are infinite matrices

with integer entries satisfying the columns property that are not KPR/ $\mathbb{N}$ , but all previously known examples of matrices KPR/ $\mathbb{N}$  had satisfied the columns property.

In the following theorem,  $B$  is an  $\omega \times (\omega + \omega)$  matrix, so the definition of KPR/ $\mathbb{N}$  requires an obvious adjustment.

**Theorem 2.3.** *Let  $A$  be the  $\omega \times \omega$  matrix such that, for  $i, j \in \omega$ ,*

$$a_{i,j} = \begin{cases} 2 & \text{if } j = i, \\ 1 & \text{if } 2^i \leq j < 2^{i+1}, \\ 0 & \text{otherwise,} \end{cases}$$

and let  $B = \begin{pmatrix} A & -I \end{pmatrix}$ , where  $I$  is the  $\omega \times \omega$  identity matrix. Then  $B$  is KPR/ $\mathbb{N}$  and  $B$  does not satisfy the columns property. In fact, no nonempty set of columns of  $B$  sum to  $\vec{0}$ .

*Proof.* [4, Theorem 2.1]. □

On the other hand, it is shown in [4, Theorem 2.2] that if  $A$  is a countably infinite matrix with integer entries and bounded row sums, then some nonempty set of columns of  $A$  do sum to  $\vec{0}$ .

A version of a problem posed in [12] asked whether, if an infinite matrix  $A$  is IPR/ $\mathbb{N}$ ,  $\langle d_n \rangle_{n=0}^\infty$  is a sequence in  $\mathbb{N}$ , and  $\mathbb{N}$  is finitely colored, must there exist  $\vec{x}$  such that the entries of  $A\vec{x}$  are monochromatic and for each  $n < \omega$ ,  $x_n \equiv 0 \pmod{d_n}$ . In [5, Proposition 5], Barber and Leader give an example showing that the answer is “no” with  $d_n = 2^n$  for each  $n$ .

All known examples of matrices that were KPR/ $\mathbb{N}$  had bounded entries in each column. (More precisely, given a matrix  $A$ , one can first multiply each row by a constant so that the smallest absolute value of a nonzero entry in that row is 1, and then ask whether there is a column with unbounded entries.)

**Theorem 2.4.** *The matrix*

$$A = \begin{pmatrix} 2 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 4 & 0 & 0 & 1 & 1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 8 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & -1 & -1 & -1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

is KPR/ $\mathbb{N}$ .

*Proof.* [2, Theorem 9]. □

### 3 Relations between image and kernel partition regularity

The new result that is the most definitive is a complete characterization of kernel partition regularity in terms of image partition regularity.

**Theorem 3.1.** *Let  $u, v \in \mathbb{N} \cup \{\omega\}$  and let  $A$  be a  $u \times v$  matrix with rational entries. Then there is a  $v \times v$  matrix  $B$  with rational entries such that for each nontrivial subsemigroup  $S$  of  $\mathbb{Q}$ ,*

$$\{\vec{x} \in (S \setminus \{0\})^v : A\vec{x} = \vec{0}\} = \{B\vec{y} : \vec{y} \in (S \setminus \{0\})^v\} \cap (S \setminus \{0\})^v.$$

*Further for each such  $S$ ,  $A$  is KPR/ $S$  if and only if  $B$  is IPR/ $S$ .*

*Proof.* [16, Theorem 2.4]. □

Notice that in Theorem 3.1, the relevant kernel members of  $A$  are exactly equal to the relevant image members of  $B$ .

There is a corresponding result starting with an image partition regular matrix, but it only applies when  $S = \mathbb{Q}$ . In fact, it is shown in [16] that if  $S$  is a nontrivial proper subsemigroup of  $\mathbb{Q}^+ = \{x \in \mathbb{Q} : x > 0\}$  or a nontrivial proper subgroup of  $\mathbb{Q}$ , then there is a  $3 \times 2$  matrix  $B$  which is IPR/ $S$  for which the conclusion of Theorem 3.2 fails.

**Theorem 3.2.** *Let  $u, v \in \mathbb{N} \cup \{\omega\}$ , let  $B$  be a  $u \times v$  matrix with rational entries. Then there exist  $J \subseteq u$  and a  $J \times u$  matrix  $A$  such that*

$$\{\vec{y} \in (\mathbb{Q} \setminus \{0\})^u : A\vec{y} = \vec{0}\} = \{B\vec{x} : \vec{x} \in \mathbb{Q}^v\} \cap (\mathbb{Q} \setminus \{0\})^u.$$

*Further,  $A$  is WIPR/ $\mathbb{Q}$  if and only if  $B$  is KPR/ $\mathbb{Q}$ .*

*Proof.* [16, Theorem 2.8]. □

Given a matrix  $A$  which is IPR/ $\mathbb{N}$  and has linearly dependent rows, there is a naturally associated matrix  $B(A)$  which is KPR/ $\mathbb{N}$ . (This matrix is constructed using the linear dependence among the rows of  $A$ . See [16, Theorem 2.6] for details.) It was known that there exists an infinite matrix  $A$  with rational entries for which  $B(A)$  is KPR/ $\mathbb{N}$  but  $A$  is not IPR/ $\mathbb{N}$ . But what happened if the entries of  $A$  were integers was not known. In [5] Barber and Leader showed that there are matrices  $A_1$  and  $A_2$  with integer entries such that  $A_1$  is IPR/ $\mathbb{N}$ ,  $A_2$  is not IPR/ $\mathbb{N}$ , but  $B(A_2) = B(A_1)$  and so  $B(A_2)$  is KPR/ $\mathbb{N}$ .

## 4 Relations among kernel partition regularity for different subsemigroups of $\mathbb{R}$

It was known ([17, Theorem 2.6]) that there exist infinite matrices which are IPR/ $\mathbb{Q}$  but not IPR/ $\mathbb{N}$ . But the corresponding question for kernel partition regularity remained open. The main motivation for [13] was the question of whether the matrix

$$A = \begin{pmatrix} 1 & 1 & -1 & 0 & 0 & 0 & 0 & \dots \\ \frac{1}{2} & 0 & 0 & 1 & -1 & 0 & 0 & \dots \\ \frac{1}{3} & 0 & 0 & 0 & 0 & 1 & -1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

was KPR/ $\mathbb{Q}$ . (It is trivial that  $A$  is not KPR/ $\mathbb{N}$ .) Unfortunately (from the point of view of trying to answer the question), it is a consequence of the main result of [13] that this matrix is not even KPR/ $\mathbb{R}$ .

A few years later, the question of whether every matrix which is KPR/ $\mathbb{Q}$  must also be KPR/ $\mathbb{N}$  was answered.

**Theorem 4.1.** *The matrix*

$$A = \begin{pmatrix} \frac{1}{2} & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ \frac{1}{4} & 0 & 0 & 1 & 1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & \dots \\ \frac{1}{8} & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & -1 & -1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

is KPR/ $\mathbb{Q}$  but not KPR/ $\mathbb{N}$ . In fact,  $A$  is KPR/ $\mathbb{D}$ , where  $\mathbb{D}$  is the set of dyadic rationals.

*Proof.* [2, Theorem 12]. □

Theorem 4.1 was significantly extended in [3].

**Definition 4.2.** Let  $P$  be the set of primes and let  $F \subseteq P$ . Then

$$\mathbb{G}_F = \{a/b : a \in \mathbb{Z}, b \in \mathbb{N} \text{ and all prime factors of } b \text{ are in } F\}.$$

It is not hard to see that  $\{\mathbb{G}_F : F \subseteq P\}$  is exactly the set of subrings  $R$  of  $\mathbb{Q}$  with  $1 \in R$ .

**Theorem 4.3.** *Let  $\langle d_n \rangle_{n=1}^\infty$  be a sequence in  $\mathbb{Q}$  and let*

$$A = \begin{pmatrix} d_1 & 1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ d_2 & 0 & 0 & 0 & 1 & 1 & 1 & -1 & 0 & 0 & 0 & 0 & \dots \\ d_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & -1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

- (1) Let  $F$  and  $H$  be subsets of  $P$  with  $H \setminus F \neq \emptyset$ , pick  $q \in H \setminus F$ , and for each  $n \in \mathbb{N}$ , let  $d_n = \frac{1}{q^n}$ . Then  $A$  is  $KPR/\mathbb{G}_H$  and  $A\vec{x} = \vec{0}$  has no solutions in  $\mathbb{G}_F$ .
- (2) Enumerate  $P$  as  $\langle p_n \rangle_{n=1}^\infty$ . For each  $n \in \mathbb{N}$ , let  $d_n = \prod_{t=1}^n \frac{1}{p_t}$ . Then  $A$  is  $KPR/\mathbb{Q}$  and  $A\vec{x} = \vec{0}$  has no solutions in  $\mathbb{G}_F$  for any proper subset  $F$  of  $P$ .

*Proof.* [3, Theorems 4.3 and 4.4]. □

## 5 Relations among image partition regularity for different subsemigroups of $\mathbb{R}$

We begin this section with an immediate consequence of results established earlier.

**Theorem 5.1.** (1) Let  $F$  and  $H$  be subsets of  $P$  with  $H \setminus F \neq \emptyset$ . There is a matrix  $B$  such that  $B$  is  $IPR/\mathbb{G}_H$  and no image of  $C$  is contained in  $\mathbb{G}_F$ .

- (2) There is a matrix  $B$  such that  $B$  is  $IPR/\mathbb{Q}$ , and for each proper subset  $F$  of  $\mathbb{Q}$ ,  $B$  has no image contained in  $\mathbb{G}_F$ .

*Proof.* Theorems 3.1 and 4.3. □

In [6] the following two notions of image partition regularity near zero were introduced.

**Definition 5.2.** Let  $S$  be a subsemigroup of  $(\mathbb{R}, +)$  with 0 in the closure of  $S$ , let  $u, v \in \mathbb{N} \cup \{\omega\}$ , and let  $A$  be a  $u \times v$  matrix with entries from  $\mathbb{Q}$ .

- (a) The matrix  $A$  is *image partition regular over  $S$  near zero* (abbreviated  $IPR/S_0$ ) if and only if, whenever  $S \setminus \{0\}$  is finitely colored and  $\delta > 0$ , there exists  $\vec{x} \in S^v$  such that the entries of  $A\vec{x}$  are monochromatic and lie in the interval  $(-\delta, \delta)$ .
- (b) If  $v = \omega$ , then  $A$  is *image partition regular over  $S$  near zero in the strong sense* (abbreviated  $IPR/S_{0s}$ ) if and only if, whenever  $S \setminus \{0\}$  is finitely colored and  $\delta > 0$ , there exists  $\vec{x} \in S^\omega$  such that  $\lim_{n \rightarrow \infty} x_n = 0$  and the entries of  $A\vec{x}$  are monochromatic and lie in the interval  $(-\delta, \delta)$ .

It is trivial that all of the implications diagrammed in Figure 1 hold. (In [6]  $IPR/S$  was defined so that, if  $S$  is a group, the notion is equivalent to what we have defined as  $WIPR/S$ .) Examples were presented in [6] showing that most of the missing implications were not valid in general. There were actually seventeen missing implications, but it was shown that if one had an example of



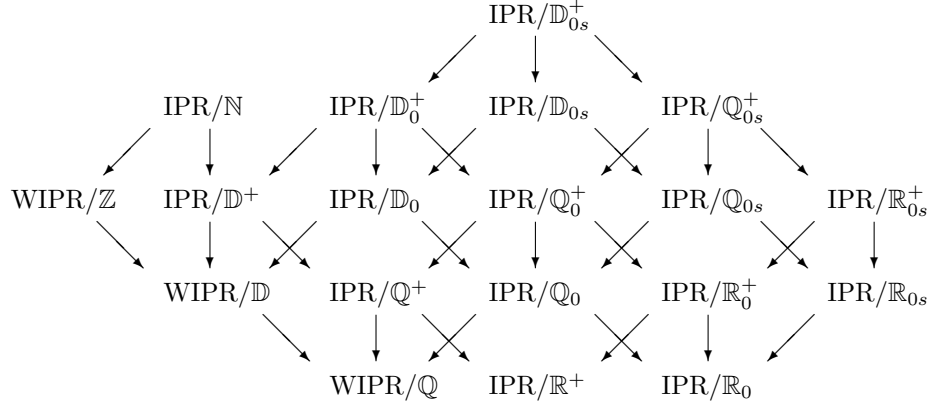


Figure 1: Diagram of Implications

a matrix which was  $\text{IPR}/\mathbb{N}$  but not  $\text{IPR}/\mathbb{R}_0$ , then none of the seventeen missing implications was valid. (For example, it was not known whether every matrix which was  $\text{IPR}/\mathbb{D}$  must be  $\text{IPR}/\mathbb{Q}_0$ . A matrix which is  $\text{IPR}/\mathbb{N}$  but not  $\text{IPR}/\mathbb{R}_0$  is  $\text{IPR}/\mathbb{D}$  and is not  $\text{IPR}/\mathbb{Q}_0$ .) It was shown in [6] that the matrix

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 4 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 4 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & \dots \\ 8 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & \dots \\ 8 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & \dots \\ 8 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & \dots \\ 8 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

is not  $\text{IPR}/\mathbb{R}_0$ , and the question was asked whether  $A$  is  $\text{IPR}/\mathbb{N}$ .

In [2] a different matrix was shown to be  $\text{IPR}/\mathbb{N}$  but not  $\text{IPR}/\mathbb{R}_0$ , establishing that none of the missing implications in Figure 1 is valid. In [1], Barber established that the matrix  $A$  above is  $\text{IPR}/\mathbb{N}$ .

It is easy to see that a matrix which is  $\text{IPR}/\mathbb{Q}$  need not be  $\text{WIPR}/\mathbb{N}$ . For

example, the matrix

$$\begin{pmatrix} 1 & 0 & 0 & \dots \\ \frac{1}{2} & 1 & 0 & \dots \\ \frac{1}{3} & 0 & 1 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

from [17, Theorem 2.6] is easily seen to be IPR/ $\mathbb{Q}$  but not WIPR/ $\mathbb{N}$ . But the question arises as to what happens if the entries of the matrix are assumed to be integers. Barber and Leader showed that the implication is still not valid.

**Theorem 5.3.** *There exists a sequence  $\langle c_n \rangle_{n=1}^\infty$  in  $\mathbb{N}$  such that the matrix*

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 & \dots \\ c_1 & 2 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 4 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 4 & 0 & 0 & 0 & \dots \\ c_2 & 0 & 4 & 4 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 8 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 8 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & 8 & \dots \\ c_3 & 0 & 0 & 0 & 8 & 8 & 8 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

is IPR/ $\mathbb{Q}$  but not WIPR/ $\mathbb{N}$ .

*Proof.* [5, p. 296]. (The definition of image partition regularity they were using gave WIPR/ $\mathbb{Q}$  rather than IPR/ $\mathbb{Q}$ , but it is easy to see that their proof establishes that the matrix is IPR/ $\mathbb{Q}$ .)  $\square$

## 6 Milliken-Taylor matrices

Among the earliest known infinite matrices that are IPR/ $\mathbb{N}$  are the *Milliken-Taylor* matrices. They are so named because the fact that they are IPR/ $\mathbb{N}$  follows easily from the Milliken-Taylor Theorem [19, Theorem 2.2], [23, Lemma 2.2].

**Definition 6.1.** Let  $k \in \omega$  and let  $\vec{a} = \langle a_0, a_1, \dots, a_k \rangle$  be a sequence in  $\mathbb{R}$  such that  $\vec{a} \neq \vec{0}$ . The sequence  $\vec{a}$  is *compressed* if and only if no  $a_i = 0$  and for each  $i \in \{0, 1, \dots, k-1\}$ ,  $a_i \neq a_{i+1}$ . The sequence  $c(\vec{a}) = \langle c_0, c_1, \dots, c_m \rangle$  is the compressed sequence obtained from  $\vec{a}$  by first deleting all occurrences of 0 and then deleting any entry which is equal to its successor. Then  $c(\vec{a})$  is called the *compressed form of  $\vec{a}$* . And  $\vec{a}$  is said to be a *compressed sequence* if  $\vec{a} = c(\vec{a})$ .

**Definition 6.2.** Let  $k \in \omega$ , let  $\vec{a} = \langle a_0, a_1, \dots, a_k \rangle$  be a compressed sequence in  $\mathbb{R} \setminus \{0\}$ , and let  $A$  be an  $\omega \times \omega$  matrix. Then  $A$  is an *MT*( $\vec{a}$ )-matrix if and

only if the rows of  $A$  are all rows  $\vec{r} \in \mathbb{Z}^\omega$  such that  $c(\vec{r}) = \vec{a}$ . The matrix  $A$  is a *Milliken-Taylor matrix* if and only if it is an  $MT(\vec{a})$ -matrix for some  $\vec{a}$ .

Let  $\vec{a}$  and  $\vec{b}$  be compressed sequences in  $\mathbb{Z} \setminus \{0\}$ , let  $A$  be an  $MT(\vec{a})$ -matrix, and let  $B$  be an  $MT(\vec{b})$ -matrix. Two basic facts were known about  $A$  and  $B$ . First,  $A$  and  $B$  are IPR/ $\mathbb{Z}$ . Second, if  $\vec{a}$  is not a rational multiple of  $\vec{b}$ , then there is a partition of  $\mathbb{Z} \setminus \{0\}$  into two cells neither of which contains an image of both  $A$  and  $B$ .

In [7], De and Paul identified what is needed for a subfamily of a semigroup to contain images of all Milliken-Taylor matrices for compressed sequences in  $\mathbb{N}$ . (Restricting to positive entries in  $\vec{a}$  was needed here since they were working with arbitrary semigroups, where  $(-1)x$  may not mean anything.)

**Theorem 6.3.** *Let  $(S, +)$  be an arbitrary (not necessarily commutative) semi-group, let  $\vec{a}$  be a compressed sequence in  $\mathbb{N}$ , and let  $\mathcal{A} \subseteq \mathcal{P}(S)$  such that*

- (a)  $(\forall A \in \mathcal{A})(\forall B \in \mathcal{A})(A \cap B \in \mathcal{A})$ ;
- (b)  $\mathcal{A} \neq \emptyset$  and  $\emptyset \notin \mathcal{A}$ ;
- (c)  $(\forall A \in \mathcal{A})(\forall a \in A)(\exists B \in \mathcal{A})(a + B \subseteq A)$ ; and
- (d)  $(\forall A \in \mathcal{A})(\exists B \in \mathcal{A})(B + B \subseteq A)$ .

*Then whenever  $S$  is finitely colored,  $A \in \mathcal{A}$ , and  $M$  is an  $MT(\vec{a})$  matrix, there exists  $\vec{x} \in S^\omega$  such that the entries of  $M\vec{x}$  are monochromatic and in  $A$ .*

*Proof.* [7, Theorem 2.6]. □

In [8], the same authors extended the image partition regularity of Milliken-Taylor matrices to allow the entries of  $\vec{a}$  to come from  $\mathbb{R} \setminus \{0\}$ .

**Theorem 6.4.** *Let  $\vec{a}$  be a compressed sequence in  $\mathbb{R} \setminus \{0\}$  with  $a_0 > 0$  and let  $M$  be an  $MT(\vec{a})$  matrix. Then  $M$  is IPR/ $\mathbb{R}_0^+$ .*

*Proof.* [8, Theorem 3.6]. □

Quite recently, the ability to separate Milliken-Taylor matrices was extended to the group  $\mathbb{Q}$ , allowing the compressed sequence to come from  $\mathbb{Q} \setminus \{0\}$ . (The fact that one can allow the terms of  $\vec{a}$  to come from  $\mathbb{Q} \setminus \{0\}$  is essentially trivial. The substance is producing the ability to appropriately color  $\mathbb{Q} \setminus \{0\}$ .)

**Theorem 6.5.** *Let  $\vec{a}$  and  $\vec{b}$  be compressed sequences in  $\mathbb{Q} \setminus \{0\}$  such that  $\vec{b}$  is not a multiple of  $\vec{a}$ , let  $A$  be an  $MT(\vec{a})$ -matrix, and let  $B$  be an  $MT(\vec{b})$  matrix. There exists a 2-coloring of  $\mathbb{Q} \setminus \{0\}$  such that there do not exist  $\vec{x}$  and  $\vec{y}$  in  $(\mathbb{Q} \setminus \{0\})^\omega$  with the entries of  $A\vec{x}$  together with the entries of  $B\vec{y}$  monochromatic.*

*Proof.* [18, Corollary 4.5]. □

The conclusion of Theorem 6.5 (minus the information about the number of colors) can be restated as saying that the matrix  $\begin{pmatrix} A & \mathbf{O} \\ \mathbf{O} & B \end{pmatrix}$  is not IPR/ $\mathbb{Q}$ . (The matrix is  $(\omega + \omega) \times (\omega + \omega)$  rather than  $\omega \times \omega$  so Definition 1.1 requires an obvious adjustment.)

Consequently, we find the following theorem to be very surprising. It says that Milliken-Taylor matrices determined by compressed sequences with final term equal to 1 are almost compatible. In this theorem,  $\bar{1}$  and  $\bar{0}$  are the length  $\omega$  column vectors with constant value 1 and 0 respectively. Also  $\mathbf{F}$  is an  $MT(\langle 1 \rangle)$ -matrix; that is a *finite sums* matrix. (The matrix  $B$  is an  $\omega \cdot (m + 2) \times \omega$  matrix.)

**Theorem 6.6.** *Let  $m \in \omega$  and for each  $i \in \{0, 1, \dots, m\}$ , let  $k(i) \in \mathbb{N}$ , let  $\bar{a}_i = \langle a_{i,0}, a_{i,1}, \dots, a_{i,k(i)} \rangle$  be a compressed sequence in  $\mathbb{Z} \setminus \{0\}$  with  $a_{i,k(i)} = 1$ , and let  $M_i$  be an  $MT(\bar{a}_i)$ -matrix. Then*

$$B = \begin{pmatrix} \bar{1} & \bar{0} & \dots & \bar{0} & M_0 \\ \bar{0} & \bar{1} & \dots & \bar{0} & M_1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \bar{0} & \bar{0} & \dots & \bar{1} & M_m \\ \bar{0} & \bar{0} & \dots & \bar{0} & \mathbf{F} \end{pmatrix}$$

is IPR/ $\mathbb{N}$ .

*Proof.* [15, Corollary 6.4]. □

## 7 Some special matrices

The first special kind of matrix with which we are concerned in this section is intimately related to the Milliken-Taylor matrices.

**Definition 7.1.** Let  $k \in \omega$  and let  $\vec{a} = \langle a_0, a_1, \dots, a_{k-1} \rangle$  be a sequence in  $\mathbb{Z} \setminus \{0\}$ . Then  $M(\vec{a})$  is an  $\omega \times \omega$  matrix which has all rows with a single 1 as the only nonzero entry as well as all rows whose nonzero entries are  $a_0, a_1, \dots, a_{k-1}$  in order, each occurring only once.

Note that we are not assuming that  $\vec{a}$  is a compressed sequence. If  $\vec{x} \in \mathbb{N}^\omega$ , then the entries of  $M(\vec{a})\vec{x}$  are the entries of  $\vec{x}$  together with all sums of the form  $\sum_{i=0}^{k-1} a_i x_{n(i)}$ , where  $n(0) < n(1) < \dots < n(k-1)$ . We would like to characterize those sequences  $\vec{a}$  for which  $M(\vec{a})$  is IPR/ $\mathbb{N}$  (and such a characterization is obtained in [9] for finite versions of  $M(\vec{a})$ ). For infinite matrices, we are only able to obtain a characterization in the case that each  $a_i$  is plus or minus a power of a fixed integer.

**Theorem 7.2.** *Let  $k, b \in \mathbb{N}$  and let  $\vec{a} = \langle a_0, a_1, \dots, a_{k-1} \rangle$  be a sequence in  $\mathbb{Z} \setminus \{0\}$  such that for each  $i \in \{0, 1, \dots, k-1\}$ , there is some  $c \in \omega$  such that  $a_i = b^c$  or  $a_i = -b^c$ . Then  $M(\vec{a})$  is IPR/ $\mathbb{N}$  if and only if one of*

- (1)  $a_0 + a_1 + \dots + a_{k-1} = 0$  and  $a_{k-1} = 1$ ;
- (2)  $a_0 + a_1 + \dots + a_{k-1} = 1$ ; or
- (3)  $a_0 = a_1 = \dots = a_{k-1} = 1$ .

*Proof.* [9, Theorem 2.2]. □

For the rest of this section we will be dealing with matrices that have constant row sums. Any such matrix will automatically be image partition regular via a constant sequence. (It is known – see [10, Theorem 4.8] – that a finite matrix  $A$  is IPR/ $\mathbb{N}$  if and only if whenever  $\mathbb{N}$  is finitely colored, there is some injective  $\vec{x}$  such that the entries of  $A\vec{x}$  are monochromatic.)

**Definition 7.3.** Let  $(S, +)$  be a semigroup, let  $\kappa$  be an infinite cardinal, and let  $A$  be a  $\kappa \times \kappa$  matrix. Then  $A$  is *injectively IPR/ $S$*  if and only if, whenever  $S$  is finitely colored, there exists an injective  $\vec{x} \in S^\kappa$  such that the entries of  $A\vec{x}$  are monochromatic.

**Definition 7.4.** For  $k \in \mathbb{N} \setminus \{1\}$  let  $R_k$  be an  $\omega \times \omega$  matrix with entries from  $\omega$  consisting of all rows, the sum of whose entries equals  $k$ .

If  $\vec{x} \in \mathbb{R}^\omega$ , then the entries of  $R_k\vec{x}$  are all sums of the form  $\sum_{i=0}^{k-1} x_{n(i)}$  where  $n(0) \leq n(1) \leq \dots \leq n(k-1)$ .

It is an old question of Owings [20] whether, whenever  $\mathbb{N}$  is 2-colored, there must exist injective  $\vec{x} \in \mathbb{N}^\omega$  with the entries of  $F_2\vec{x}$  monochromatic. And it was known (and not hard to see) that for any  $k \in \mathbb{N} \setminus \{1\}$ ,  $R_k$  is not injectively IPR/ $\mathbb{N}$ . It was not known whether  $R_k$  is injectively IPR/ $\mathbb{Q}$  or injectively IPR/ $\mathbb{R}$ . (The latter is still not known without some special set theoretic assumptions.)

Since the entries of  $R_k$  are nonnegative integers, there is an obvious extension of the definition of injective image partition regularity to any semigroup  $(S, +)$ . We denote the  $n^{\text{th}}$  infinite cardinal by  $\omega_n$ , where as usual, a cardinal is the first ordinal of a given size, and we denote the cardinal of  $\mathbb{R}$  by  $\mathfrak{c}$ .

**Theorem 7.5.** *Let  $n \in \omega$ , let  $k \in \mathbb{N} \setminus \{1\}$ , and let  $G$  be the direct sum of  $\omega_n$  copies of  $\mathbb{Q}$ . Then  $R_k$  is not injectively IPR/ $G$ . In particular, if  $\mathfrak{c} < \omega_{\omega+1}$ , then  $R_k$  is not injectively IPR/ $\mathbb{R}$ .*

*Proof.* [14, Theorem 2.8]. □

**Definition 7.6.** For  $k \in \mathbb{N} \setminus \{1\}$  and an infinite cardinal  $\kappa$ ,  $F_k^\kappa$  is a  $\kappa \times \kappa$  matrix with entries from  $\{0, 1\}$  consisting of all rows with exactly  $k$  occurrences of 1.

Of course for any  $k \in \mathbb{N} \setminus \{1\}$ ,  $F_k^\omega$  is injectively IPR/ $\mathbb{N}$  and therefore IPR/ $\mathbb{R}$ . Also, if  $\kappa > \omega$ , it is impossible for a  $\kappa \times \kappa$  matrix to be injectively IPR/ $\mathbb{Q}$ . We do not know whether for some or all  $k \in \mathbb{N} \setminus \{1\}$ ,  $F_k^{\omega_1}$  is injectively IPR/ $\mathbb{R}$ , but the following result shows that  $F_k^c$  is not. The coloring involved does not even depend on  $k$ .

**Theorem 7.7.** *There is a 2-coloring of  $\mathbb{R}$  such that given any  $k \in \mathbb{N} \setminus \{1\}$ , there is no injective  $\vec{x} \in \mathbb{R}^c$  with the entries of  $(F_k^c)\vec{x}$  monochromatic.*

*Proof.* [14, Theorem 3.2]. □

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<sup>1</sup>Currently available at <http://nhindman.us/preprint.html>.