

# Image partition regularity of matrices over commutative semigroups

Neil Hindman<sup>1</sup>

*Department of Mathematics  
Howard University  
Washington, DC 20059  
USA*

Dona Strauss

*Department of Pure Mathematics  
University of Leeds  
Leeds LS2 9J2  
UK*

---

## Abstract

Let  $(S, +)$  be an infinite commutative semigroup with identity 0. Let  $u, v \in \mathbb{N}$  and let  $A$  be a  $u \times v$  matrix with nonnegative integer entries. If  $S$  is cancellative, let the entries of  $A$  come from  $\mathbb{Z}$ . Then  $A$  is *image partition regular over  $S$*  ( $IPR/S$ ) iff whenever  $S \setminus \{0\}$  is finitely colored, there exists  $\vec{x} \in (S \setminus \{0\})^v$  such that the entries of  $A\vec{x}$  are monochromatic. The matrix  $A$  is *centrally image partition regular over  $S$*  ( $CIPR/S$ ) iff whenever  $C$  is a central subset of  $S$ , there exists  $\vec{x} \in (S \setminus \{0\})^v$  such that  $A\vec{x} \in C^u$ . These notions have been extensively studied for subsemigroups of  $(\mathbb{R}, +)$  or  $(\mathbb{R}, \cdot)$ . We obtain some necessary and some sufficient conditions for  $A$  to be  $IPR/S$  or  $CIPR/S$ . For example, if  $G$  is an infinite divisible group, then  $A$  is  $CIPR/G$  iff  $A$  is  $IPR/\mathbb{Z}$ . If for all  $c \in \mathbb{N}$ ,  $cS \neq \{0\}$  and  $A$  is  $IPR/\mathbb{N}$ , then  $A$  is  $IPR/S$ . If  $S$  is cancellative,  $c \in \mathbb{N}$ , and  $cS = \{0\}$ , we obtain a simple sufficient condition for  $A$  to be  $IPR/S$ . It is well-known that  $A$  is  $IPR/S$  if  $A$  is a first entries matrix with the property that  $cS$  is a central\* subset of  $S$  for every first entry  $c$  of  $A$ . We extend this theorem to first entries matrices whose first entries may not satisfy this condition. We discuss whether, if  $S$  is finitely colored, there exists  $\vec{x} \in (S \setminus \{0\})^v$ , with distinct entries, for which the entries of  $A\vec{x}$  are monochromatic and distinct. Along the way, we obtain several new results about the algebra of  $\beta S$ , the Stone-Čech compactification of the discrete semigroup  $S$ .

---

*Email addresses:* [nhindman@aol.com](mailto:nhindman@aol.com) (Neil Hindman), [d.strauss@hull.ac.uk](mailto:d.strauss@hull.ac.uk) (Dona Strauss)

*URL:* <http://nhindman.us> (Neil Hindman)

<sup>1</sup>This author acknowledges support received from the National Science Foundation (USA) via Grant DMS-1460023.

*Key words:* image partition regularity, Stone-Čech compactification,  
commutative semigroup  
*2010 MSC:* 05D10, 54D35

---

## 1. Introduction

We investigate in this paper the image partition regularity of finite matrices with integer or nonnegative integer entries over commutative semigroups. We observe that this is a subject with important applications in Ramsey Theory, as some classical theorems – such as Shur’s Theorem or van der Waerden’s Theorem – are equivalent to the statement that a certain finite matrix is *IPR*/ $\mathbb{N}$ . We restrict our attention to commutative semigroups because we don’t have any interesting results for noncommutative semigroups. A fundamental tool is that if  $(S, +)$  is a commutative semigroup,  $u, v \in \mathbb{N}$ , the set of positive integers, and  $A$  is a  $u \times v$  matrix with entries from  $\omega = \mathbb{N} \cup \{0\}$ , then the mapping  $\vec{x} \mapsto A\vec{x}$  from  $S^v \rightarrow S^u$  is a homomorphism. (And of course, if  $S$  is cancellative, then the mapping  $x \mapsto 2x$  from  $S$  to  $S$  is a homomorphism if and only if  $S$  is commutative.)

We often restrict the entries of  $A$  to  $\omega$  because for  $x \in S$ ,  $-x$  may not mean anything. However, if  $S$  is cancellative (so that it can be embedded in its group  $G$  of differences) we allow the entries of  $A$  to come from  $\mathbb{Z}$  and, if  $\vec{x} \in S^v$ , we regard  $A\vec{x}$  as having entries in  $G$ . When multiplication by a fraction makes sense, we may allow the entries to come from  $\mathbb{Q}$ .

We shall commonly assume that  $(S, +)$  has an identity 0. If a semigroup, such as  $(\mathbb{N}, +)$ , does not have an identity, one may be adjoined. None of the matrices we consider will have a row or column consisting entirely of zeroes. Such matrices are uninteresting from a Ramsey Theoretic point of view.

**Definition 1.1.** Let  $u, v \in \mathbb{N}$  and let  $A$  be a  $u \times v$  matrix with entries from  $\mathbb{Q}$ . Then  $A$  is *admissible* if and only if it has no row or column consisting entirely of zeroes. Given a commutative semigroup  $S$ , the matrix  $A$  is *appropriate for*  $S$  provided it is admissible and has entries from  $\omega$  if  $S$  is not cancellative and has entries from  $\mathbb{Z}$  if  $S$  is cancellative.

The study of partition regularity of matrices over  $\mathbb{N}$  was begun by R. Rado [11], and later [12] over other subsemigroups of  $\mathbb{R}$ . He characterized those systems of homogeneous linear equations which have the property that whenever  $\mathbb{N}$  is partitioned into finitely many classes (or “finitely colored”), one of these classes contains a solution to the system (or “there is a monochromatic solution”). In terminology due to W. Deuber, the coefficient matrix of such a system is said to be *kernel partition regular*.

**Definition 1.2.** Let  $u, v \in \mathbb{N}$  and let  $A$  be an admissible matrix with entries from  $\mathbb{Q}$ . Then  $A$  is *kernel partition regular over*  $\mathbb{N}$  if and only if whenever  $\mathbb{N}$  is finitely colored, there exists monochromatic  $\vec{x} \in \mathbb{N}^v$  such that  $A\vec{x} = \vec{0}$ .

**Definition 1.3.** Let  $u, v \in \mathbb{N}$  and let  $A$  be a  $u \times v$  matrix with entries from  $\mathbb{Q}$ . Let  $F = \mathbb{Q}$  or  $F = \mathbb{Z}$ . Denote the columns of  $A$  by  $\vec{c}_1, \vec{c}_2, \dots, \vec{c}_v$ . Then  $A$  satisfies the *columns condition over  $F$*  if and only if there exist  $m \in \{1, 2, \dots, v\}$  and a partition  $\{I_1, I_2, \dots, I_m\}$  of  $\{1, 2, \dots, v\}$  into nonempty sets such that

- (a)  $\sum_{i \in I_1} \vec{c}_i = \vec{0}$  and
- (b) for each  $t \in \{2, 3, \dots, m\}$  (if any),  $\sum_{i \in I_t} \vec{c}_i$  is a linear combination with coefficients from  $F$  of  $\{\vec{c}_i : i \in \bigcup_{j=1}^{t-1} I_j\}$ .

**Theorem 1.4.** Let  $u, v \in \mathbb{N}$  and let  $A$  be a  $u \times v$  matrix with entries from  $\mathbb{Q}$ . Then  $A$  is kernel partition regular over  $\mathbb{N}$  if and only if  $A$  satisfies the columns condition over  $\mathbb{Q}$ .

*Proof.* [11, Satz IV]. □

In [2], W. Deuber proved what was known as *Rado's Conjecture*.

**Theorem 1.5** (Deuber). Call a subset  $E$  of  $\mathbb{N}$  large if whenever  $u, v \in \mathbb{N}$  and  $A$  is a  $u \times v$  matrix which is kernel partition regular, there exists  $\vec{x} \in E^v$  such that  $A\vec{x} = \vec{0}$ . If a large set  $E$  is finitely colored, then there is a monochromatic large set.

The proof of Theorem 1.5 used the fact that certain sets, called by Deuber  $(m, p, c)$  sets, are partition regular over  $\mathbb{N}$  in the sense that whenever  $\mathbb{N}$  is finitely colored, there must be a monochromatic  $(m, p, c)$  set. Deuber's  $(m, p, c)$  sets are images of certain matrices which satisfy the *first entries condition*. We follow the convention of denoting the entries of a matrix by the lower case of the capital letter which denotes the matrix.

**Definition 1.6.** Let  $u, v \in \mathbb{N}$  and let  $A$  be an admissible  $u \times v$  matrix with entries from  $\mathbb{Q}$ . Then  $A$  satisfies the *first entries condition* if and only if

- (1) the first (leftmost) nonzero entry of each row is positive and
- (2) if  $i, j \in \{1, 2, \dots, u\}$ ,  $k \in \{1, 2, \dots, v\}$ ,  $a_{ik}$  is the first nonzero entry in row  $i$ , and  $a_{jk}$  is the first nonzero entry in row  $j$ , then  $a_{ik} = a_{jk}$ .

A number  $b$  is a *first entry* of  $A$  if and only if there is some row  $i \in \{1, 2, \dots, v\}$  such that  $b$  is the first nonzero entry in row  $i$ .

We will call a matrix which satisfies the first entries condition a *first entries matrix*.

Notice that a  $u \times v$  matrix  $A$  satisfies the columns condition over  $\mathbb{Q}$  if and only if there exist  $m \in \{1, 2, \dots, v\}$  and a  $v \times m$  first entries matrix  $B$  with entries from  $\mathbb{Q}$  and all first entries equal to 1 such that  $AB = \mathbf{O}$ , where  $\mathbf{O}$  is the  $u \times m$  matrix with all entries equal to 0.

**Definition 1.7.** Let  $u, v \in \mathbb{N}$  and let  $A$  be an admissible  $u \times v$  matrix with entries from  $\mathbb{Q}$ . Then  $A$  is image partition regular over  $\mathbb{N}$  (*IPR/ $\mathbb{N}$* ) if and only if whenever  $\mathbb{N}$  is finitely colored, there exists  $\vec{x} \in \mathbb{N}^v$  such that the entries of  $A\vec{x}$  are monochromatic.

Deuber showed that the first entries matrices that produced his  $(m, p, c)$  sets are  $IPR/\mathbb{N}$ . Rado's conjecture dates from the 1930's and Deuber's proof of Rado's conjecture was published in 1973. But it was not until the publication of [8] in 1993 that any characterizations of matrices that are  $IPR/\mathbb{N}$  were found. Now [10, Theorem 15.24] includes 12 statements equivalent to the assertion that  $A$  is  $IPR/\mathbb{N}$ .

Since then, several investigations into image partition regularity over other semigroups have been made. And unfortunately, the definitions vary. (The first author of this paper is responsible for at least three of these versions. A fact for which he apologizes.)

- (1) In [10],  $IPR/S$  is defined for an arbitrary commutative semigroup  $(S, +)$  and an admissible  $u \times v$  matrix  $A$  with entries from  $\omega$ . There  $A$  is  $IPR/S$  if and only if whenever  $S$  is finitely colored, there exists  $\vec{x} \in (S \setminus \{0\})^v$  with the entries of  $A\vec{x}$  monochromatic.
- (2) In [6],  $IPR/S$  is defined for a subsemigroup of  $(\mathbb{R}, +)$  and an admissible  $u \times v$  matrix  $A$  with entries from  $\mathbb{Q}$ . There  $A$  is  $IPR/S$  if and only if whenever  $S \setminus \{0\}$  is finitely colored, there exists  $\vec{x} \in S^v$  with the entries of  $A\vec{x}$  monochromatic.
- (3) In [9],  $IPR/S$  is defined for a subsemigroup of  $(\mathbb{R}, +)$  and an admissible  $u \times v$  matrix  $A$  with entries from  $\mathbb{Q}$ . There  $A$  is  $IPR/S$  if and only if whenever  $S \setminus \{0\}$  is finitely colored, there exists  $\vec{x} \in (S \setminus \{0\})^v$  with the entries of  $A\vec{x}$  monochromatic.

Notice that from the point of view of Ramsey Theory, one wants to have no zero entries in  $\vec{x}$ . (Consider how uninteresting van der Waerden's Theorem would be, if one had to allow the increment to be 0.)

All three definitions apply to  $S = \mathbb{Z}$ . It is a consequence of [8, Theorem 2.2, (I)  $\Leftrightarrow$  (V)] that definitions (2) and (3) are equivalent for  $S = \mathbb{Z}$ . And if the entries of  $A$  are restricted to  $\omega$ , then all three versions are equivalent for  $S = \mathbb{Z}$ . But, if one allows the entries of  $A$  to come from  $\mathbb{Z}$  (as we shall do in this paper when  $S$  is a cancellative semigroup), then the matrix  $A = \begin{pmatrix} 1 & -1 \\ 2 & -2 \end{pmatrix}$  satisfies definition (1) but not definitions (2) or (3) because it is easy to color  $\mathbb{Z} \setminus \{0\}$  so that  $x$  and  $2x$  are never the same color.

Having said all of that, we introduce the versions we will use in this paper.

**Definition 1.8.** Let  $(S, +)$  be a commutative semigroup with identity 0, let  $u, v \in \mathbb{N}$ , and let  $A$  be a  $u \times v$  matrix which is appropriate for  $S$ .

- (a)  $A$  is *image partition regular over  $S$*  ( $IPR/S$ ) if and only if whenever  $S \setminus \{0\}$  is finitely colored, there exists  $\vec{x} \in (S \setminus \{0\})^v$  such that the entries of  $A\vec{x}$  are monochromatic.
- (b) If  $S$  is cancellative, let  $G$  be its group of differences. Then  $A$  is *weakly image partition regular over  $S$*  ( $WIPR/S$ ) if and only if whenever  $S \setminus \{0\}$

is finitely colored, there exists  $\vec{x} \in (G \setminus \{0\})^v$  such that the entries of  $A\vec{x}$  are monochromatic.

It is shown in [8, Pages 461-462] that the matrix

$$\begin{pmatrix} 1 & -1 \\ 3 & 2 \\ 4 & 6 \end{pmatrix}$$

is  $WIPR/\mathbb{N}$  but not  $IPR/\mathbb{N}$ .

In Section 3 we will obtain sufficient conditions for a matrix to be  $IPR/S$  or  $WIPR/S$ . For example, if for each  $c \in \mathbb{N}$ ,  $cS \neq \{0\}$ , and  $A$  is  $IPR/\mathbb{N}$ , then  $A$  is  $IPR/S$ . (Here  $cS = \{cx : x \in S\}$ .) The results of Section 3 depend on results about *centrally image partition regular* matrices, which we shall present in Section 2.

The results about central sets utilize the algebraic structure of the Stone-Ćech compactification  $\beta S$  of the discrete space  $S$ . We take the points of  $S$  to be the ultrafilters on  $S$ , identifying the points of  $S$  with the principal ultrafilters. Given  $B \subseteq S$ ,  $\bar{B} = \{p \in \beta S : B \in p\}$  and  $\{\bar{B} : B \subseteq S\}$  is a basis for the open sets and a basis for the closed sets of  $\beta S$ . The operation  $+$  on  $S$  extends to  $\beta S$  so that  $\beta S$  is a right topological semigroup (meaning that the function  $\rho_p : \beta S \rightarrow \beta S$  defined by  $\rho_p(q) = q + p$  is continuous for each  $p \in \beta S$ ). Also  $S$  is contained in the topological center of  $\beta S$  (meaning that the function  $\lambda_x : \beta S \rightarrow \beta S$  defined by  $\lambda_x(q) = x + q$  is continuous for each  $x \in S$ ). The reader should be cautioned that  $(\beta S, +)$  is not likely to be commutative. In fact if  $S$  is cancellative, then the center of  $\beta S$  is equal to the center of  $S$ . Given  $p, q \in \beta S$  and  $B \subseteq S$  one has  $B \in p + q$  if and only if  $\{x \in S : -x + B \in q\} \in p$ , where  $-x + B = \{y \in S : x + y \in B\}$ . We write  $S^* = \beta S \setminus S$ .

As does any compact Hausdorff right topological semigroup,  $\beta S$  has a smallest two sided ideal,  $K(\beta S)$ , which is the union of all of the minimal right ideals and the union of all of the minimal left ideals. The intersection of any minimal right ideal with any minimal left ideal is a group. If  $L$  is a minimal left ideal,  $R$  is a minimal right ideal, and  $p$  is the identity of  $L \cap R$ , then  $L \cap R = p + \beta S + p$ .

An idempotent  $p \in \beta S$  is *minimal* if and only if  $p \in K(\beta S)$ . And the idempotent  $p$  is minimal if and only if it is minimal with respect to the ordering of idempotents wherein  $p \leq q$  if and only if  $p = p + q = q + p$ . A subset  $A$  of  $S$  is an *IP set* if and only if it is a member of some idempotent in  $\beta S$ . Equivalently by [10, Theorem 5.12],  $A$  is an IP set if and only if there is some sequence  $\langle x_n \rangle_{n=1}^{\infty}$  in  $S$  such that  $FS(\langle x_n \rangle_{n=1}^{\infty}) \subseteq A$ , where  $FS(\langle x_n \rangle_{n=1}^{\infty}) = \{\sum_{n \in F} x_n : F \in \mathcal{P}_f(\mathbb{N})\}$  and for any set  $X$ ,  $\mathcal{P}_f(X)$  is the set of finite nonempty subsets of  $X$ . See [10, Part I] for an elementary introduction to the algebra of  $\beta S$ .

**Definition 1.9.** Let  $S$  be an infinite commutative semigroup with identity 0. If  $S$  is cancellative, let  $R$  denote the group of differences of  $S$ . Otherwise, let  $R = S$ . Let  $u, v \in \mathbb{N}$  and let  $A$  be a  $u \times v$  matrix which is appropriate for  $S$ . Then  $\vec{x} \mapsto A\vec{x}$  is a homomorphism from  $S^v$  to  $R^u$  which extends to a continuous homomorphism from  $\beta(S^v)$  to  $(\beta R)^u$  by [10, Theorem 4.8]. We denote this continuous extension by  $\tilde{A}$ .

In the following lemma, we state a basic fact which we shall use frequently.

**Lemma 1.10.** *Let  $(S, +)$  be an infinite commutative semigroup with identity 0, let  $u, v \in \mathbb{N}$ , and let  $A$  be a  $u \times v$  matrix which is appropriate for  $S$ . Let  $p$  be an idempotent in  $\beta S$  and let  $\bar{p} = (p, p, p, \dots, p) \in (\beta S)^u$ . Assume that for every  $P \in p$ , there exists  $\vec{x} \in S^v$  such that  $A\vec{x} \in P^u$ . Then  $\tilde{A}^{-1}[\{\bar{p}\}]$  is a compact subsemigroup of  $\beta(S^v)$ . In particular,  $\tilde{A}^{-1}[\{\bar{p}\}]$  contains an idempotent. If  $q \in \tilde{A}^{-1}[\{\bar{p}\}]$ , then, for every  $P \in p$ , there exists  $Q \in q$  such that  $A\vec{x} \in P^u$  for every  $\vec{x} \in Q$ .*

*Proof.* If  $S$  is cancellative, let  $R$  be its group of differences. Otherwise, let  $R = S$ . By assumption,  $\bar{p} \in \text{cl}_{(\beta R)^u} \tilde{A}[S^v] = \tilde{A}[\beta(S^v)]$ , so  $\tilde{A}^{-1}[\{\bar{p}\}] \neq \emptyset$ . Since  $\tilde{A}$  is a homomorphism and  $p$  is an idempotent,  $\tilde{A}^{-1}[\{\bar{p}\}]$  is a compact semigroup. If  $P \in p$ , then  $\bar{P}^u$  is a neighborhood of  $\bar{p}$  so  $\tilde{A}^{-1}[\bar{P}^u]$  is a neighborhood of  $q$  and so there exists  $Q \in q$  such that  $\bar{Q} \subseteq \tilde{A}^{-1}[\bar{P}^u]$ .  $\square$

Central subsets of  $\mathbb{N}$  were introduced by Furstenberg in [4], defined in terms of topological dynamics. It was an idea of V. Bergelson that we might be able to show that a subset of  $\mathbb{N}$  is central if and only if it is a member of a minimal idempotent, and (with the assistance of B. Weiss) we were able to prove this in [1, Corollary 6.12]. In [13], H. Shi and H. Yang established that for an arbitrary semigroup  $S$ , a subset satisfies the dynamical definition of a central set if and only if it is a member of a minimal idempotent in  $\beta S$ .

**Definition 1.11.** Let  $(S, +)$  be a (not necessarily commutative) semigroup.

- (a) A set  $A \subseteq S$  is *central* if and only if  $A$  is a member of a minimal idempotent in  $\beta S$ .
- (b) A set  $A \subseteq S$  is *central\** if and only if  $A$  has nonempty intersection with every central subset of  $S$ .

Equivalently,  $A$  is *central\** if and only if  $A$  is a member of every minimal idempotent in  $\beta S$ .

We present now some results about central sets and some related notions which we will use later in the paper, beginning with two simple results which do not seem to have been noted before.

**Lemma 1.12.** *Let  $(S, +)$  be a commutative cancellative semigroup and let  $G$  be its group of differences. If  $p$  is a minimal idempotent in  $\beta S$ , then  $\beta S + p = \beta G + p$ ,  $\beta G + p$  is a minimal left ideal of  $\beta G$ , and consequently  $p$  is minimal in  $\beta G$ . In particular, any set central in  $S$  is central in  $G$  and  $K(\beta S) = K(\beta G) \cap \beta S$ .*

*Proof.* Let  $p$  be a minimal idempotent in  $\beta S$ . To see that  $\beta S + p = \beta G + p$ , it suffices to show that  $\beta G + p \subseteq \beta S + p$ . For this it in turn suffices to show that  $G + p \subseteq \beta S + p$  because  $\beta G + p = \text{cl}(G + p)$ . So let  $x \in G$  and pick  $s, t \in S$  such that  $x = t - s$ . Let  $q$  be the inverse of  $s + p = p + s + p$  in the group  $p + \beta S + p$ . Then  $s + q + p = q + s + p = p$  so  $q + p = -s + p$  and thus

$-s + p \in \beta S + p$  so that  $x + p = t - s + p \in \beta S + p$  as required. If  $L$  is a left ideal of  $\beta G$  with  $L \subseteq \beta G + p$ , then  $L$  is a left ideal of  $\beta S$  and so  $L = \beta S + p$ . Therefore  $\beta G + p$  is a minimal left ideal of  $\beta G$  and so  $p$  is minimal in  $\beta G$ . To see that  $K(\beta S) = K(\beta G) \cap \beta S$ , pick any minimal idempotent  $p$  in  $\beta S$ . Then  $p \in K(\beta S) \cap K(\beta G)$  so [10, Theorem 1.65] applies.  $\square$

The following two lemmas do not need the assumption that  $S$  is commutative. However, we shall use additive notation because this fits in better with the theorems in this paper.

**Lemma 1.13.** *Let  $(S, +)$  be an arbitrary semigroup with identity 0. If  $E$  is a compact subsemigroup of  $\beta S$  which contains 0 as a minimal idempotent, then  $E$  is a finite group. In particular, if  $S$  is infinite, then 0 is not a minimal idempotent of  $\beta S$ .*

*Proof.*  $E$  is a group, because  $E = 0 + E + 0$ . Now 0 is an isolated point of  $E$ .  $E$  is homogeneous because, for every  $x \in E$ ,  $\rho_x : E \rightarrow E$  is a homeomorphism. So every point of  $E$  is isolated and  $E$  must be finite.  $\square$

**Lemma 1.14.** *Let  $(S, +)$  be an arbitrary semigroup with identity 0. Let  $V$  be a compact right topological semigroup and let  $h : V \rightarrow \beta S$  be a continuous homomorphism. If  $h(q) = 0$  for some  $q \in K(V)$ , then  $h[V]$  is a finite group.*

*Proof.* By [10, Exercise 1.7.3],  $h[K(V)] = K(h[V])$ . So 0 is an idempotent in  $K(h[V])$ .  $\square$

**Definition 1.15.** Let  $(S, +)$  be a (not necessarily commutative) semigroup.

- (a) A set  $A \subseteq S$  is  $IP^*$  if and only if  $A$  is a member of every idempotent in  $\beta S$ .
- (b) A set  $A \subseteq S$  is *piecewise syndetic* if and only if  $\overline{A} \cap K(\beta S) \neq \emptyset$ .

We have taken an algebraic definition of piecewise syndetic. By [10, Theorem 4.40],  $A$  is piecewise syndetic if and only if there exists  $G \in \mathcal{P}_f(S)$  such that for all  $F \in \mathcal{P}_f(S)$ , there exists  $x \in S$  such that  $F + x \subseteq \bigcup_{t \in G} (-t + A)$ .

Notice that if  $(S, +)$  is a commutative group and  $c \in \mathbb{N}$ , then the following theorem applies with  $H = cS$ .

**Theorem 1.16.** *Let  $(S, +)$  be a commutative group and let  $H$  be a subgroup of  $S$ . The following statements are equivalent.*

- (1)  $H$  is  $IP^*$  in  $S$ .
- (2)  $H$  is central\* in  $S$ .
- (3)  $H$  is central in  $S$ .
- (4)  $H$  is piecewise syndetic in  $S$ .

*Proof.* It is trivial that (1) implies (2), (2) implies (3), and (3) implies (4). To see that (4) implies (1), assume that  $H$  is piecewise syndetic in  $S$  and pick  $p \in \overline{H} \cap K(\beta S)$ . Let  $L$  be the minimal left ideal of  $\beta S$  to which  $p$  belongs. To see that  $H$  is IP\* in  $S$ , let  $s$  be an idempotent in  $\beta S$ . Pick a minimal left ideal  $M$  of  $\beta S$  with  $M \subseteq \beta S + s$  and let  $r$  be an idempotent in  $M$ . Let  $R = r + \beta S$ . Then  $R$  is a minimal right ideal of  $\beta S$  by [10, Theorem 1.59], so  $R \cap L$  is a group. Let  $q$  be the identity of  $R \cap L$ . Since  $p \in L = \beta S + q$ , by [10, Lemma 1.30],  $p + q = p$ . Therefore  $H \in p + q$  so  $\{x \in S : -x + H \in q\} \in p$ . Pick  $x \in H$  such that  $-x + H \in q$ . That is,  $H \in q$ . Now  $q \in r + \beta S$  so  $r + q = q$  and therefore  $\{x \in S : -x + H \in q\} \in r$ . Observe that  $\{x \in S : -x + H \in q\} \subseteq H$  so that  $H \in r$ . Since  $r \in \beta S + s$ ,  $r + s = r$ . As we saw above, this implies that  $H \in s$  as required.  $\square$

**Corollary 1.17.** *Let  $(S, +)$  be a commutative cancellative semigroup, let  $G$  be its group of differences, and let  $c \in \mathbb{N}$ . If  $cS$  is piecewise syndetic in  $S$ , then  $cG$  is IP\* in  $G$ .*

*Proof.* Assume  $cS$  is piecewise syndetic in  $S$ . Then by Lemma 1.12,  $\overline{cS} \cap K(\beta G) \neq \emptyset$  so  $cS$  is piecewise syndetic in  $G$  and therefore  $cG$  is piecewise syndetic in  $G$ . By Theorem 1.16,  $cG$  is IP\* in  $G$ .  $\square$

It is a consequence of Theorem 1.20 below that one cannot conclude that  $cS$  is IP\* in  $S$  in Corollary 1.17.

**Definition 1.18.** Let  $(S, +)$  be an infinite commutative semigroup and let  $c \in \mathbb{N}$ . Then  $l_c : S \rightarrow S$  is defined by  $l_c(x) = cx$  and  $\tilde{l}_c : \beta S \rightarrow \beta S$  is the continuous extension of  $l_c$ . For  $q \in \beta S$ ,  $cq = \tilde{l}_c(q)$ .

Note that, for example,  $2q$  is not likely to equal  $q + q$  if  $q \in S^*$ .

**Theorem 1.19.** *Let  $(S, +)$  be an infinite commutative semigroup with identity 0 and let  $c \in \mathbb{N}$ . The following statements are equivalent.*

- (1)  $cS$  is piecewise syndetic.
- (2)  $\{p \in \beta S : cp \in K(\beta S)\} \neq \emptyset$ .
- (3)  $\{p \in \beta S : cp \in K(\beta S)\}$  is an ideal of  $\beta S$ .
- (4)  $\tilde{l}_c[K(\beta S)] \subseteq K(\beta S)$ .
- (5)  $\tilde{l}_c[E(K(\beta S))] \subseteq E(K(\beta S))$ , where  $E(K(\beta S))$  is the set of idempotents in  $K(\beta S)$ .
- (6)  $cS$  is central.

*Proof.* To see that (1) implies (2), pick  $q \in K(\beta S) \cap \overline{cS}$ . Let  $\mathcal{A} = \{l_c^{-1}[A] : A \in q\}$ . We claim that  $\mathcal{A}$  has the finite intersection property, for which it suffices that  $l_c^{-1}[A] \neq \emptyset$  for each  $A \in q$ , and this follows from the fact that  $A \cap cS \in q$ . Pick  $p \in \beta S$  such that  $\mathcal{A} \subseteq p$ . Then  $cp = \tilde{l}_c(p) = q$ .



That (2) implies (3) follows immediately from the fact ([10, Corollary 4.22]) that  $\tilde{l}_c$  is a homomorphism.

That (3) implies (4) follows from the fact that  $K(\beta S) \subseteq \{p \in \beta S : cp \in K(\beta S)\}$ .

Since  $\tilde{l}_c$  is a homomorphism, one has immediately that (4) implies (5).

To see that (5) implies (6), pick an idempotent  $p \in K(\beta S)$ . Then  $\tilde{l}_c(p)$  is an idempotent in  $K(\beta S)$  and  $cS \in \tilde{l}_c(p)$  because  $S \in p$ .

It is trivial that (6) implies (1).  $\square$

A set  $A \subseteq S$  is *syndetic* provided that for every minimal left ideal  $L$  of  $\beta S$ ,  $\overline{A} \cap L \neq \emptyset$ . Equivalently, there exists  $F \in \mathcal{P}_f(S)$  such that  $S \subseteq \bigcup_{x \in F} (-x + A)$ . The set  $A$  is *thick* if and only if for every  $F \in \mathcal{P}_f(S)$ , there exists  $x \in S$  such that  $F + x \subseteq A$ . Thus  $A$  is syndetic if and only if  $S \setminus A$  is not thick. For a diagram showing the relationships among these (and several other) notions of size, see [7, Section 2].

Since every central\* set is syndetic, if  $S$  is a group and  $cS$  is piecewise syndetic, then  $cS$  is syndetic by Theorem 1.16. By way of contrast we have the following.

**Theorem 1.20.** *There is a countable, cancellative, and commutative semigroup  $(S, +)$  with identity 0 such that  $\bigcap_{c=1}^{\infty} cS$  is thick (and therefore central and piecewise syndetic) but  $\bigcup_{c=2}^{\infty} cS$  is not syndetic.*

*Proof.* Let  $T = \bigoplus_{n=1}^{\infty} \mathbb{Q}$ . For  $t \in \mathbb{N}$ , define  $e_t \in T$  by  $e_t(t) = 1$  and  $e_t(i) = 0$  if  $i \neq t$ . For  $x \in T \setminus \{0\}$ , let  $\text{supp}(x) = \{t \in \mathbb{N} : x(t) \neq 0\}$  and, for notational convenience, let  $\text{supp}(0) = \{0\}$ . Let

$$S = \{x \in T : \text{if } t = \max \text{supp}(x), \text{ then } x(t) > 0 \\ \text{and if } t \text{ is odd, then } x(t) \in \mathbb{N}\}.$$

Since  $S$  is a subsemigroup of  $T$  we have immediately that  $S$  is countable, cancellative, and commutative. To see that  $\bigcap_{c=1}^{\infty} cS$  is thick, let  $F \in \mathcal{P}_f(S)$  be given. Pick even  $t \in \mathbb{N}$  such that  $t > \max \text{supp}(x)$  for each  $x \in F$ . We claim that  $F + e_t \subseteq \bigcap_{c=1}^{\infty} cS$ . So let  $c \in \mathbb{N}$  be given and let  $x \in F$ . Then  $t = \max \text{supp}(x + e_t)$  so  $\frac{1}{c}(x + e_t) \in S$ .

Now suppose that  $D = \bigcup_{c=2}^{\infty} cS$  is syndetic and pick  $F \in \mathcal{P}_f(S)$  such that  $S \subseteq \bigcup_{x \in F} (-x + D)$ . Pick odd  $t \in \mathbb{N}$  such that  $t > \max \text{supp}(x)$  for each  $x \in F$ . Pick  $x \in F$  and  $c \in \mathbb{N} \setminus \{1\}$  such that  $x + e_t \in cS$ . Pick  $y \in S$  such that  $x + e_t = cy$ . Then  $t = \max \text{supp}(y)$  so  $(cy)(t) \geq c > 1 = (x + e_t)(t)$ , a contradiction.  $\square$

The equivalence of (1) and (6) in Theorem 1.19 holds in a more general setting.

**Theorem 1.21.** *Let  $(S, +)$  be an infinite semigroup, not necessarily commutative, and let  $V$  be a subsemigroup of  $S$ . Then  $V$  is piecewise syndetic if and only if  $V$  is central.*

*Proof.* The sufficiency is trivial, so assume  $V$  is piecewise syndetic. One has  $\overline{V}$  is a subsemigroup of  $\beta S$  by [10, Exercise 2.3.2]. Since  $V$  is piecewise syndetic,  $\overline{V} \cap K(\beta S) \neq \emptyset$  so by [10, Theorem 1.65]  $K(\overline{V}) = \overline{V} \cap K(\beta S)$  and therefore any idempotent in  $K(\overline{V})$  is minimal in  $\beta S$ .  $\square$

Some of our results in Sections 4 and 5 will have as part of their hypothesis that certain subsets of  $\mathbb{N}$  are closed under multiplication.

**Corollary 1.22.** *Let  $(S, +)$  be an infinite commutative semigroup with identity 0 and let  $c, d \in \mathbb{N}$ . If  $cS$  is piecewise syndetic and  $dS$  is piecewise syndetic, then  $(cd)S$  is piecewise syndetic.*

*Proof.*  $\widetilde{l}_c \circ \widetilde{l}_d$  and  $\widetilde{l}_{cd}$  are continuous functions agreeing on  $S$ , hence on  $\beta S$ . Therefore, using Theorem 1.19,  $(cd)K(\beta S) = c(dK(\beta S)) \subseteq cK(\beta S) \subseteq K(\beta S)$ .  $\square$

**Definition 1.23.** Let  $(S, +)$  be a commutative semigroup with identity 0, let  $u, v \in \mathbb{N}$ , and let  $A$  be a  $u \times v$  matrix which is appropriate for  $S$ .

- (a)  $A$  is *centrally image partition regular over  $S$  (CIPR/ $S$ )* if and only if whenever  $C$  is a central subset of  $S$ , there exists  $\vec{x} \in (S \setminus \{0\})^v$  such that  $A\vec{x} \in C^u$ .
- (b) If  $S$  is cancellative, let  $G$  be its group of differences. Then  $A$  is *centrally weakly image partition regular over  $S$  (CWIPR/ $S$ )* if and only if whenever  $C$  is a central subset of  $S$ , there exists  $\vec{x} \in (G \setminus \{0\})^v$  such that  $A\vec{x} \in C^u$ .

We have seen that if  $(S, +)$  is an infinite semigroup with identity 0, then 0 is not a minimal idempotent of  $\beta S$ . So whenever  $S \setminus \{0\}$  is finitely colored, one of the color classes must be central, so CIPR/ $S$  implies IPR/ $S$ .

It is known [10, Theorem 15.24] that CIPR/ $\mathbb{N}$  and IPR/ $\mathbb{N}$  are equivalent and [6, Theorem 4.13] that CWIPR/ $\mathbb{N}$  and WIPR/ $\mathbb{N}$  (which is the same as IPR/ $\mathbb{Z}$ ) are equivalent. We shall establish in Section 2 conditions on  $S$  that guarantee that if  $A$  is IPR/ $\mathbb{N}$ , then  $A$  is CIPR/ $S$  and if  $A$  is WIPR/ $\mathbb{N}$ , then  $A$  is CWIPR/ $S$ .

In Section 4 we will prove what we call the “ $A\vec{x} + B\vec{y}$  theorem” and derive some of its consequences. Among these consequences are conditions guaranteeing that certain first entries matrices are IPR/ $S$ . It is a well-known fact [10, Theorem 15.5] that a first entries matrix  $A$  with entries from  $\omega$  is IPR/ $S$  if  $cS$  is a central\* subset of  $S$  for every first entry  $c$  of  $A$ . The requirement that  $cS$  is central\* can be very restrictive. For example, in the semigroup  $(\mathbb{N}, \cdot)$ , there is no positive integer  $c \neq 1$  such that  $\{x^c : x \in \mathbb{N}\}$  (the multiplicative analogue of  $cS$ ) is piecewise syndetic. (See Corollary 2.8.)

The proof of [10, Theorem 15.5] can be easily modified to prove that if  $A$  is a first entries matrix which is appropriate for  $S$  and  $p$  is a minimal idempotent in  $\beta S$  such that  $cS \in p$  for every first entry  $c$  of  $A$ , then every member of  $p$  contains an image of  $A$ . We present this result (with a different proof) as part of Corollary 4.9.

It is a consequence of [10, Theorem 15.24] that whenever  $A$  is a  $u \times v$  matrix which is  $IPR/\mathbb{N}$  and  $\mathbb{N}$  is finitely colored, there exist  $\vec{x} \in \mathbb{N}^v$  and  $\vec{y} \in \mathbb{N}^u$  such that  $\vec{y} = A\vec{x}$ , the entries of  $\vec{x}$  are distinct, and the entries of  $\vec{y}$  are monochromatic and distinct. We shall obtain a similar result for  $S$  in Theorem 3.3 under the assumption that  $cS \neq \{0\}$  for all  $c \in \mathbb{N}$ . In Section 5 we restrict our attention to cancellative semigroups and obtain results about conditions that guarantee that one can get images with distinct entries in members of certain idempotents, including cases where one does have  $cS = \{0\}$  for some  $c \in \mathbb{N}$ .

## 2. Centrally image partition regular matrices

We begin this section with a lemma which will allow us to establish that for a central set  $C$ , the ability to find  $\vec{x} \in S^v$  with  $A\vec{x} \in C^u$  suffices to show that  $A$  is  $CIPR/S$ .

**Lemma 2.1.** *Let  $S$  be an infinite commutative semigroup. If  $S$  is cancellative, let  $R$  be its group of differences. Otherwise let  $R = S$ . Assume that either  $N = R$  or  $N = S$ . Let  $p$  be a minimal idempotent in  $\beta S$ . Let  $u, v \in \mathbb{N}$ , let  $A$  be a  $u \times v$  matrix which is appropriate for  $S$ , and assume that for every  $C \in p$ , there exists  $\vec{x} \in N^v$  such that  $A\vec{x} \in C^u$ . View  $A$  as mapping  $N^v$  to  $R^u$ , so that  $\tilde{A} : \beta(N^v) \rightarrow (\beta R)^u$ . Let  $\vec{p} = (p, p, \dots, p) \in (\beta S)^u$ . Let  $q$  be a minimal idempotent in the compact semigroup  $\tilde{A}^{-1}\{\vec{p}\}$ . Then  $q$  is minimal in  $\beta(N^v)$  and  $(N \setminus \{0\})^v \in q$ .*

*Proof.* By Lemma 1.10,  $\tilde{A}^{-1}\{\vec{p}\}$  is a compact subsemigroup of  $\beta(N^v)$ . If  $S$  is cancellative, then by Lemma 1.12,  $p$  is minimal in  $\beta R$ . (And, of course, if  $S$  is not cancellative, then  $p$  is minimal in  $\beta R$ .) By [10, Theorem 2.23],  $\vec{p}$  is minimal in  $(\beta R)^u$ . To see that  $q$  is minimal in  $\beta(N^v)$ , let  $r$  be an idempotent in  $\beta(N^v)$  such that  $r \leq q$ . Since  $\tilde{A}$  is a homomorphism,  $\tilde{A}(r) \leq \tilde{A}(q) = \vec{p}$  so  $\tilde{A}(r) = \vec{p}$  because  $\vec{p}$  is minimal in  $(\beta R)^u$ . Then  $r \in \tilde{A}^{-1}\{\vec{p}\}$  and so  $r = q$  as required.

For  $i \in \{1, 2, \dots, v\}$ , let  $B_i = \{x \in N^v : x_i \neq 0\}$ . We claim each  $B_i \in q$ . So suppose instead we have  $B_i \notin q$  and let  $\tilde{\pi}_i : \beta(N^v) \rightarrow \beta N$  be the continuous extension of the projection  $\pi_i$ . Then  $\tilde{\pi}_i$  is constantly equal to 0 on a member of  $q$  and so  $\tilde{\pi}_i(q) = 0$ . But then by [10, Exercise 1.7.3], 0 is a minimal idempotent in  $\beta N$ , contradicting Lemma 1.13. Therefore  $(N \setminus \{0\})^v = \bigcap_{i=1}^v B_i \in q$ .  $\square$

In Definition 1.23, we defined a matrix  $A$  to be  $CIPR/S$  if and only if whenever  $C$  is a central subset of  $S$ , there exists  $\vec{x} \in (S \setminus \{0\})^v$  such that  $A\vec{x} \in C^u$ . As we mentioned in the introduction, another choice would be to only require that  $\vec{x} \in S^v \setminus \{\vec{0}\}$ . We see now that this choice would be equivalent to the one we made.

**Theorem 2.2.** *Let  $(S, +)$  be an infinite commutative semigroup with identity 0. Let  $u, v \in \mathbb{N}$  and let  $A$  be a  $u \times v$  matrix which is appropriate for  $S$ . Let  $p$  be a minimal idempotent in  $\beta S$ . The following statements are equivalent.*

- (1) *Whenever  $C \in p$ , there exists  $\vec{x} \in S^v \setminus \{\vec{0}\}$  such that  $A\vec{x} \in C^u$ .*

(2) Whenever  $C \in p$ , there exists  $\vec{x} \in (S \setminus \{0\})^v$  such that  $A\vec{x} \in C^u$ .

*Proof.* It is trivial that (2) implies (1). To see that (1) implies (2), assume that (1) holds and let  $C \in p$ . Then every member of  $p$  is central, so by assumption,  $p$  satisfies the hypotheses of Lemma 2.1 with  $N = S$ , and therefore there exists  $\vec{x} \in (S \setminus \{0\})^v$  such that  $A\vec{x} \in C^u$ .  $\square$

A similar situation applies to the notion of  $CWIPR/S$ .

**Theorem 2.3.** *Let  $(S, +)$  be an infinite commutative cancellative semigroup with identity 0 and let  $G$  be its group of differences. Let  $u, v \in \mathbb{N}$  and let  $A$  be a  $u \times v$  matrix which is appropriate for  $S$ . The following statements are equivalent.*

- (1) *Whenever  $C$  is a central subset of  $S$ , there exists  $\vec{x} \in G^v \setminus \{\vec{0}\}$  such that  $A\vec{x} \in C^u$ .*
- (2)  *$A$  is  $CWIPR/S$ . That is, whenever  $C$  is a central subset of  $S$ , there exists  $\vec{x} \in (G \setminus \{0\})^v$  such that  $A\vec{x} \in C^u$ .*

*Proof.* Apply Lemma 2.1 with  $N = R = G$ .  $\square$

By [10, Theorem 15.24(g)], if  $u, v \in \mathbb{N}$  and  $A$  is an admissible  $u \times v$  matrix with entries from  $\mathbb{Q}$ , then  $A$  is  $IPR/\mathbb{N}$  if and only if there exist  $m \in \mathbb{N}$ , an admissible  $v \times m$  matrix  $H$  with entries from  $\omega$ ,  $c \in \mathbb{N}$ , and a  $u \times m$  first entries matrix  $B$  with entries from  $\omega$  and all first entries equal to  $c$  such that  $AH = B$ . We see now that there is a similar characterization of  $IPR/\mathbb{Z}$ , an easy fact that seems not to have been noted before.

**Lemma 2.4.** *Let  $u, v \in \mathbb{N}$  and let  $A$  be a  $u \times v$  matrix which is appropriate for  $\mathbb{Z}$ . Then  $A$  is  $IPR/\mathbb{Z}$  if and only if there exist  $m \in \mathbb{N}$ , an admissible  $v \times m$  matrix  $H$  with entries from  $\mathbb{Z}$ ,  $c \in \mathbb{N}$ , and a  $u \times m$  first entries matrix  $B$  with entries from  $\omega$  and all first entries equal to  $c$  such that  $AH = B$ .*

*Proof.* The sufficiency will follow from Theorem 2.6(c) below, so assume that  $A$  is  $IPR/\mathbb{Z}$ . Then  $A$  is  $WIPR/\mathbb{N}$  so by (I)  $\Rightarrow$  (II) in [8, Theorem 2.2], pick  $t_1, t_2, \dots, t_v$  in  $\mathbb{Q} \setminus \{0\}$  such that

$$\left( \begin{array}{cc} AT & -I \end{array} \right)$$

is kernel partition regular over  $\mathbb{N}$ , where  $I$  is the  $u \times u$  identity matrix and

$$T = \begin{pmatrix} t_1 & 0 & \dots & 0 \\ 0 & t_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & t_v \end{pmatrix}.$$

Pick  $n \in \mathbb{N}$  such that all entries of  $nT$  are integers. By [10, Lemma 15.15] and its proof, pick  $m, d \in \mathbb{N}$  and a  $(u+v) \times m$  first entries matrix  $D$  with entries

from  $\omega$  and all first entries equal to  $d$  such that  $\begin{pmatrix} A(nT) & -nI \end{pmatrix} D = \mathbf{O}$ , where  $\mathbf{O}$  is the  $u \times m$  matrix with all entries equal to 0. Let  $E$  consist of the first  $v$  rows of  $D$  and let  $F$  consist of the last  $u$  rows of  $D$ . Let  $c = nd$ , let  $H = (nT)E$ , and let  $B = nF$ .  $\square$

**Definition 2.5.** Let  $u, v \in \mathbb{N}$  and let  $A$  be an admissible  $u \times v$  matrix with integer entries. Let

$$\gamma(A) = \{c \in \mathbb{N} : (\exists m \in \mathbb{N})(\exists H)(\exists B)(H \text{ is an admissible } v \times m \text{ matrix with entries from } \omega, B \text{ is a } u \times m \text{ first entries matrix with entries from } \omega \text{ and all first entries equal to } c \text{ and } AH = B)\}.$$

and let

$$\gamma'(A) = \{c \in \mathbb{N} : (\exists m \in \mathbb{N})(\exists H)(\exists B)(H \text{ is an admissible } v \times m \text{ matrix with entries from } \mathbb{Z}, B \text{ is a } u \times m \text{ first entries matrix with entries from } \omega \text{ and all first entries equal to } c \text{ and } AH = B)\}.$$

**Theorem 2.6.** Let  $(S, +)$  be an infinite commutative semigroup with identity 0. Let  $u, v \in \mathbb{N}$  and let  $A$  be a  $u \times v$  matrix which is appropriate for  $S$ .

- (a) If there is some  $c \in \gamma(A)$  such that  $cS$  is central\* in  $S$ , then  $A$  is CIPR/ $S$ .
- (b) If  $S$  is cancellative and there is some  $c \in \gamma'(A)$  such that  $cS$  is central\* in  $S$ , then  $A$  is CWIPR/ $S$ .
- (c) If  $S$  is a group and there is some  $c \in \gamma'(A)$  such that  $cS$  is central\* in  $S$ , then  $A$  is CIPR/ $S$ .

*Proof.* (a). Assume that  $c \in \gamma(A)$  and  $cS$  is central\* in  $S$ . Pick  $m, H$ , and  $B$  as guaranteed by the fact that  $c \in \gamma(A)$ . By [10, Theorem 15.5], for each central set  $C$  in  $S$ , there exists  $\vec{y} \in S^m$  such that  $B\vec{y} \in C^u$  so if  $\vec{x} = H\vec{y}$ , then  $A\vec{x} \in C^u$ .

Now let  $C$  be a central set in  $S$  and pick a minimal idempotent  $p$  in  $\beta S$  with  $C \in p$ . Then by Lemma 2.1 with  $N = S$ , there is some  $\vec{x} \in (S \setminus \{0\})^v$  such that  $A\vec{x} \in C^u$ .

The proof of (b) is nearly identical, noting that since the entries of  $H$  are allowed to be negative if  $c \in \gamma'(A)$ , one has  $\vec{x} \in G^v$ , where  $G$  is the group of differences of  $S$ .

Conclusion (c) is an immediate consequence of (b).  $\square$

**Corollary 2.7.** Let  $(S, +)$  be an infinite commutative semigroup with identity 0. Let  $u, v \in \mathbb{N}$  and let  $A$  be a  $u \times v$  matrix which is appropriate for  $S$ . Assume that for every  $c \in \mathbb{N}$ ,  $cS$  is central\* in  $S$ .

- (a) If  $A$  is IPR/ $\mathbb{N}$ , then  $A$  is CIPR/ $S$ .
- (b) If  $S$  is cancellative and  $A$  is IPR/ $\mathbb{Z}$ , then  $A$  is CWIPR/ $S$ .
- (c) If  $S$  is a group and  $A$  is IPR/ $\mathbb{Z}$ , then  $A$  is CIPR/ $S$ .

*Proof.* (a). By [10, Theorem 15.24(g)],  $\gamma(A) \neq \emptyset$ .  
 (b) and (c). By Lemma 2.4,  $\gamma'(A) \neq \emptyset$ . □

The condition of Corollary 2.7 is not satisfied by some familiar semigroups such as  $(\mathbb{N}, \cdot)$ . This is not hard to prove directly, but it follows immediately from the following corollary to Theorem 1.16. We write  $\mathbb{Q}^+$  for  $\{x \in \mathbb{Q} : x > 0\}$ .

**Corollary 2.8.** *Let  $c \in \mathbb{N} \setminus \{1\}$ . Then  $\{x^c : x \in \mathbb{N}\}$  is not piecewise syndetic in  $(\mathbb{N}, \cdot)$  and  $\{x^c : x \in \mathbb{Q}^+\}$  is not piecewise syndetic in  $(\mathbb{Q}^+, \cdot)$ .*

*Proof.* By Corollary 1.17 it suffices to show that  $\{x^c : x \in \mathbb{Q}^+\}$  is not IP\* in  $(\mathbb{Q}^+, \cdot)$ . Suppose instead that  $\{x^c : x \in \mathbb{Q}^+\}$  is IP\* in  $(\mathbb{Q}^+, \cdot)$ . Let  $\langle y_t \rangle_{t=1}^\infty$  be the sequence of primes in increasing order and let  $FP(\langle y_t \rangle_{t=1}^\infty) = \{\prod_{t \in F} y_t : F \text{ is a finite nonempty subset of } \mathbb{N}\}$ . By [10, Lemma 5.11], pick an idempotent  $p$  in  $(\beta\mathbb{Q}^+, \cdot)$  such that  $FP(\langle y_t \rangle_{t=1}^\infty) \in p$ . (Here  $\beta\mathbb{Q}^+$  is the Stone-Ćech compactification of  $\mathbb{Q}^+$  with the discrete topology.) Since also  $\{x^c : x \in \mathbb{Q}^+\} \in p$ , there exist some finite nonempty subset  $F$  of  $\mathbb{N}$  and some  $x \in \mathbb{Q}^+$  such that  $\prod_{t \in F} y_t = x^c$ . Since  $x^c \in \mathbb{N}$ , we must have  $x \in \mathbb{N}$ . But there are no repeated prime factors in  $\prod_{t \in F} y_t = x^c$ , a contradiction. □

We saw in Corollary 2.7(c) that if  $S$  is a commutative group and  $cS$  is central\* for every  $c \in \mathbb{N}$ , then every admissible matrix with integer entries which is  $IPR/\mathbb{Z}$  is  $CIPR/S$ . We shall see in Theorem 2.11 that if  $G$  is an infinite divisible group, then an admissible matrix with integer entries is  $CIPR/G$  if and only if it is  $IPR/\mathbb{Z}$ .

**Lemma 2.9.** *Let  $(S, +)$  and  $(T, +)$  be infinite commutative semigroups with identities. Let  $u, v \in \mathbb{N}$  and let  $A$  be an admissible  $u \times v$  matrix with entries from  $\omega$ . If  $S$  and  $T$  are cancellative, allow the entries of  $A$  to come from  $\mathbb{Z}$ . If there is a surjective homomorphism from  $S$  to  $T$  and  $A$  is  $CIPR/S$ , then  $A$  is  $CIPR/T$ .*

*Proof.* Let  $\varphi : S \rightarrow T$  be a surjective homomorphism and let  $\tilde{\varphi} : \beta S \rightarrow \beta T$  be its continuous extension. By [10, Exercise 3.4.1 and Corollary 4.22],  $\tilde{\varphi}$  is a surjective homomorphism.

Let  $C$  be a central subset of  $T$  and pick a minimal idempotent  $p \in \overline{C}$ . By [10, Exercise 1.7.3] pick a minimal idempotent  $q$  in  $\beta S$  such that  $\tilde{\varphi}(q) = p$ . Pick  $B \in q$  such that  $\tilde{\varphi}[\overline{B}] \subseteq \overline{C}$ . Pick  $\vec{x} \in (S \setminus \{0\})^v$  such that  $A\vec{x} \in B^u$ . Given  $i \in \{1, 2, \dots, u\}$ ,  $\varphi(\sum_{j=1}^v a_{ij}x_j) \in C$  and  $\varphi(\sum_{j=1}^v a_{ij}x_j) = \sum_{j=1}^v a_{ij}\varphi(x_j)$ , so if

$$\vec{y} = \begin{pmatrix} \varphi(x_1) \\ \varphi(x_2) \\ \vdots \\ \varphi(x_v) \end{pmatrix},$$

then  $A\vec{y} \in C^u$ . □

**Lemma 2.10.** *Let  $(G, +)$  be a divisible group. Let  $u, v \in \mathbb{N}$  and let  $A$  be a  $u \times v$  matrix which is appropriate for  $G$ . If  $A$  is  $CIPR/G$ , then  $A$  is  $IPR/\mathbb{Z}$ .*

*Proof.* Assume that  $A$  is  $CIPR/G$ . By [9, Theorem 2.4(II)],  $A$  is  $IPR/\mathbb{Z}$  if and only if  $A$  is  $IPR/\mathbb{Q}$ , so it suffices to show that  $A$  is  $IPR/\mathbb{Q}$ .

By [3, Theorem 19.1],  $G$  is a direct sum of copies of  $\mathbb{Q}$  and quasicyclic groups. Since each direct summand is a homomorphic image of  $G$ , by Lemma 2.9  $A$  is centrally image partition regular over each direct summand. If one of these summands is  $\mathbb{Q}$ , we are done. So it suffices to let  $H$  be a quasicyclic group such that  $A$  is  $CIPR/H$ , and show that  $A$  is  $IPR/\mathbb{Q}$ .

We represent  $H$  as  $\{\mathbb{Z} + \frac{a}{p^n} : a \in \mathbb{Z} \text{ and } n \in \mathbb{N}\}$  for some prime  $p$ , so  $H$  is a subgroup of  $\mathbb{Q}/\mathbb{Z}$ . To see that  $A$  is  $IPR/\mathbb{Q}$ , let  $r \in \mathbb{N}$  and let  $\psi : \mathbb{Q} \rightarrow \{1, 2, \dots, r\}$ . Let  $f : \mathbb{Q} \cap [-\frac{1}{2}, \frac{1}{2}] \xrightarrow{1-1} \mathbb{Q}/\mathbb{Z}$  be the restriction of the projection; that is for  $x \in \mathbb{Q} \cap [-\frac{1}{2}, \frac{1}{2}]$ ,  $f(x) = \mathbb{Z} + x$ . Let  $\varphi$  be the restriction of  $\psi \circ f^{-1}$  to  $H$ . Pick  $k \in \{1, 2, \dots, r\}$  such that  $\varphi^{-1}[\{k\}]$  is central in  $H$  and let  $p$  be a minimal idempotent in  $\beta H$  such that  $\varphi^{-1}[\{k\}] \in p$ . (Here we are giving  $H$  the discrete topology.) Pick by Lemma 2.1 (with  $N = R = H$ ) an idempotent  $q \in \beta(H^v)$  such that  $(H \setminus \{0\})^v \in q$  and for every  $C \in p$ , there exists  $W \in q$  such that for all  $\vec{x} \in W$ ,  $A\vec{x} \in C^u$ . Pick  $W \in q$  such that for all  $\vec{x} \in W$ ,  $A\vec{x} \in \varphi^{-1}[\{k\}]^u$ .

Note that if  $a, b \in \mathbb{Z} \setminus \{0\}$ ,  $\epsilon \leq \frac{1}{2(|a| + |b|)}$ , and  $x, y \in (-\epsilon, \epsilon)$ , then  $ax + by \in (-\frac{1}{2}, \frac{1}{2})$  and  $f(ax + by) = af(x) + bf(y)$ . Consequently, there is a neighborhood  $U$  of  $\vec{0}$  in  $(\mathbb{Q}/\mathbb{Z})^v$  such that, if  $\vec{x} \in U$  and  $\vec{y} = A\vec{x}$ , then

$$A \cdot \begin{pmatrix} f^{-1}(x_1) \\ f^{-1}(x_2) \\ \vdots \\ f^{-1}(x_v) \end{pmatrix} = \begin{pmatrix} f^{-1}(y_1) \\ f^{-1}(y_2) \\ \vdots \\ f^{-1}(y_u) \end{pmatrix}.$$

(Here we are giving  $\mathbb{Q}$  its usual topology,  $\mathbb{Q}/\mathbb{Z}$  the quotient topology, and  $(\mathbb{Q}/\mathbb{Z})^v$  the product topology.)

Since  $q$  is an idempotent and  $U$  is a neighborhood of  $\vec{0}$ ,  $U \in q$ . Pick  $\vec{x} \in W \cap U$  and let  $\vec{y} = A\vec{x}$ . Then  $\vec{y} = A\vec{x} \in \varphi^{-1}[\{k\}]^u$  so for each  $i \in \{1, 2, \dots, u\}$ ,  $\psi(f^{-1}(y_i)) = k$  as required.  $\square$

**Theorem 2.11.** *Let  $(G, +)$  be a divisible group. Let  $u, v \in \mathbb{N}$  and let  $A$  be a  $u \times v$  matrix which is appropriate for  $G$ . Then  $A$  is  $CIPR/G$  if and only if  $A$  is  $IPR/\mathbb{Z}$ .*

*Proof.* The necessity is Lemma 2.10. Since  $G$  is divisible, for each  $c \in \mathbb{N}$ ,  $cG = G$ , so the sufficiency follows from Corollary 2.7(c).  $\square$

### 3. Image partition regular matrices

We saw in Corollary 2.7 that for a commutative semigroup  $(S, +)$ , the assumption that  $cS$  is central\* in  $S$  for each  $c \in \mathbb{N}$  yields the conclusion that an admissible matrix being  $IPR/\mathbb{N}$  implies that it is  $CIPR/S$  and, if  $S$  is cancellative, then  $IPR/\mathbb{Z}$  implies  $CWIPR/S$ . We have also seen that the assumption that  $cS$  is central\* can fail in some very civilized semigroups.

We shall show in Theorem 3.3 that the much weaker assumption that  $cS \neq \{0\}$  for each  $c \in \mathbb{N}$  yields the facts that  $IPR/\mathbb{N}$  implies  $IPR/S$  and, if  $S$  is cancellative, then  $IPR/\mathbb{Z}$  implies  $WIPR/S$ .

If  $A = \begin{pmatrix} 1 & 3 \\ 2 & 2 \end{pmatrix}$ , then  $A$  is trivially  $IPR/\mathbb{N}$ . As a consequence of [10, Theorem 15.24(m)], such a matrix is also nontrivially  $IPR/\mathbb{N}$ . That is, if  $u, v \in \mathbb{N}$  and  $A$  is a  $u \times v$  matrix with entries from  $\mathbb{Q}$  which is  $IPR/\mathbb{N}$ , then given any central subset  $C$  of  $\mathbb{N}$ , there exists  $\vec{x} \in \mathbb{N}^v$  such that the entries of  $\vec{x}$  are distinct,  $A\vec{x} \in C^u$ , and entries of  $A\vec{x}$  corresponding to distinct rows of  $A$  are distinct. The corresponding fact for matrices that are  $IPR/\mathbb{Z}$  as we are defining that notion in this paper, does not seem to have been spelled out before.

**Lemma 3.1.** *Let  $u, v \in \mathbb{N}$  and let  $A$  be an admissible  $u \times v$  matrix with entries from  $\mathbb{Q}$  which is  $IPR/\mathbb{Z}$ . Then given any central subset  $C$  of  $\mathbb{N}$ , there exists  $\vec{x} \in (\mathbb{Z} \setminus \{0\})^v$  such that the entries of  $\vec{x}$  are distinct,  $A\vec{x} \in C^u$ , and entries of  $A\vec{x}$  corresponding to distinct rows of  $A$  are distinct.*

*Proof.* In [6],  $IPR/\mathbb{Z}$  was defined by coloring  $\mathbb{Z} \setminus \{0\}$  and asking for  $\vec{x}$  in  $\mathbb{Z}^v$  with  $A\vec{x}$  monochromatic. For the purposes of this proof, let us say that a  $u \times v$  matrix  $B$  is  $IPR^*/\mathbb{Z}$  if and only if, whenever  $\mathbb{Z} \setminus \{0\}$  is finitely colored, there exists  $\vec{x} \in \mathbb{Z}^v$  such that the entries of  $B\vec{x}$  are monochromatic.

In [6, Theorem 4.13], it is shown that if  $A$  is  $IPR^*/\mathbb{Z}$ , then there exist  $b_1, b_2, \dots, b_v \in \mathbb{Q} \setminus \{0\}$  such that the matrix

$$B = \begin{pmatrix} b_1 & 0 & 0 & \dots & 0 \\ 0 & b_2 & 0 & \dots & 0 \\ 0 & 0 & b_3 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & b_v \\ & & & & A \end{pmatrix}$$

is  $IPR^*/\mathbb{Z}$ , and that given any central subset  $C$  of  $\mathbb{N}$ , there exists  $\vec{x} \in \mathbb{Z}^v$  such that the entries of  $\vec{x}$  are distinct,  $A\vec{x} \in C^u$ , and entries of  $A\vec{x}$  corresponding to distinct rows of  $A$  are distinct.

Note that our given matrix  $A$  is  $IPR^*/\mathbb{Z}$ . First apply [6, Theorem 4.13] to  $A$ , getting  $b_1, b_2, \dots, b_v \in \mathbb{Q} \setminus \{0\}$  so that  $B$  is  $IPR^*/\mathbb{Z}$ , then apply that theorem to  $B$  to get  $\vec{x} \in \mathbb{Z}^v$  such that the entries of  $\vec{x}$  are distinct,  $B\vec{x} \in C^{v+u}$ , and entries of  $B\vec{x}$  corresponding to distinct rows of  $B$  are distinct. Since each  $b_i x_i \in \mathbb{N}$ ,  $\vec{x} \in (\mathbb{Z} \setminus \{0\})^v$ .  $\square$

**Lemma 3.2.** *Let  $u, v \in \mathbb{N}$  and let  $A$  be an admissible  $u \times v$  matrix with entries from  $\mathbb{Z}$ .*

- (a) *If  $A$  is  $IPR/\mathbb{N}$ , then for each  $r \in \mathbb{N}$  there exists  $k \in \mathbb{N}$  such that whenever  $\varphi : \{1, 2, \dots, k\} \rightarrow \{1, 2, \dots, r\}$ , there exists  $\vec{x} \in \{1, 2, \dots, k\}^v$  such that  $\varphi$  is constant on  $A\vec{x}$ , the entries of  $\vec{x}$  are distinct, and the entries of  $A\vec{x}$  corresponding to distinct rows of  $A$  are distinct.*



- (b) If  $A$  is  $IPR/\mathbb{Z}$ , then for each  $r \in \mathbb{N}$  there exists  $k \in \mathbb{N}$  such that whenever  $\varphi : \{1, 2, \dots, k\} \rightarrow \{1, 2, \dots, r\}$ , there exists  $\vec{x} \in \{t \in \mathbb{Z} : 0 < |t| \leq k\}^v$  such that  $\varphi$  is constant on  $A\vec{x}$ , the entries of  $\vec{x}$  are distinct, and the entries of  $A\vec{x}$  corresponding to distinct rows of  $A$  are distinct.

*Proof.* These are both standard compactness arguments. We will spell out the details for (b). So assume  $A$  is  $IPR/\mathbb{Z}$ , let  $r \in \mathbb{N}$ , and suppose the conclusion fails. For each  $k \in \mathbb{N}$ , pick  $\varphi_k : \{1, 2, \dots, k\} \rightarrow \{1, 2, \dots, r\}$  such that there is no  $\vec{x} \in \{t \in \mathbb{Z} : 0 < |t| \leq k\}^v$  such that  $\varphi_k$  is constant on  $A\vec{x}$ , the entries of  $\vec{x}$  are distinct, and the entries of  $A\vec{x}$  corresponding to distinct rows of  $A$  are distinct. Define  $\psi_k : \mathbb{N} \rightarrow \{1, 2, \dots, r\}$  by  $\psi_k(t) = \varphi_k(t)$  if  $t \leq k$  and  $\psi_k(t) = 1$  if  $t > k$ .

Let  $\tau$  be a cluster point of  $\langle \psi_k \rangle_{k=1}^\infty$  in the product space  $\times_{n=1}^\infty \{1, 2, \dots, r\}$ , where  $\{1, 2, \dots, r\}$  has the discrete topology. Pick  $m \in \{1, 2, \dots, r\}$  such that  $\tau^{-1}[\{m\}]$  is central in  $\mathbb{N}$  and let  $C = \tau^{-1}[\{m\}]$ . By Lemma 3.1, pick  $\vec{x} \in (\mathbb{Z} \setminus \{0\})^v$  such that  $A\vec{x} \in C^u$ , the entries of  $\vec{x}$  are distinct, and entries of  $A\vec{x}$  corresponding to distinct rows of  $A$  are distinct. Let  $\vec{y} = A\vec{x}$  and let  $l = \max(\{|x_i| : i \in \{1, 2, \dots, v\}\} \cup \{y_j : j \in \{1, 2, \dots, u\}\})$ . Let

$$U = \{\gamma \in \times_{n=1}^\infty \{1, 2, \dots, r\} : \text{for all } t \in \{1, 2, \dots, l\}, \gamma(t) = \tau(t)\}.$$

Then  $U$  is a neighborhood of  $\tau$  so pick  $k > l$  such that  $\psi_k \in U$ . Since the entries of  $\vec{x}$  are distinct and entries of  $\vec{y}$  corresponding to distinct rows of  $A$  are distinct, there must be some  $i$  and  $s$  in  $\{1, 2, \dots, u\}$  such that  $\varphi_k(y_i) \neq \varphi_k(y_s)$ . Since  $y_i \leq l < k$  and  $y_s \leq l < k$ , we have  $m = \tau(y_i) = \psi_k(y_i) = \varphi_k(y_i) \neq \varphi_k(y_s) = \psi_k(y_s) = \tau(y_s) = m$ , a contradiction.  $\square$

**Theorem 3.3.** *Let  $(S, +)$  be an infinite commutative semigroup with identity 0. Let  $u, v \in \mathbb{N}$  and let  $A$  be a  $u \times v$  matrix which is appropriate for  $S$ . Assume that for all  $c \in \mathbb{N}$ ,  $cS \neq \{0\}$ .*

- (a) *If  $A$  is  $IPR/\mathbb{N}$ , then  $A$  is  $IPR/S$ . If  $S$  is cancellative, then whenever  $S \setminus \{0\}$  is finitely colored, there exists  $\vec{x} \in (S \setminus \{0\})^v$  such that the entries of  $\vec{x}$  are distinct, the entries of  $A\vec{x}$  are monochromatic, and entries of  $A\vec{x}$  corresponding to distinct rows of  $A$  are distinct.*
- (b) *If  $S$  is cancellative and  $A$  is  $IPR/\mathbb{Z}$ , then  $A$  is  $WIPR/S$ . In fact, if  $S \setminus \{0\}$  is finitely colored and  $G$  is the group of differences of  $S$ , then there exists  $\vec{x} \in (G \setminus \{0\})^v$  such that the entries of  $\vec{x}$  are distinct, the entries of  $A\vec{x}$  are monochromatic, and entries of  $A\vec{x}$  corresponding to distinct rows of  $A$  are distinct.*
- (c) *If  $S$  is a group and  $A$  is  $IPR/\mathbb{Z}$ , then  $A$  is  $IPR/S$ . In fact, if  $S \setminus \{0\}$  is finitely colored, then there exists  $\vec{x} \in (S \setminus \{0\})^v$  such that the entries of  $\vec{x}$  are distinct, the entries of  $A\vec{x}$  are monochromatic, and entries of  $A\vec{x}$  corresponding to distinct rows of  $A$  are distinct.*

*Proof.* To verify (a), assume that  $A$  is  $IPR/\mathbb{N}$ . Let  $r \in \mathbb{N}$  and let  $\psi : S \setminus \{0\} \rightarrow \{1, 2, \dots, r\}$ . Pick  $k \in \mathbb{N}$  as guaranteed by Lemma 3.2(a). Pick  $z \in S$

such that  $\{z, 2z, \dots, kz\} \cap \{0\} = \emptyset$ . (If no such  $z$  existed, one would have  $(k!)S = \{0\}$ .) Define  $\varphi : \{1, 2, \dots, k\} \rightarrow \{1, 2, \dots, r\}$  by  $\varphi(t) = \psi(tz)$ . Pick by Lemma 3.2(a)  $\vec{w} \in \{1, 2, \dots, k\}^v$  such that  $\varphi$  is constant on  $A\vec{w}$ , the entries of  $\vec{w}$  are distinct, and entries of  $A\vec{w}$  corresponding to distinct rows of  $A$  are distinct. Define  $\vec{x} \in (S \setminus \{0\})^v$  by, for  $j \in \{1, 2, \dots, v\}$ ,  $x_j = w_j z$ . Given  $i \in \{1, 2, \dots, u\}$ ,  $\sum_{j=1}^v a_{ij} x_j = \sum_{j=1}^v a_{ij} w_j z = (\sum_{j=1}^v a_{ij} w_j) z$  so  $\psi(\sum_{j=1}^v a_{ij} x_j) = \varphi(\sum_{j=1}^v a_{ij} w_j)$  so  $\psi$  is constant on  $A\vec{x}$ . We have shown that  $A$  is *IPR/S*.

Now assume that  $S$  is cancellative. Then if  $l \neq j \in \{1, 2, \dots, v\}$ , then  $x_l = w_l z \neq w_j z = x_j$ . Finally, assume that  $i, s \in \{1, 2, \dots, u\}$  and rows  $i$  and  $s$  of  $A$  are distinct. If we had  $\sum_{j=1}^v a_{ij} x_j = \sum_{j=1}^v a_{sj} x_j$ , then we would have  $\sum_{j=1}^v a_{ij} w_j = \sum_{j=1}^v a_{sj} w_j$ .

The proof of (b) is identical to the proof of (a) under the assumption that  $S$  is cancellative except that Lemma 3.2(b) is invoked instead of Lemma 3.2(a). Statement (c) is an immediate consequence of statement (b).  $\square$

We note that we cannot guarantee either of the additional conclusions in Theorem 3.3(a) without some additional assumptions. To see this let  $T$  be an infinite set, let  $a \in T$ , and define  $x+y = a$  for all  $x, y \in T$ . Let  $S = T \cup \{0\}$ , where

$0$  is an identity adjoined to  $T$ . Let  $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}$ , and define  $\varphi : T \rightarrow \{0, 1\}$

by  $\varphi(a) = 0$  and  $\varphi(x) = 1$  for  $x \in T \setminus \{a\}$ . If  $\vec{x} \in (S \setminus \{0\})^2$ ,  $A\vec{x} = \vec{y}$ , and the entries of  $\vec{y}$  are monochromatic, then  $x_1 = x_2 = y_1 = y_2 = y_3 = a$ .

We remark that if  $S$  is an abelian group,  $cS = \{0\}$  and  $A$  is a first entries matrix with entries from  $\omega$ , all of whose first entries are relatively prime to  $c$ , then  $A$  is *IPR/S*. This is because, by [3, Theorem 2.1],  $S$  is a direct sum of  $p$ -groups, where each  $p$  divides  $c$ . Consequently, if  $(m, c) = 1$ , then  $mS = S$ , so  $mS$  is central\* and a modification of the proof of [10, Theorem 15.5] to allow negative entries applies. We shall get a stronger result in Theorem 5.6, assuming just that  $S$  is a cancellative commutative semigroup.

Notice that if  $S$  is a commutative semigroup, and  $c \in \mathbb{N}$  such that  $cS \neq \{0\}$  but  $cS$  is not central in  $S$ , then the  $1 \times 1$  matrix  $(c)$  is *IPR/S* (since one may pick  $x \in S$  such that  $cx \neq 0$ ) but not *CIPR/S* because  $S \setminus cS$  is a central set.

As is shown in [10, Theorem 15.10], if  $(S, +)$  is a commutative semigroup with identity  $0$ ,  $c \in \mathbb{N}$ , and  $cS = \{0\}$ , then the matrix

$\begin{pmatrix} 1 & 1 \\ 1 & c \\ 0 & c \end{pmatrix}$  is *IPR/N*, but does not satisfy even the weakest possible form

of image partition regularity over  $S$  (in which  $S$  is finitely colored and one asks for  $\vec{x} \in S^2 \setminus \{\vec{0}\}$ ).

We saw in Theorems 2.2 and 2.3 that the two most reasonable possible definitions of centrally image partition regular are equivalent. If  $S = \mathbb{N}$  or  $S = \mathbb{Z}$ , we know that *IPR/S* and *CIPR/S* are equivalent.

**Question 3.4.** *Do there exist a commutative semigroup  $(S, +)$ ,  $u, v \in \mathbb{N}$ , and a  $u \times v$  matrix  $A$  which is appropriate for  $S$  and is not *IPR/S* but has the*

property that whenever  $S \setminus \{0\}$  is finitely colored, there must exist  $\vec{x} \in S^v \setminus \{\vec{0}\}$  such that the entries of  $A\vec{x}$  are monochromatic?

We show now that  $(\omega, +)$  does not provide an affirmative answer to Question 3.4. The proof is similar to the proof of Lemma 2.1.

**Theorem 3.5.** *Let  $u, v \in \mathbb{N}$  and let  $A$  be an admissible  $u \times v$  matrix with integer entries. Assume that whenever  $\omega \setminus \{0\}$  is finitely colored, there exists  $\vec{x} \in \omega^v \setminus \{\vec{0}\}$  such that the entries of  $A\vec{x}$  are monochromatic. Then  $A$  is  $IPR/\omega$ .*

*Proof.* By [10, Theorem 5.7], we may pick  $p \in \beta\omega$  such that for all  $C \in p$ , there exists  $\vec{x} \in \omega^v \setminus \{\vec{0}\}$  such that  $A\vec{x} \in C^u$ . Let  $\vec{p} = (p, p, \dots, p) \in (\beta\omega)^u$ .

Then  $\vec{p} \in \text{cl}\tilde{A}[\omega^v \setminus \{\vec{0}\}] = \tilde{A}[\text{cl}(\omega^v \setminus \{\vec{0}\})] = \tilde{A}[\beta(\omega^v) \setminus \{\vec{0}\}]$  so pick  $q \in \beta(\omega^v) \setminus \{\vec{0}\}$  such that  $\tilde{A}(q) = \vec{p}$ .

For each  $F$  such that  $\emptyset \neq F \subseteq \{1, 2, \dots, v\}$ , let  $B_F = \{x \in S^v \setminus \{\vec{0}\} : F = \{i \in \{1, 2, \dots, v\} : x_i \neq 0\}\}$ . Pick  $F$  such that  $B_F \in q$ . By reordering the columns of  $A$ , we may presume we have some  $r \in \{1, 2, \dots, v\}$  such that  $F = \{1, 2, \dots, r\}$ . Let  $D$  consist of the first  $r$  columns of  $A$ . We claim that  $D$  is  $IPR/\mathbb{N}$ . To see this, let  $C \in p$ . Pick  $E \in q$  such that  $\tilde{A}[\overline{E}] \subseteq \overline{C}^u$ . Pick  $\vec{x} \in B_F \cap E$ . Let  $\vec{z} \in \omega^r$  be defined by  $z_i = x_i$  for  $i \in \{1, 2, \dots, r\}$ . Since  $\vec{x} \in B_F$ , we have  $\vec{z} \in (\omega \setminus \{0\})^r$ . And  $D\vec{z} = A\vec{x} \in \overline{C}^u$ .

Since  $D$  is  $IPR/\mathbb{N}$  we have by [10, Theorem 15.24(k)] that  $A$  is  $IPR/\mathbb{N}$  and therefore  $IPR/\omega$ .  $\square$

#### 4. The $A\vec{x} + B\vec{y}$ theorem

This section consists mostly of the proof of Theorem 4.4 and some of its consequences.

**Lemma 4.1.** *Let  $(S, +)$  be an infinite commutative cancellative semigroup with identity 0, let  $G$  be its group of differences, and let  $M$  be a subsemigroup of  $(\mathbb{N}, \cdot)$ . Let  $T = \bigcap_{m \in M} \text{cl}_{\beta S}(mS)$  and let  $V = \bigcap_{m \in M} \text{cl}_{\beta G}(mG)$ . Then  $K(T) = T \cap K(V)$ .*

*Proof.* By [10, Theorem 1.65] it suffices to show that  $T \cap K(V) \neq \emptyset$ . For each  $F \in \mathcal{P}_f(G)$  and each  $m \in M$ , we claim we can choose some  $t_{F,m} \in mS$  such that for every  $x \in F \cap mG$ ,  $x + t_{F,m} \in mS$ . To see this, if  $F \cap mG = \emptyset$ , let  $t_{F,m} = 0$ . Otherwise, enumerate  $F \cap mG$  as  $x_1, x_2, \dots, x_n$  and for each  $i \in \{1, 2, \dots, n\}$ , pick  $s_i$  and  $u_i$  in  $S$  such that  $x_i = m(s_i - u_i)$ . Let  $t_{F,m} = m(u_1 + u_2 + \dots + u_n)$ .

Let  $D = \{(F, m) : F \in \mathcal{P}_f(G) \text{ and } m \in M\}$  and direct  $D$  by agreeing that  $(F, m) \leq (H, r)$  if and only if  $F \subseteq H$  and  $m$  divides  $r$ . Let  $p$  be a limit point in  $\beta S$  of the net  $\langle t_{F,m} \rangle_{(F,m) \in D}$  and let  $q \in K(V)$ . Then for every  $m \in M$ ,  $mG \subseteq \{x \in G : -x + mS \in p\}$  so  $mS \in q + p$  and thus  $q + p \in T \cap K(V)$  as required.  $\square$

**Lemma 4.2.** *Let  $(X, +)$  be a compact Hausdorff right topological semigroup, and let  $\langle E_\alpha \rangle_{\alpha \in A}$  and  $\langle J_\beta \rangle_{\beta \in B}$  be decreasing nets of nonempty subsets of  $X$*

contained in the algebraic center of  $X$ . Assume that, for every  $\alpha \in A$  and every  $x \in E_\alpha$ , there exists  $\alpha' \in A$  such that  $x + E_{\alpha'} \subseteq E_\alpha$ . Assume also that, for every  $\beta \in B$  and every  $y \in J_\beta$ , there exists  $\beta' \in B$  such that  $y + J_{\beta'} \subseteq J_\beta$ . Then, if  $E = \bigcap_{\alpha \in A} \mathcal{C}\ell(E_\alpha)$  and  $I = \bigcap_{\alpha \in A} \bigcap_{\beta \in B} \mathcal{C}\ell(E_\alpha + J_\beta)$ ,  $E \cup I$  is a compact subsemigroup of  $X$  and  $I$  is an ideal of  $E \cup I$ .

*Proof.* Note that the algebraic center of  $X$  is contained in the topological center of  $X$  since  $\lambda_x = \rho_x$  for every  $x$  in the algebraic center. Since the nets are decreasing and  $X$  is compact, we have that  $E \neq \emptyset$  and  $I \neq \emptyset$ .

To see that  $I$  is a subsemigroup of  $X$ , let  $p, q \in I$ . To see that  $p + q \in I$ , let  $U$  be an open neighborhood of  $p + q$ , let  $\alpha \in A$ , and let  $\beta \in B$ . Pick a neighborhood  $W$  of  $p$  such that  $W + q \subseteq U$ , pick  $x \in E_\alpha$  and  $y \in J_\beta$  such that  $x + y \in W$ . Pick  $\alpha' \in A$  and  $\beta' \in B$  such that  $x + E_{\alpha'} \subseteq E_\alpha$  and  $y + J_{\beta'} \subseteq J_\beta$ . Pick a neighborhood  $V$  of  $q$  such that  $x + y + V \subseteq U$ . Pick  $w \in E_{\alpha'}$  and  $z \in J_{\beta'}$  such that  $w + z \in V$ . Then  $x + y + w + z = x + w + y + z \in U \cap (E_\alpha + J_\beta)$  as required.

The proof that  $E$  is a subsemigroup of  $X$  is similar and slightly simpler since one does not need to use the fact that  $J_\beta$  is contained in the center of  $X$ .

We shall show that  $I + E \subseteq I$  and  $E + I \subseteq I$ . It will follow that  $E \cup I$  is a subsemigroup of  $X$  and that  $I$  is an ideal of  $E \cup I$ .

Let  $p \in I$  and  $q \in E$ . To see that  $p + q \in I$ , let  $U$  be an open neighborhood of  $p + q$  and let  $\alpha \in A$  and  $\beta \in B$  be given. Pick a neighborhood  $W$  of  $p$  such that  $W + q \subseteq U$ . Pick  $x \in E_\alpha$  and  $y \in J_\beta$  such that  $x + y \in W$ . Pick  $\alpha' \in A$  such that  $x + E_{\alpha'} \subseteq E_\alpha$ . Pick a neighborhood  $V$  of  $q$  and pick  $z \in E_{\alpha'} \cap V$ . Then  $x + y + z = x + z + y \in E_\alpha + J_\beta \cap U$ .

The proof that  $q + p \in I$  is similar. □

**Definition 4.3.** Let  $S$  be an arbitrary semigroup, let  $p$  be an idempotent in  $\beta S$ , and let  $P \in p$ . Then  $P^*(p)$  denotes  $\{s \in P : -s + P \in p\}$ .

Given an idempotent  $p$  and  $P \in p$ , by [10, Lemma 4.14],  $P^*(p) \in p$  and  $-s + P^*(p) \in p$  for every  $s \in P^*(p)$ .

Note that in the following theorem, one might have  $T = \{0\}$ , but in any event  $T$  is a subsemigroup of  $\beta S$ . (If  $M$  is a subsemigroup of  $(\mathbb{N}, \cdot)$ , then  $T = \{0\}$  if and only if there is some  $m \in M$  such that  $mS = \{0\}$  because, if  $mS \neq \{0\}$  for each  $m \in M$ , then one has  $\{mS \setminus \{0\} : m \in M\}$  has the finite intersection property.) The proof of the following theorem uses an argument due to Furstenberg and Katznelson in [5].

For some of our applications we will need to assume that  $T$  is infinite.

We remark that the following theorem extends the well-known fact that every central subset of  $S$  contains an arbitrarily long arithmetic progression, whose increment can be chosen in an arbitrary  $IP$  subset of  $S$ . This fact is the

simple special case of Theorem 4.4 in which  $A = \begin{pmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$  and  $B = \begin{pmatrix} 0 \\ 1 \\ 2 \\ \vdots \\ k \end{pmatrix}$ .

**Theorem 4.4.** *Let  $(S, +)$  be an infinite commutative semigroup with identity 0, let  $u, v, n \in \mathbb{N}$ , let  $A$  be a  $u \times v$  matrix which is appropriate for  $S$  and let  $B$  be a  $u \times n$  matrix which is appropriate for  $S$ . Let  $M$  be a nonempty subset of  $\mathbb{N}$  and let  $T = \bigcap_{m \in M} \text{cl}_{\beta S}(mS)$ . Assume that  $p$  is a minimal idempotent of  $T$  and that for all  $P \in p$  there exists  $\vec{x} \in S^v$  such that  $A\vec{x} \in P^u$ . Let  $\vec{p} = (p, p, \dots, p) \in (\beta S)^u$ . Assume that one of the following conditions holds:*

- (1) *the entries of  $B$  come from  $\omega$ ;*
- (2)  *$p \in K(\beta S)$ ;*
- (3)  *$S$  is a group; or*
- (4)  *$M$  is a subsemigroup of  $(\mathbb{N}, \cdot)$ .*

*If  $S$  is cancellative and (2), (3), or (4) holds, let  $R$  be the group of differences of  $S$ . If either  $S$  is cancellative but none of (2), (3), or (4) hold or  $S$  is not cancellative, let  $R = S$ . Let  $V = \bigcap_{m \in M} \text{cl}_{\beta R}(mR)$ , let  $q$  be an idempotent in  $\tilde{A}^{-1}[\{\vec{p}\}]$ , and let  $r$  be an idempotent in  $\bigcap_{m \in M} \text{cl}_{\beta(R^n)}(mR)^n$ . Let  $U \in r$ . Then for all  $C \in p$  and all  $Q \in q$ , there exist  $\vec{x} \in Q$  and  $\vec{y} \in U$  such that  $A\vec{x} \in C^u$  and  $A\vec{x} + B\vec{y} \in C^u$ .*

*Proof.* By Lemma 1.10,  $\tilde{A}^{-1}[\{\vec{p}\}]$  is a compact subsemigroup of  $\beta(S^v)$  so it contains an idempotent. Let  $C \in p$  and  $Q \in q$  be given. Since  $\tilde{A}(q) = \vec{p}$ , we may presume that  $A[Q] \subseteq C^u$ . Let  $Q^* = Q^*(q)$  and  $U^* = U^*(r)$ .

For  $K \in \mathcal{P}_f(Q^*)$  and  $F \in \mathcal{P}_f(M)$ , let

$$E(K, F) = \{A\vec{x} : \vec{x} \in Q^* \cap (\bigcap_{\vec{s} \in K} (-\vec{s} + Q^*)) \cap \bigcap_{m \in F} (mS)^u\}.$$

Given  $K \in \mathcal{P}_f(Q^*)$  and  $F \in \mathcal{P}_f(M)$ , since  $\bigcap_{m \in F} \times_{i=1}^u \overline{mS}$  is a neighborhood of  $\vec{p}$  and  $\tilde{A}(q) = \vec{p}$ , we have that  $E(K, F) \neq \emptyset$ .

For every  $D \in \mathcal{P}_f(U^*)$  and  $F \in \mathcal{P}_f(M)$ , let

$$J(D, F) = \{B\vec{y} : \vec{y} \in U^* \cap \bigcap_{\vec{t} \in D} (-\vec{t} + U^*) \cap \bigcap_{m \in F} (mR)^n\}.$$

Each  $J(D, F)$  is nonempty, because it is the image of the intersection of a finite number of members of  $r$ . Note that each of these sets is contained in  $\bigcap_{m \in F} (mR)^u$ .

Direct  $\mathcal{P}_f(Q^*) \times \mathcal{P}_f(M)$  by  $(K, F) \leq (K', F')$  if and only if  $K \subseteq K'$  and  $F \subseteq F'$ . Direct  $\mathcal{P}_f(U^*) \times \mathcal{P}_f(M)$  by  $(D, F) \leq (D', F')$  if and only if  $D \subseteq D'$  and  $F \subseteq F'$ . We shall show that the nets  $\langle E(K, F) \rangle_{(K, F) \in \mathcal{P}_f(Q^*) \times \mathcal{P}_f(M)}$  and  $\langle J(D, F) \rangle_{(D, F) \in \mathcal{P}_f(U^*) \times \mathcal{P}_f(M)}$  satisfy the hypotheses of Lemma 4.2 with  $X = (\beta R)^u$ . It is trivial that both nets are decreasing.

Let  $(K, F) \in \mathcal{P}_f(Q^*) \times \mathcal{P}_f(M)$  and let  $\vec{y} \in E(K, F)$ . Pick  $\vec{x} \in Q^* \cap \bigcap_{\vec{s} \in K} (-\vec{s} + Q^*)$  such that  $\vec{y} = A\vec{x}$ . Let  $K' = \{\vec{x}\} \cup \{\vec{s} + \vec{x} : \vec{s} \in K\}$ . Then  $K' \in \mathcal{P}_f(Q^*)$ . We claim that  $\vec{y} + E(K', F) \subseteq E(K, F)$ . So let  $\vec{z} \in E(K', F)$  and pick  $\vec{w} \in Q^* \cap (\bigcap_{\vec{t} \in K'} (-\vec{t} + Q^*))$  such that  $\vec{z} = A\vec{w}$ . Then  $\vec{y} + \vec{z} \in \bigcap_{m \in F} (mS)^u$ ,  $\vec{y} + \vec{z} = A(\vec{x} + \vec{w})$ , and  $\vec{x} + \vec{w} \in Q^* \cap \bigcap_{\vec{s} \in K} (-\vec{s} + Q^*)$ .

Now let  $(D, F) \in \mathcal{P}_f(U^*) \times \mathcal{P}_f(M)$  and let  $\vec{x} \in J(D, F)$ . Pick  $\vec{w} \in U^* \cap \bigcap_{\vec{t} \in D} (-\vec{t} + U^*) \cap \bigcap_{m \in F} (mR)^n$  such that  $\vec{z} = B\vec{w}$ . Let  $D' = \{\vec{w}\} \cup \{\vec{t} + \vec{w} : \vec{t} \in D\}$ . Then  $\vec{x} + J(D', F) \subseteq J(D, F)$ .

Now let  $E = \bigcap_{(K, F) \in \mathcal{P}_f(Q^*) \times \mathcal{P}_f(M)} \text{cl}_{(\beta R)^u}(E(K, F))$  and let

$$I = \left( \bigcap_{(K, F) \in \mathcal{P}_f(Q^*) \times \mathcal{P}_f(M)} \bigcap_{(D, H) \in \mathcal{P}_f(U^*) \times \mathcal{P}_f(M)} \text{cl}_{(\beta R)^u}(E(K, F) + J(D, H)) \right).$$

By Lemma 4.2,  $E \cup I$  is a compact subsemigroup of  $(\beta R)^u$  and  $I$  is an ideal of  $E \cup I$ . Observe that  $E$  and  $I$  are contained in  $V^u$ .

We claim that  $\bar{p} \in K(V^u)$ . By [10, Theorem 2.23],  $K(V^u) = K(V)^u$ ,  $K(T^u) = K(T)^u$ , and  $K((\beta R)^u) = K(\beta R)^u$ . Assume first that either  $S$  is cancellative but none of (2), (3), or (4) hold or  $S$  is not cancellative. Then  $T = V$  so  $\bar{p} \in K(V)^u = K(V^u)$ . Now assume that  $S$  is cancellative and one of (2), (3), or (4) holds. If (3) holds, then again  $T = V$  so  $\bar{p} \in K(V^u)$ . If (4) holds, then by Lemma 4.1,  $K(T) \subseteq K(V)$  so  $\bar{p} \in K(T)^u \subseteq K(V)^u = K(V^u)$ . Finally, assume that  $p \in K(\beta S)$ . By Lemma 1.12,  $K(\beta S) \subseteq K(\beta R)$ . Then  $p \in T \cap K(\beta S)$  so by [10, Theorem 1.65],  $K(T) = T \cap K(\beta S)$  and thus  $\bar{p} \in K(T)^u = (T \cap K(\beta S))^u \subseteq (V \cap K(\beta R))^u = V^u \cap K(\beta R)^u = V^u \cap K((\beta R)^u)$ . By [10, Theorem 1.65],  $K(V^u) = V^u \cap K((\beta R)^u)$  so  $\bar{p} \in K(V^u)$  as claimed.

Now we claim that  $\bar{p} \in E$ , so let  $K \in \mathcal{P}_f(Q^*)$ ,  $F \in \mathcal{P}_f(M)$ , and a neighborhood  $U$  of  $\bar{p}$  be given. Then  $U \cap \times_{i=1}^u \overline{\bigcap_{m \in F} (mS)}$  is a neighborhood of  $\bar{p} = \tilde{A}(q)$  so pick  $\vec{x} \in Q^* \cap \bigcap_{\vec{s} \in K} (-\vec{s} + Q^*)$  such that  $A\vec{x} \in U \cap \times_{i=1}^u \overline{\bigcap_{m \in F} (mS)}$ . Then  $A\vec{x} \in U \cap E(K, F)$ .

Therefore  $\bar{p} \in K(E \cup I)$ , by [10, Theorem 1.65], and hence  $p \in I$ . Since  $(\bar{C})^u$  is a neighborhood of  $\bar{p}$  in  $(\beta R)^u$ ,  $(\bar{C})^u$  meets  $E(K, F) + J(D, F)$  for every  $K \in \mathcal{P}_f(Q^*)$ , every  $F \in \mathcal{P}_f(M)$  and every  $D \in \mathcal{P}_f(U^*)$ . It follows that there exist  $\vec{x} \in Q$  and  $\vec{y} \in U$  such that  $A\vec{x} \in C^u$  and  $A\vec{x} + B\vec{y} \in C^u$ , as claimed.  $\square$

**Lemma 4.5.** *Let  $(S, +)$  be an infinite commutative semigroup with identity 0, let  $u, v \in \mathbb{N}$ , and let  $A$  be a  $u \times v$  matrix which is appropriate for  $S$ . Let  $M$  be a nonempty subset of  $\mathbb{N}$  and let  $T = \bigcap_{m \in M} \text{cl}_{\beta S}(mS)$ . Assume that  $T$  is infinite,  $p$  is a minimal idempotent of  $T$ , and that for all  $P \in p$  there exists  $\vec{x} \in S^v$  such that  $A\vec{x} \in P^u$ . Assume that one of the following conditions holds:*

- (1) *the entries of  $A$  come from  $\omega$ ;*
- (2)  *$p \in K(\beta S)$ ;*
- (3)  *$S$  is a group; or*
- (4)  *$M$  is a subsemigroup of  $(\mathbb{N}, \cdot)$ .*

*Let  $\bar{p} = (p, p, \dots, p) \in (\beta S)^u$  and let  $q$  be a minimal idempotent in  $\tilde{A}^{-1}[\{\bar{p}\}]$ . Then  $(S \setminus \{0\})^v \in q$ . If  $S$  is cancellative, then, for every distinct  $i, j \in \{1, 2, \dots, v\}$ ,  $\{\vec{x} \in \beta(S^v) : x_i \neq x_j\} \in q$ .*

*Proof.* By Lemma 1.13,  $p \neq 0$ . For each  $i \in \{1, 2, \dots, v\}$ , define a homomorphism  $\pi_i : S^v \rightarrow S$  by  $\pi_i(\vec{x}) = x_i$  and let  $\tilde{\pi}_i : \beta(S^v) \rightarrow \beta S$  be its continuous extension, which is a homomorphism by [10, Corollary 4.22]. To see that  $(S \setminus \{0\})^v \in q$  it suffices to show that for each  $i \in \{1, 2, \dots, v\}$ ,  $\tilde{\pi}_i(q) \neq 0$ , since then  $(S \setminus \{0\})^v = \bigcap_{i=1}^v \pi_i^{-1}[S \setminus \{0\}] \in q$ .

So let  $i \in \{1, 2, \dots, v\}$  and suppose that  $\tilde{\pi}_i(q) = 0$ . Then by Lemma 1.14,  $\tilde{\pi}_i[\tilde{A}^{-1}[\{\bar{p}\}]]$  is a finite group with identity 0. Let  $Q = \pi_i^{-1}[\{0\}]$  and note that  $Q \in q$ .

We claim that for each  $P \in p$ , there exists  $\vec{w} \in S^v$  such that  $A\vec{w} \in P^u$  and  $\pi_i(\vec{w}) \in P$ . So let  $P \in p$  be given. Let  $B$  denote the  $v \times 1$  matrix whose entries are those in the  $i^{\text{th}}$  column of  $A$ . By Theorem 4.4 with  $r = p$ , pick  $\vec{x} \in Q$  and  $y \in P$  such that  $A\vec{x} + By \in P^u$ . Define  $\vec{w} \in S^v$  by  $w_i = y$  and  $w_j = x_j$  if  $j \in \{1, 2, \dots, v\} \setminus \{i\}$ . Since  $\vec{x} \in Q$ ,  $x_i = 0$  so  $A\vec{w} = A\vec{x} + By \in P^u$  so our claim is established.

For each  $P \in p$ , pick  $\vec{w}(P) \in S^v$  such that  $A\vec{w}(P) \in P^u$  and  $\pi_i(\vec{w}(P)) \in P$ . Direct  $p$  by reverse inclusion and let  $r$  be a limit point of the net  $\langle \vec{w}(P) \rangle_{P \in p}$  in  $\beta(S^v)$ . Then  $\tilde{A}(r) = \bar{p}$  and  $\pi(r) = p$ . This is a contradiction, because it implies that  $p$  is an idempotent in the group  $\tilde{\pi}_i[\tilde{A}^{-1}[\{\bar{p}\}]]$  which is not equal to the identity of the group.

Now assume that  $S$  is cancellative and let  $G$  denote the group of differences of  $S$ . Assume that there exist distinct  $i, j \in \{1, 2, \dots, v\}$  such that  $Q = \{\vec{x} \in S^v : x_i = x_j\} \in q$ . Define a homomorphism  $h : S^v \rightarrow G$  by  $h(\vec{x}) = x_i - x_j$  and let  $\tilde{h} : \beta(S^v) \rightarrow \beta G$  be its continuous extension. Since  $\tilde{h}(q) = 0$ , by Lemma 1.14,  $\tilde{h}[\tilde{A}^{-1}[\{\bar{p}\}]]$  is a finite group with identity 0.

We claim that for each  $P \in p$ , there exists  $\vec{w} \in S^v$  such that  $A\vec{w} \in P^u$  and  $h(\vec{w}) \in P$ . Once we have established this, we obtain a contradiction exactly as before. So let  $P \in p$  be given. As before, let  $B$  denote the  $v \times 1$  matrix whose entries are those in the  $i^{\text{th}}$  column of  $A$ . By Theorem 4.4, pick  $\vec{x} \in Q$  and  $y \in P$  such that  $A\vec{x} + By \in P^u$ . Define  $\vec{w} \in S^v$  by  $w_i = x_i + y$  and  $w_t = x_t$  if  $t \in \{1, 2, \dots, v\} \setminus \{i\}$ . Then  $A\vec{w} = A\vec{x} + By \in P^u$  and  $h(\vec{w}) = y \in P$  as required.  $\square$

**Corollary 4.6.** *Assume that the hypotheses of Theorem 4.4 hold and  $T$  is infinite. Then, for every  $C \in p$ , there exists  $\vec{x} \in (S \setminus 0)^v$  and  $\vec{y} \in U$  such that  $A\vec{x} \in C^u$  and  $A\vec{x} + B\vec{y} \in C^u$ . If  $S$  is cancellative, we can also choose  $\vec{x}$  to have distinct entries.*

*Proof.* This follows immediately from Theorem 4.4 and Lemma 4.5.  $\square$

Notice that to invoke Corollary 4.6, we need that  $U$  is a member of an idempotent in  $\bigcap_{m \in M} \text{cl}_{\beta(R^n)}(mR)^n$ . If  $p$  is minimal in  $\beta S$ , we see that we can guarantee that the entries of  $\vec{x}$  are nonzero for any IP subset  $U$  of  $S^n$ .

**Corollary 4.7.** *Let  $(S, +)$  be an infinite commutative semigroup with identity 0, let  $u, v \in \mathbb{N}$ , let  $A$  be a  $u \times v$  matrix which is appropriate for  $S$ , and assume that  $p$  is a minimal idempotent in  $\beta S$  such that for all  $C \in p$  there exists  $\vec{x} \in S^v$  such that  $A\vec{x} \in C^u$ . Let  $n \in \mathbb{N}$  and let  $B$  be a  $u \times n$  matrix which is appropriate for*

*S.* Let  $U$  be an IP subset of  $S^n$ . Then for each  $C \in p$ , there exist  $\vec{x} \in (S \setminus \{0\})^v$  and  $\vec{y} \in U$  such that  $A\vec{x} \in C^u$  and  $A\vec{x} + B\vec{y} \in C^u$ . If  $S$  is cancellative, we can also choose  $\vec{x}$  to have distinct entries.

*Proof.* Let  $M = \{1\}$ . Then in Theorem 4.4,  $T = \beta S$ , so  $p$  is minimal in  $T$ . Pick an idempotent  $r \in \beta(S^n)$  such that  $U \in r$ . Then  $r \in \bigcap_{m \in M} \text{cl}_{\beta(R^n)}(mR)^n$  so Corollary 4.6 applies.  $\square$

**Lemma 4.8.** *Let  $(S, +)$  be a commutative semigroup with identity 0. Let  $M$  be a nonempty subset of  $\mathbb{N}$ , let  $T = \bigcap_{m \in M} \text{cl}_{\beta S}(mS)$ , and let  $p$  be an idempotent in  $T$ . For each  $c \in M$ , there exists an idempotent  $q \in \beta S$  such that  $cq = p$ . If  $M$  is a subsemigroup of  $(\mathbb{N}, \cdot)$ , we can choose such  $q \in T$ .*

*Proof.* Let  $c \in M$ . Then  $cS \in p$ . For each  $P \in p$ , pick  $s_P \in S$  such that  $cs_P \in P$ . If  $r$  is a limit point of the net  $\langle s_P \rangle_{P \in p}$  in  $\beta S$ , then  $cr = p$ . Since  $\{w \in \beta S : cw = p\}$  is nonempty, it is a compact subsemigroup of  $\beta S$  and therefore contains an idempotent.

Now assume that  $M$  is a subsemigroup of  $(\mathbb{N}, \cdot)$ . If  $M = \{1\}$ , then  $T = \beta S$ . So assume that  $M$  is infinite. For each  $n \in \mathbb{N}$ , let  $m_n$  denote the product of the first  $n$  elements of  $M$ . For each  $n \in \mathbb{N}$ , we can choose  $r_n \in \beta S$  for which  $cm_n r_n = p$ . Let  $z$  be a limit point of the sequence  $\langle m_n r_n \rangle_{n=1}^\infty$  in  $\beta S$ . Then  $z \in T$  and  $cz = p$ . It follows that  $\{z \in T : cz = p\}$  is a compact subsemigroup of  $\beta S$  and therefore contains an idempotent.  $\square$

In the proof of the following corollary we will use the fact that if  $X$  is an IP set in  $S$ , then  $X^v$  is an IP set in  $S^v$ . The easiest way to see this is to pick a sequence  $\langle x_n \rangle_{n=1}^\infty$  such that  $FS(\langle x_n \rangle_{n=1}^\infty) \subseteq X$  and note that  $FS(\langle (x_n, x_n, \dots, x_n) \rangle_{n=1}^\infty) \subseteq X^v$ .

**Corollary 4.9.** *Let  $(S, +)$  be an infinite commutative semigroup with identity 0. Let  $M$  be a nonempty subset of  $(\mathbb{N}, \cdot)$ , let  $T = \bigcap_{m \in M} \text{cl}_{\beta S}(mS)$ , and let  $p$  be a minimal idempotent in  $T$ . Assume that  $T$  is infinite. Let  $u, v \in \mathbb{N}$  and let  $D$  be a  $u \times v$  matrix which is appropriate for  $S$ . Assume that  $D$  is a first entries matrix and every first entry of  $D$  is in  $M$ . If  $p \in K(\beta S)$  or if  $M$  is a subsemigroup of  $(\mathbb{N}, \cdot)$ , then, for every  $C \in p$ , there exists  $\vec{x} \in (S \setminus \{0\})^v$  such that  $D\vec{x} \in C^u$ .*

*Proof.* We observe that Lemma 1.13 implies that  $p \neq 0$ .

We proceed by induction on  $u + v$ . We may presume that  $D$  has no repeated rows. If  $v = 1$ , then  $D = (c)$  for some  $c \in M$ . Since  $cS \in p$  and  $p \neq 0$ , there exists  $x \in S \setminus \{0\}$  such that  $cx \in C$ .

Now assume that  $v > 1$  and the conclusion holds for smaller values of  $u + v$ .

Case 1. There is no first entry of  $D$  in column  $v$ . Let  $A$  be the first  $v - 1$  columns of  $D$  and let  $\vec{b}$  be the last column. Let  $r = p$  and let  $U = S \setminus \{0\}$  noting that  $r \in T \subseteq \bigcap_{m \in M} \text{cl}_{\beta R}(mR)$ , where  $R$  is the group of differences of  $S$  if  $S$  is cancellative and  $R = S$  otherwise. By the induction hypothesis we have that for all  $C \in p$ , there exists  $\vec{x} \in S^{v-1}$  such that  $A\vec{x} \in C^u$ . By



Corollary 4.6, given  $C \in p$ , there exist  $\vec{x} \in (S \setminus \{0\})^{v-1}$  and  $y \in S \setminus \{0\}$  such that  $D \begin{pmatrix} \vec{x} \\ y \end{pmatrix} = A\vec{x} + \vec{b}y \in C$ .

Case 2. There is a first entry in column  $v$ . Note that since  $D$  is admissible, we must have  $u > 1$ . By permuting the rows of  $D$ , we may assume that there exist  $c \in M$ , a  $u - 1$  entry column vector  $\vec{b}$ , a  $v - 1$  entry row vector  $\vec{0}$ , and a  $(u - 1) \times (v - 1)$  first entries matrix  $A$  such that

$$D = \begin{pmatrix} A & \vec{b} \\ \vec{0} & c \end{pmatrix}.$$

By the induction hypothesis we have that for all  $C \in p$ , there exists  $\vec{x} \in S^{v-1}$  such that  $A\vec{x} \in C^{u-1}$ . By Lemma 4.8, pick an idempotent  $r \in \beta S$  such that  $cr = p$ , choosing  $r \in T$  in the case in which  $M$  is a subsemigroup of  $(\mathbb{N}, \cdot)$ . Let  $C \in p$  be given. Since  $r \neq 0$ , we can choose a member  $U$  of  $r$  with  $U \subseteq S \setminus \{0\}$ , such that  $cU \subseteq C$ . If  $p$  is a minimal idempotent in  $\beta S$ , it follows from Corollary 4.7 that there exist  $\vec{x} \in (S \setminus 0)^{v-1}$  and  $y \in U$  such that  $A\vec{x} + \vec{b}y \in C^{u-1}$ . If  $M$  is a subsemigroup of  $(\mathbb{N}, \cdot)$ , the same conclusion follows from Corollary 4.6. Hence, in either case,  $D \begin{pmatrix} \vec{x} \\ y \end{pmatrix} \in C^u$ .  $\square$

Before continuing our list of corollaries to Theorem 4.4, we pause to observe that the requirement that  $M$  is a subsemigroup of  $(\mathbb{N}, \cdot)$  cannot be simply deleted from Corollary 4.9.

**Theorem 4.10.** *Let  $M = \{2\}$  and let  $D = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$ . Then  $D$  is a first entries matrix with all first entries in  $M$  and there exists an infinite commutative semigroup  $(S, +)$  with identity  $0$ , a minimal idempotent  $p \in T = \text{cl}_{\beta S}(2S)$ , and  $C \in p$  such that there is no  $\vec{x} \in S^2$  with  $D\vec{x} \in C$ .*

*Proof.* Let  $S$  be an isomorphic copy of  $(\mathbb{N}, \cdot)$  converted to additive notation. Then, in terms of  $(\mathbb{N}, \cdot)$ , we are letting  $X = \{x^2 : x \in \mathbb{N}\}$ , letting  $T = \text{cl}_{\beta \mathbb{N}} X$ , and letting  $p$  be a minimal idempotent in  $(T, \cdot)$ . Let  $B = \{x^4 : x \in \mathbb{N}\}$ . We will show that  $X \setminus B \in p$  and there does not exist  $\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{N}^2$  such that

$$\begin{pmatrix} x^2 y \\ y^2 \end{pmatrix} \in (X \setminus B)^2.$$

To see that  $X \setminus B \in p$ , we will show that  $B$  is not piecewise syndetic in  $X$  (and therefore not a member of a minimal idempotent in  $T$ ). We use the combinatorial characterization of piecewise syndetic, stated immediately after Definition 1.15. Let  $G \in \mathcal{P}_f(X)$  be given and pick a prime  $p$  which is not a factor of any member of  $G$ . Let  $F = \{p^2, p^4\}$  and suppose one has  $x \in X$  and  $t$  and  $s$  in  $G$  such that  $tp^2x \in B$  and  $sp^4x \in B$ . This is impossible, since then the number of factors of  $p$  in  $p^2x$  and the number of factors of  $p$  in  $p^4x$  are both divisible by 4.

Now suppose one has  $x, y \in \mathbb{N}$  such that  $\begin{pmatrix} x^2y \\ y^2 \end{pmatrix} \in (X \setminus B)^2$ . Since  $x^2y \in X$ , we have  $y \in X$ . But then  $y^2 \in B$ .  $\square$

Central sets in any semigroup have much in common. Each is a member of a minimal idempotent. Any minimal idempotent is the identity of a maximal group contained in  $K(\beta S)$ , and any two such groups are isomorphic. We observe now that central sets can also be significantly different. Let  $S$  be the semigroup of Theorem 1.20, so that both  $\bigcap_{c=1}^{\infty} cS$  and  $S \setminus \bigcup_{c=2}^{\infty} cS$  are thick and therefore central. If  $p$  is any minimal idempotent in  $\bigcap_{c=1}^{\infty} cS$ , then by Corollary 4.9 any member of  $p$  contains images of any first entries matrix with entries from  $\mathbb{Z}$ . For any  $d \in \mathbb{N} \setminus \{1\}$ ,  $(d)$  is a first entries matrix which has no images in  $S \setminus \bigcup_{c=2}^{\infty} cS$ .

**Corollary 4.11.** *Let  $(S, +)$  be an infinite commutative semigroup with identity 0. Let  $M$  be a subsemigroup of  $(\mathbb{N}, \cdot)$ . If  $mS$  is infinite for every  $m \in M$ , then every first entries matrix which is appropriate for  $S$  and has all its first entries in  $M$  is IPR/ $S$ .*

*Proof.* Since  $\{cl(mS) \cap S^* : m \in M\}$  is a family of compact sets with the finite intersection property, if  $T = \bigcap_{m \in M} cl(mS)$ , then  $T \cap S^* \neq \emptyset$ . It follows that  $T$  is infinite, because a  $G_\delta$  subset of  $S^*$  cannot be finite by [10, Theorem 3.36]. Thus our claim follows from Corollary 4.9.  $\square$

**Corollary 4.12.** *Let  $(S, +)$  be an infinite commutative semigroup with identity 0. Then either*

- (a) *every first entries matrix which is appropriate for  $S$  is IPR/ $S$  or*
- (b) *there is a subsemigroup  $M$  of  $(\mathbb{N}, \cdot)$  such that  $M$  has positive additive density and every first entries matrix which is appropriate for  $S$  and has all first entries in  $M$ , is CIPR/ $S$ .*

*Proof.* If  $cS \neq \{0\}$  for every  $c \in \mathbb{N}$ , then by Theorem 3.3(a), every matrix which is IPR/ $\mathbb{N}$  is IPR/ $S$ . So assume that  $cS = \{0\}$  for some  $c$  and pick the least such  $c$ , noting that  $c > 1$ . Let  $M = \{m \in \mathbb{N} : m \equiv 1 \pmod{c}\}$ . Then  $mS = S$  for every  $m \in M$ . (Given  $m \in M$  pick  $n \in \omega$  such that  $m = nc + 1$ . Then for  $x \in S$ ,  $mx = ncx + x = x$ .) Thus, if  $T = \bigcap_{m \in M} cl_{\beta S}(mS)$ , we have  $T = \beta S$  and so, if  $p$  is any minimal idempotent in  $\beta S$ , we have  $p \in T$ . So our claim follows from Corollary 4.9.  $\square$

In the event that there is some  $c \in \mathbb{N} \setminus \{1\}$  such that  $cS$  is piecewise syndetic, we can identify a specific example of an infinite subsemigroup of  $(\mathbb{N}, \cdot)$  with the property that every first entries matrix which is appropriate for  $S$  and has all its first entries in  $M$  is IPR/ $S$ .

**Corollary 4.13.** *Let  $(S, +)$  be an infinite commutative semigroup with identity 0 and let  $M = \{c \in \mathbb{N} : cS \text{ is piecewise syndetic}\}$ . Every first entries matrix  $A$  with all its first entries in  $M$ , which is appropriate for  $S$ , is IPR/ $S$ .*

*Proof.* If  $M = \{1\}$ , then  $T = \beta S$  and every minimal idempotent of  $\beta S$  is in  $T$ . So every first entries matrix which is appropriate for  $S$  and has all its first entries in  $M$ , is *CIPR/S* by Corollary 4.9.

Now assume that  $|M| > 1$ . Let  $m$  denote the product of the first entries of  $A$ . By Theorem 1.19, for any minimal idempotent  $r$  in  $\beta S$ ,  $p = mr$  is also a minimal idempotent in  $\beta S$ . Since  $cS \in p$  for every first entry  $c$  of  $A$ , our claim follows from Corollary 4.9.  $\square$

We have an extension of Corollary 4.9 along the lines of [10, Theorem 15.5].

**Corollary 4.14.** *Let  $(S, +)$  be an infinite commutative semigroup with identity 0. Let  $M$  be a nonempty subset of  $\mathbb{N}$ , let  $T = \bigcap_{m \in M} c\ell_{\beta S}(mS)$ , assume that  $T$  is infinite, and let  $p$  be a minimal idempotent in  $T$ . Let  $u, v \in \mathbb{N}$  and let  $A$  be a  $u \times v$  first entries matrix which is appropriate for  $S$  such that every first entry of  $A$  is in  $M$ . If  $p \in K(\beta S)$  or  $M$  is a subsemigroup of  $(\mathbb{N}, \cdot)$ , then for all  $C \in p$ , there exists a sequence  $\langle \vec{x}_n \rangle_{n=1}^\infty$  in  $(S \setminus \{0\})^v$  such that for each  $F \in \mathcal{P}_f(\mathbb{N})$ ,  $A\vec{x}_F \in C^u$ , where  $\vec{x}_F = \sum_{t \in F} \vec{x}_t$ . If  $S$  is cancellative, the sequence  $\langle \vec{x}_n \rangle_{n=1}^\infty$  can be chosen so that the entries of each  $\vec{x}_F$  are distinct.*

*Proof.* By Corollary 4.9, for every  $C \in p$  there exists  $\vec{x} \in S^v$  such that  $A\vec{x} \in C^u$ . Let  $\bar{p} = (p, p, \dots, p) \in (\beta S)^u$  and let  $q$  be a minimal idempotent in  $\bar{A}^{-1}\{\bar{p}\}$ . By Corollary 4.5,  $(S \setminus \{0\})^v \in q$  and if  $S$  is cancellative, then, for every distinct  $i, j \in \{1, 2, \dots, v\}$ ,  $\{\vec{x} \in \beta(S^v) : x_i \neq x_j\} \in q$ . If  $S$  is cancellative, let  $Q = (S \setminus \{0\})^v \cap \bigcap_{i=1}^{v-1} \bigcap_{j=i+1}^v \{\vec{x} \in \beta(S^v) : x_i \neq x_j\}$ . Otherwise, let  $Q = (S \setminus \{0\})^v$ . By [10, Theorem 5.8], pick  $\langle \vec{x}_n \rangle_{n=1}^\infty$  such that  $FS(\langle \vec{x}_n \rangle_{n=1}^\infty) \subseteq Q$ .  $\square$

## 5. Images with distinct entries

We saw in Theorem 3.3 that if  $S$  is cancellative,  $cS \neq \{0\}$  for every  $c \in \mathbb{N}$ , and  $A$  is a  $u \times v$  matrix which is appropriate for  $S$  and is *IPR/N*, then  $A$  is *IPR/S* in the strong sense that, given any finite coloring of  $S \setminus \{0\}$ , there exists  $\vec{x} \in (S \setminus \{0\})^v$  such that  $A\vec{x}$  is monochromatic,  $x_i \neq x_j$  if  $i \neq j$ , and entries of  $A\vec{x}$  corresponding to distinct rows of  $A$  are not equal. And we recall that all first entries matrices are *IPR/N*. In this section we investigate when we can get similar conclusions for all members of certain idempotents. In particular, we shall see in Theorem 5.6 that we can get a similar result for a large class of first entries matrices if  $S$  is a cancellative semigroup and  $cS = \{0\}$ .

We start by determining conditions under which one can get distinct entries for all members of a given minimal idempotent when  $S$  is cancellative.

**Lemma 5.1.** *Let  $(S, +)$  be an infinite commutative cancellative semigroup with identity 0, let  $u, v \in \mathbb{N}$  with  $u > 1$ , and let  $A$  be an admissible  $u \times v$  matrix with entries from  $\mathbb{Z}$ . Assume that whenever  $S \setminus \{0\}$  is finitely colored, there exists  $\vec{x} \in S^v$  such that if  $\vec{y} = A\vec{x}$ , then the entries of  $\vec{y}$  are monochromatic and  $y_i \neq y_j$  when  $1 \leq i < j \leq u$ . Then whenever  $1 \leq i < j \leq u$ , there exists  $k \in \{1, 2, \dots, v\}$  such that  $(a_{ik} - a_{jk})S$  is infinite.*

*Proof.* Let  $G$  be the group of differences of  $S$ . Let  $\mathcal{G} = \{C \subseteq S \setminus \{0\} : \text{there exists } \vec{x} \in S^v \text{ such that if } \vec{y} = A\vec{x}, \text{ then } \vec{y} \in C^u \text{ and if } 1 \leq i < j \leq u, \text{ then } y_i \neq y_j\}$ . Then by assumption, whenever  $S \setminus \{0\}$  is finitely colored, there is a monochromatic member of  $\mathcal{G}$  so by [10, Theorem 5.7], pick  $p \in \beta S$  such that for all  $C \in p$ ,  $C \setminus \{0\} \in \mathcal{G}$ .

Let  $1 \leq i < j \leq u$  and suppose that for each  $k \in \{1, 2, \dots, v\}$ ,  $(a_{ik} - a_{jk})S$  is finite. For  $k \in \{1, 2, \dots, v\}$ , let  $H_k = (a_{ik} - a_{jk})S$  and let  $F = \sum_{k=1}^v H_k$ . Note that  $F$  is finite. Given  $a \in F \setminus \{0\}$ , we have that  $a + p \neq p$  by [10, Corollary 8.2] so pick  $B_a \in p \setminus (a + p)$  and let  $C_a = B_a \cap (-a + (G \setminus B_a))$ . Then  $C_a \in p$  and there is no  $x \in C_a$  such that  $a + x \in C_a$ .

Let  $C = \bigcap_{a \in F \setminus \{0\}} C_a$ . Then  $C \in p$  so pick  $\vec{x} \in S^v$  such that if  $\vec{y} = A\vec{x}$ , then  $\vec{y} \in C^u$  and  $y_i \neq y_j$ . Let  $a = y_i - y_j$ . Then  $a = \sum_{k=1}^v a_{ik}x_k - \sum_{k=1}^v a_{jk}x_k = \sum_{k=1}^v (a_{ik} - a_{jk})x_k \in \sum_{k=1}^v H_k = F$  and  $a \neq 0$ . Then  $y_j \in C_a$  and  $a + y_j = y_i \in C_a$ , a contradiction.  $\square$

**Theorem 5.2.** *Let  $(S, +)$  be an infinite commutative cancellative semigroup with identity 0, let  $v \in \mathbb{N} \setminus \{1\}$ , let  $1 \leq i < j \leq v$ , and let  $E = \{\vec{x} \in S^v : x_i = x_j\}$ . Then  $E$  is not piecewise syndetic in  $S^v$ .*

*Proof.* Assume that  $E$  is piecewise syndetic in  $S^v$ . Pick  $q \in \overline{E} \cap K(\beta(S^v))$  and let  $G$  be the group of differences of  $S$ . Then  $G^v$  is the group of differences of  $S^v$  so by Lemma 1.12,  $q \in K(\beta(G^v))$ . Define a surjective homomorphism  $h : G^v \rightarrow G$  by  $h(\vec{x}) = x_i - x_j$  and let  $\tilde{h} : \beta(G^v) \rightarrow \beta G$  be its continuous extension. By [10, Exercise 3.4.1 and Corollary 4.22],  $\tilde{h}$  is a surjective homomorphism. Since  $q \in \overline{E}$ ,  $\tilde{h}(q) = 0$  so by Lemma 1.14,  $\tilde{h}[\beta(G^v)]$  is finite, which is impossible.  $\square$

**Theorem 5.3.** *Let  $(S, +)$  be an infinite commutative cancellative semigroup with identity 0, let  $u, v \in \mathbb{N}$  with  $u > 1$ , and let  $A$  be an admissible  $u \times v$  matrix with entries from  $\mathbb{Z}$ . Let  $M$  be a subsemigroup of  $(\mathbb{N}, \cdot)$  for which  $T = \bigcap_{m \in M} \text{cl}_{\beta S}(mS)$  is infinite. Let  $p$  be a minimal idempotent in  $T$  and assume that for all  $C \in p$ , there exists  $\vec{x} \in S^v$  such that  $A\vec{x} \in C^u$ . Assume also that, for every  $m \in M$  and all distinct  $i, j \in \{1, 2, \dots, u\}$ , there exists  $k \in \{1, 2, \dots, v\}$  such that  $(a_{ik} - a_{jk})mS$  is infinite. Then, for every  $C \in p$ , there exists  $\vec{x} \in (S \setminus \{0\})^v$ , with distinct entries, such that the entries of  $A\vec{x}$  are distinct elements of  $C$ .*

*Proof.* Let  $\bar{p} = (p, p, \dots, p) \in (\beta S)^u$  and let  $q$  be a minimal idempotent in  $\tilde{A}^{-1}[\{\bar{p}\}]$ . By Lemma 4.5,  $(S \setminus \{0\})^v \in q$  and for each distinct  $i$  and  $j$  in  $\{1, 2, \dots, v\}$ ,  $\{\vec{x} \in S^v : x_i \neq x_j\} \in q$ . It thus suffices to show that for each distinct  $i$  and  $j$  in  $\{1, 2, \dots, u\}$ ,  $\{\vec{x} \in S^v : (A\vec{x})_i \neq (A\vec{x})_j\} \in q$ . For then, given  $C \in p$ , there exists  $D \in q$  such that  $A[D] \subseteq C^u$  and  $D \subseteq (S \setminus \{0\})^v \cap \bigcap_{i=1}^{v-1} \bigcap_{j=i+1}^v \{\vec{x} \in S^v : x_i \neq x_j\} \cap \bigcap_{i=1}^{u-1} \bigcap_{j=i+1}^u \{\vec{x} \in S^v : (A\vec{x})_i \neq (A\vec{x})_j\}$ .

So let  $i$  and  $j$  be distinct elements of  $\{1, 2, \dots, u\}$ , let  $Q = \{\vec{x} \in S^v : (A\vec{x})_i = (A\vec{x})_j\}$  and suppose that  $Q \in q$ . Let  $G$  be the group of differences of  $S$ , let  $\theta : S^v \rightarrow G$  be defined by  $\theta(\vec{x}) = (A\vec{x})_i - (A\vec{x})_j$ , and let  $\tilde{\theta} : \beta(S^v) \rightarrow \beta G$  be the continuous extension of  $\theta$ . Then  $\tilde{\theta}(q) = 0$  and by Lemma 1.14,  $\tilde{\theta}[\tilde{A}^{-1}[\{\bar{p}\}]]$

is a finite group. We can choose  $k \in \{1, 2, \dots, v\}$  such that  $(a_{ik} - a_{jk})mS$  is infinite for every  $m \in S$ . (If for each  $k \in \{1, 2, \dots, v\}$ , there were some  $m_k \in M$  such that  $(a_{ik} - a_{jk})m_k S$  is finite and  $m = m_1 \cdot m_2 \cdots m_v$ , then we would have  $(a_{ik} - a_{jk})mS$  is finite for each  $k$ .) Since the intersection of any finite number of these sets is infinite,  $\bigcap_{m \in M} \mathcal{C}_{\beta G}((a_{ik} - a_{jk})mS) \cap G^* = (a_{ik} - a_{jk})T \cap G^*$  is a compact subsemigroup of  $\beta G$ , and so it contains an idempotent  $e$ . Since  $\{r \in T : (a_{ik} - a_{jk})r = e\}$  is a compact subsemigroup of  $T$ , we can choose an idempotent  $r \in T$  for which  $(a_{ik} - a_{jk})r = e$ .

Let  $B$  denote the  $u \times 1$  matrix equal to the  $k^{\text{th}}$  column of  $A$ . By Theorem 4.4, for every  $C \in p$  and every  $U \in r$ , there exists  $\vec{x}_{C,U} \in Q$  and  $s_{C,U} \in U \cap S$  such that  $A\vec{x}_{C,U} + Bs_{C,U} \subseteq C^u$ . Let  $\vec{z}_{C,U} \in S^v$  denote the vector for which  $\vec{z}_{C,U}(k) = \vec{x}_{C,U}(k) + s_{C,U}$  and  $\vec{z}_{C,U}(t) = \vec{x}_{C,U}(t)$  for  $t \in \{1, 2, \dots, v\} \setminus \{k\}$ . Then  $A\vec{z}_{C,U} = A\vec{x}_{C,U} + Bs_{C,U}$ . Observe that  $\theta(\vec{z}_{C,U}) = (a_{ik} - a_{jk})s_{C,U}$ . We give  $p \times r$  the natural directed set ordering by stating that  $(C, U) \prec (C', U')$  if  $C' \subseteq C$  and  $U' \subseteq U$ . Let  $q'$  denote a limit point in  $\beta(S^v)$  of the net  $\langle \vec{z}_{C,U} \rangle_{(C,U) \in p \times r}$ . Then  $\tilde{A}(q') = \tilde{p}$  and  $\tilde{\theta}(q') = e$ . This is a contradiction, because it implies that the group  $\tilde{\theta}[\tilde{A}^{-1}[\{\tilde{p}\}]]$  contains an idempotent  $e$  distinct from 0.  $\square$

**Theorem 5.4.** *Let  $(S, +)$  be an infinite commutative cancellative semigroup with identity 0, let  $u, v \in \mathbb{N}$  with  $u > 1$ , and let  $A$  be an admissible  $u \times v$  matrix with entries from  $\mathbb{Z}$ . Let  $p$  be a minimal idempotent in  $\beta S$  and assume that for all  $C \in p$ , there exists  $\vec{x} \in S^v$  such that  $A\vec{x} \in C^u$ . The following statements are equivalent.*

- (1) *For all  $C \in p$ , there exists  $\vec{x} \in (S \setminus \{0\})^v$  such that  $x_i \neq x_j$  if  $1 \leq i < j \leq v$  and if  $\vec{y} = A\vec{x}$ , then  $\vec{y} \in C^u$  and  $y_i \neq y_j$  if  $1 \leq i < j \leq u$ .*
- (2) *There exists  $r \in \beta S$  such that for all  $C \in r$ , there exists  $\vec{x} \in S^v$  such that  $A\vec{x} \in C^u$  and if  $\vec{y} = A\vec{x}$  and  $1 \leq i < j \leq u$ , then  $y_i \neq y_j$ .*
- (3) *Whenever  $S \setminus \{0\}$  is finitely colored, there exists  $\vec{x} \in S^v$  such that if  $\vec{y} = A\vec{x}$ , then the entries of  $\vec{y}$  are monochromatic and  $y_i \neq y_j$  when  $1 \leq i < j \leq u$ .*
- (4) *Whenever  $S \setminus \{0\}$  is finitely colored, there exists  $\vec{x} \in (S \setminus \{0\})^v$  such that  $x_i \neq x_j$  if  $1 \leq i < j \leq v$  and if  $\vec{y} = A\vec{x}$ , then the entries of  $\vec{y}$  are monochromatic and  $y_i \neq y_j$  when  $1 \leq i < j \leq u$ .*
- (5) *Whenever  $1 \leq i < j \leq u$ , there exists  $k \in \{1, 2, \dots, v\}$  such that  $(a_{ik} - a_{jk})S$  is infinite.*

*Proof.* It is trivial that (1) implies (2). If  $r$  is as in statement (2), then one can't have  $\{0\} \in r$  so if  $S \setminus \{0\}$  is finitely colored, then some color class is in  $r$  and thus (2) implies (3). The fact that (3) implies (5) is Lemma 5.1. It is trivial that (1) implies (4) and (4) implies (3). It remains to show that (5) implies (1). But this follows from Theorem 5.3 with  $M = \{1\}$ .  $\square$

Statements (2) through (5) of Theorem 5.4 do not mention the given idempotent  $p$ , however, our proof that (5) implies (1), and therefore (5) implies (4),

strongly uses the fact that  $p$  is minimal in  $\beta S$ . We see that this assumption is required.

**Theorem 5.5.** *Let  $S_1 = \bigoplus_{n=1}^{\infty} \mathbb{Z}_2$ ,  $S_2 = \bigoplus_{n=1}^{\infty} \mathbb{Z}_3$ , and  $S = S_1 \times S_2$ . Let  $A = \begin{pmatrix} 2 & 0 \\ 2 & 3 \end{pmatrix}$ . There is an idempotent  $p \in \beta S$  such that for all  $C \in p$ , there exists  $\vec{x} \in S^2$  such that  $A\vec{x} \in C^2$ , conclusion (5) of Theorem 5.4 holds, and conclusion (4) does not.*

*Proof.* Pick an idempotent  $p \in \beta S$  such that  $\{(\vec{0}, \vec{b}) : \vec{b} \in S_2\} \in p$ . Let  $C \in p$  and pick  $\vec{b} \in S_2$  such that  $(\vec{0}, \vec{b}) \in C$ . Let  $\vec{x} = \begin{pmatrix} (\vec{0}, 2\vec{b}) \\ (\vec{0}, \vec{b}) \end{pmatrix}$ . Then  $A\vec{x} = \begin{pmatrix} (\vec{0}, \vec{b}) \\ (\vec{0}, \vec{b}) \end{pmatrix} \in C^2$ .

Since  $\{-3\vec{s} : \vec{s} \in S\}$  is infinite, conclusion (5) is satisfied. To see that conclusion (4) is not satisfied, let  $C_1 = \{(\vec{0}, \vec{b}) : \vec{b} \in S_2\}$  and let  $C_2 = S \setminus C_1$ . Suppose we have  $\vec{x} = \begin{pmatrix} (\vec{a}, \vec{b}) \\ (\vec{c}, \vec{d}) \end{pmatrix} \in S^2$  such that the entries of  $A\vec{x}$  are monochromatic and distinct. Then  $A\vec{x} = \begin{pmatrix} (\vec{0}, 2\vec{b}) \\ (3\vec{c}, 2\vec{b}) \end{pmatrix}$ . Since the entries are monochromatic,  $3\vec{c} = \vec{0}$ , so the entries are not distinct.  $\square$

We conclude with our promised extension of parts of [10, Theorem 5.5].

**Theorem 5.6.** *Let  $(S, +)$  be an infinite commutative cancellative semigroup with identity 0 and assume that  $c \in \mathbb{N}$  and  $cS = \{0\}$ . Let  $M = \{m \in \mathbb{N} : (c, m) = 1\}$ . Let  $u, v \in \mathbb{N}$  and let  $A$  be a  $u \times v$  first entries matrix which is appropriate for  $S$  and all of whose first entries are in  $M$ . Let  $T = \bigcap_{m \in M} cl_{\beta S}(mS)$ . Then  $T \cap S^*$  is a subsemigroup of  $\beta S$  and if  $p$  is a minimal idempotent in  $T$ , then for every  $C \in p$ , there exists  $\vec{x} \in (S \setminus \{0\})^v$  such that  $A\vec{x} \in C^u$  and  $x_i \neq x_j$  whenever  $i \neq j$  in  $\{1, 2, \dots, v\}$ .*

*Proof.* Observe that  $S$  is a group, since every element of  $S$  has finite order. If  $m \in M$ ,  $am + bc = 1$  for some  $a, b \in \mathbb{Z}$ . Given  $x \in S$ ,  $x = (am + bc)x = m(ax) + 0$  and so  $mS = S$ . Consequently  $T = \beta S$  and  $p$  is a minimal idempotent in  $\beta S$ . Therefore Corollary 4.6 applies.  $\square$

## References

## References

- [1] V. Bergelson and N. Hindman, *Nonmetrizable topological dynamics and Ramsey Theory*, Trans. Amer. Math. Soc. **320** (1990), 293-320.
- [2] W. Deuber, *Partitionen und lineare Gleichungssysteme*, Math. Z. **133** (1973), 109-123.

- [3] L. Fuchs, *Abelian groups*, Pergamon Press, New York-Oxford-London-Paris, 1960.
- [4] H. Furstenberg, *Recurrence in ergodic theory and combinatorial number theory*, Princeton University Press, Princeton, 1981.
- [5] H. Furstenberg and Y. Katznelson, *Idempotents in compact semigroups and Ramsey Theory*, Israel J. Math. **68** (1989), 257-270.
- [6] N. Hindman, *Partition regularity of matrices*, in Combinatorial Number Theory, B. Landman, M. Nathanson, J. Nešetřil, R. Nowakowski, and C. Pomerance, editors, deGruyter, Berlin, 2007, 265-298. Also published as *Integers* **7(2)** (2007), A-18. <http://www.integers-ejcnt.org/vol7-2.html>
- [7] N. Hindman and J. Johnson, *Images of  $C$  sets and related large sets under nonhomogeneous spectra*, *Integers* **12A** (2012), Article 2. <http://integers-ejcnt.org/vol12b.html>
- [8] N. Hindman and I. Leader, *Image partition regularity of matrices*, *Combin. Probab. Comput.* **2** (1993), 437-463.
- [9] N. Hindman and D. Strauss, *Image partition regularity over the integers, rationals, and reals*, *New York J. Math.* **11** (2005) 519-538.
- [10] N. Hindman and D. Strauss, *Algebra in the Stone-Čech compactification: theory and applications, 2nd edition*, Walter de Gruyter & Co., Berlin, 2012.
- [11] R. Rado, *Studien zur Kombinatorik*, *Math. Z.* **36** (1933), 242-280.
- [12] R. Rado, *Note on combinatorial analysis*, *Proc. Lond. Math. Soc.* **48** (1943), 122-160.
- [13] H. Shi and H. Yang, *Nonmetrizable topological dynamical characterization of central sets*, *Fund. Math.* **150** (1996), 1-9.