This paper was published in *Annals of the New York Academy of Sciences* **806** (1996), 28-41. To the best of my knowledge, this is the final version as it was submitted to the publisher.–NH

IP* Sets in Product Spaces

Vitaly Bergelson[1]

and

Neil Hindman[1]

Abstract. An IP* set in a semigroup $(S,\cdot)$ is a set which meets every set of the form $FP(\langle x_n\rangle_{n=1}^\infty) = \{\Pi_{n\in F}\ x_n : F$ is a finite nonempty subset of $\mathbb{N}\}$, where the products are taken in increasing order of indices. We show here, using the Stone-Čech compactification of the product space $S_1 \times S_2 \times \ldots \times S_\ell$, that if each S_i is commutative, then whenever C is an IP* set in $S_1 \times S_2 \times \ldots \times S_\ell$ and for each $i \in \{1, 2, \ldots, \ell\}$, $\langle x_{i,n}\rangle_{n=1}^\infty$ is a sequence in S_i, C contains cartesian products of arbitrarily large finite substructures of $FP(\langle x_{1,n}\rangle_{n=1}^\infty) \times FP(\langle x_{2,n}\rangle_{n=1}^\infty) \times \ldots \times FP(\langle x_{\ell,n}\rangle_{n=1}^\infty)$. (The notion of "substructure" is made precise in Definition 2.4.) We show further that C need not contain any product of infinite substructures and that the commutativity hypothesis may not be omitted. Similar results apply to arbitrary finite products of semigroups. By way of contrast, we show in Theorem 2.3 that a much stronger conclusion holds for some cell of any finite partition of $S_1 \times S_2 \times \ldots \times S_\ell$ without even any commutativity assumptions.

1. Introduction.

In a semigroup $(S,\cdot)$, we write $\Pi_{n\in F}\ x_n$ for the product written in increasing order of indices. (Thus $\Pi_{n\in\{1,5,7\}}\ x_n = x_1 \cdot x_5 \cdot x_7$.) We further write $FP(\langle x_n\rangle_{n=1}^\infty) = \{\Pi_{n\in F}\ x_n : F \in \mathcal{P}_f(\mathbb{N})\}$ where $\mathcal{P}_f(\mathbb{N}) = \{A : A$ is a finite nonempty subset of $\mathbb{N}\}$ and $\mathbb{N}$ is the set of positive integers. (Similarly, if the operation of the semigroup is denoted by $+$, we write $FS(\langle x_n\rangle_{n=1}^\infty) = \{\Sigma_{n\in F}\ x_n : F \in \mathcal{P}_f(\mathbb{N})\}$.) Loosely following Furstenberg [5] we say that a set $A \subseteq S$ is an IP set if and only if there is a sequence $\langle x_n\rangle_{n=1}^\infty$ in S with $FP(\langle x_n\rangle_{n=1}^\infty) \subseteq A$. A set $C \subseteq S$ is then an IP* set if and only if $C \cap A \neq \emptyset$ for every IP set A (equivalently if and only if $C \cap FP(\langle x_n\rangle_{n=1}^\infty) \neq \emptyset$ for every sequence $\langle x_n\rangle_{n=1}^\infty$ in S).

[1] The authors acknowledge support received from the National Science Foundation via grants DMS 9401093 and DMS 9424421 respectively.
AMS Subject Classifications: Primary 05D10, 54B10; *Secondary* 22A30.
Key Words and Phrases: IP* sets, idempotents, product spaces, finite products.

Now IP* sets not only must meet every IP set; a much stronger statement is true. This statement involves the notion of a *product subsystem* which we pause now to define.

1.1 Definition. Let $(S, \cdot)$ be a semigroup and let $\langle y_n \rangle_{n=1}^{\infty}$ be a sequence in S. The sequence $\langle x_n \rangle_{n=1}^{\infty}$ is a *product subsystem* of $\langle y_n \rangle_{n=1}^{\infty}$ if and only if there is a sequence $\langle H_n \rangle_{n=1}^{\infty}$ in $\mathcal{P}_f(\mathbb{N})$ such that

(a) for each $n \in \mathbb{N}$, $\max H_n < \min H_{n+1}$ and

(b) for each $n \in \mathbb{N}$, $x_n = \Pi_{t \in H_n}\ y_t$.

Note that if $\langle x_n \rangle_{n=1}^{\infty}$ is a product subsystem of $\langle y_n \rangle_{n=1}^{\infty}$, then $FP(\langle x_n \rangle_{n=1}^{\infty}) \subseteq FP(\langle y_n \rangle_{n=1}^{\infty})$. (If requirement (a) of the definition is replaced by the requirement that $H_n \cap H_m = \emptyset$ when $n \neq m$, this conclusion may fail if S is not commutative. For example, if $x_1 = y_1 \cdot y_3$ and $x_2 = y_2$, then $x_1 \cdot x_2$ need not be in $FP(\langle y_n \rangle_{n=1}^{\infty})$.) In the event the operation in S is denoted by $+$, we change products to sums and refer to a *sum subsystem.*

The much stronger statement to which we referred is that given any IP* set $C \subseteq S$ and any sequence $\langle y_n \rangle_{n=1}^{\infty}$ in S, there is a product subsystem $\langle x_n \rangle_{n=1}^{\infty}$ of $\langle y_n \rangle_{n=1}^{\infty}$ such that $FP(\langle x_n \rangle_{n=1}^{\infty}) \subseteq C$. This statement will be proved as Corollary 1.7 below. (It is well known, but we don't have a convenient reference for it.)

It was shown in [2] that IP* sets may be large in unexpected ways. For example [2, Theorem 2.6] an IP* set in $(\mathbb{N}, +)$ must contain an infinite sequence with all of its sums (as expected) *and* its products as well. We investigate in this paper the extent to which IP* sets in product semigroups must be large in terms of sequences in the coordinates. One may ask for example, given semigroups S_1 and S_2, an IP* set C in $S_1 \times S_2$, and sequences $\langle w_n \rangle_{n=1}^{\infty}$ in S_1 and $\langle z_n \rangle_{n=1}^{\infty}$ in S_2, whether there must be product subsystems $\langle x_n \rangle_{n=1}^{\infty}$ of $\langle w_n \rangle_{n=1}^{\infty}$ and $\langle z_n \rangle_{n=1}^{\infty}$ of $\langle z_n \rangle_{n=1}^{\infty}$ with $FP(\langle x_n \rangle_{n=1}^{\infty}) \times FP(\langle y_n \rangle_{n=1}^{\infty}) \subseteq C$.

We give a strong negative answer to this question in Section 2. One does find however that if S_1 and S_2 are commutative semigroups, then any IP* set in $S_1 \times S_2$ must contain the product of arbitrarily large finite substructures of $\langle w_n \rangle_{n=1}^{\infty}$ and $\langle z_n \rangle_{n=1}^{\infty}$ and a similar statement applies to any finite product of semigroups (Theorem 2.6).

We also see a surprising turning of the tables. One is accustomed to finding properties that must hold for some cell of any finite partition and automatically expecting at least such a conclusion for IP* sets. For example, as we have already mentioned, while one cell of a partition of $\mathbb{N}$ must contain $FS(\langle x_n\rangle_{n=1}^{\infty})$ for some sequence $\langle x_n\rangle_{n=1}^{\infty}$, any IP* set must contain $FS(\langle x_n\rangle_{n=1}^{\infty}) \cup FP(\langle x_n\rangle_{n=1}^{\infty})$. Similarly, we will see below that given any discrete semigroup S and any idempotent p in βS, while trivially p must belong to the closure of some cell of any partition of S, p must belong also to the closure of *every* IP* set.

By contrast we will see in Theorems 2.2 and 2.7 that one can guarantee a much stronger conclusion for one cell of any partition of the product of two semigroups than can be guaranteed for IP* sets in the same product.

A special semigroup is of significant interest for these problems, namely the semigroup $(\mathcal{P}_f(\mathbb{N}), \cup)$ because a version of Theorem 2.6 for this semigroup is sufficient to imply the validity of Theorem 2.6 for all semigroups. We present this derivation in Section 3.

Our proofs utilize the algebraic structure of the Stone-Čech compactification βS of a discrete semigroup $(S, \cdot)$. We take βS to be the set of all ultrafilters on S, identifying the principal ultrafilters with the points of S. We denote also by $\cdot$ the operation on βS making $(\beta S, \cdot)$ a right topological semigroup with S contained in its topological center. That is, for all $p \in \beta S$, the function $\rho_p : \beta S \longrightarrow \beta S$ defined by $\rho_p(q) = q \cdot p$ is continuous and for all $x \in S$, the function $\lambda_x : \beta S \longrightarrow \beta S$ defined by $\lambda_x(q) = x \cdot q$ is continuous. The reader is referred to [6] and [7] for an elementary introduction to this operation, with the caution that there $(\beta S, \cdot)$ is taken to be left rather than right topological. (We have made the switch to conform to majority usage, at least among our collaborators.) The basic fact characterizing the right continuous operation on βS is, given $p, q \in \beta S$ and $A \subseteq S$, $A \in p \cdot q$ if and only if $\{x \in S : x^{-1}A \in q\} \in p$ where $x^{-1}A = \{y \in S : x \cdot y \in A\}$. In the event that the operation is denoted by $+$, the characterization above becomes $A \in p + q$ if and only if $\{x \in S : -x + A \in q\} \in p$ where $-x + A = \{y \in S : x + y \in A\}$.

We will use only a few basic facts about $(\beta S, \cdot)$ which we present here.

1.2 Theorem. *Any compact right topological semigroup has an idempotent.*

Proof. [4, Corollary 2.10]. □

The proof of the following theorem is the Galvin-Glazer proof of the Finite Sum Theorem, which was never published by them, though it has appeared in surveys. Those places where it has appeared (to our knowledge) however use left continuity so that the resulting products are in decreasing order of indices. To minimize confusion, we present the proof here.

1.3 Theorem. *Let $(S,\cdot)$ be a semigroup, let p be an idempotent in βS, and let $A \in p$. There is a sequence $\langle x_n \rangle_{n=1}^\infty$ in S such that $FP(\langle x_n \rangle_{n=1}^\infty) \subseteq A$.*

Proof. Let $A_1 = A$ and let $B_1 = \{x \in S : x^{-1}A_1 \in p\}$. Since $A_1 \in p = p \cdot p$, $B_1 \in p$. Pick $x_1 \in B_1 \cap A_1$, let $A_2 = A_1 \cap ({x_1}^{-1}A_1)$, and note that $A_2 \in p$. Inductively given $A_n \in p$, let $B_n = \{x \in S : x^{-1}A_n \in p\}$. Since $A_n \in p = p \cdot p$, $B_n \in p$. Pick $x_n \in B_n \cap A_n$ and let $A_{n+1} = A_n \cap ({x_n}^{-1}A_n)$.

To see for example why $x_2 \cdot x_4 \cdot x_5 \cdot x_7 \in A$, note that $x_7 \in A_7 \subseteq A_6 \subseteq {x_5}^{-1}A_5$ so that $x_5 \cdot x_7 \in A_5 \subseteq {x_4}^{-1}A_4$. Thus $x_4 \cdot x_5 \cdot x_7 \in A_4 \subseteq A_3 \subseteq {x_2}^{-1}A_2$ so that $x_2 \cdot x_4 \cdot x_5 \cdot x_7 \in A_2 \subseteq A_1 = A$.

More formally, we show by induction on $|F|$ that if $F \in \mathcal{P}_f(\mathbb{N})$ and $m = \min F$ then $\Pi_{n\in F}\, x_n \in A_m$. If $|F| = 1$, then $\Pi_{n\in F}\, x_n = x_m \in A_m$. Assume $|F| > 1$, let $G = F\backslash\{m\}$, and let $k = \min G$. Note that since $k > m$, $A_k \subseteq A_{m+1}$. Then by the induction hypothesis, $\Pi_{n\in G}\, x_n \in A_k \subseteq A_{m+1} \subseteq {x_m}^{-1}A_m$ so $\Pi_{n\in F}\, x_n = x_m \cdot \Pi_{n\in G}\, x_n \in A_m$. □

1.4 Theorem. *Let S be a semigroup and let $\langle x_n \rangle_{n=1}^\infty$ be a sequence in S. Then $\bigcap_{m=1}^\infty c\ell FP(\langle x_n \rangle_{n=m}^\infty)$ is a subsemigroup of βS. In particular, there is an idempotent p in βS such that for each $m \in \mathbb{N}$, $FP(\langle x_n \rangle_{n=m}^\infty) \in p$.*

Proof. Let $T = \bigcap_{m=1}^\infty c\ell FP(\langle x_n \rangle_{n=m}^\infty)$. Then T is the intersection of a nested collection of closed nonempty subsets of βS so $T \neq \emptyset$. To see that T is a semigroup, let $p, q \in T$ be given and let $m \in \mathbb{N}$. To see that $FP(\langle x_n \rangle_{n=m}^\infty) \in p \cdot q$ we show that $FP(\langle x_n \rangle_{n=m}^\infty) \subseteq \{s \in S : s^{-1}FP(\langle x_n \rangle_{n=m}^\infty) \in q\}$. (Since $FP(\langle x_n \rangle_{n=m}^\infty) \in p$, this will suffice.) To this end, let $s \in FP(\langle x_n \rangle_{n=m}^\infty)$ be given and pick $F \in \mathcal{P}_f(\mathbb{N})$ with $\min F \geq m$ such that $s = \Pi_{n\in F}\ x_n$. Let $k = \max F + 1$. Then $FP(\langle x_n\rangle_{n=k}^\infty) \in q$ so it suffices to show that $FP(\langle x_n\rangle_{n=k}^\infty) \subseteq s^{-1}FP(\langle x_n\rangle_{n=m}^\infty)$. So, let $t \in FP(\langle x_n\rangle_{n=k}^\infty)$ be given and pick $G \in \mathcal{P}_f(\mathbb{N})$ with $\min G \geq k$ such that $t = \Pi_{n\in G}\ x_t$. Then $\max F < \min G$ so $st = \Pi_{n\in F\cup G}\ x_n \in FP(\langle x_n\rangle_{n=m}^\infty)$.

For the "in particular" conclusion note that by Theorem 1.2, T has an idempotent. □

The following simple characterization of IP* sets also provides some explanation of the origin of "IP".

1.5 Theorem. *Let $(S,\cdot)$ be a semigroup and let $C \subseteq S$. Then C is an IP* set if and only if for every idempotent $p \in \beta S$, $C \in p$.*

Proof. Necessity. Let $p\cdot p = p \in \beta S$ and suppose $C \notin p$. Then $S\backslash C \in p$ so by Theorem 1.3 there is a sequence $\langle x_n\rangle_{n=1}^\infty$ with $FP(\langle x_n\rangle_{n=1}^\infty) \subseteq S\backslash C$, a contradiction.

Sufficiency. Let $\langle x_n\rangle_{n=1}^\infty$ be a sequence in S and pick by Theorem 1.4 an idempotent p with $FP(\langle x_n\rangle_{n=1}^\infty) \in p$. Then $C \cap FP(\langle x_n\rangle_{n=1}^\infty) \in p$ so $C \cap FP(\langle x_n\rangle_{n=1}^\infty) \neq \emptyset$. □

1.6 Theorem. *Let $(S,\cdot)$ be a semigroup, let $\langle y_n\rangle_{n=1}^\infty$ be a sequence in S, and let p be an idempotent in $\bigcap_{m=1}^\infty c\ell(FP(\langle y_n\rangle_{n=m}^\infty))$. If $A \in p$, then there is a product subsystem $\langle x_n\rangle_{n=1}^\infty$ of $\langle y_n\rangle_{n=1}^\infty$ with $FP(\langle x_n\rangle_{n=1}^\infty) \subseteq A$.*

Proof. Pick by Theorem 1.4 an idempotent $p \in \beta S$ such that for each $m \in \mathbb{N}$, $FP(\langle x_n\rangle_{n=m}^\infty) \in p$. (The rest of the proof is now a modification of the proof of Theorem 1.3.)

Let $C_1 = A$, let $B_1 = \{y \in S : y^{-1}C_1 \in p\}$, and pick $y_1 \in B_1 \cap C_1$. Pick $H_1 \in \mathcal{P}_f(\mathbb{N})$ such that $y_1 = \Pi_{t\in H_1}\ x_t$. Let $k_1 = \max H_1 + 1$ and let $C_2 = C_1 \cap {y_1}^{-1}C_1 \cap FP(\langle x_t\rangle_{t=k_1}^\infty)$.

Inductively, given $n > 1$ and $k_{n-1} \in \mathbb{N}$ and $C_n \in p$ with $C_n \subseteq FP(\langle x_t\rangle_{t=k_{n-1}}^\infty)$, let $B_n = \{y \in S : y^{-1}C_n \in p\}$. Pick $y_n \in B_n \cap C_n$ and pick $H_n \in \mathcal{P}_f(\mathbb{N})$ with $\max H_n \geq k_{n-1}$ such that $y_n = \Pi_{t\in H_n}\ x_t$. Let $k_n = \max H_n + 1$ and let $C_{n+1} = C_n \cap {y_n}^{-1}C_n \cap FP(\langle x_t\rangle_{t=k_n}^\infty)$.

Then one immediately sees that $\langle y_n\rangle_{n=1}^\infty$ is a product subsystem of $\langle x_n\rangle_{n=1}^\infty$. Just as in the proof of Theorem 1.3 one sees that $FP(\langle y_n\rangle_{n=1}^\infty) \subseteq A$. □

The following corollary will be used in the proof of Theorem 2.9.

1.7 Corollary. *Given any IP* set $C \subseteq S$ and any sequence $\langle y_n\rangle_{n=1}^\infty$ in S, there is a product subsystem $\langle x_n\rangle_{n=1}^\infty$ of $\langle y_n\rangle_{n=1}^\infty$ such that $FP(\langle x_n\rangle_{n=1}^\infty) \subseteq C$.*

Proof. Let $T = \bigcap_{m=1}^\infty c\ell FP(\langle y_n\rangle_{n=m}^\infty)$. By Theorem 1.4 T is a compact subsemigroup of βS, so pick by Theorem 1.2 an idempotent $p \in T$. By Theorem 1.5 $C \in p$, so by Theorem 1.6 there is a product subsystem $\langle x_n\rangle_{n=1}^\infty$ of $\langle y_n\rangle_{n=1}^\infty$ such that $FP(\langle x_n\rangle_{n=1}^\infty) \subseteq C$. □

1.8 Theorem. *Let X be a set and let $\mathcal{G} \subseteq \mathcal{P}(X)$. The following statements are equivalent.*

(a) For each $r \in \mathbb{N}$, if $X = \bigcup_{i=1}^r A_i$, there exist $i \in \{1, 2, \ldots, r\}$ and $G \in \mathcal{G}$ with $G \subseteq A_i$.

(b) There is an ultrafilter p on X such that for each $A \in p$, there exists $G \in \mathcal{G}$ with $G \subseteq A$.

Proof. [7, Theorem 6.7]. □

We write $\mathbb{N}$ for the set of positive integers and ω for the set of nonnegative integers.

Acknowledgement: The authors would like to thank Paul Milnes for some helpful conversations.

2. IP* Sets in Products.

We show in this section that IP* sets in any finite product of commutative semigroups contain products of arbitrarily large finite subsystems of sequences in the coordinates. We first modify the notion of product subsystems (Definition 1.1) to apply to finite sequences. (Requirement (a) below is of course a triviality, but see the discussion following Definition 1.1 for the fact that requirement (b) can be important.)

2.1 Definition. Let $\langle y_n\rangle_{n=1}^\infty$ be a sequence in a semigroup S and let $k, m \in \mathbb{N}$. Then $\langle x_n\rangle_{n=1}^m$ is a *product subsystem* of $\langle y_n\rangle_{n=k}^\infty$ if and only if there exists a sequence $\langle H_n\rangle_{n=1}^m$ in $\mathcal{P}_f(\mathbb{N})$ such that

(a) $\min H_1 \geq k$,

(b) $\max H_n < \min H_{n+1}$ for each $n \in \{1, 2, \ldots, m-1\}$, and

(c) $x_n = \Pi_{t \in H_n} y_t$ for each $n \in \{1, 2, \ldots, m\}$.

We establish, using an argument that we have used before [1], a purely Ramsey Theoretic property in the product of arbitrary semigroups which is of interest in its own right, independent of IP* sets. It is similar in flavor to the Milliken-Taylor Theorem [8, 9]. We first present the two dimensional version, partly because it is simpler but also to present a stronger conclusion than will be needed here which will contrast sharply with Theorem 2.7 below.

2.2 Theorem. *Let S_1 and S_2 be semigroups and let $\langle y_{1,n}\rangle_{n=1}^{\infty}$ and $\langle y_{2,n}\rangle_{n=1}^{\infty}$ be sequences in S_1 and S_2 respectively. Let $k, r \in \mathbb{N}$ and let $S_1 \times S_2 = \bigcup_{j=1}^{r} C_j$. There exist $j \in \{1, 2, \ldots, r\}$ and a product subsystem $\langle x_{1,n}\rangle_{n=1}^{\infty}$ of $\langle y_{1,n}\rangle_{n=k}^{\infty}$ such that for each $m \in \mathbb{N}$ there is a product subsystem $\langle x_{2,n}\rangle_{n=1}^{\infty}$ of $\langle y_{2,n}\rangle_{n=k}^{\infty}$ such that*

$$FP(\langle x_{1,n}\rangle_{n=1}^{m}) \times FP(\langle x_{2,n}\rangle_{n=1}^{\infty}) \subseteq C_i.$$

Proof. Given $i \in \{1, 2\}$, pick by Theorem 1.4 an idempotent

$$p_i \in \bigcap_{k=1}^{\infty} c\ell_{\beta S_i} FP(\langle y_{i,n}\rangle_{n=k}^{\infty}).$$

For $x \in S_1$ and $j \in \{1, 2, \ldots, r\}$, define $B_2(x, j) = \{y \in S_2 : (x, y) \in C_j\}$, and for $j \in \{1, 2, \ldots, r\}$, let $B_1(j) = \{x \in S_1 : B_2(x, j) \in p_2\}$.

Now for each $x \in S_1$, $S_2 = \bigcup_{j=1}^{r} B_2(x, j)$ so there exists $j \in \{1, 2, \ldots, r\}$ such that $B_2(x, j) \in p_2$. Consequently, $S_i = \bigcup_{j=1}^{r} B_1(j)$ so pick $j \in \{1, 2, \ldots, r\}$ such that $B_1(j) \in p_1$. Pick by Theorem 1.6 a product subsystem $\langle x_{1,n}\rangle_{n=1}^{\infty}$ of $\langle y_{1,n}\rangle_{n=k}^{\infty}$ such that $FP(\langle x_{1,n}\rangle_{n=1}^{\infty}) \subseteq B_1(j)$. Let $m \in \mathbb{N}$. Then $FP(\langle x_{1,n}\rangle_{n=1}^{m}) \subseteq B_1(j)$ so

$$\bigcap\{B_2(a, j) : a \in FP(\langle x_{1,n}\rangle_{n=1}^{m})\} \in p_2.$$

Pick by Theorem 1.6 a product subsystem $\langle x_{2,n}\rangle_{n=1}^{\infty}$ of $\langle y_{2,n}\rangle_{n=k}^{\infty}$ such that

$$FP(\langle x_{2,n}\rangle_{n=1}^{\infty}) \subseteq \bigcap\{B_2(a, j) : a \in FP(\langle x_{1,n}\rangle_{n=1}^{m})\}.$$

Then $FP(\langle x_{1,n}\rangle_{n=1}^{m}) \times FP(\langle x_{2,n}\rangle_{n=1}^{\infty}) \subseteq C_i$. □

It is a consequence of Theorem 2.7 below that one cannot get a product of full infinite subsystems, even if the cells of the partition $\{C_1, C_2, \ldots, C_r\}$ are required to be symmetric. (Recall that a subset C of a cartesian product is *symmetric* provided $(a, b) \in C$ implies $(b, a) \in C$.)

At first glance it is not clear how to generalize the proof of Theorem 2.2 to higher dimensions, because the two coordinates are treated so differently. It turns out that intermediate coordinates are treated more like the first than the last. One can in fact replace the last coordinate $FP(\langle x_{\ell,n}\rangle_{n=1}^{m})$ in the following lemma by $FP(\langle x_{\ell,n}\rangle_{n=1}^{\infty})$.

2.3 Theorem. *Let $\ell \in \mathbb{N}$ and for each $i \in \{1, 2, \ldots, \ell\}$ let S_i be a semigroup and let $\langle y_{i,n}\rangle_{n=1}^{\infty}$ be a sequence in S_i. Let $m, k, r \in \mathbb{N}$ and let $\times_{i=1}^{\ell} S_i = \bigcup_{j=1}^{r} C_j$. There exists $j \in \{1, 2, \ldots, r\}$ and for each $i \in \{1, 2, \ldots, \ell\}$, there exists a product subsystem $\langle x_{i,n}\rangle_{n=1}^{m}$ of $\langle y_{i,n}\rangle_{n=k}^{\infty}$ such that*

$$\times_{i=1}^{\ell} FP(\langle x_{i,n}\rangle_{n=1}^{m}) \subseteq C_j \ .$$

Proof. Given $i \in \{1, 2, \ldots, \ell\}$, pick by Theorem 1.4 an idempotent

$$p_i \in \bigcap_{k=1}^{\infty} c\ell_{\beta S_i} FP(\langle y_{i,n}\rangle_{n=k}^{\infty}).$$

For $(x_1, x_2, \ldots, x_{\ell-1}) \in \times_{i=1}^{\ell-1} S_i$ and $j \in \{1, 2, \ldots, r\}$, let

$$B_\ell(x_1, x_2, \ldots, x_{\ell-1}, j) = \{y \in S_\ell : (x_1, x_2, \ldots, x_{\ell-1}, y) \in C_j\} \ .$$

Now given $t \in \{2, 3, \ldots, \ell-1\}$, assume that $B_{t+1}(x_1, x_2, \ldots, x_t, j)$ has been defined for each $(x_1, x_2, \ldots, x_t) \in \times_{i=1}^{t} S_i$ and each $j \in \{1, 2, \ldots, r\}$. Given $(x_1, x_2, \ldots, x_{t-1}) \in \times_{i=1}^{t-1} S_i$ and $j \in \{1, 2, \ldots, r\}$, let

$$B_t(x_1, x_2, \ldots, x_{t-1}, j) = \{y \in S_t : B_{t+1}(x_1, x_2, \ldots, x_{t-1}, y, j) \in p_{t+1}\} \ .$$

Finally, given that $B_2(x, j)$ has been defined for each $x \in S_1$ and each $j \in \{1, 2, \ldots, r\}$, let $B_1(j) = \{x \in S_1 : B_2(x, j) \in p_2\}$.

We show by downward induction on t that for each $t \in \{2, 3, \ldots, \ell\}$ and each $(x_1, x_2, \ldots, x_{t-1}) \in \times_{i=1}^{t-1} S_i$, $S_t = \bigcup_{j=1}^{r} B_t(x_1, x_2, \ldots, x_{t-1}, j)$. This is trivially true for $t = \ell$. Assume $t \in \{2, 3, \ldots, \ell - 1\}$ and the statement is true for $t + 1$. Let $(x_1, x_2, \ldots, x_{t-1}) \in \times_{i=1}^{t-1} S_i$. Given $y \in S_t$, one has that

$$S_{t+1} = \bigcup_{j=1}^{r} B_{t+1}(x_1, x_2, \ldots, x_{t-1}, y, j)$$

so one may pick $j \in \{1, 2, \ldots, r\}$ such that $B_{t+1}(x_1, x_2, \ldots, x_{t-1}, y, j) \in p_{t+1}$. Then $y \in B_t(x_1, x_2, \ldots, x_{t-1}, j)$.

Since for each $x \in S_1$, $S_2 = \bigcup_{j=1}^{r} B_2(x, j)$, one sees similarly that $S_1 = \bigcup_{j=1}^{r} B_1(j)$. Pick $j \in \{1, 2, \ldots, r\}$ such that $B_1(j) \in p_1$.

Pick by Theorem 1.6 a product subsystem $\langle x_{1,n} \rangle_{n=1}^{\infty}$ of $\langle y_{1,n} \rangle_{n=1}^{\infty}$ such that $FP(\langle x_{1,n} \rangle_{n=1}^{\infty}) \subseteq B_1(j)$. Let

$$D_2 = \bigcap \{B_2(a, j) : a \in FP(\langle x_{1,n} \rangle_{n=1}^{m})\} .$$

Since $FP(\langle x_{1,n} \rangle_{n=1}^{m})$ is finite, we have $D_2 \in p_2$ so pick by Theorem 1.6 a product subsystem $\langle x_{2,n} \rangle_{n=1}^{\infty}$ of $\langle y_{2,n} \rangle_{n=1}^{\infty}$ such that $FP(\langle x_{2,n} \rangle_{n=1}^{\infty} \subseteq D_2$.

Let $t \in \{2, 3, \ldots, \ell - 1\}$ and assume $\langle x_{t,n} \rangle_{n=1}^{\infty}$ has been chosen. Let

$$D_{t+1} = \bigcap \{B_{t+1}(a_1, a_2, \ldots, a_t, j) : (a_1, a_2, \ldots, a_t) \in \times_{i=1}^{t} FP(\langle x_{i,n} \rangle_{n=1}^{m})\} .$$

Then $D_{t+1} \in p_{t+1}$ so pick by Theorem 1.6 a product subsystem $\langle x_{t+1,n} \rangle_{n=1}^{\infty}$ of $\langle y_{t+1,n} \rangle_{n=1}^{\infty}$ such that $FP(\langle x_{t+1,n} \rangle_{n=1}^{\infty} \subseteq D_{t+1}$.

Then $\times_{t=1}^{\ell} FP(\langle x_{t,n} \rangle_{n=1}^{m}) \subseteq C_j$, as required. □

In contrast with Theorem 2.3, where the partition conclusion applied to products of arbitrary semigroups, we now restrict ourselves to products of commutative semigroups. We will see in Theorem 2.8 that without this restriction one is not guaranteed products of any subsystems at all.

2.4 Definition. Let S be a semigroup, let $\langle y_n \rangle_{n=1}^{\infty}$ be a sequence in S, and let $m \in \mathbb{N}$. The sequence $\langle x_n \rangle_{n=1}^{m}$ is a *weak product subsystem* of $\langle y_n \rangle_{n=1}^{\infty}$ if and only if there exists a sequence $\langle H_n \rangle_{n=1}^{m}$ in $\mathcal{P}_f(\mathbb{N})$ such that $H_n \cap H_k = \emptyset$ when $n \neq k$ in $\{1, 2, \ldots, m\}$ and $x_n = \Pi_{t \in H_n} y_t$ for each $n \in \{1, 2, \ldots, m\}$.

Recall by way of contrast, that in a *product subsystem* one requires that $\max H_n < \min H_{n+1}$.

2.5 Lemma. *Let $\ell \in \mathbb{N}$ and for each $i \in \{1,2,\ldots,\ell\}$, let S_i be a commutative semigroup and let $\langle y_{i,n}\rangle_{n=1}^\infty$ be a sequence in S_i. Let*

$$\begin{aligned}\mathcal{L} = \{p \in \beta(\times_{i=1}^{\ell} S_i) : &\ \textit{for each } A \in p \textit{ and each } m,k \in \mathbb{N} \\ &\textit{there exist for each } i \in \{1,2,\ldots,\ell\} \textit{ a weak product subsystem} \\ &\langle x_{i,n}\rangle_{n=1}^m \textit{ of } \langle y_{i,n}\rangle_{n=k}^\infty \textit{ such that } \times_{i=1}^{\ell} FP(\langle x_{i,n}\rangle_{n=1}^m) \subseteq A\}\ .\end{aligned}$$

Then $\mathcal{L}$ is a compact subsemigroup of $\beta(\times_{i=1}^{\ell}\mathbb{N})$.

Proof. Since any product subsystem is also a weak product subsystem, we have by Theorems 2.3 and 1.8 that $\mathcal{L} \neq \emptyset$. Since $\mathcal{L}$ is defined as the set of ultrafilters all of whose members satisfy a given property, $\mathcal{L}$ is closed, hence compact. To see that $\mathcal{L}$ is a semigroup, let $p,q \in \mathcal{L}$, let $A \in p\cdot q$ and let $m,k \in \mathbb{N}$. Then $\{\vec{a} \in \times_{i=1}^{\ell} S_i : \vec{a}^{-1}A \in q\} \in p$ (where, recall, $\vec{a}^{-1}A = \{\vec{b} : \vec{a}\cdot\vec{b} \in A\}$) so choose for each $i \in \{1,2,\ldots,\ell\}$ a weak product subsystem $\langle x_{i,n}\rangle_{n=1}^m$ of $\langle y_{i,n}\rangle_{n=k}^\infty$ such that

$$\times_{i=1}^{\ell} FP(\langle x_{i,n}\rangle_{n=1}^m) \subseteq \{\vec{a} \in \times_{i=1}^{\ell} S_i : \vec{a}^{-1}A \in q\}\ .$$

Given $i \in \{1,2,\ldots,\ell\}$ and $n \in \{1,2,\ldots,m\}$, pick $H_{i,n} \in \mathcal{P}_f(\mathbb{N})$ with $\min H_{i,n} \geq k$ such that $x_{i,n} = \Pi_{t\in H_{i,n}} y_{i,t}$ and if $1 \leq n < s \leq m$, then $H_{i,n} \cap H_{i,s} = \emptyset$.

Let $r = \max(\bigcup_{i=1}^{\ell}\bigcup_{n=1}^m H_{i,n}) + 1$ and let

$$B = \bigcap\{\vec{a}^{-1}A : \vec{a} \in \times_{i=1}^{\ell} FP(\langle x_{i,n}\rangle_{n=1}^m)\}\ .$$

Then $B \in q$ so choose for each $i \in \{1,2,\ldots,\ell\}$ a weak product subsystem $\langle z_{i,n}\rangle_{n=1}^m$ of $\langle y_{i,n}\rangle_{n=r}^\infty$ such that $\times_{i=1}^{\ell} FP(\langle x_{i,n}\rangle_{n=1}^m) \subseteq B$. Given $i \in \{1,2,\ldots,\ell\}$ and $n \in \{1,2,\ldots,m\}$, pick $K_{i,n} \in \mathcal{P}_f(\mathbb{N})$ with $\min K_{i,n} \geq r$ such that $z_{i,n} = \Pi_{t\in K_{i,n}} y_{i,t}$ and if $1 \leq n < s \leq m$, then $K_{i,n} \cap K_{i,s} = \emptyset$.

For $i \in \{1,2,\ldots,\ell\}$ and $n \in \{1,2,\ldots,m\}$, let $L_{i,n} = H_{i,n} \cup K_{i,n}$. Then $\Pi_{t\in L_{i,n}} y_{i,t} = \Pi_{t\in H_{i,n}} y_{i,t} \cdot \Pi_{t\in K_{i,n}} y_{i,t} = x_{i,t}\cdot z_{i,t}$ and if $1 \leq n < s \leq m$, then $L_{i,n} \cap L_{i,s} = \emptyset$. Thus for each $i \in \{1,2,\ldots,\ell\}$, $\langle x_{i,t}\cdot z_{i,t}\rangle_{n=1}^m$ is a weak product subsystem of $\langle y_{i,n}\rangle_{n=k}^\infty$.

Finally we claim that

$$\times_{i=1}^{\ell} FP(\langle x_{i,n}\cdot z_{i,n}\rangle_{n=1}^m) \subseteq A\ .$$

To this end let $\vec{c} \in \times_{i=1}^{\ell} FP(\langle x_{i,n} \cdot z_{i,n} \rangle_{n=1}^{m})$ be given. For each $i \in \{1,2,\ldots,\ell\}$, pick $F_i \subseteq \{1,2,\ldots,m\}$ such that $c_i = \Pi_{t\in F_i}(x_{i,t} \cdot z_{i,t})$ and let $a_i = \Pi_{t\in F_i} x_{i,t}$ and $b_i = \Pi_{t\in F_i} z_{i,t}$. Then $\vec{b} \in \times_{i=1}^{\ell} FP(\langle z_{i,n} \rangle_{n=1}^{m})$ so $\vec{b} \in B$. Since $\vec{a} \in \times_{i=1}^{\ell} FP(\langle x_{i,n} \rangle_{n=1}^{m})$ one has that $\vec{b} \in \vec{a}^{-1}A$ so that $\vec{a} \cdot \vec{b} \in A$. Since each S_i is commutative, we have for each $i \in \{1,2,\ldots,\ell\}$ that $\Pi_{t\in F_i}(x_{i,t} \cdot z_{i,t}) = (\Pi_{t\in F_i} x_{i,t}) \cdot (\Pi_{t\in F_i} z_{i,t})$ so that $\vec{c} = \vec{a} \cdot \vec{b}$ as required. □

2.6 Theorem. *Let $\ell \in \mathbb{N}$ and for each $i \in \{1,2,\ldots,\ell\}$, let S_i be a commutative semigroup and let $\langle y_{i,n} \rangle_{n=1}^{\infty}$ be a sequence in S_i. Let C be an IP* set in $\times_{i=1}^{\ell} S_i$ and let $m \in \mathbb{N}$. Then for each $i \in \{1,2,\ldots,\ell\}$ there is a weak product subsystem $\langle x_{i,n} \rangle_{n=1}^{m}$ of $\langle y_{i,n} \rangle_{n=k}^{\infty}$ such that $\times_{i=1}^{\ell} FP(\langle x_{i,n} \rangle_{n=1}^{m}) \subseteq C$.*

Proof. Let $\mathcal{L}$ be as in Lemma 2.5. Then $\mathcal{L}$ is a compact subsemigroup of $\times_{i=1}^{\ell} S_i$ so by Theorem 1.2 there is an idempotent $p \in \mathcal{L}$. By Theorem 1.5, $C \in p$. Thus by the definition of $\mathcal{L}$ for each $i \in \{1,2,\ldots,\ell\}$ there is a weak product subsystem $\langle x_{i,n} \rangle_{n=1}^{m}$ of $\langle y_{i,n} \rangle_{n=k}^{\infty}$ such that $\times_{i=1}^{\ell} FP(\langle x_{i,n} \rangle_{n=1}^{m}) \subseteq C$. □

Three natural questions are raised by Theorem 2.6. (1) Can one obtain infinite weak product subsystems (defined in the obvious fashion) such that $\times_{i=1}^{\ell} FP(\langle x_{i,n} \rangle_{n=1}^{\infty}) \subseteq C$? (2) Can one replace "weak product subsystems" with "product subsystems"? (3) Can one omit the requirement that the semigroups S_i be commutative? We answer all three of these questions in the negative.

The first two questions are answered in Theorem 2.7 using the semigroup $(\mathbb{N}, +)$. Since the operation is addition we refer to "sum subsystems" rather than "product subsystems".

2.7 Theorem. *There is an IP* set C in $\mathbb{N} \times \mathbb{N}$ such that:*

(a) There do not exist $z \in \mathbb{N}$ and a sequence $\langle x_n \rangle_{n=1}^{\infty}$ in $\mathbb{N}$ such that either $\{z\} \times FS(\langle x_n \rangle_{n=1}^{\infty}) \subseteq C$ or $FS(\langle x_n \rangle_{n=1}^{\infty}) \times \{z\} \subseteq C$. (In particular there do not exist infinite weak sum subsystems $\langle x_{1,n} \rangle_{n=1}^{\infty}$ and $\langle x_{2,n} \rangle_{n=1}^{\infty}$ of $\langle 2^n \rangle_{n=1}^{\infty}$ with $FS(\langle x_{1,n} \rangle_{n=1}^{\infty}) \times FS(\langle x_{2,n} \rangle_{n=1}^{\infty}) \subseteq C$).

(b) There do not exist sum subsystems $\langle x_n \rangle_{n=1}^{2}$ and $\langle y_n \rangle_{n=1}^{2}$ of $\langle 2^n \rangle_{n=1}^{\infty}$ with

$\{x_1, x_2\} \times \{y_1, y_2\} \subseteq C$.

Proof. Let

$$C = (\mathbb{N} \times \mathbb{N}) \backslash \{(\Sigma_{n \in F}\ 2^n, \Sigma_{n \in G}\ 2^n) : F, G \in \mathcal{P}(\omega) \text{ and } \max F < \min G \text{ or } \max G < \min F\}\ .$$

To see that C is an IP* set in $\mathbb{N} \times \mathbb{N}$, suppose instead that we have a sequence $\langle (x_n, y_n) \rangle_{n=1}^{\infty}$ in $\mathbb{N} \times \mathbb{N}$ with

$$FS(\langle (x_n, y_n) \rangle_{n=1}^{\infty}) \subseteq \{(\Sigma_{n \in F}\ 2^n, \Sigma_{n \in G}\ 2^n) : F, G \in \mathcal{P}(\omega) \text{ and } \max F < \min G \text{ or } \max G < \min F\}\ .$$

Pick F_1 and G_1 in $\mathcal{P}(\omega)$ such that $x_1 = \Sigma_{t \in F_1}\ 2^t$ and $y_1 = \Sigma_{t \in G_1}\ 2^t$. Let $k = \max(F_1 \cup G_1)$. Choose $H \in \mathcal{P}_f(\mathbb{N})$ such that $\min H > 1$, $2^{k+1} | \Sigma_{n \in H}\ x_n$, and $2^{k+1} | \Sigma_{n \in H}\ y_n$. (Consider congruence classes mod 2^{k+1} to see that one can do this.) Pick $F', G' \in \mathcal{P}(\omega)$ such that $\Sigma_{n \in H}\ x_n = \Sigma_{n \in F'}\ 2^n$ and $\Sigma_{n \in H}\ y_n = \Sigma_{n \in G'}\ 2^n$. Then $k + 1 \leq \min(F' \cup G')$, so $x_1 + \Sigma_{n \in H}\ x_n = \Sigma_{n \in F_1 \cup F'}\ 2^n$ and $y_1 + \Sigma_{n \in H}\ y_n = \Sigma_{n \in G_1 \cup G'}\ 2^n$. Also

$$\begin{aligned} \Sigma_{n \in \{1\} \cup H}(x_n, y_n) &= (x_1 + \Sigma_{n \in H}\ x_n, y_1 + \Sigma_{n \in H}\ y_n) \\ &\in \{(\Sigma_{n \in F}\ 2^n, \Sigma_{n \in G}\ 2^n) : F, G \in \mathcal{P}(\omega) \\ &\qquad \text{and } \max F < \min G \text{ or } \max G < \min F\}\ . \end{aligned}$$

Thus $k+1 \leq \max(F_1 \cup F') < \min(G_1 \cup G') \leq k$ or $k+1 \leq \max(G_1 \cup G') < \min(F_1 \cup F') \leq k$, a contradiction.

To establish (a), suppose that one has $z \in \mathbb{N}$ and a sequence $\langle x_n \rangle_{n=1}^{\infty}$ in $\mathbb{N}$ such that either $\{z\} \times FS(\langle x_n \rangle_{n=1}^{\infty}) \subseteq C$ or $FS(\langle x_n \rangle_{n=1}^{\infty}) \times \{z\} \subseteq C$ and assume without loss of generality that $\{z\} \times FS(\langle x_n \rangle_{n=1}^{\infty}) \subseteq C$. Pick $F \in \mathcal{P}(\omega)$ such that $z = \Sigma_{t \in F}\ 2^t$ and let $k = \max F$. Pick $H \in \mathcal{P}_f(\mathbb{N})$ such that $2^{k+1} | \Sigma_{n \in H}\ x_n$. Pick $G \in \mathcal{P}(\omega)$ such that $\Sigma_{n \in H}\ x_n = \Sigma_{t \in G}\ 2^t$. Then $\max F < \min G$ so $(z, \Sigma_{n \in H}\ x_n) \notin C$.

To establish (b) suppose that one has sum subsystems $\langle x_n \rangle_{n=1}^{2}$ and $\langle y_n \rangle_{n=1}^{2}$ of $\langle 2^n \rangle_{n=1}^{\infty}$ with $\{x_1, x_2\} \times \{y_1, y_2\} \subseteq C$. Pick $F_1, G_1, F_2, G_2 \in \mathcal{P}(\omega)$ such that $x_1 = \Sigma_{n \in F_1} 2^n$, $x_2 = \Sigma_{n \in F_2} 2^n$, $y_1 = \Sigma_{n \in G_1} 2^n$, $y_2 = \Sigma n \in G_2 2^n$, $\max F_1 < \min F_2$, and $\max G_1 < \min G_2$. Without loss of generality, $\max F_1 \geq \max G_1$. But then we have $\max G_1 \leq \max F_1 < \min F_2$ so $(x_2, y_1) \notin C$, a contradiction. □

A striking contrast is provided by Theorems 2.2 and 2.7. It is easy to divide most semigroups into two classes, neither of which is an IP* set. Consequently it is not too surprising when one finds a property that must be satisfied by an IP* set in a semigroup S (such as containing a sequence with all of its sums and products when $S = \mathbb{N}$) which need not be satisfied by any cell of a partition of S. In this case we have a property, namely that of containing $FS(\langle x_{1,n}\rangle_{n=1}^{m}) \times FS(\langle x_{2,n}\rangle_{n=1}^{\infty})$ for some sum subsystems of any given sequences, which must be satisfied by some cell of a partition of $\mathbb{N} \times \mathbb{N}$, but need not be satisfied by IP* sets in $\mathbb{N} \times \mathbb{N}$.

The third question raised by Theorem 2.6 is answered with an example very similar to that used in the proof of Theorem 2.7.

2.8 Theorem. *Let S be the free semigroup on the alphabet $\{y_1, y_2, y_3, \ldots\}$. There is an IP* set C in $S \times S$ such that there do not exist weak product subsystems $\langle x_n\rangle_{n=1}^{2}$ and $\langle w_n\rangle_{n=1}^{2}$ of $\langle y_n\rangle_{n=1}^{\infty}$ such that $\{x_1, x_2\} \times \{w_1, w_2\} \subseteq C$.*

Proof. Let

$$C = (S \times S)\backslash\{(\Pi_{n\in F}\ y_n, \Pi_{n\in G}\ y_n) : F, G \in \mathcal{P}_f(\mathbb{N}) \text{ and } \max F < \min G \text{ or } \max G < \min F\}\ .$$

To see that C is an IP* set, suppose one has a sequence $\langle (x_n, w_n)\rangle_{n=1}^{\infty}$ with

$$FP(\langle (x_n, w_n)\rangle_{n=1}^{\infty}) \subseteq \{(\Pi_{n\in F}\ y_n, \Pi_{n\in G}\ y_n) : F, G \in \mathcal{P}_f(\mathbb{N}) \text{ and } \max F < \min G \text{ or } \max G < \min F\}\ .$$

Given any $i < j$ in $\mathbb{N}$, pick $F_i, F_j, G_i, G_j, F_{i,j}, G_{i,j} \in \mathcal{P}_f(\mathbb{N})$ such that $x_i = \Pi_{n\in F_i} y_n$, $x_j = \Pi_{n\in F_j} y_n$, $w_i = \Pi_{n\in G_i} y_n$, $w_j = \Pi_{n\in G_j} y_n$, $x_i x_j = \Pi_{n\in F_{i,j}} y_n$, and $w_i w_j = \Pi_{n\in F_i} y_n$. Since $x_i x_j = \Pi_{n\in F_{i,j}} y_n$, we have that $\max F_i < \min F_j$ and $F_{i,j} = F_i \cup F_j$ and similarly $\max G_i < \min G_j$ and $G_{i,j} = G_i \cup G_j$. Then we may pick $j \in \mathbb{N}$ such that $\max(F_1 \cup G_1) < \min(F_j \cup G_j)$. Then

$$(x_1 x_j, w_1 w_j) \notin \{(\Pi_{n\in F}\ y_n, \Pi_{n\in G}\ y_n) : F, G \in \mathcal{P}_f(\mathbb{N}) \text{ and } \max F < \min G \text{ or } \max G < \min F\}\ ,$$

a contradiction.

Now suppose we have weak product subsystems $\langle x_n \rangle_{n=1}^2$ and $\langle w_n \rangle_{n=1}^2$ of $\langle y_n \rangle_{n=1}^\infty$ such that $\{x_1, x_2\} \times \{w_1, w_2\} \subseteq C$. Pick $F_1, F_2, G_1, G_2 \in \mathcal{P}_f(\mathbb{N})$ such that $x_1 = \Pi_{n \in F_1} y_n$, $x_2 = \Pi_{n \in F_2} y_n$, $w_1 = \Pi_{n \in G_1} y_n$, and $w_2 = \Pi_{n \in G_2} y_n$. Since $x_1 x_2 \in FP(\langle y_n \rangle_{n=1}^\infty)$, we must have $\max F_1 < \min F_2$ and similarly $\max G_1 < \min G_2$. Without loss of generality, $\max F_1 \geq \max G_1$. But then we have $\max G_1 \leq \max F_1 < \min F_2$ so $(x_2, w_1) \notin C$, a contradiction. □

We know that there are certain IP* sets, namely sets of returns in a dynamical system, which satisfy stronger conclusions than arbitrary IP* sets. (See, for example, [3].) It is possible to show that if C is such a dynamically defined IP* set and S has an identity, one can choose infinite product subsystems $\langle x_{1,n} \rangle_{n=1}^\infty$ and $\langle x_{2,n} \rangle_{n=1}^\infty$ of given $\langle y_{i,n} \rangle_{n=1}^\infty$ *uniformly* with $FP(\langle x_{1,n} \rangle_{n=1}^\infty) \times FP(\langle x_{2,n} \rangle_{n=1}^\infty) \subseteq C$. That is, one has $\langle H_n \rangle_{n=1}^\infty$ in $\mathcal{P}_f(\mathbb{N})$ such that for each n and each $i \in \{1, 2\}$, $x_{i,n} = \Pi_{t \in H_n} y_{i,t}$.

We do not know if we can impose such uniformity on the weak product subsystems guaranteed by Theorem 2.6. We do have the following simple result establishing a certain amount of uniformity for arbitrary semigroups. It has an obvious generalization to any finite dimension, but for the sake of simplicity, we restrict our attention to two dimensions.

2.9 Theorem. *Let S_1 and S_2 be semigroups and let C be an IP* set in $S_1 \times S_2$. Let $\ell, m \in \mathbb{N}$ and let $\big\{\langle w_{i,n} \rangle_{n=1}^\infty : i \in \{1, 2, \ldots, \ell\}\big\}$ be a set of sequences in S_1 and let $\big\{\langle z_{j,n} \rangle_{n=1}^\infty : j \in \{1, 2, \ldots, m\}\big\}$ be a set of sequences in S_2. Then there is a sequence $\langle H_n \rangle_{n=1}^\infty$ in $\mathcal{P}_f(\mathbb{N})$ such that:*

(a) for each $n \in \mathbb{N}$, $\max H_n < \min H_{n+1}$ and

(b) if for each $n \in \mathbb{N}$, each $i \in \{1, 2, \ldots, \ell\}$ and each $j \in \{1, 2, \ldots, m\}$, $x_{i,n} = \Pi_{t \in H_n} w_{i,t}$ and $y_{j,n} = \Pi_{t \in H_n} z_{j,t}$, then for each $i \in \{1, 2, \ldots, \ell\}$ and each $j \in \{1, 2, \ldots, m\}$, $FP(\langle (x_{i,n}, y_{j,n}) \rangle_{n=1}^\infty) \subseteq C$.

Proof. Enumerate $\{1, 2, \ldots, \ell\} \times \{1, 2, \ldots, m\}$ as $\langle \big(i(k), j(k)\big) \rangle_{k=1}^{\ell m}$. Let for each $n \in \mathbb{N}$, $a_{1,n} = (w_{i(1),n}, z_{j(1),n})$. Pick by Corollary 1.7 a sequence $\langle H_{1,n} \rangle_{n=1}^\infty$ such that if $b_{1,n} = \Pi_{t \in H_{1,n}} a_{1,n}$, then $FP(\langle b_{1,n} \rangle_{n=1}^\infty) \subseteq C$. Let $a_{2,n} = (\Pi_{t \in H_{1,n}} w_{i(2),t}, \Pi_{t \in H_{1,n}} z_{j(2),t})$

Inductively, given $a_{k,n} = (\Pi_{t \in H_{k-1,n}} w_{i(k),t}, \Pi_{t \in H_{k-1,n}} z_{j(k),t})$ pick by Corollary 1.7 a sequence $\langle F_{k,n}\rangle_{n=1}^\infty$ in $\mathcal{P}_f(\mathbb{N})$ such that if $b_{k,n}=\Pi_{t\in F_{k,n}}a_{k,n}$, then $FP(\langle b_{k,n}\rangle_{n=1}^\infty)\subseteq C$. For each n let $H_{k,n}=\bigcup_{t\in F_{k,n}}H_{k-1,t}$ and let (if $k<\ell m$)

$$a_{k+1,n}=(\Pi_{t\in H_{k,n}}w_{i(k+1),t}\ ,\Pi_{t\in H_{k,n}}z_{j(k+1),t}).$$

The induction being complete, let $H_n=H_{\ell m,n}$ for each n. For each $n\in\mathbb{N}$, each $i\in\{1,2,\ldots,\ell\}$ and each $j\in\{1,2,\ldots,m\}$, let $x_{i,n}=\Pi_{t\in H_n}w_{i,t}$ and $y_{j,n}=\Pi_{t\in H_n}z_{j,t}$. Then, given $i\in\{1,2,\ldots,\ell\}$ and $j\in\{1,2,\ldots,m\}$, pick $k\in\mathbb{N}$ such that $(i,j)=\big(i(k),j(k)\big)$. Then $FP(\langle(x_{i,n},y_{j,n})\rangle_{n=1}^\infty\subseteq FP(\langle b_{k,n}\rangle_{n=1}^\infty)\subseteq C$. □

3. An Alternate Derivation of Finite Substructures.

We show here that a version of Theorem 2.6 for the semigroup $(\mathcal{P}_f(\mathbb{N}),\cup)$ suffices to derive Theorem 2.6 in its entirety. Since the operation in this semigroup is $\cup$ we write $FU(\langle H_n\rangle_{n=1}^m)=\{\bigcup_{n\in F}\ H_n:\emptyset\neq F\subseteq\{1,2,\ldots,m\}\}$. Similarly, given a sequence $\langle(H_{1,n},H_{2,n})\rangle_{n=1}^\infty$ in $\mathcal{P}_f(\mathbb{N})\times\mathcal{P}_f(\mathbb{N})$ we write $FU(\langle(F_{1,n},F_{2,n})\rangle_{n=1}^m)=\{(\bigcup_{n\in F}\ H_{1,n},\bigcup_{n\in F}\ H_{2,n}):\emptyset\neq F\subseteq\{1,2,\ldots,m\}\}$.

For the semigroup $(\mathcal{P}_f(\mathbb{N}),\cup)$, Theorem 2.6 is completely trivial because the only IP* set in $\mathcal{P}_f(\mathbb{N})\times\mathcal{P}_f(\mathbb{N})$ is $\mathcal{P}_f(\mathbb{N})\times\mathcal{P}_f(\mathbb{N})$ itself. (This is because every element of $\mathcal{P}_f(\mathbb{N})$ is an idempotent.) We need a weaker notion of IP* set in order to obtain useful results. We will say that a sequence $\langle H_{i,n}\rangle_{n=1}^\infty$ is a *disjoint sequence* provided $H_{i,n}\cap H_{i,m}=\emptyset$ whenever $n\neq m$.

3.1 Definition. *Let $\ell\in\mathbb{N}$. A set $C\subseteq\times_{i=1}^\ell\mathcal{P}_f(\mathbb{N})$ is a weak IP* set if and only if for any disjoint sequences $\langle H_{1,n}\rangle_{n=1}^\infty,\langle H_{2,n}\rangle_{n=1}^\infty,\ldots,\langle H_{\ell,n}\rangle_{n=1}^\infty$ in $\mathcal{P}_f(\mathbb{N})$, one has $C\cap FU(\langle(H_{1,n},H_{2,n},\ldots,H_{\ell,n})\rangle_{n=1}^\infty)\neq\emptyset$.*

We will also need to concern ourselves with a restricted subsemigroup of $\beta\big(\times_{i=1}^\ell\mathcal{P}_f(\mathbb{N})\big)$.

3.2 Definition. *Let $\ell\in\mathbb{N}$. Then $\mathcal{I}_\ell=\{p\in\beta\big(\times_{i=1}^\ell\mathcal{P}_f(\mathbb{N})\big)$: for each $n\in\mathbb{N}$, $\{(H_1,H_2,\ldots,H_\ell):(\bigcup_{i=1}^\ell H_i)\cap\{1,2,\ldots,n\}=\emptyset\}\in p\}$.*

It is routine to show that $\mathcal{I}_\ell$ is a subsemigroup of $\beta\big(\times_{i=1}^\ell\mathcal{P}_f(\mathbb{N})\big)$. We modify Theorem 1.5.

3.3 Theorem. *Let $\ell \in \mathbb{N}$, let $C \subseteq \times_{i=1}^{\ell}\mathcal{P}_f(\mathbb{N})$ be a weak IP* set, and let p be an idempotent in $\mathcal{I}_\ell$. Then $C \in p$.*

Proof. As in the proof of Theorem 1.6 one shows that given any $A \in p$ there exist disjoint sequences $\langle H_{1,n}\rangle_{n=1}^{\infty}, \langle H_{2,n}\rangle_{n=1}^{\infty}, \ldots, \langle H_{\ell,n}\rangle_{n=1}^{\infty}$ in $\mathcal{P}_f(\mathbb{N})$, such that

$$FU(\langle (H_{1,n}, H_{2,n}, \ldots, H_{\ell,n})\rangle_{n=1}^{\infty}) \subseteq A \ .$$

Consequently, one cannot have $\times_{i=1}^{\ell}\mathcal{P}_f(\mathbb{N})\backslash C \in p$. □

One can define a *strong IP set* in $\times_{i=1}^{\ell}\mathcal{P}_f(\mathbb{N})$ by requiring that this set contain $FU(\langle (H_{1,n}, H_{2,n}, \ldots, H_{\ell,n})\rangle_{n=1}^{\infty}$ where each $\langle H_{i,n}\rangle_{n=1}^{\infty}$ is a disjoint sequence. One can then show in a fashion similar to the proof of Corollary 1.7 that any weak IP* set meets any strong IP set along a strong IP set.

Next we modify Lemma 2.5.

3.4 Lemma. *Let $\ell \in \mathbb{N}$ and for each $i \in \{1, 2, \ldots, \ell\}$ and each $n \in \mathbb{N}$, let $Y_{i,n} = \{n\}$. Let*

$$\begin{aligned} \mathcal{M} = \{p \in \mathcal{I}_\ell : & \text{ for each } A \in p \text{ and each } m, k \in \mathbb{N} \\ & \text{there exist for each } i \in \{1, 2, \ldots, \ell\} \text{ a weak product subsystem} \\ & \langle H_{i,n}\rangle_{n=1}^{m} \text{ of } \langle Y_{i,n}\rangle_{n=k}^{\infty} \text{ such that } \times_{i=1}^{\ell} FU(\langle H_{i,n}\rangle_{n=1}^{m}) \subseteq A\} \ . \end{aligned}$$

Then $\mathcal{M}$ is a compact subsemigroup of $\beta(\times_{i=1}^{\ell}\mathbb{N})$.

Proof. Let $\mathcal{L}$ be as in Lemma 2.5. Then $\mathcal{M} = \mathcal{L} \cap \mathcal{I}_\ell$. Since both $\mathcal{L}$ and $\mathcal{I}_\ell$ are semigroups, it suffices to show that $\mathcal{M} \cap \mathcal{L} \neq \emptyset$.

For each $n \in \mathbb{N}$, let $A_n = \{(H_1, H_2, \ldots, H_\ell) \in \times_{i=1}^{\ell}\mathcal{P}_f(\mathbb{N}) : (\bigcup_{i=1}^{\ell} H_i) \cap \{1, 2, \ldots, n\} = \emptyset\}$. Let $\mathcal{B} = \{C \subseteq \times_{i=1}^{\ell}\mathcal{P}_f(\mathbb{N})$: for each $m, k \in \mathbb{N}$ and each choice of a weak product subsystem $\langle H_{i,n}\rangle_{n=1}^{m}$ of $\langle Y_{i,n}\rangle_{n=k}^{\infty}$, $\times_{i=1}^{\ell} FU(\langle H_{i,n}\rangle_{n=1}^{m}) \cap C \neq \emptyset\}$. Now if $p \in \beta\big(\times_{i=1}^{\ell}\mathcal{P}_f(\mathbb{N})\big)$ and $\{A_k : k \in \mathbb{N}\} \cup \mathcal{B} \subseteq p$, then $p \in \mathcal{L} \cap \mathcal{I}_\ell$ (since, given any $A \in p$, $\times_{i=1}^{\ell}\mathcal{P}_f(\mathbb{N})\backslash A \notin \mathcal{B}$). Thus it suffices to show that $\{A_k : k \in \mathbb{N}\} \cup \mathcal{B}$ has the finite intersection property.

To see this, suppose instead we have some $k \in \mathbb{N}$ and some $C_1, C_2, \ldots, C_r \in \mathcal{B}$ such that $A_k \cap \bigcap_{j=1}^{r} C_i = \emptyset$. Then $(\times_{i=1}^{\ell}\mathcal{P}_f(\mathbb{N})\backslash A_k) \cup \bigcup_{j=1}^{r}(\times_{i=1}^{\ell}\mathcal{P}_f(\mathbb{N})\backslash C_j)$ so by

Theorem 2.3, some one of these sets contains $\times_{i=1}^{\ell} FU(\langle H_{i,n}\rangle_{n=1}^{m})$ for some choice of weak product subsystems $\langle H_{i,n}\rangle_{n=1}^{m}$ of $\langle Y_{i,n}\rangle_{n=k}^{\infty}$, which is impossible. $\square$

3.5 Theorem. *Let $\ell \in \mathbb{N}$, let $C \subseteq \times_{i=1}^{\ell} \mathcal{P}_f(\mathbb{N})$ be a weak IP* set, and let $m \in \mathbb{N}$. Then for each $i \in \{1, 2, \ldots, \ell\}$ there is a disjoint sequence $\langle H_{i,n}\rangle_{n=1}^{m}$ such that $\times_{i=1}^{\ell} FU(\langle H_{i,n}\rangle_{n=1}^{m}) \subseteq C$.*

Proof. Let $\mathcal{M}$ be as in Lemma 3.4. Then $\mathcal{M}$ is a compact subsemigroup of $\mathcal{I}_\ell$ so pick an idempotent $p \in \mathcal{M}$. By Theorem 3.3 $C \in p$ so by the definition of $\mathcal{M}$ we have for each $i \in \{1, 2, \ldots, \ell\}$ some disjoint sequence $\langle H_{i,n}\rangle_{n=1}^{m}$ such that $\times_{i=1}^{\ell} FU(\langle H_{i,n}\rangle_{n=1}^{m}) \subseteq C$. $\square$

Finally we show how Theorem 3.5 suffices to yield Theorem 2.6 for any semigroups. (We reprint its formulation for the convenience of the reader.)

2.6 Theorem. *Let $\ell \in \mathbb{N}$ and for each $i \in \{1, 2, \ldots, \ell\}$, let S_i be a commutative semigroup and let $\langle y_{i,n}\rangle_{n=1}^{\infty}$ be a sequence in S_i. Let C be an IP* set in $\times_{i=1}^{\ell} S_i$ and let $m \in \mathbb{N}$. Then for each $i \in \{1, 2, \ldots, \ell\}$ there is a weak product subsystem $\langle x_{i,n}\rangle_{n=1}^{m}$ of $\langle y_{i,n}\rangle_{n=k}^{\infty}$ such that $\times_{i=1}^{\ell} FP(\langle x_{i,n}\rangle_{n=1}^{m}) \subseteq C$.*

Proof. Let

$$D = \{(H_1, H_2, \ldots, H_\ell) \in \times_{i=1}^{\ell} \mathcal{P}_f(\mathbb{N}) : (\Pi_{t \in H_1}\ y_{1,t}, \Pi_{t \in H_2}\ y_{2,t}, \ldots, \Pi_{t \in H_\ell}\ y_{\ell,t}) \in C\}\ .$$

Then it is easy to see that D is a weak IP* set. So pick for each $i \in \{1, 2, \ldots, \ell\}$ a disjoint sequence $\langle H_{i,n}\rangle_{n=1}^{m}$ as guaranteed by Theorem 3.5. For each $n \in \{1, 2, \ldots, m\}$ and each $i \in \{1, 2, \ldots, \ell\}$, let $x_{i,n} = \Pi_{t \in H_{i,n}}\ y_{i,t}$. Then $\times_{i=1}^{\ell} FP(\langle x_{i,n}\rangle_{n=1}^{m}) \subseteq C$. $\square$

References

[1] V. Bergelson and N. Hindman, *Ultrafilters and multidimensional Ramsey theorems*, Combinatorica **9** (1989), 1-7.

[2] V. Bergelson and N. Hindman, *IP*-sets and central sets*, Combinatorica **14** (1994), 269-277.

[3] V. Bergelson, N. Hindman, and B. Kra, *Iterated spectra of numbers — elementary, dynamical, and algebraic approaches*, Trans. Amer. Math. Soc., to appear.

[4] R. Ellis, *Lectures on topological dynamics*, Benjamin, New York, 1969.

[5] H. Furstenberg, *Recurrence in ergodic theory and combinatorial number theory*, Princeton Univ. Press, Princeton, N.J., 1981.

[6] N. Hindman, *Ultrafilters and combinatorial number theory*, in Number Theory Carbondale 1979, M. Nathanson ed., Lecture Notes in Math. **751** (1979), 119-184.

[7] N. Hindman, *Ultrafilters and Ramsey Theory – an update*, in Set Theory and its Applications, J. Steprāns and S. Watson eds., Lecture Notes in Math. **1401** (1989), 97-118.

[8] K. Milliken, *Ramsey's Theorem with sums or unions*, J. Comb. Theory (Series A) **18** (1975), 276-290.

[9] A. Taylor, *A canonical partition relation for finite subsets of* ω, J. Comb. Theory (Series A) **21** (1976), 137-146.