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Problems and New Results on the Algebra of $\beta\mathbb{N}$ and its Application to Ramsey Theory

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Abstract. We present a new result on the algebraic structure of $\beta\mathbb{N}$, the Stone-Čech compactification of $\mathbb{N}$, and some new applications of that structure to the part of combinatorics known as "Ramsey Theory". We then discuss some difficult open problems about the algebraic structure of $\beta\mathbb{N}$, and some related difficult open problems in Ramsey Theory.

1 Introduction

Given a discrete semigroup $(S,\cdot)$, there is a natural extension of the operation (also denoted by $\cdot$) to the Stone-Čech compactification βS of S making $(\beta S,\cdot)$ a right topological semigroup with S contained in its topological center. (That is, for each $p\in\beta S$, the function $\rho_p:\beta S\to\beta S$ is continuous, and for each $s\in S$, the function $\lambda_s:\beta S\to\beta S$ is continuous, where $\rho_p(q)=q\cdot p$ and $\lambda_s(q)=s\cdot q$.) We shall be concerned in this paper almost exclusively with the semigroups $(\mathbb{N},+)$ and $(\beta\mathbb{N},+)$, but the reader should be cautioned that, in spite of the fact that the operation on $\beta\mathbb{N}$ is denoted by $+$, it is highly non-commutative.

Beginning with the discovery in 1975 by F. Galvin and S. Glazer of the simple proof of the Finite Sums Theorem using the algebra of $\beta\mathbb{N}$, there has been extensive interaction between Ramsey Theory and the algebraic structure of βS. An elementary introduction to the algebra of βS and its applications to Ramsey Theory and other structures can be found in the book [9], which covers developments up through the middle of 1997.

In this paper we shall present in Section 2 a new algebraic result obtained by E. Zelenuk [13]. In Section 3 we shall present a new algebraic proof of a Ramsey Theoretic result of V. Bergelson and A. Leibman [2]. In Section 4 we shall discuss some open problems dealing with the algebraic structure of $\beta\mathbb{N}$ and in Section 5 we shall discuss some open problems in Ramsey Theory that appear to be related to the algebra of $\beta\mathbb{N}$.

We remind the reader that, like any compact Hausdorff right topological semigroup, $(\beta\mathbb{N},+)$ has a smallest two sided ideal $K(\beta\mathbb{N})$ which is the union of all minimal right ideals as well as the union of all minimal left ideals, and that the intersection of any minimal left ideal with any minimal right ideal is a group. See [9] for these facts as well as any other unfamiliar algebraic assertions encountered in this paper.

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2 Sums of Idempotents

Answering a question of J. Lawson and A. Lisan, J. Berglund and I showed [3] that there are idempotents in $K(\beta\mathbb{N})$ whose sum is not idempotent. (In fact, there is a set of $2^{\mathfrak{c}}$ idempotents, no two of which sum to an idempotent.) It is a trivial fact that if p and q are idempotents in the same minimal left ideal of $\beta\mathbb{N}$, then $p+q=p$ and $q+p=q$. Similarly, if p and q are idempotents in the same minimal right ideal of $\beta\mathbb{N}$, then $p+q=q$ and $q+p=p$. In [3], we asked whether there are any idempotents in $K(\beta\mathbb{N})$ that do not lie in the same minimal left nor the same minimal right ideal but whose sum is idempotent. (Equivalently, are there any idempotents in $K(\beta\mathbb{N})$ whose sum is an idempotent but not equal to either of them?) We present in this section, with his kind permission, E. Zelenuk's affirmative answer to this question. (This is a fragment of his classification of finite regular subsemigroups in βG.)

The main idea of the proof that there are idempotents in $K(\beta\mathbb{N})$ whose sum is not idempotent is quite simple. One takes a finite semigroup S which has idempotents in $K(S)$ whose sum is not an idempotent and constructs a continuous function φ from $\beta\mathbb{N}$ to S whose restriction to a compact subsemigroup T, with $K(\beta\mathbb{N})\subseteq T$, is a homomorphism. The inverse image of any idempotent is then a compact subsemigroup which therefore has idempotents. If $\varphi(p)+\varphi(q)$ is not idempotent, then $p+q$ is not idempotent.

The idea behind Zelenuk's proof is similar, but more complicated. One cannot conclude that $p+q$ is idempotent from the fact that $\varphi(p)+\varphi(q)$ is idempotent.

We leave the routine proof of the following lemma to the reader.

Lemma 2.1 *Let $S=\{e,a_{11},a_{12},a_{21},a_{22}\}$ with product defined by the table:*

$\cdot$	e	a_{11}	a_{12}	a_{21}	a_{22}
e	e	a_{11}	a_{12}	a_{21}	a_{22}
a_{11}	a_{12}	a_{11}	a_{12}	a_{11}	a_{12}
a_{12}	a_{12}	a_{11}	a_{12}	a_{11}	a_{12}
a_{21}	a_{22}	a_{21}	a_{22}	a_{21}	a_{22}
a_{22}	a_{22}	a_{21}	a_{22}	a_{21}	a_{22}

Then S is a semigroup. (That is, the operation is associative.)

We take $\omega=\mathbb{N}\cup\{0\}$.

Definition 2.2 Fix $g:\omega\to S$ such that for all $x\in S$, $g^{-1}[\{x\}]$ is infinite. Define

$$f:\mathbb{N}\to S \text{ by } f(\textstyle\sum_{n\in F} 2^n)=\prod_{n\in F} g(n)$$

where the product $\prod_{n\in F} g(n)$ is computed in increasing order of indices.

Recall from [9] that the semigroup $\mathbb{H} = \bigcap_{n=1}^{\infty} \overline{\mathbb{N}2^n}$ holds all of the idempotents and much of the algebraic structure of $\beta\mathbb{N}$.

Lemma 2.3 *Let $\widetilde{f} : \beta\mathbb{N} \to S$ be the continuous extension of f. Then the restriction of $\widetilde{f}$ to $\mathbb{H}$ is a homomorphism.*

Proof. By [9, Theorem 4.21] it suffices to show that for each $x \in \mathbb{N}$ there exists $m \in \mathbb{N}$ such that for all $y \in \mathbb{N}2^m$, $f(x+y) = f(x) \cdot f(y)$. Let $x = \sum_{n\in F} 2^n$ and let $m = \max F + 1$. Let $y \in \mathbb{N}2^m$. Then $y = \sum_{n\in G} 2^n$ where $\min G \geq m > \max F$. Thus $f(x+y) = f(\sum_{n\in F\cup G} 2^n) = \prod_{n\in F\cup G} g(n) = \left(\prod_{n\in F} g(n)\right) \cdot \left(\prod_{n\in G} g(n)\right) = f(x) \cdot f(y)$. $\square$

Notice that $K(S) = \{a_{11}, a_{12}, a_{21}, a_{22}\}$ so that, by [9, Exercise 1.7.3], the idempotent p produced in Theorem 2.4 below cannot be in $K(\beta\mathbb{N})$.

Theorem 2.4 *There exist $p \in \mathbb{H}$ and $\{\alpha_{11}, \alpha_{12}, \alpha_{21}, \alpha_{22}\} \subseteq K(\mathbb{H}) = K(\beta\mathbb{N}) \cap \mathbb{H}$ such that $\widetilde{f}(p) = e$, $\widetilde{f}(\alpha_{ij}) = a_{ij}$ for $i, j \in \{1, 2\}$ (so that the listed elements are all distinct), and the operation $+$ satifies*

$+$	p	α_{11}	α_{12}	α_{21}	α_{22}
p	p	α_{11}	α_{12}	α_{21}	α_{22}
α_{11}	α_{12}	α_{11}	α_{12}	α_{11}	α_{12}
α_{12}	α_{12}	α_{11}	α_{12}	α_{11}	α_{12}
α_{21}	α_{22}	α_{21}	α_{22}	α_{21}	α_{22}
α_{22}	α_{22}	α_{21}	α_{22}	α_{21}	α_{22}

In particular, α_{11}, α_{22}, and α_{12} are idempotents in $K(\beta\mathbb{N})$ and $\alpha_{11} + \alpha_{22} = \alpha_{12}$.

Proof. That $K(\mathbb{H}) = K(\beta\mathbb{N}) \cap \mathbb{H}$ follows from [9, Theorem 1.65 and Lemma 6.8].

Let $h = \widetilde{f}_{|\mathbb{H}}$ and note that $h[\mathbb{H}] = S$. (Given $x \in S$, if $p \in \mathbb{H}$ and $\{2^n : g(n) = x\} \in p$, then $h(p) = x$.) The minimal left ideals of S are $\{a_{11}, a_{21}\}$ and $\{a_{12}, a_{22}\}$ and the minimal right ideals of S are $\{a_{11}, a_{12}\}$ and $\{a_{21}, a_{22}\}$.

Let $A = h^{-1}[\{a_{11}, a_{21}\}]$. Then A is a left ideal of $\mathbb{H}$ so pick a minimal left ideal L of $\mathbb{H}$ such that $L \subseteq A$. Then $h[L]$ is a left ideal of S and so $h[L] = \{a_{11}, a_{21}\}$. Similarly pick minimal right ideals R_1' and R_2' of $\mathbb{H}$ such that $h[R_1'] = \{a_{11}, a_{12}\}$ and $h[R_2'] = \{a_{21}, a_{22}\}$.

Now $h^{-1}[\{e\}]$ is a compact subsemigroup of $\mathbb{H}$ so pick $p \in \mathbb{H}$ such that $p = p + p$ and $h(p) = e$. Let $R_1 = p + R_1'$ and $R_2 = p + R_2'$. By [9, Theorem 1.46] R_1 and R_2 are minimal right ideals of $\mathbb{H}$. Further, if $x \in \mathbb{H}$, then $h(p + x) = e \cdot h(x) = h(x)$, so $h[R_1] = \{a_{11}, a_{12}\}$ and $h[R_2] = \{a_{21}, a_{22}\}$.

Let α_{11} be the identity of the group $R_1 \cap L$ and note that $h(\alpha_{11}) \in \{a_{11}, a_{12}\} \cap \{a_{11}, a_{21}\}$ and thus $h(\alpha_{11}) = a_{11}$. Similarly, letting α_{21} be the identity of the group $R_2 \cap L$, we have that $h(\alpha_{21}) = a_{21}$.

Let $\alpha_{12} = \alpha_{11} + p$ and let $\alpha_{22} = \alpha_{21} + p$. Then $h(\alpha_{12}) = a_{11} \cdot e = a_{12}$ and $h(\alpha_{22}) = a_{21} \cdot e = a_{22}$.

Since $\{\alpha_{11}, \alpha_{12}, \alpha_{21}, \alpha_{22}\} \subseteq p + \mathbb{H}$ we have that $p + \alpha_{ij} = \alpha_{ij}$ for all $i, j \in \{1, 2\}$. Next

$$\alpha_{12} + \alpha_{12} = \alpha_{11} + p + \alpha_{11} + p = \alpha_{11} + \alpha_{11} + p = \alpha_{11} + p = \alpha_{12}$$

and similarly $\alpha_{22} + \alpha_{22} = \alpha_{22}$. Next note that

$$\alpha_{11} + \alpha_{12} = \alpha_{11} + \alpha_{11} + p = \alpha_{11} + p = \alpha_{12}$$

and

$$\alpha_{12} + \alpha_{11} = \alpha_{11} + p + \alpha_{11} = \alpha_{11} + \alpha_{11} = \alpha_{11}\,.$$

Similarly, $\alpha_{21} + \alpha_{22} = \alpha_{22}$ and $\alpha_{22} + \alpha_{21} = \alpha_{21}$.

Now α_{11} and α_{21} are idempotents in L and so $L = L + \alpha_{11} = L + \alpha_{21}$ and thus $\alpha_{11} + \alpha_{21} = \alpha_{11}$ and $\alpha_{21} + \alpha_{11} = \alpha_{21}$. Also,

$$\alpha_{12} + \alpha_{22} = \alpha_{11} + p + \alpha_{21} + p = \alpha_{11} + \alpha_{21} + p = \alpha_{11} + p = \alpha_{12}$$

and

$$\alpha_{22} + \alpha_{12} = \alpha_{21} + p + \alpha_{11} + p = \alpha_{21} + \alpha_{11} + p = \alpha_{21} + p = \alpha_{22}\,.$$

We have $\alpha_{11} + \alpha_{22} = \alpha_{11} + \alpha_{21} + p = \alpha_{11} + p = \alpha_{12}$ and $\alpha_{21} + \alpha_{12} = \alpha_{21} + \alpha_{11} + p = \alpha_{21} + p = \alpha_{22}$. Next $\alpha_{12} + \alpha_{21} + \alpha_{12} + \alpha_{21} = \alpha_{12} + \alpha_{22} + \alpha_{21} = \alpha_{12} + \alpha_{21}$ so $\alpha_{12} + \alpha_{21}$ is an idempotent in the group $R_1 \cap L$ and consequently $\alpha_{12} + \alpha_{21} = \alpha_{11}$. Similarly, $\alpha_{22} + \alpha_{11}$ is an idempotent in the group $R_2 \cap L$ and thus $\alpha_{22} + \alpha_{11} = \alpha_{21}$. $\square$

3 Polynomial Progressions

We remind the reader of the following notions of largeness. We write $\mathcal{P}_f(\mathbb{N}) = \{H \subseteq \mathbb{N} : H$ is nonempty and finite$\}$.

Definition 3.1 (a) A subset A of $\mathbb{N}$ is *syndetic* if and only if there is some $H \in \mathcal{P}_f(\mathbb{N})$ such that $\mathbb{N} = \bigcup_{t\in H} (-t + A)$

(b) A subset A of $\mathbb{N}$ is *piecewise syndetic* if and only if there is some $H \in \mathcal{P}_f(\mathbb{N})$ such that, for every $F \in \mathcal{P}_f(\mathbb{N})$ there is some $x \in \mathbb{N}$ with $F + x \subseteq \bigcup_{t\in H} (-t + A)$.

In [2], V. Bergelson and A. Leibman proved a result [Theorem C], one of whose consequences is the following: *Given any finite set R of polynomials that take on integer values at integers and have zero constant term, given any piecewise syndetic subset A of $\mathbb{N}$, and given any sequence $\langle z_i\rangle_{i=1}^\infty$ in $\mathbb{Z}$, there exist $y \in FS(\langle z_i\rangle_{i=1}^\infty)$ and $a \in A$ such that $\{a + p(y) : p \in R\} \subseteq A$.* (Here $FS(\langle z_i\rangle_{i=1}^\infty) = \{\sum_{i\in F} z_i : F$ is a finite nonempty subset of $\mathbb{N}\}$.) Notice that if $R = \{x, 2x, 3x, \ldots, kx\}$, this assertion is van der Waerden's Theorem for length $k+1$ arithmetic progressions. We present here an algebraic derivation of this assertion.

While the results presented here are not as general as those in [2], we claim that our proof of Corollary 3.8 is considerably simpler than the proof of the corresponding fact in [2]. There, starting with $\epsilon > 0$, one chooses a sequence of points $x_0, x_1, \ldots$ in a compact metric space, and concludes that some two are within $\epsilon/2$ of each other. Further, when x_{m+1} is chosen, one needs to work with an ϵ_m chosen based on uniform continuity, and one needs to use the fact that the induction hypothesis is valid on a dense set of points, based on the fact that it is valid somewhere.

By contrast, in the proof presented here, one works entirely with one fixed syndetic set B and the corresponding finite set $H \subseteq \mathbb{N}$ such that $\mathbb{N} = \bigcup_{t\in H} (-t + B)$, and chooses $t_0, t_1, \ldots$ in H, never having to introduce any other finite partition of $\mathbb{N}$ (which would correspond to the ϵ_m's), and concludes that some two are equal. (Here $-t+B = \{x \in \mathbb{N} : t+x \in B\}$.) (I must, however, emphasize that several crucial parts of the proof are taken directly from [2].)

Definition 3.2 (a). A polynomial $p \in \mathbb{Q}[n]$ is an *integral polynomial* provided $p(n) \in \mathbb{Z}$ whenever $n \in \mathbb{Z}$ and $p(0) = 0$.

(b). $\mathcal{R} = \{R : R$ is a finite set of integral polynomials$\}$.

We shall need the following fundamental facts.

Lemma 3.3 *(a). Let $A \subseteq \mathbb{N}$. Then $\overline{A} \cap K(\beta\mathbb{N}) \neq \emptyset$ if and only if A is piecewise syndetic.*

(b) Let $q \in \beta\mathbb{N}$. Then $q \in K(\beta\mathbb{N})$ if and only if for every $A \in q$, $\{x \in \mathbb{N} : -x+A \in q\}$ is syndetic.

(c) If $v + v = v \in \beta\mathbb{N}$, $A \in v$, and $A^\star = \{x \in A : -x+A \in v\}$, then for all $x \in A^\star$, $-x + A^\star \in v$.

Proof. (a). [9, Theorem 4.40].

(b). [9, Theorem 4.39].

(c). [9, Lemma 4.14]. $\square$

Definition 3.4 Order $\bigoplus_{i=1}^{\infty} \omega$ lexicographically based on the largest coordinate on which elements differ, denoting this order by $<$. Define $\varphi : \mathcal{R} \to \bigoplus_{i=1}^{\infty} \omega$ by

$$\varphi(R) = (w_1, w_2, w_3, \ldots)$$

where for each $i \in \mathbb{N}$, $w_i = |\{a \in \mathbb{Q} :$ there exists $p \in R$ such that $\deg p = i$ and the leading coefficient of p is $a\}|$.

Notice that $\bigoplus_{i=1}^{\infty} \omega$ is well ordered by the lexicographic ordering. Notice also that if $R \in \mathcal{R}$, then $\varphi(R) = (0, 0, 0, \ldots)$ if and only if $R \subseteq \{\overline{0}\}$.

Lemma 3.5 *Let $R \in \mathcal{R}$ such that $R \neq \emptyset$ and $\overline{0} \notin R$ and pick $f \in R$ of smallest degree. For $x \in \mathbb{Z}$ and $p \in R$, define $g(p,x) \in \mathbb{Q}[n]$ by $g(p,x)(n) = p(x+n) - p(x) - f(n)$. Let $L \in \mathcal{P}_f(\mathbb{Z})$ and let $S = \{g(p,x) : p \in R$ and $x \in L\}$. Then $S \in \mathcal{R}$ and $\varphi(S) < \varphi(R)$.*

Proof. Trivially $S \in \mathcal{R}$. Let $j = \deg f$, let $\varphi(R) = (w_1, w_2, w_3, \ldots)$, and let $\varphi(S) = (v_1, v_2, v_3, \ldots)$. We claim that for $i > j$, $v_i = w_i$, and that $v_j = w_j - 1$, so that $\varphi(S) < \varphi(R)$ as required.

Indeed, it is routine to check that if $i = \deg p > j$ and $x \in \mathbb{Z}$, then $\deg g(p,x) = \deg p$ and the leading coefficients of $g(p,x)$ and p are the same, so that $v_i = w_i$. To complete the proof, let b be the leading coefficient of f and observe that $\{c \in \mathbb{Z} :$ there exists $u \in S$ such that $\deg u = j$ and the leading coefficient of u is $c\} = \{a - b :$ there exists $p \in R$ such that $\deg p = j$ and the leading coefficient of p is $a\} \setminus \{0\}$. Consequently $v_j = w_j - 1$ as required. $\square$

Theorem 3.6 *Let $R \in \mathcal{R}$, let $v + v = v \in \beta\mathbb{N}$, let A be a piecewise syndetic subset of $\mathbb{N}$ and let L be a minimal left ideal of $\beta\mathbb{N}$ such that $\overline{A} \cap L \neq \emptyset$. Then*

$$\{n \in \mathbb{N} : \overline{A} \cap L \cap \bigcap_{p \in R} \overline{-p(n) + A} \neq \emptyset\} \in v\,.$$

Proof. Suppose not, and pick R such that $\varphi(R)$ is minimal among all counterexamples. Notice that $R \neq \emptyset$ and $R \neq \{\overline{0}\}$ because the statement is trivially true for both of these sets. We may in fact assume that $\overline{0} \notin R$ because $R\backslash\{\overline{0}\}$ is also a counterexample and $\varphi(R\backslash\{\overline{0}\}) = \varphi(R)$. Pick $v = v + v$ and a piecewise syndetic set A and a minimal left ideal L of $\beta\mathbb{N}$ for which the conclusion of the theorem fails. (Note that there is a minimal left ideal L of $\beta\mathbb{N}$ such that $\overline{A} \cap L \neq \emptyset$ by Lemma 3.3(a).) Let

$$D = \mathbb{N}\backslash\{n \in \mathbb{N} : \overline{A} \cap L \cap \bigcap_{p \in R} \overline{-p(n) + A} \neq \emptyset\}$$

and note that $D \in v$. Notice also that L is in fact a left ideal of $\beta\mathbb{Z}$. (It is an easy exercise, which is [9, Exercise 4.3.5], that $\mathbb{N}^*$ is a left ideal of $\beta\mathbb{Z}$ so [9, Lemma 1.43(c)] applies.)

Pick $f \in R$ of smallest degree. For $x \in \mathbb{Z}$ and $p \in R$, define $g(p,x) \in \mathbb{Q}[n]$ by $g(p,x)(n) = p(x+n) - p(x) - f(n)$. Pick $q_0 \in \overline{A} \cap L$ and let $B = \{x \in \mathbb{N} : -x + A \in q_0\}$. Then by Lemma 3.3(b), B is syndetic so pick $H \in \mathcal{P}_f(\mathbb{N})$ such that $\mathbb{N} = \bigcup_{t \in H}(-t + B)$. Pick $t_0 \in H$ such that $-t_0 + B \in q_0$ and let $C_0 = -t_0 + B$. Since $C_0 \in q_0$, $\overline{C_0} \cap L \neq \emptyset$.

Let $S_0 = \{g(p,0) : p \in R\}$ and let $E_0 = \{n \in \mathbb{N} : \overline{C_0} \cap L \cap \bigcap_{p \in S_0} \overline{-p(n) + C_0} \neq \emptyset\}$. By Lemma 3.5, $S_0 \in \mathcal{R}$ and $\varphi(S_0) < \varphi(R)$ so $E_0 \in v$. Pick $y_1 \in E_0 \cap D^\star$ and pick $r_1 \in \overline{C_0} \cap L \cap \bigcap_{p \in S_0} \overline{-p(y_1) + C_0}$. Let $q_1 = -f(y_1) + r_1$ and note that, since L is a left ideal of $\beta\mathbb{Z}$, $q_1 \in L$. Pick $t_1 \in H$ such that $-t_1 + B \in q_1$.

Inductively, assume that we have $m \in \mathbb{N}$ and have chosen $\langle q_i \rangle_{i=0}^m$ in L, $\langle t_i \rangle_{i=0}^m$ in H, and $\langle y_i \rangle_{i=1}^m$ in $\mathbb{N}$ such that

(1) for $j \in \{0, 1, \ldots, m\}$, $-t_j + B \in q_j$,
(2) for $l \in \{1, 2, \ldots, m\}$, $y_l + y_{l+1} + \ldots + y_m \in D^\star$, and
(3) for $l \in \{0, 1, \ldots, m-1\}$ and $p \in R$, $-\big(t_l + p(y_{l+1} + y_{l+2} + \ldots + y_m)\big) + B \in q_m$.

Hypotheses (1) and (2) trivially hold for $m = 1$. To verify hypothesis (3), let $p \in R$. We need to show that $-\big(t_0 + p(y_1)\big) + B \in q_1$. Now $r_1 + g(p,0)(y_1) \in \overline{C_0}$ and so $-t_0 + B \in r_1 + g(p,0)(y_1) = r_1 + p(y_1) - f(y_1) = q_1 + p(y_1)$ as required.

Now let $T_m = \Big\{\{y_{l+1} + y_{l+2} + \ldots + y_m\} : l \in \{0, 1, \ldots, m-1\}\Big\} \cup \{0\}$ and let $S_m = \{g(p,x) : p \in R \text{ and } x \in T_m\}$. Let

$$C_m = (-t_m + B) \cap \bigcap_{p \in R} \bigcap_{l=0}^{m-1} \Big(-\big(t_l + p(y_{l+1} + y_{l+2} + \ldots + y_m)\big) + B\Big)\,.$$

Then by hypotheses (1) and (3), $C_m \in q_m$ and so $C_m \cap L \neq \emptyset$. By Lemma 3.5, $S_m \in \mathcal{R}$ and $\varphi(S_m) < \varphi(R)$ and consequently the statement of the current theorem is valid for S_m and C_m.

Let $E_m = \{n \in \mathbb{N} : \overline{C_m} \cap L \cap \bigcap_{p \in S_m} \overline{-p(n) + C_m} \neq \emptyset\}$. Then $E_m \in v$ and, by hypothesis (2) and Lemma 3.3(c), for each $l \in \{1, 2, \ldots, m\}$, $-(y_l + y_{l+1} + \ldots + y_m) + D^\star \in v$. Pick

$$y_{m+1} \in E_m \cap \bigcap_{l=1}^m -(y_l + y_{l+1} + \ldots + y_m) + D^\star$$

and pick $r_{m+1} \in \overline{C_m} \cap L \cap \bigcap_{p \in S_m} \overline{-p(y_{m+1}) + C_m}$. Let $q_{m+1} = -f(y_{m+1}) + r_{m+1}$ and note that $q_{m+1} \in L$. Pick $t_{m+1} \in H$ such that $-t_{m+1} + B \in q_{m+1}$.

Hypotheses (1) and (2) hold directly. To verify hypothesis (3), let $l \in \{0, 1, \ldots, m\}$ and let $p \in R$. Assume first that $l = m$. Then $r_{m+1} + g(p, 0)(y_{m+1}) \in \overline{C_m}$ and so $-t_m + B \in r_{m+1} + g(p, 0)(y_{m+1}) = r_{m+1} + p(y_{m+1}) - f(y_{m+1}) = q_{m+1} + p(y_{m+1})$ so that $-\big(t_m + p(y_{m+1})\big) + B \in q_{m+1}$ as required.

Now assume that $l < m$, let $x = y_{l+1} + y_{l+2} + \ldots + y_m$, and notice that $x \in T_m$. Then

$$r_{m+1} + g(p, x)(y_{m+1}) \in \overline{C_m} \subseteq \overline{-\big(t_l + p(x)\big) + B}$$

and so $-\big(t_l + p(x)\big) + B \in r_{m+1} + g(p, x)(y_{m+1}) = r_{m+1} + p(x + y_{m+1}) - p(x) - f(y_{m+1}) = q_{m+1} + p(x + y_{m+1}) - p(x)$. Thus $-\big(t_l + p(x + y_{m+1})\big) + B \in q_{m+1}$ as required.

The induction being complete we may choose $l < m$ such that $t_l = t_m$, because H is finite. Let $y = y_{l+1} + y_{l+2} + \ldots + y_m$. By hypothesis (2), $y \in D^\star$. We have that

$$(-t_m + B) \cap \bigcap_{p \in R} \Big(-\big(t_m + p(y)\big) + B\Big) \in q_m$$

so pick $a \in (-t_m + B) \cap \bigcap_{p \in R} \Big(-\big(t_m + p(y)\big) + B\Big)$. Let $r = a + t_m + q_0$ and notice that $r \in \overline{A} \cap L \cap \bigcap_{p \in R} \overline{-p(y) + A}$, contradicting the fact that $y \in D$. □

Corollary 3.7 *Let $R \in \mathcal{R}$, let $\langle z_i \rangle_{i=1}^{\infty}$ be a sequence in $\mathbb{Z}$, and let A be a piecewise syndetic subset of $\mathbb{N}$. Then there exist $r \in \overline{A} \cap K(\beta\mathbb{N})$ and $F \in \mathcal{P}_f(\mathbb{N})$ such that $\{r + p(\Sigma_{i \in F}\, z_i) : p \in R\} \subseteq \overline{A}$.*

Proof. Pick by [9, Lemma 5.11] some $v = v + v$ in $\beta\mathbb{N}$ such that $FS(\langle z_i \rangle_{i=1}^{\infty}) \in v$. Pick by Lemma 3.3(a) a minimal left ideal L of $\beta\mathbb{N}$ such that $A \cap L \neq \emptyset$. Then by Theorem 3.6 $\{n \in \mathbb{N} : \overline{A} \cap L \cap \bigcap_{p \in R} \overline{-p(n) + A} \neq \emptyset\} \in v$ so pick $n \in FS(\langle z_i \rangle_{i=1}^{\infty})$ such that $\overline{A} \cap L \cap \bigcap_{p \in R} \overline{-p(n) + A} \neq \emptyset$. Pick $F \in \mathcal{P}_f(\mathbb{N})$ such that $n = \Sigma_{i \in F}\, z_i$. □

Corollary 3.8 *Let $R \in \mathcal{R}$, let $\langle z_i \rangle_{i=1}^{\infty}$ be a sequence in $\mathbb{Z}$, and let A be a piecewise syndetic subset of $\mathbb{N}$. Then there exists $F \in \mathcal{P}_f(\mathbb{N})$ such that $\{a \in A : \{a + p(\Sigma_{i \in F} z_i) : p \in R\} \subseteq A\}$ is piecewise syndetic.*

Proof. Pick by Corollary 3.7 some $r \in \overline{A} \cap K(\beta\mathbb{N})$ and $F \in \mathcal{P}_f(\mathbb{N})$ such that $\{r + p(\Sigma_{i \in F} z_i) : p \in R\} \subseteq \overline{A}$. Then $A \cap \bigcap_{p \in R}(-p(\Sigma_{i \in F} z_i) + A) \in r$ and $r \in K(\beta\mathbb{N})$ and so, by Lemma 3.3, $A \cap \bigcap_{p \in R}(-p(\Sigma_{i \in F} z_i) + A)$ is piecewise syndetic. If $a \in A \cap \bigcap_{p \in R}(-p(\Sigma_{i \in F} z_i) + A)$, then $\{a + p(\Sigma_{i \in F} z_i) : p \in R\} \subseteq A$. □

4 Some Open Problems in the Algebra of $\beta\mathbb{N}$ and $\beta\mathbb{Z}$

Some of the most difficult problems in the algebra of $\beta\mathbb{N}$ involve the relationship between the operations $+$ and $\cdot$ on $\beta\mathbb{N}$. It is easy to see that if $n \in \mathbb{N}$ and $p, q \in \beta\mathbb{N}$, then $n \cdot (p + q) = n \cdot p + n \cdot q$ [9, Lemma 13.1]. This is the only nontrivial instance of the distributive law which is known to hold.

Theorem 4.1 (van Douwen) *The set* $\{p \in \mathbb{N}^* : \text{for all } q, r \in \mathbb{N}^*,\ (q+r)\cdot p \neq q\cdot p + r\cdot p$ *and* $r \cdot (q+p) \neq r \cdot q + r \cdot p\}$ *has dense interior in* $\mathbb{N}^*$.

Proof. This is [4, Corollary 6.6], or see [9, Corollary 13.27]. □

It is also known [9, Theorem 13.18] that there do not exist $p \in \mathbb{N}^*$ and $m, n \in \mathbb{N}$ such that $p \cdot (m+n) = p \cdot m + p \cdot n$.

Question 4.2 *Do there exist* $p \in \mathbb{N}^*$ *and* $q, r \in \beta\mathbb{N}$ *such that* $p \cdot (q+r) = p \cdot q + p \cdot r$ *or* $(q+r) \cdot p = q \cdot p + r \cdot p$*?*

It is also known that $K(\beta\mathbb{N}, \cdot) \cap K(\beta\mathbb{N}, +) = \emptyset$ and $K(\beta\mathbb{N}, \cdot) \cap c\ell K(\beta\mathbb{N}, +) \neq \emptyset$ [9, Corollaries 13.15 and 16.25].

Theorem 4.3 *Let* $p, q, r, s \in \mathbb{N}^*$. *If* $\{a \in \mathbb{N} : a\mathbb{N} \in r\}$ *is infinite, then* $q + p \neq s \cdot r$.

Proof. This often discovered result is [9, Theorem 13.14]. □

Question 4.4 *Do there exist* $p, q, r, s \in \mathbb{N}^*$ *such that* $q + p = s \cdot r$*?*

In [12] D. Strauss showed that there are no nontrivial continuous isomomorphisms from $\beta\mathbb{N}$ to $\mathbb{N}^*$, answering a question of E. van Douwen in [4]. In fact she showed something stronger.

Theorem 4.5 *If* $\phi : \beta\mathbb{N} \to \mathbb{N}^*$ *is a continuous homomorphism, then* $\phi[\beta\mathbb{N}]$ *is finite and* $|\phi[\mathbb{N}^*]| = 1$.

Proof. [9, Theorem 10.18]. □

Question 4.6 *Are there any nontrivial continuous homomorphisms from* $\beta\mathbb{N}$ *to* $\mathbb{N}^*$*?*

We note that there are some equivalent versions of Question 4.6.

Theorem 4.7 *The following statements are equivalent.*

(a) There is a nontrivial continuous homomorphism from $\beta\mathbb{N}$ *to* $\mathbb{N}^*$.
(b) There exist $p \neq q$ *in* $\mathbb{N}^*$ *such that* $p + p = q + p = p + q = q + q = q$.
(c) There is a finite subsemigroup of $\mathbb{N}^*$ *whose elements are not all idempotent.*
(d) There exists $p \in \mathbb{N}^*$ *such that* $p + p + p = p + p \neq p$.

Proof. The equivalence of (a), (b), and (c) is [9, Corollary 10.20]. It is routine to verify that (b) and (d) are equivalent with $q = p + p$. □

By contrast with statement (d) of Theorem 4.7, it is known that if $p + p + p = p$, then $p + p = p$ [9, Exercise 7.1.3].

We close this section with what is probably the oldest unsolved problem in the algebra of $\beta\mathbb{Z}$. (The question, phrased in terms of the shift map on $\mathbb{Z}$ predates the systematic study of the algebra of $\beta\mathbb{Z}$. See the notes to Chapter 6 of [9] for a discussion of the origins of this question.)

Question 4.8 *Does there exist a sequence* $\langle p_n \rangle_{n=1}^{\infty}$ *in* $\beta\mathbb{Z}$ *such that for each* $n \in \mathbb{N}$, $\beta\mathbb{Z} + p_n \subsetneq \beta\mathbb{Z} + p_{n+1}$.

5 Some Open Problems in Ramsey Theory Related to $\beta\mathbb{N}$

For some time after the proof of the Finite Sums Theorem, it was an open question as to whether an analogous Finite Sums and Products Theorem were valid. That is, was it true that *whenever* $r \in \mathbb{N}$ *and* $\mathbb{N} = \bigcup_{i=1}^{r} A_i$*, there must exist* $i \in \{1, 2, \ldots, r\}$ *and a sequence* $\langle x_n \rangle_{n=1}^{\infty}$ *such that* $FS(\langle x_n \rangle_{n=1}^{\infty}) \cup FP(\langle x_n \rangle_{n=1}^{\infty}) \subseteq A_i$? (Here, analogously with FS, we have $FP(\langle x_n \rangle_{n=1}^{\infty}) = \{\prod_{n \in F} z_n : F$ is a finite nonempty subset of $\mathbb{N}\}$.) The first partial result on this question was the following.

Theorem 5.1 *Let* $r \in \mathbb{N}$ *and let* $\mathbb{N} = \bigcup_{i=1}^{r} A_i$*. There exist* $i \in \{1, 2, \ldots, r\}$ *and sequences* $\langle x_n \rangle_{n=1}^{\infty}$ *and* $\langle y_n \rangle_{n=1}^{\infty}$ *in* $\mathbb{N}$ *with* $FS(\langle x_n \rangle_{n=1}^{\infty}) \cup FP(\langle y_n \rangle_{n=1}^{\infty}) \subseteq A_i$*.*

Proof. [9, Corollary 5.22]. □

However, hopes for an affirmative answer to the infinite sums and products problems were dashed by the following result. (Here $PS(\langle x_n \rangle_{n=1}^{\infty}) = \{x_n + x_m : n, m \in \mathbb{N}$ and $n \neq m\}$ and $PP(\langle x_n \rangle_{n=1}^{\infty}) = \{x_n \cdot x_m : n, m \in \mathbb{N}$ and $n \neq m\}$.)

Theorem 5.2 *There exist* $r \in \mathbb{N}$ *and sets* $A_1, A_2, \ldots, A_r$ *such that* $\mathbb{N} = \bigcup_{i=1}^{r} A_i$ *but there do not exist* $i \in \{1, 2, \ldots, r\}$ *and a sequence* $\langle x_n \rangle_{n=1}^{\infty}$ *in* $\mathbb{N}$ *with* $PS(\langle x_n \rangle_{n=1}^{\infty}) \cup PP(\langle x_n \rangle_{n=1}^{\infty}) \subseteq A_i$*.*

Proof. [9, Theorem 17.16]. □

It is a corollary of Theorem 5.2 that there does not exist $p \in \mathbb{N}^*$ such that $p+p = p \cdot p$ [9, Corollary 17.17].

I would strongly conjecture that the answer to the following question is "yes" for every r and n. However, the only nontrivial case in which the answer is known to be "yes" is the case $n = r = 2$, which is a result of R. Graham. (See the notes to Chapter 17 of [9].)

Question 5.3 *Let* $r, n \in \mathbb{N}$*. If* $\mathbb{N} = \bigcup_{i=1}^{r} A_i$*, must there exist* $i \in \{1, 2, \ldots, r\}$ *and a one-to-one sequence* $\langle x_t \rangle_{t=1}^{n}$ *such that* $FS(\langle x_t \rangle_{t=1}^{n}) \cup FP(\langle x_t \rangle_{t=1}^{n}) \subseteq A_i$*?*

A curious Ramsey Theoretic fact is that one may have sets that are partition regular for certain structures of one size but contain none of the same kind of structures of the next size. Consider, for example, the following two theorems.

An $k \times m$ matrix C is *kernel partition regular* if and only if, whenever $r \in \mathbb{N}$ and $\mathbb{N} = \bigcup_{i=1}^{r} A_i$, there exist $i \in \{1, 2, \ldots, r\}$ and $\vec{x} \in {A_i}^m$ such that $C\vec{x} = \vec{0}$.

Theorem 5.4 (Bergelson, Hindman, and Leader) *Let* $k \in \mathbb{N}$*. There exist a set* $B \subseteq \mathbb{N}$ *and a kernel partition regular* $(k+1) \times (k+3)$ *matrix* D *such that*

(1) for every $m \in \mathbb{N}$ *and every kernel partition regular* $k \times m$ *matrix* C*, whenever* $r \in \mathbb{N}$ *and* $B = \bigcup_{i=1}^{r} A_i$*, there exist* $i \in \{1, 2, \ldots, r\}$ *and* $\vec{x} \in {A_i}^m$ *such that* $C\vec{x} = \vec{0}$ *and*

(2) there does not exist $\vec{x} \in B^{k+3}$ *such that* $D\vec{x} = \vec{0}$*.*

Proof. [1, Corollary 3.7]. □

Theorem 5.5 (Nešetřil, and Rödl) *Let $k \in \mathbb{N}$. There is a set $B \subseteq \mathbb{N}$ such that*

(1) whenever $r \in \mathbb{N}$ and $B = \bigcup_{i=1}^{r} A_i$, there must exist $i \in \{1, 2, \ldots, r\}$ and a sequence $\langle x_n \rangle_{n=1}^{k}$ with $FS(\langle x_n \rangle_{n=1}^{k}) \subseteq A_i$ but

(2) there does not exist a length $k+1$ sequence $\langle y_n \rangle_{n=1}^{k+1}$ with $FS(\langle y_n \rangle_{n=1}^{k+1}) \subseteq B$.

Proof. This follows from [10, Theorem 1.1], which is the analogous statement for finite unions. (For a detailed derivation of this theorem from [10, Theorem 1.1], see [8, Corollary 3.8].) □

These sorts of results led P. Erdős, J. Nešetřil, and V. Rödl to ask the following question dealing with infinite sequences, but with sums restricted to a fixed number of terms.

Definition 5.6 Let $k \in \mathbb{N}$ and let $\langle x_n \rangle_{n=1}^{\infty}$ be a sequence in $\mathbb{N}$. Then $FS_{\leq k}(\langle x_n \rangle_{n=1}^{\infty}) = \{\sum_{t \in F} x_t : \emptyset \neq F \subseteq \mathbb{N} \text{ and } |F| \leq k\}$.

Question 5.7 *Let $k \in \mathbb{N}$. Does there exist a set $B \subseteq \mathbb{N}$ such that*

(1) whenever $r \in \mathbb{N}$ and $B = \bigcup_{i=1}^{r} A_i$, there must exist $i \in \{1, 2, \ldots, r\}$ and a sequence $\langle x_n \rangle_{n=1}^{\infty}$ such that $FS_{\leq k}(\langle x_n \rangle_{n=1}^{\infty}) \subseteq A_i$ but

(2) there does not exist a length $k+1$ sequence $\langle y_n \rangle_{n=1}^{k+1}$ with $FS(\langle y_n \rangle_{n=1}^{k+1}) \subseteq B$?

Of course, the answer to Question 5.7 is "yes" for $k = 1$, but it remains open for all other values of k.

It is a fact [9, Corollary 5.15] that for any sequence $\langle y_n \rangle_{n=1}^{\infty}$ in $\mathbb{N}$, if $r \in \mathbb{N}$ and $FS(\langle y_n \rangle_{n=1}^{\infty}) = \bigcup_{i=1}^{r} A_i$, then there must exist $i \in \{1, 2, \ldots, r\}$ and a sequence $\langle x_n \rangle_{n=1}^{\infty}$ such that $FS(\langle x_n \rangle_{n=1}^{\infty}) \subseteq A_i$. Currently, the only known proof that there is any set B satisfying condition (1) of Question 5.7 is the Finite Sums Theorem (which yields in fact that $FS(\langle x_n \rangle_{n=1}^{\infty}) \subseteq A_i$). Accordingly, I have been led to conjecture an affirmative answer to the following question (which would imply a negative answer to Question 5.7 for every $k \in \mathbb{N}$).

Question 5.8 *Let $B \subseteq \mathbb{N}$ such that whenever $r \in \mathbb{N}$ and $B = \bigcup_{i=1}^{r} A_i$, there must exist $i \in \{1, 2, \ldots, r\}$ and a sequence $\langle x_n \rangle_{n=1}^{\infty}$ such that $FS_{\leq 2}(\langle x_n \rangle_{n=1}^{\infty}) \subseteq A_i$. Must there exist some sequence $\langle y_n \rangle_{n=1}^{\infty}$ such that $FS(\langle y_n \rangle_{n=1}^{\infty}) \subseteq B$?*

A question intermediate between Questions 5.7 and 5.8 is the following.

Question 5.9 *Let $k \in \mathbb{N}$. Does there exist a set $B \subseteq \mathbb{N}$ such that*

(1) whenever $r \in \mathbb{N}$ and $B = \bigcup_{i=1}^{r} A_i$, there must exist $i \in \{1, 2, \ldots, r\}$ and a sequence $\langle x_n \rangle_{n=1}^{\infty}$ such that $FS_{\leq k}(\langle x_n \rangle_{n=1}^{\infty}) \subseteq A_i$ but

(2) there does not exist a sequence $\langle y_n \rangle_{n=1}^{\infty}$ with $FS_{\leq k+1}(\langle y_n \rangle_{n=1}^{\infty}) \subseteq B$?

Our last open problem deals with characterizing infinite image partition regular matrices.

In 1933, R. Rado [11] characterized those (finite) matrices with rational entries which are kernel partition regular. He showed that A is kernel partition regular if and only if A has a computable property called the *columns condition*. (See [9, Section 15.3] for a presentation and proof of Rado's Theorem.)

Sixty years later, I. Leader and I obtained several characterizations of (finite) *image partition regular* matrices [7]. These are the matrices A with the property that whenever $\mathbb{N}$ is finitely colored, there will be some $\vec{x}$ (with entries from $\mathbb{N}$) such that the entries of $A\vec{x}$ are monochrome. Image partition regular matrices are of special interest because many of the classical theorems of Ramsey Theory are naturally stated as statements about image partition regular matrices. For example, Schur's Theorem and the length 4 version of van der Waerden's Theorem amount to the assertions that the matrices

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{pmatrix}$$

are image partition regular.

Infinite image partition regular matrices are also of significant interest. For example, the Finite Sums Theorem is the assertion that the matrix

$$\begin{pmatrix} 1 & 0 & 0 & \cdots \\ 0 & 1 & 0 & \cdots \\ 1 & 1 & 0 & \cdots \\ 0 & 0 & 1 & \cdots \\ 0 & 1 & 1 & \cdots \\ 1 & 1 & 1 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

(whose rows are all vectors with entries from $\{0, 1\}$ with only finitely many 1's and not all 0's) is image partition regular.

The question of which infinite matrices are image partition regular seems to be significantly more complicated than the finite case. For example, it was shown in [5] that there are infinite image partition regular matrices A and B and a coloring of $\mathbb{N}$ in two colors neither of which contains all entries of $A\vec{x}$ and $B\vec{y}$ for any $\vec{x}, \vec{y} \in \mathbb{N}^\omega$.

Problem 5.10 *Characterize those infinite matrices with rational (or integer) coefficients and only finitely many nonzero entries on each row that are image partition regular. (Or characterize those infinite matrices that are kernel partition regular.)*

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