Minimal left ideals of βS with isolated points

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Abstract. The smallest ideal $K(\beta S)$ of the Stone-Čech compactification of a discrete semigroup S is the union of pairwise isomorphic and homeomorphic minimal left ideals. We provide characterizations of semigroups S that have the property that the minimal left ideals of βS have isolated points, provide details about the structure of $K(\beta S)$ for such semgiroups, and in some instances provide explicit descriptions of $K(\beta S)$.

Contents

1. Introduction

Our subject is the algebraic and topological structure of the smallest ideal $K(\beta S)$ of the Stone-Čech compactification of a discrete semigroup S.

We take the points of βS to be the ultrafilters on S, identifying a point $x \in S$ with the principal ultrafilter $e(x) = \{A \subseteq S : x \in A\}$. We let $S^* = \beta S \setminus S$, the set of nonprincipal ultrafilters on S. Given $A \subseteq S$, $\overline{A} = \{p \in \beta S : A \in p\}$. The set $\{\overline{A} : A \subseteq S\}$ is a basis for the open sets (as well as a basis for the closed sets) in βS. With this topology βS is a compact Hausdorff space with the property that if X is any compact Hausdorff space and $f : S \to X$, there is a continuous function $\widetilde{f} : \beta S \to X$ which extends f.

The operation $\cdot$ on S extends uniquely to an operation on βS so that $(\beta S, \cdot)$ is a right topological semigroup with S contained in its topological center. That is for each $p \in \beta S$, the function $\rho_p : \beta S \to \beta S$ defined by $\rho_p(q) = q \cdot p$ is continuous and for each $x \in S$, the function $\lambda_x : \beta S \to \beta S$

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defined by $\lambda_x(q) = x\cdot q$ is continuous. Given points p and q in βS and $A \subseteq S$, $A \in p\cdot q$ if and only if $\{s \in S : s^{-1}A \in q\} \in p$ where $s^{-1}A = \{t \in S : st \in A\}$. The product is also characterized in terms of limits by $p \cdot q = \lim_{s\to p} \lim_{t\to q} st$ where s and t denote members of S. In many cases, the p-limit notation is useful. Given $p \in \beta S$, a compact Hausdorff space X, a set $D \in p$, a function $f : D \to X$, and $y \in X$, $p\text{-}\lim_{s\in D} f(s) = y$ if and only if, for every neighborhood U of y, $\{s \in D : f(s) \in U\} \in p$. We observe that $p\text{-}\lim_{s\in D} f(s) = y$ if and only if $\lim_{s\to p} f(s) = y$ where s denotes a member of D. This is equivalent to the statement that $\widetilde{f}(p) = y$ where $\widetilde{f} : c\ell_{\beta S}(D) \to X$ denotes the continuous extension of f. If $A \in p$ and for each $s \in A$, $B_s \in q$, then $p \cdot q = p\text{-}\lim_{s\in A} q\text{-}\lim_{t\in B_s} st$. See [2, Part I] for an elementary introduction to the algebra and topology of βS.

Any compact Hausdorff right topological semigroup T has idempotents. If $A \subseteq T$, then $E(A)$ is the set of idempotents in A. If T is any compact Hausdorff right topological semigroup, then T has a smallest two sided ideal $K(T)$. Minimal left ideals are compact, and therefore have idempotents. So T satisfies the hypothesis of the Structure Theorem for semigroups. The Structure Theorem is due to A. Suschkewitsch [8] for finite semigroups and to D. Rees [7] in the general case.

Theorem 1.1 (The Structure Theorem). *Let $(T, \cdot)$ be a semigroup and assume that T has a minimal left ideal which has an idempotent. Let R be a minimal right ideal of T, let L be a minimal left ideal of T, let $X = E(L)$, let $Y = E(R)$, and let $G = RL$. Then $G = R \cap L$ and G is a maximal subgroup of $K(T)$. Define an operation $\cdot$ on $X\times G\times Y$ by $(x, g, y)\cdot(x', g', y') = (x, gyx'g', y')$. Then the function $\varphi : X\times G\times Y \to K(T)$ defined by $\varphi(x, g, y) = xgy$ is an isomorphism. Further*

1. *The minimal right ideals of T partition $K(T)$ and the minimal left ideals of T partition $K(T)$.*
2. *The maximal groups in $K(T)$ partition $K(T)$.*
(c) *All minimal right ideals of T are isomorphic and all minimal left ideals of T are isomorphic.*
(4) *All maximal groups in $K(T)$ are isomorphic.*

Proof. [2, Theorem 1.64]. □

In this context, it is worth noting that since $(\beta S, \cdot)$ is a right topological semigroup, then by [2, Theorem 2.11(c)] all minimal left ideals of βS are homeomorphic.

A subset A of a semigroup S is *piecewise syndetic* if and only if $\overline{A} \cap K(\beta S) \neq \emptyset$. Given $p \in \beta S$, we have by [2, Corollary 4.41] that $p \in c\ell K(\beta S)$ if and only if every member of p is piecewise syndetic.

Minimal left ideals of βS may or may not have isolated points. We are concerned in this paper with semigroups S with the property that the minimal left ideals of βS do have isolated points.

In Section 2 we will derive some results about the structure of $K(\beta S)$ that hold if the minimal left ideals of βS have isolated points as well as several characterizations of such semigroups.

In Section 3 we address the situation in which βS has finitely many minimal right ideals. In that case minimal left ideals are finite and so must have isolated points. This has additional implications for the structure of $K(\beta S)$ which we investigate. An amusing consequence is that if S is commutative, then βS either has a unique minimal right ideal or else has at least $2^{\mathfrak{c}}$ minimal right ideals, while in general βS may have any finite number of minimal right ideals.

In Section 4 we derive several properties of rectangular semigroups, and construct a class of examples similar to rectangular semigroups. If S is any member of this class, it has the property that its minimal left ideals all have isolated points, but not for either of the two obvious reasons, namely that the smallest ideal meets S or the minimal left ideals are finite.

2. General results about minimal left ideals with isolated points

Lemma 2.1. *Let $(S,\cdot)$ be an infinite semigroup.and assume that there is a minimal left ideal L of βS which has an idolated point q.*

(1) *For every idempotent $e \in L$, there exists $Q \in q$ such that $\overline{Q} \cap L = \overline{Q}e = \{q\}$.*

(2) *Every point of the group $q\beta S \cap L$ is isolated in L.*

Proof. (1) Pick $P \in q$ such that $\overline{P} \cap L = \{q\}$. Let e be an idempotent in L. Then e is a right identity for L so $qe = q$. Let $Q = P \cap \{s \in S : s^{-1}P \in e\}$. If $p \in \overline{Q} \cap L$, then $pe = p$ so $p \in \overline{Q}e$. If $s \in Q$, then $P \in se$ so $Qe \subseteq \overline{P} \cap L = \{q\}$ so $\overline{Q}e \subseteq \{q\}$. Therefore $q \in \overline{Q} \cap L \subseteq \overline{Q}e \subseteq \{q\}$.

(2) This is [4, Theorem 3.2(3)]. □

Theorem 2.2. *Let $(S,\cdot)$ be an infinite semigroup and assume that βS has a minimal left ideal L with a point q which is isolated in L. Let $R = q\beta S$, let $G = R \cap L$, and let $Y = E(R)$. Then G is a finite group, R is a compact topological semigroup, Y is a compact right zero topological semigroup, and the function $f : G \times Y \to R$ defined by $f(g,p) = gp$ is an isomorphism and a homeomorphism.*

Proof. It was proved in [4, Theorem 3.2] that R is compact and G is finite. Let e be the identity of G and note that $R = e\beta S$. Each element of Y is a left identity for R so Y is a right zero semigroup. To see that Y is compact, let $\langle p_\iota \rangle_{\iota \in I}$ be a net in Y which converges to a point $q \in \beta S$. Since R is compact,

$q \in R$. Then $qe = (\lim_{\iota \in I} p_\iota) \cdot e = \lim_{\iota \in I}(p_\iota e) = \lim_{\iota \in I} e = e$ so $qq = qeq = eq = q$ and thus $q \in Y$.

We claim now that for each $g \in G$, the restriction of λ_g to R is continuous. So let $g \in G$ be given and let h be the inverse of g in G. By Lemma 2.1(2), e is isolated in L so pick $P \in e$ such that $\overline{P} \cap L = \{e\}$. Pick $A \in g$ such that $\rho_h[\,\overline{A}\,] \subseteq \overline{P}$. Pick $t \in A$. Then $th \in \overline{P} \cap L$ so $th = e$ and therefore $te = g$. Thus for every $x \in R$, $gx = tex = tx$ so λ_g and λ_t agree on R.

The fact that f is an isomorphism onto R is a consequence of the proof of [2, Theorem 1.64], but the direct proof is short and simple, so we present it here. To see that f is a homomorphism, let $(g,p),(h,q) \in G \times Y$. Then $f(g,p) \cdot f(h,q) = gphq = ghq = f(gh,pq)$. To see that f is injective, let $(g,p),(h,q) \in G \times Y$ and assume that $f(g,p) = f(h,q)$. Then $h^{-1}gp = q$ so p and q are idempotents in the same minimal left ideal (as well as the same minimal right ideal) so $p = q$. Also $g = ge = gpe = hqe = he = h$. To see that $f[G \times Y] = R$, let $x \in R$. Then for some $p \in Y$, $x \in R \cap \beta Sp = p\beta Sp$. Then $xe = pxe = epxe$ so $xe \in e\beta Se = G$. And $f(xe,p) = xep = xp = x$.

Now we claim that f is a homeomorphism. Since $G \times Y$ is compact, it suffices to show that f is continuous. So let $(g,p) \in G \times Y$ and let $A \in gp$. Since λ_g is continuous on R, pick $B \in p$ such that $\lambda_g[\,\overline{B} \cap R\,] \subseteq \overline{A}$. Then $\{g\} \times (\,\overline{B} \cap Y)$ is a neighborhood of (g,p) in $G \times Y$ and $f[\{g\} \times (\overline{B} \cap Y)] \subseteq \overline{A}$.

Finally, we claim that $G \times Y$ is a topological semigroup (so that R and Y are topological semigroups). That is, we claim that the operation on $G \times Y$ is jointly continuous. So let $(g,p),(h,q) \in G \times Y$ and let $\{gh\} \times (\overline{A} \cap Y)$ be a basic neighborhood of $(g,p) \cdot (h,q) = (gh,q)$ in $G \times Y$. Then $(\{g\} \times Y) \cdot (\{h\} \times (\overline{A} \cap Y) \subseteq \{gh\} \times (\overline{A} \cap Y)$. □

One of the equivalences in the following theorem involves the notion of a *dynamical system*. We take a dynamical system to consist of a pair $(X, \langle T_s \rangle_{s \in S})$ where X is a nonempty compact Hausdorff space, S is a discrete semigroup, for each $s \in S$, T_s is a continuous function from X to itself, and for each $s,t \in S$, $T_s \circ T_t = T_{st}$.

Another of the equivalences involves the notion of a *QC-set*.

Definition 2.3. Let $(S,\cdot)$ be a discrete semigroup. A subset A of S is a $\hat{\text{Q}}$C-set if and only if there exist w and z in βS such that $s \cdot w = z$ for all $s \in A$.

One of the reasons that QC-sets are interesting is that they are related to algebraic products of tensor products – a subject investigated in [6].

Theorem 2.4. *Let $(S,\cdot)$ be an infinite semigroup. Statements (a) through (g) are equivalent and imply statement (h).*

(a) *There is a minimal left ideal L of βS which has a point that is isolated in L.*
(b) *Each minimal left ideal of βS has an isolated point.*
(c) *There is a piecewise syndetic QC-set in S.*

(d) *Every minimal left ideal L of βS has a dense set of points that are isolated in L.*
(e) *Every minimal left ideal L of βS has an idempotent which is isolated in L.*
(f) *For $s \in S$, let $B_s = \{t \in S : st = t\}$. There is a piecewise syndetic set $D \subseteq S$ such that $\{B_s : s \in D\}$ has the finite intersection property.*
(g) *For any dynamical system $(X, \langle T_s \rangle_{s \in S})$ with S as the acting semigroup, there exist $x \in X$ and a minimal right ideal R of βS such that for every $p \in E(R)$, $\{s \in S : T_s(x) = x\} \in p$.*
(h) *There exist a minimal left ideal L of βS and a minimal right ideal R of βS such that $R \cap L$ is a finite group, R is a compact topological semigroup, $E(R)$ is a compact right zero semigroup, and the function $f : (R \cap L) \times E(R) \to R$ defined by $f(g, p) = gp$ is an isomorphism and a homeomorphism.*

Proof. Since all minimal left ideals of βS are homeomorphic we have that (a) and (b) are equivalent.

$(a) \Rightarrow (c)$. Pick a left ideal L of βS and a point $q \in L$ which is isolated in L. Pick an idempotent $e \in L$. By Lemma 2.1(1), pick $Q \in q$ such that $\overline{Q}e = \{q\}$. Since $Q \in q$, Q is piecewise syndetic. Since $se = q$ for every $s \in Q$, Q is a QC-set.

$(c) \Rightarrow (d)$. Let Q be a piecewise syndetic QC-set in S. We can choose a minimal left ideal L of βS for which $\overline{Q} \cap L \neq \emptyset$. Let $R_Q = \{w \in \beta S : sw = tw$ for every $s, t \in Q\}$. Since Q is a QC-set, $R_Q \neq \emptyset$ so R_Q is a right ideal of βS. Therefore there is an idempotent $e \in L \cap R_Q$. Now e is a right identity for L so $\overline{Q} \cap L \subseteq \overline{Q}e$. Since $e \in R_Q$, $se = te$ for every $s, t \in Q$. So $|Qe| = 1$, and hence $|\overline{Q}e| = 1$, because $\overline{Q}e = c\ell_{\beta S}(Qe)$. It follows that $\overline{Q} \cap L = \{q\}$ for some element $q \in L$, which is an isolated point of L.

Choose any $s \in Q$ and any $t \in S$. We claim that tse is an isolated point of L. To see this, observe that $see = se = q$. Pick $A \in e$ such that $s\overline{A}e \subseteq \overline{Q}$. Then for every $a \in A$, $sae \in \overline{Q} \cap L = \{q\}$. So $tsae = tq$ for every $a \in A$. Since $ts\overline{A} \cap L \subseteq ts\overline{A}e$, it follows that $ts\overline{A} \cap L$ is a singleton subset of L, and hence that tse is an isolated point of L. Now Sse is dense in L, because $\beta Sse = L$, and so the set of isolated points of L is dense in L.

Using the fact already noted that all minimal left ideals of βS are homeomorphic we have that each minimal left ideal of βS has a dense set of isolated points.

It is trivial that (d) implies (a) and (e) implies (a). That (b) implies (e) follows from Lemma 2.1(2).

The equivalence of (b) and (f) is [4, Theorem 3.7(2)] and the equivalence of (b) and (g) is [4, Theorem 3.13]. That (a) implies (h) is a consequence of Theorem 2.2. □

Question 2.5. *Does statement (h) of Theorem 2.4 imply the other statememts?*

We remark that one may have $K(\beta S)$ and $K(\beta T)$ isomorphic but not homeomorphic. To see this, let $x \vee y = \max\{x, y\}$ and let $*$ be the right zero operation on $\mathbb{N}$. Then $K(\beta\mathbb{N}, \vee) = \mathbb{N}^*$ and $K(\beta\mathbb{N}, *) = \beta\mathbb{N}$, while both are algebraically the right zero semigroup on $2^{\mathfrak{c}}$ elements.

Recall that a semigroup S is *weakly right cancellative* if and only if for all a and b in S, $\{s \in S : sa = b\}$ is finite.

Theorem 2.6. *Let $(S, \cdot)$ be an infinite semigroup which is left cancellative. Assume that there is a minimal left ideal L of βS that has an isolated point. Then S is not weakly right cancellative.*

Proof. Suppose that S is weakly right cancellative. Pick by Theorem 2.4 a minimal left ideal L of βS that has an idempotent e which is isolated in L. By [2, Theorem 4.36] S^* is an ideal of βS so $L \subseteq S^*$.

By Lemma 2.1(1), pick $Q \in e$ such that $\overline{Q}e = \{e\}$. Let $E = \{s \in S : se = e\}$. Then $Q \subseteq E$ so $E \in e$. Let $A = \{t \in S : t$ has finite order$\}$. We claim that $E \subseteq A$ so that $A \in e$. To this end, let $s \in E$. By [2, Theorem 3.35], $B = \{t \in S : st = t\} \in e$ so pick $t \in B \cap E$. Let $C = \{v \in S : vt = t\}$. Then $s \in C$ so C is a subsemigroup of S, and since S is weakly right cancellative, C is finite. Thus $s \in C \subseteq A$ as claimed.

Note that since S is left cancellative, any idempotent in S is a left identity for S. We now claim that every $s \in E$ is a left identity for S. To see this, let $s \in E$ and as above pick $t \in E$ such that $st = t$. Then $t \in A$ so $\{t^n : n \in \mathbb{N}\}$ is a finite semigroup and thus there is some $n \in \mathbb{N}$ such that t^n is an idempotent. Now let $x \in S$ be given. Then $x = t^n x$ so $sx = st^n x = t^n x = x$. We thus have that E is a right zero semigroup, which is infinite since $e \in S^*$. This contradicts the assumption that S is weakly right cancellative. □

We now consider the implications of S having a QC-set of positive density. (Here *density* means *Følner density*. See [1, Sections 1 and 4] for an introduction to Følner density.)

We shall say that a Borel measure defined on βS is left invariant if $\mu(B) = \mu(s^{-1}B)$ for every Borel subset B of βS and every $s \in S$. In the case in which S is left cancellative and μ is a left invariant measure on βS, $\mu(sB) = \mu(s^{-1}sB) = \mu(B)$ for every $s \in S$ and every Borel subset B of βS. We shall use the well-known fact that, if S is a semigroup which is left cancellative and left amenable, then a subset A of S has positive density if and only if there is a left invariant probability measure μ on βS such that $\mu(\overline{A}) > 0$. (For a proof of the sufficiency of that statement, which is the only part that we will use here, see [1, Theorems 4.7 and 4.16].)

Theorem 2.7. *Let S be a semigroup which is left cancellative and left amenable. Assume that S contains a QC-set Q of positive density. Then the minimal left ideals of βS are finite, and so the conclusions of Theorem 2.2 hold.*

Proof. $\mathcal{L}$ will denote the set of left invariant probability measures on βS. We note that, for every right ideal R of S and every $\mu \in \mathcal{L}$, $\mu(\overline{R}) = 1$, because, for every $s \in R$, $s^{-1}R = S$ and so $\mu(\overline{R}) = \mu(s^{-1}\overline{R}) = \mu(\beta S) = 1$.

We can choose an element $w \in \beta S$ such that $sw = tw$ for every $s, t \in Q$. By [2, Lemma 8.5], for every $s, t \in Q$, $V_{s,t} = \{v \in S : sv = tv\} \in w$. Since $V_{s,t}$ is nonempty, it is a closed right ideal in βS and so $\mu(\overline{V_{s,t}}) = 1$ for every $\mu \in \mathcal{L}$. It follows from the fact that μ is regular, that $\mu(R_Q) = \mu(\bigcap_{s,t\in Q}(\overline{V_{s,t}}) = 1$, because each open neighborhood of $\bigcap_{s,t\in Q}\overline{V_{s,t}}$ contains the intersection of a finite number of the sets of the form $\overline{V_{s,t}}$ with $s, t \in Q$. So the support of every measure in $\mathcal{L}$ is contained in R_Q.

There exists $\mu \in \mathcal{L}$ such that $\mu(\overline{Q}) > 0$. Let M denote the support of μ and let $q \in \overline{Q} \cap M$. Observe that M is a left ideal in βS. For every $s \in Q$, sS is a right ideal in S and so $M \subseteq \overline{sS} = s\beta S$. Thus $q = su$ for some $u \in M$. Since $u \in R_Q$, $su = tu$ for every $t \in Q$. It follows that $qu = q$, since $Q \in q$. Now $\{u \in M : qu = q\}$ is a subsemigroup of M. It is closed, because $u \in R_Q$ and so $qu = q = su$ if and only if $tu = q$ for every $t \in Q$. It follows that there is an idempotent $e \in M$ for which $qe = q$. Since $e \in R_Q$, $|Qe| = 1$ and so $|\overline{Q}e| = 1$ and therefore $\overline{Q}e = \{q\}$.

We can define a probability measure ν on βS by putting $\nu(B) = \rho_e^{-1}[B]$ for every Borel subset B of βS. For every $t \in S$ and every Borel subset B of βS, $t^{-1}\rho_e^{-1}[B] = \rho_e^{-1}[t^{-1}B]$. Thus $\nu(t^{-1}B) = \nu(B)$ and so $\nu \in \mathcal{L}$. Let $L = \beta S e$. Observe that e is a right identity for L. Since $\rho_e^{-1}[L] = \beta S$, $\nu[L] = 1$ and thus L is contained in the support of ν. If U is any open neighborhood of q, then $\rho_e^{-1}[U]$ is also an open neighborhood of q because $q = qe \in U$. Since q is in the support of μ, $\mu(\rho_e^{-1}[U]) > 0$ and so $\nu(U) > 0$. Thus q is in the support of ν.

Now $L \cap \overline{Q} \subseteq \overline{Q}e = \{q\}$. So $\{q\}$ is an isolated point of L. It follows that, for every $t \in S$, $\nu(\{q\}) = \nu(\{tq\}) > 0$. So Sq is finite and therefore $\beta S q = Sq$ is finite. Since βS contains a finite left ideal, its minimal left ideals are finite. □

3. Finitely many minimal right ideals

Lemma 3.1. *Let $(S, \cdot)$ be an infinite semigroup and assume that βS has finitely many minimal right ideals. Then the minimal left ideals of βS are finite and for each $q \in K(\beta S)$, the restriction of λ_q to $K(\beta S)$ is continuous.*

Proof. By [2, Theorem 6.39] the minimal left ideals of βS are finite. Let $q \in K(\beta S)$ be given and let L be the minimal left ideal with $q \in L$. Enumerate the minimal right ideals of βS as $R_1, R_2, \ldots, R_n$ and for $i \in \{1, 2, \ldots, n\}$, let e_i be the identity of $L \cap R_i$. Since q is isolated in L, pick by Lemma 2.1(1) $Q_i \in q$ for each $i \in \{1, 2, \ldots, n\}$ such that $\overline{Q_i}e_i = \{q\}$. Pick $s \in \bigcap_{i=1}^n Q_i$. We show that λ_q and λ_s agree on $K(\beta S)$. To see this, let $x \in K(\beta S)$ be given and pick $i \in \{1, 2, \ldots, n\}$ such that $x \in R_i$. Then $qx = se_i x = sx$. □

In the following theorem $G \times Y$ has the product topology and the coordinatewise operation and $X \times G \times Y$ has the product topology and the operation defined by $(x, h, y) \cdot (x', h', y') = (x, hyx'h', y')$.

Theorem 3.2. *Let $(S, \cdot)$ be an infinite semigroup and assume that βS has finitely many minimal right ideals. Let L be a minimal left ideal of βS, let R be a minimal right ideal of βS, let $X = E(L)$, let $G = R \cap L$, and let $Y = E(R)$. Then*

(1) *X is a finite left zero semigroup;*
(2) *G is a finite group;*
(3) *R is a compact topological semigroup;*
(4) *Y is a compact right zero topological semigroup;*
(5) *$K(\beta S)$ is a compact topological semigroup;*
(6) *the function $f : G \times Y \to R$ defined by $f(h, y) = hy$ is an isomorphism and a homeomorphism; and*
(7) *the function $\varphi : X \times G \times Y \to K(\beta S)$ defined by $\varphi(x, h, y) = xhy$ is an isomorphism and a homeomorphism.*

Proof. By Lemma 3.1, L is finite so all points of L are isolated in L. Pick $q \in R \cap L$. Then $R = q\beta S$ so Theorem 2.2 applies. It only remains to verify conclusions (1), (5), and (7). Since L is finite and $X \neq \emptyset$, conclusion (1) holds.

(5) Since there are finitely many minimal right ideals, and each minimal right ideal is compact, we have that $K(\beta S)$ is compact and each minimal right ideal is open in $K(\beta S)$. To see that $K(\beta S)$ is a topological semigroup, let $x, y \in K(\beta S)$ and let U be an open neighborhood of xy in $K(\beta S)$. Pick minimal right ideals R_1 and R_2 such that $x \in R_1$ and $y \in R_2$. (We are not assuming that $R_1 \neq R_2$.) Since $xy \in R_1$ and R_1 is open in $K(\beta S)$, we may assume that $U \subseteq R_1$.

We have chosen a minimal left ideal L of βS. Let e_1 and e_2 be the identities of $R_1 \cap L$ and $R_2 \cap L$ respectively. By Lemma 3.1, the restrictions of λ_{e_1} and λ_{e_2} to $K(\beta S)$ are continuous. Let ψ_1 be the restriction of λ_{e_1} to R_2 and let ψ_2 be the restriction of λ_{e_2} to R_1. Then $\psi_1[R_2] = R_1$, $\psi_2[R_1] = R_2$, $\psi_1 \circ \psi_2$ is the identity on R_1, and $\psi_2 \circ \psi_1$ is the identity on R_2. Thus ψ_1 is a homeomorphism from R_2 onto R_1 and ψ_2 is a homeomorphism from R_1 onto R_2.

We have that e_2U is a neighborhood of e_2xy contained in R_2. Since R_2 is a topological semigroup, we have open neighborhoods V of e_2x and W of y in R_2 with $VW \subseteq e_2U$. Then e_1V is an open neighborhood of x in R_1 and $e_1VW \subseteq U$. Since R_1 and R_2 are open in $K(\beta S)$, e_1V and W are open in $K(\beta S)$.

(7) The fact that φ is an isomorphism onto $K(\beta S)$ is part of the proof of [2, Theorem 1.64]. Since $K(\beta S)$ is compact, to see that φ is a homeomorphism, it suffices to show that φ is continuous. So let $(x, h, y) \in X \times G \times Y$ and let U be a neighborhood of $\varphi(x, h, y)$ in $K(\beta S)$. Since λ_{xh} is continuous on $K(\beta S)$,

pick $B \in y$ such that $\lambda_{xh}[\,\overline{B} \cap K(\beta S)\,] \subseteq U$. Let $V = \{x\} \times \{h\} \times (\overline{B} \cap Y)$. Then V is a neighborhood of (x, h, y) and $\varphi[V] \subseteq U$. $\square$

For any semigroup S, if L is a minimal left ideal of βS, R is a minimal right ideal of βS, $X = E(L)$, $G = R \cap L$, and $Y = E(R)$, then by the Structure Theorem (Theorem 1.1) the function φ of Theorem 3.2 is an isomorphism. If X is not closed in βS (as is the case in $\beta\mathbb{N}$ by [2, Theorem 6.15.2]), then φ is not a homeomorphism. Indeed, if e is the identity of G, then $X \times \{e\} \times \{e\}$ is closed in $X \times G \times Y$, but $\varphi[X \times \{e\} \times \{e\}] = X$ which is not closed in $K(\beta S)$. (If X were closed in $K(\beta S)$, it would be closed in L which is compact.)

Theorem 3.3. *Let $(S, \cdot)$ be an infinite commutative semigroup. The function φ of Theorem 3.2 is a homeomorphism if and only if βS has a unique minimal right ideal.*

Proof. Let X, G, and Y be as in the statement of Theorem 3.2 and let e be the identity of G. If βS has a unique minimal right ideal, then Theorem 3.2(7) applies. (This part does not use the assumption that S is commutative.)

Now assume that $\varphi : X \times G \times Y \to K(\beta S)$ is a homeomorphism. Now $\{e\} \times G \times \{e\}$ is closed in $X \times G \times Y$ so $G = \varphi[\{e\} \times G \times \{e\}]$ is closed in $K(\beta S)$. Now $L = \beta S e = c\ell_{\beta S}(Se) = c\ell_{K(\beta S)}(Se) = c\ell_{K(\beta S)}(eSe)$, since S is commutative. And $c\ell_{K(\beta S)}(eSe) \subseteq e\beta S e = G$ so $L \subseteq G = L \cap R$ and thus $L \subseteq R$. If βS had another minimal right ideal, it would miss L. $\square$

Note that if the number of minimal right ideals of βS is finite, then the minimal left ideals are finite, while by [2, Theorem 6.39] if the minimal left ideals of βS are infinite, the number of minimal right ideals is at least $2^{\mathfrak{c}}$.

Corollary 3.4. *Let S be an infinite commutative semigroup. Then βS either has a unique minimal right ideal or βS has at least $2^{\mathfrak{c}}$ minimal right ideals.*

Proof. Assume that βS has more than one minimal right ideal. By Theorem 3.3 the function φ of Theorem 3.2 is not a homeomorphism so by Theorem 3.2, βS has infinitely many minimal right ideals. Since they each have nonempty intersection with a given minimal left ideal, the minimal left ideals of βS are infinite, so by [2, Theorem 6.39], there must be at least $2^{\mathfrak{c}}$ minimal right ideals. $\square$

We see next that it is possible for βS to have any finite number of minimal right ideals.

Definition 3.5. Let A and B be nonempty sets. Then $\mathbf{R}(A, B)$ is $A \times B$ with the rectangular semigroup operation defined by $(a, b) \cdot (c, d) = (a, d)$.

Theorem 3.6. *Let $n \in \mathbb{N}$, let $|A| = n$, let B be an infinite set, and let $S = \mathbf{R}(A, B)$. Let $\widetilde{\pi_1} : \beta S \to A$ be the continuous extension of the projection function. For $a \in A$, let $R_a = \{p \in \beta S : \widetilde{\pi_1}(p) = a\}$. The minimal right ideals of βS are the sets R_a for $a \in A$.*

Proof. By [4, Theorem 2.2] each R_a is a minimal right ideal of βS. Let R be a minimal right ideal of βS and pick $p \in R$. Let $a = \widetilde{\pi_1}(p)$. Then $R = p\beta S$ and so by [4, Theorem 2.1(3)], $R \subseteq R_a$. $\square$

Theorem 3.7. *Let $(S, \cdot)$ be an infinite semigroup such that βS has a minimal left ideal L with an isolated point. If S is either left cancellative or commutative, then βS has a unique minimal right ideal.*

Proof. Suppose that βS has distinct minimal right ideals R and R', let e be the identity of $R \cap L$, and let e' be the identity of $R' \cap L$. Pick an isolated point q of L. By Lemma 2.1(1) pick Q and Q' in q such that $\overline{Q} \cap L = \overline{Q}e = \{q\}$ and $\overline{Q'} \cap L = \overline{Q'}e' = \{q\}$. Pick $s \in Q \cap Q'$. Then $se = se'$. If S is left cancellative, then by [2, Lemma 8.1], $e = e'$. If S is commutative, then by [2, Theorem 4.23], $es = e's$ so $R = R'$. $\square$

4. Rectangular semigroups and a variation

In this section we present examples of semigroups S with the property that βS has minimal left ideals with isolated points. The structure of $K(\beta S)$ in these examples involves the *tensor product* of ultrafilters.

Definition 4.1. Let A and B be nonempty discrete sets, let $p \in \beta A$, and let $q \in \beta B$. The *tensor product* $p \otimes q$ is the ultrafilter on $A \times B$ such that for every $C \subseteq A \times B$, $C \in p \otimes q$ if and only if $\{x \in A : \{y \in B : (x, y) \in C\} \in q\} \in p$.

The tensor product can be characterized as $p \otimes q = \lim\limits_{s \to p} \lim\limits_{t \to q} (s, t)$, where (s, t) denotes an element of $A \times B$.

We observe that, for every $q \in \beta B$, the map $p \mapsto p \otimes q$ is a continuous map from βA to $\beta(A \times B)$, and, for every $s \in A$, the map $q \mapsto s \otimes q$ is a continuous map from βB to $\beta(A \times B)$.

Lemma 4.2. *Let A and B be nonempty discrete sets, let $p \in \beta A$, and let $q \in \beta B$. Let $\widetilde{\pi_1} : \beta(A \times B) \to \beta A$ and $\widetilde{\pi_2} : \beta(A \times B) \to \beta B$ be the continuous extensions of the projection functions.*

1. *$\widetilde{\pi_1}(p \otimes q) = p$ and $\widetilde{\pi_2}(p \otimes q) = q$.*
2. *If $C \in p$ and $D \in q$, then $C \times D \in p \otimes q$.*
3. *If $r \in \beta(A \times B)$ and either $\widetilde{\pi_1}(r) \in A$ or $\widetilde{\pi_2}(r) \in B$, then $r = \widetilde{\pi_1}(r) \otimes \widetilde{\pi_2}(r)$.*

Proof. The verification of each of these assertions is a routine application of the definition of $\otimes$. $\square$

In the following theorem we extend the results of [4, Theorem 2.1] about rectangular semigroups.

Theorem 4.3. *Let A and B be nonempty discrete sets and let $S = \mathbf{R}(A, B)$. Let $\widetilde{\pi_1} : \beta S \to \beta A$ be the continuous extension of $\pi_1 : S \to A$ and let $\widetilde{\pi_2} : \beta S \to \beta B$ be the continuous extension of $\pi_2 : S \to B$. For $x \in \beta A$ and for $y \in \beta B$, let $R_x = \{p \in \beta S : \widetilde{\pi_1}(p) = x\}$ and let $L_y = \{p \in \beta S : \widetilde{\pi_2}(p) = y\}$.*

(1) *If $p, q \in \beta S$, then $\widetilde{\pi_1}(pq) = \widetilde{\pi_1}(p)$ and $\widetilde{\pi_2}(pq) = \widetilde{\pi_2}(p)$.*
(2) *If $p, q \in \beta S$, then $pq = \widetilde{\pi_1}(p) \otimes \widetilde{\pi_2}(q)$.*
(3) *If $x \in \beta A$ and $y \in \beta B$, then R_x is a right ideal of βS and L_y is a left ideal of βS.*
(4) *If $p, q, r \in \beta S$, then $pqr = pr$.*
(5) *Let $p \in \beta S$. The following statements are equivalent.*
 (a) *$p \in K(\beta S)$.*
 (b) *$p \in (\beta S)(\beta S)$.*
 (c) *$p = \widetilde{\pi_1}(p) \otimes \widetilde{\pi_2}(p)$.*
 (d) *$p \in \beta A \otimes \beta B$.*
 (e) *p is an idempotent.*
(6) *If $x \in \beta A$ and $y \in \beta B$, then $R_x \cap L_y \cap K(\beta S) = \{x \otimes y\}$.*
(7) *If $x \in \beta A$ and $y \in \beta B$, then R_x contains a unique minimal right ideal and L_y contains a unique minimal left ideal.*
(8) *Let $y \in \beta B$ and let L be the unique minimal left ideal contained in L_y. Then $\{p \in L : p \text{ is isolated in } L\} = A \otimes y$.*
(9) *If $x \in \beta A$ and $y \in \beta B$, then the minimal right ideal contained in R_x is $x \otimes \beta B$ and the minimal left ideal contained in L_y is $\beta A \otimes y$. So the minimal right ideals of βS are the sets of the form $x \otimes \beta B$ for $x \in \beta A$ and the minimal left ideals of βS are the sets of the form $\beta A \otimes y$ for $y \in \beta B$.*
(10) *If A and B are infinite, then $K(\beta S)$ is not a Borel subset of βS. If either A or B is finite, then $K(\beta S)$ is compact.*
(11) *Define $\phi : K(\beta S) \to \mathbf{R}(\beta A, \beta B)$ by $\phi(p) = \big(\widetilde{\pi_1}(p), \widetilde{\pi_2}(p)\big)$. Then ϕ is a continuous isomorphism. Also ϕ is a homeomorphism if and only if either A or B is finite.*

Proof. Statements (1), (3), and (4) follow from statements (3), (4), and (1) of [4, Theorem 2.1] respectively.

(2) Since pq and $\widetilde{\pi_1}(p) \otimes \widetilde{\pi_2}(q)$ are ultrafilters, it suffices to show that $pq \subseteq \widetilde{\pi_1}(p) \otimes \widetilde{\pi_2}(q)$ so let $C \in pq$ and let $D = \{(a,b) \in S : (a,b)^{-1}C \in q\}$. Then $D \in p$ so $\pi_1[D] \in \widetilde{\pi_1}(p)$. We claim that $\pi_1[D] \subseteq \{a \in A : \{d \in B : (a,d) \in C\} \in \widetilde{\pi_2}(q)\}$. So let $a \in \pi_1[D]$ and pick b such that $(a,b) \in D$. Then $(a,b)^{-1}C \in q$ so $\pi_2[(a,b)^{-1}C] \in \widetilde{\pi_2}(q)$ and $\pi_2[(a,b)^{-1}C] \subseteq \{d \in B : (a,d) \in C\}$.

(5) $(a) \Rightarrow (b)$. p is a member of some minimal right ideal R of βS and if e is an idempotent in R, then $p = eP$.

$(b) \Rightarrow (c)$. Pick $q, r \in \beta S$ such that $p = qr$. By (2), $qr = \widetilde{\pi_1}(q) \otimes \widetilde{\pi_2}(r)$. By (1) $\widetilde{\pi_1}(p) = \widetilde{\pi_1}(q)$ and $\widetilde{\pi_2}(p) = \widetilde{\pi_2}(r)$.

It is trivial that (c) implies (d).

$(d) \Rightarrow (e)$. Pick $x \in \beta A$ and $y \in \beta B$ such that $p = x \otimes y$. By (2) $pp = (x \otimes y) \cdot (x \otimes y) = \widetilde{\pi_1}(x \otimes y) \otimes \widetilde{\pi_2}(x \otimes y) = x \otimes y = p$.

$(e) \Rightarrow (a)$. Pick $q \in K(\beta S)$. By (4) $p = pp = pqp \in K(\beta S)$.

(6) R_x contains a minimal right ideal and L_y contains a minimal left ideal so $R_x \cap L_y \cap K(\beta S) \neq \emptyset$. Let $p \in R_x \cap L_y \cap K(\beta S)$. By (5) $p = \widetilde{\pi_1}(p) \otimes \widetilde{\pi_2}(p) = x \otimes y$.

(7) Suppose L_1 and L_2 are distinct minimal left ideals contained in L_y. Then $R_x \cap L_1 \cap K(\beta S) \neq \emptyset$ and $R_x \cap L_2 \cap K(\beta S) \neq \emptyset$ while $|R_x \cap L_y \cap K(\beta S)| = 1$, a contradiction. Similarly R_x does not contain two distinct minimal right ideals.

(8) Let p be an isolated point of L and pick $C \subseteq S$ such that $\overline{C} \cap L = \{p\}$. Let $x = \widetilde{\pi_1}(p)$. Then $p = x \otimes y$ by (5). Let $D = \{a \in A : \{b \in B : (a,b) \in C\} \in y\}$. If $D = \{a\}$, then (recalling that we identify points of A with the principal ultrafilters generated by those points) we have $p = a \otimes y$. If we had distinct a_1 and a_2 in D, we would have $a_1 \otimes y$ and $a_2 \otimes y$ in $\overline{C} \cap L$.

For the other inclusion, let $a \in A$. Then $\widetilde{\pi_1}^{-1}[\{a\}] \cap L = R_a \cap L \neq \emptyset$ and $R_a \cap L \subseteq R_a \cap L_y \cap K(\beta S) = \{a \otimes y\}$ by (6) so $\widetilde{\pi_1}^{-1}[\{a\}] \cap L = \{a \otimes y\}$.

(9) We establish the conclusion for L_y, the proof for R_x being essentially the same. (Or look ahead to the proof of Theorem 4.4(12).) So let L be the minimal left ideal contained in L_y and let $p \in L$. By (5) $p = \widetilde{\pi_1}(p) \otimes y \in \beta A \otimes y$.

Now let $u \in \beta A$. Then $u \otimes y \in K(\beta S)$ by (5) so $u \otimes y \in L'$ for some minimal left ideal L'. But then $\widetilde{\pi_2}[L'] = \{y\}$ so $L' \subseteq L_y$ so $L' = L$.

(10) By (5) $K(\beta S) = \beta A \otimes \beta B$. Assume first that either A or B is finite. Then by Lemma 4.2(3), $\beta A \otimes \beta B = \beta(A \times B)$.

Now assume that A and B are infinite and pick countably infinite subsets A_0 of A and B_0 of B. It is an immediate consequence of [5, Theorem 2.4] that $\beta A_0 \otimes \beta B_0$ is not a Borel subset of $\beta(A_0 \times B_0)$. It is routine to verify from the definition of $\otimes$ that $(\beta A \otimes \beta B) \cap \beta(A_0 \times B_0) = \beta A_0 \otimes \beta B_0$. Thus, since $\beta(A_0 \times B_0)$ is a clopen subset of $\beta(A \times B)$, if $\beta A \otimes \beta B$ were a Borel subset of $\beta S = \beta(A \times B)$, one would have that $\beta A_0 \otimes \beta B_0$ is a Borel subset of $\beta(A_0 \times B_0)$.

(11) Given $p \in K(\beta S)$ we have $\widetilde{\pi_1}(p) \in \beta A$ and $\widetilde{\pi_2}(p) \in \beta B$ so $\phi : K(\beta S) \to \mathbf{R}(\beta A, \beta B)$. By (3), $\phi[K(\beta S)] = \mathbf{R}(\beta A, \beta B)$. Let $p, q \in K(\beta S)$. Then

$$\begin{aligned}\phi(pq) &= \big(\widetilde{\pi_1}(pq), \widetilde{\pi_2}(pq)\big) \\ &= \big(\widetilde{\pi_1}(p), \widetilde{\pi_2}(q)\big) \\ &= \big(\widetilde{\pi_1}(p), \widetilde{\pi_2}(p)\big) \cdot \big(\widetilde{\pi_1}(q), \widetilde{\pi_2}(q)\big) \\ &= \phi(p) \cdot \phi(q)\,.\end{aligned}$$

By (6) we have that ϕ is injective.

To see that ϕ is continuous, let $p \in K(\beta S)$ and let W be a neighborhood of $\phi(p)$ in $\beta A \times \beta B$. Pick $C \subseteq A$ and $D \subseteq B$ such that $\big(\widetilde{\pi_1}(p), \widetilde{\pi_2}(p)\big) \in \overline{C} \times \overline{D} \subseteq W$. Pick $E \in p$ such that $\widetilde{\pi_1}[\,\overline{E}\,] \subseteq \overline{C}$ and $\widetilde{\pi_2}[\,\overline{E}\,] \subseteq \overline{D}$. Then $\overline{E} \cap K(\beta S)$ is a neighborhood of p in $K(\beta S)$ and $\phi[\,\overline{E} \cap K(\beta S)] \subseteq W$.

If A and B are infinite, then since $\beta A \times \beta B$ is compact, the fact that ϕ is not a homeomorphism follows from (10).

If either A or B is finite, then $K(\beta S)$ is compact and ϕ is a continuous bijection, so ϕ is a homeomorphism. □

In the semigroups of Theorem 4.3, the minimal left ideals have isolated points because at least some of them have points that are isolated in βS. As we have seen, one often concludes that minimal left ideals have isolated points because they are finite. In the semigroups of the following theorem we have that $K(\beta S) \cap S = \emptyset$ unless B has a largest member and, if A is infinite, then the minimal left ideals have as many elements as βA.

Recall that a *semilattice* $(B, \vee)$ is a set partially ordered by a reflexive, transitive, and antisymmetric relation with the property that any two $x, y \in B$ have a least upper bound $x \vee y$.

If κ is a regular infinite cardinal and $B = \kappa$, then the set U in the following theorem is the set of κ-uniform ultrafilters on B. If κ is singular, then U properly contains the set of κ-uniform ultrafilters on B.

Theorem 4.4. *Let A be a nonempty set and let $(B, \vee)$ be a nonempty semilattice. Let $S = A \times B$ and for $(a, b), (c, d) \in S$, define $(a, b) \cdot (c, d) = (a, b \vee d)$. Let $\widetilde{\pi_1} : \beta S \to \beta A$ and $\widetilde{\pi_2} : \beta S \to \beta B$ be the continuous extensions of the projection functions. Let $U = \{q \in \beta B : (\forall s \in B)(\{t \in B : s \leq t\} \in q)\}$ and let $V = \widetilde{\pi_2}^{-1}[U]$. For $x \in \beta A$ and for $y \in U$, let $R_x = \{p \in \beta S : \widetilde{\pi_1}(p) = x\}$ and let $L_y = \{p \in \beta S : \widetilde{\pi_2}(p) = y\}$.*

(1) *$U \neq \emptyset$. If B does not have a maximum element, $|U| \geq 2^{\mathfrak{c}}$.*
(2) *If $p, q \in \beta S$, then $\widetilde{\pi_1}(pq) = \widetilde{\pi_1}(p)$.*
(3) *If $p \in \beta S$ and $q \in V$, then $\widetilde{\pi_2}(pq) = \widetilde{\pi_2}(q)$.*
(4) *V is an ideal of βS and so $K(\beta S) \subseteq V$. If B does not have a maximum element, $K(\beta S) \cap S = \emptyset$.*
(5) *If $p \in \beta S$ and $q \in V$, then $pq = \widetilde{\pi_1}(p) \otimes \widetilde{\pi_2}(q)$.*
(6) *If $x \in \beta A$ and $y \in U$, then R_x is a left ideal of βS and L_y is a left ideal of βS.*
(7) *If $p, q \in \beta S$ and $r \in V$, then $pqr = pr$.*
(8) *Let $p \in \beta S$. The following statements are equivalent.*
   (a) *$p \in K(\beta S)$.*
   (b) *$p \in (\beta S) \cdot V$.*
   (c) *$p \in V$ and $p = \widetilde{\pi_1}(p) \otimes \widetilde{\pi_2}(p)$.*
   (d) *$p \in \beta A \otimes U$.*
   (e) *$p \in V$ and p is an idempotent.*
(9) *If $x \in \beta A$ and $y \in U$, then $R_x \cap L_y \cap K(\beta S) = \{x \otimes y\}$.*
(10) *If $x \in \beta A$ and $y \in U$, then R_x contains a unique minimal right ideal and L_y contains a unique minimal left ideal.*
(11) *Let $y \in U$ and let L be the unique minimal left ideal contained in L_y. Then $\{p \in L : p$ is isolated in $L\} = A \otimes y$.*
(12) *If $x \in \beta A$ and $y \in U$, then the minimal right ideal contained in R_x is $x \otimes U$ and the minimal left ideal contained in L_y is $\beta A \otimes y$. So the minimal right ideals of βS are the sets of the form $x \otimes U$ for $x \in \beta A$*

and the minimal left ideals of βS are the sets of the form $\beta A\otimes y$ for $y\in U$.

(13) *$K(\beta S)$ is compact if and only if A is finite or B has a maximum element.*

(14) *Define $\phi: K(\beta S)\to \mathbf{R}(\beta A,U)$ by $\phi(p)=\big(\widetilde{\pi_1}(p),\widetilde{\pi_2}(p)\big)$. Then ϕ is a continuous isomorphism. And ϕ is a homeomorphism if and only if either A is finite or B has a maximum element.*

Proof. (1) For $s\in B$, let $C_s=\{t\in B: s\leq t\}$. It is obvious that U is nonempty. If B does not have a maximum element, then, for every finite nonempty subset F of B, $\bigcap_{s\in F} C_s$ is infinite. So by [2, Theorem 3.62], $|U|\geq 2^{\mathfrak{c}}$.

(2) $\widetilde{\pi_1}(pq)=\widetilde{\pi_1}(p\text{-}\lim_{s\in S} q\text{-}\lim_{t\in S} st)=p\text{-}\lim_{s\in S} q\text{-}\lim_{t\in S}\pi_1(st)=p\text{-}\lim_{s\in S} q\text{-}\lim_{t\in S}\pi_1(s)=p\text{-}\lim_{s\in S}\pi_1(s)=\widetilde{\pi_1}(p)\,.$

(3) For $b\in B$, let $C_b=\{(c,d)\in S: d\geq b\}$ and note that $C_b\in q$. Note also that if $(a,b)\in S$ and $(c,d)\in C_b$, then $\pi_2\big((a,b)(c,d)\big)=\pi_2(c,d)$. Then $\widetilde{\pi_2}(pq)=\widetilde{\pi_2}\big(p\text{-}\lim_{(a,b)\in S} q\text{-}\lim_{(c,d)\in C_b}(a,b)(c,d)\big)=p\text{-}\lim_{(a,b)\in S} q\text{-}\lim_{(c,d)\in C_b}\pi_2(c,d)=p\text{-}\lim_{(a,b)\in S}\widetilde{\pi_2}\big(q\text{-}\lim_{(c,d)\in C_b}(c,d)\big)=p\text{-}\lim_{(a,b)\in S}\widetilde{\pi_2}(q)=\widetilde{\pi_2}(q)\,.$

(4) By (3) V is a left ideal of βS. To see that V is a right ideal, let $p\in V$, let $q\in\beta S$, let $s\in B$, and let $C=\{t\in B: t\geq s\}$. We need to show that $C\in\widetilde{\pi_2}(pq)$. Suppose instead that $B\setminus C\in\widetilde{\pi_2}(pq)$. Pick $D\in pq$ such that $\widetilde{\pi_2}[\,\overline{D}\,]\subseteq\overline{B\setminus C}$. Let $E=\{x\in S: x^{-1}D\in q\}$. Then $E\in p$ and ${\pi_2}^{-1}[C]\in p$ so pick $(a,b)\in{\pi_2}^{-1}[C]$ such that $(a,b)^{-1}D\in q$. Pick $(c,d)\in(a,b)^{-1}D$. Then $(a,b\vee d)\in D$ so $b\vee d\in B\setminus C$. But $b\in C$ so $b\vee d\in C$, a contradiction.

If B does not have a maximum element, $U\subseteq B^*$ and so $V\cap S=\emptyset$.

(5) If $p\in\beta S$ and $q\in V$, then

$$pq=\lim_{(a,b)\to p}\lim_{(c,d)\to q}(a,b)(c,d)=\lim_{(a,b)\to p}\lim_{(c,d)\to q}(a,b\vee d)\,,$$

where a,c denote elements of A and b,d denote elements of B. Since $c\vee d=d$ if $c\leq d$, $pq=\lim_{a\to\widetilde{\pi}_1(p)}\lim_{d\to\widetilde{\pi}_2(q)}(a,d)=\widetilde{\pi}_1(p)\otimes\widetilde{\pi}_2(q)$.

(6) This is an immediate consequence of statements (2) and (3).

(7) For $b\in B$, let $C_b=\{(c,d)\in S: d>b\}$ and note that $C_b\in r$. Now

$$\begin{aligned}pqr&=p\text{-}\lim_{(a,b)\in S} q\text{-}\lim_{(c,d)\in S} r\text{-}\lim_{(u,v)\in C_{b\vee d}}(a,b)(c,d)(u,v)\\&=p\text{-}\lim_{(a,b)\in S} q\text{-}\lim_{(c,d)\in S} r\text{-}\lim_{(u,v)\in C_{b\vee d}}(a,b\vee d)(u,v)\\&=p\text{-}\lim_{(a,b)\in S} q\text{-}\lim_{(c,d)\in S} r\text{-}\lim_{(u,v)\in C_{b\vee d}}(a,v)\\&=p\text{-}\lim_{(a,b)\in S} r\text{-}\lim_{(u,v)\in C_b}(a,v)=pr\,.\end{aligned}$$

(8) $(a)\Rightarrow(b)$. By (4) $p\in V$. Also p is a member of some minimal right ideal R of βS and if e is an idempotent in R, then $p=ep$.

$(b) \Rightarrow (c)$. Pick $q \in \beta S$ and $r \in V$ such that $p = qr$. By (5), $qr = \widetilde{\pi_1}(q) \otimes \widetilde{\pi_2}(r)$. By (2) and (3) $\widetilde{\pi_1}(p) = \widetilde{\pi_1}(q)$ and $\widetilde{\pi_2}(p) = \widetilde{\pi_2}(r)$.

It is trivial that (c) implies (d).

$(d) \Rightarrow (e)$. Pick $x \in \beta A$ and $y \in U$ such that $p = x \otimes y$. Then $\widetilde{\pi_2}(p) = y$ so $p \in V$. By (5) $pp = (x \otimes y) \cdot (x \otimes y) = \widetilde{\pi_1}(x \otimes y) \otimes \widetilde{\pi_2}(x \otimes y) = x \otimes y = p$.

$(e) \Rightarrow (a)$. Pick $q \in K(\beta S)$. By (7) $p = pp = pqp \in K(\beta S)$.

(9) R_x contains a minimal right ideal and L_y contains a minimal left ideal so $R_x \cap L_y \cap K(\beta S) \neq \emptyset$. Let $p \in R_x \cap L_y \cap K(\beta S)$. By (8) $p = \widetilde{\pi_1}(p) \otimes \widetilde{\pi_2}(p) = x \otimes y$.

(10) Suppose R_1 and R_2 are distinct minimal right ideals contained in R_x. Then $R_1 \cap L_y \cap K(\beta S) \neq \emptyset$ and $R_2 \cap L_y \cap K(\beta S) \neq \emptyset$ while $|R_x \cap L_y \cap K(\beta S)| = 1$, a contradiction. Similarly L_y does not contain two distinct minimal left ideals.

(11) Let p be an isolated point of L and pick $C \subseteq S$ such that $\overline{C} \cap L = \{p\}$. Let $x = \widetilde{\pi_1}(p)$. Then $p = x \otimes y$ by (8). Let $D = \{a \in A : \{b \in B : (a,b) \in C\} \in y\}$. If $D = \{a\}$, then we have $p = a \otimes y$. If we had distinct a_1 and a_2 in D, we would have $a_1 \otimes y$ and $a_2 \otimes y$ in $\overline{C} \cap L$.

For the other inclusion, let $a \in A$. Then $\widetilde{\pi_1}^{-1}[\{a\}] \cap L = R_a \cap L \neq \emptyset$ and $R_a \cap L \subseteq R_a \cap L_y \cap K(\beta S) = \{a \otimes y\}$ by (9) so $\widetilde{\pi_1}^{-1}[\{a\}] \cap L = \{a \otimes y\}$.

(12) In the proof of Theorem 4.3 we established the conclusion for L_y. Here we establish the conclusion for R_x. So let R be the minimal right ideal contained in R_x. First let $p \in R$. By (8) $p = x \otimes \widetilde{\pi_2}(p) \in x \otimes U$ since $p \in V$ by (4).

Now let $v \in U$. Then $x \otimes v \in K(\beta S)$ by (8) so $x \otimes v \in R'$ for some minimal right ideal R'. But then $\widetilde{\pi_1}[R'] = \{x\}$ so $R' \subseteq R_x$ so $R' = R$.

(13) Assume first that A is finite. Then by (8), $K(\beta S) = A \otimes U$. Since U is a compact subset of βS, $a \otimes U$ is compact for every $a \in A$ and so $K(\beta S)$ is compact.

If B has a maximum element u, then $U = \{u\}$ and $K(\beta S) = \beta A \otimes u$, which is compact.

Now suppose that A is infinite and that B does not have a maximum element. Assume that $K(\beta S)$ is compact. Let $\langle a_n \rangle_{n=1}^{\infty}$ be an injective sequence in A and pick a (strongly) discrete sequence $\langle y_n \rangle_{n=1}^{\infty}$ in U and a sequence $\langle Y_n \rangle_{n=1}^{\infty}$ of pairwise disjoint subsets of B such that $Y_n \in y_n$ for each $n \in \mathbb{N}$. Then $\langle a_n \otimes y_n \rangle_{n=1}^{\infty}$ is a sequence in $K(\beta S)$. Pick a cluster point $p \otimes q$ of this sequence.

Note first that $\{a_n : n \in \mathbb{N}\} \in p$ since otherwise $\overline{(A \setminus \{a_n : n \in \mathbb{N}\}) \times B}$ would be a neighborhood of $p \otimes q$ which no $a_n \otimes y_n$ is a meber of. Next note that for each $n \in \mathbb{N}$, $B \setminus Y_n \in q$ since otherwise $\overline{A \times Y_n}$ would be a neighborhood of $p \otimes q$ containing only one member of the given sequence. Let $E = \bigcup_{n=1}^{\infty} \big(\{a_n\} \times (B \setminus Y_n)\big)$. Then $E \in p \otimes q$ so pick n such that $E \in a_n \otimes y_n$. This is a contradiction.

(14) By (8) $\phi[K(\beta S)] = \mathbf{R}(\beta A, U)$. If $p, q \in K(\beta S)$ and $\phi(p) = \phi(q)$, then by (8), $p = \widetilde{\pi_1}(p) \otimes \widetilde{\pi_2}(p) = \widetilde{\pi_1}(q) \otimes \widetilde{\pi_2}(q) = q$. To see that ϕ is

a homomorphism, let $p,q\in K(\beta S)$. Thenn $\phi(p,q)=\big(\widetilde{\pi_1}(pq),\widetilde{\pi_2}(pq)\big)=\big(\widetilde{\pi_1}(p),\widetilde{\pi_2}(q)\big)=\big(\widetilde{\pi_1}(p),\widetilde{\pi_2}(p)\big)\cdot\big(\widetilde{\pi_1}(q),\widetilde{\pi_2}(q)\big)=\phi(p)\phi(q)$.

To see that ϕ is continuous, let $p\in K(\beta S)$ and let W be a neighborhood of $\phi(p)$ in $\beta A\times U$. Pick $C\subseteq A$ and $D\subseteq B$ such that $\big(\widetilde{\pi_1}(p),\widetilde{\pi_2}(p)\big)\in\overline{C}\times(\overline{D}\cap U)\subseteq W$. Pick $E\in p$ such that $\widetilde{\pi_1}[\,\overline{E}\,]\subseteq\overline{C}$ and $\widetilde{\pi_2}[\,\overline{E}\,]\subseteq\overline{D}$. Then $\overline{E}\cap K(\beta S)$ is a neighborhood of p in $K(\beta S)$ and $\phi[\,\overline{E}\cap K(\beta S)]\subseteq W$.

If A is infinite and B does not have a largest member, then $K(\beta S)$ is not compact while $\mathbf{R}(\beta A,U)$ is compact so ϕ is not a homeomorphism. If A is finite or B has a maximum element, then $K(\beta S)$ is compact and ϕ is a continuous bijection, so ϕ is a homeomorphism. □

We are mainly interested in the case in which B does not have a maximum element, because the study of the semigroup βS defined in Theorem 4.4 was motivated by the fact that, in this case, the minimal left ideals of βS are infinite, do not meet S, and have isolated points. However, since $\vee$-semigroups with maximum elements are abundant, it is worth remarking on the case in which B does have a maximum element u. In this case, $U=\{u\}$, $K(\beta S)=\beta A\otimes u=(\beta S)u$ (which is compact), every minimal right ideal of βS is a singleton and the unique minimal left ideal of βS is $(\beta S)u$.

It is also worth remarking on the case which A is a singleton. In this case, we can identify S with B, since S is isomorphic to B. So, if B is an arbitrary $\vee$-semilattice, $K(\beta B)$ is compact and is topologically isomorphic to the compact right zero subsemigroup U of βS, the minimal right ideals of βB are singletons and U is the unique minimal left ideal of βB.

A consideration of tensor products highlights a significant effect of cancellation. It was shown in [6] that, if T is countable and cancellative and $S=T\times T$, then no element of $T^*\otimes T^*$ can be an idempotent in βS, a member of the smallest ideal of βS or a member of S^*S^*. Whereas if S is the semigroup of Theorem 4.4, an element of V is an idempotent if and only if it is a tensor product and every member of $K(\beta S)$ is a tensor product.

We can strengthen Theorem 4.4(13) in certain cases, by proving that $K(\beta S)$ is not Borel. This includes the case in which A and B are countably infinite and B does not have a maximum element, as well as the case in which κ is an infinite cardinal, $|A|=\kappa$ and $B=\kappa$. We shall need the folllowing lemma, which was previously proved for a countable discrete space S in [3, Lemma 3.1]. The proof given there extends to arbitrary infinite discrete spaces.

Lemma 4.5. *Let S denote an infinite discrete space of cardinality κ. Every Borel subset of βS is the union of a family of compact subsets of βS of cardinality at most 2^κ.*

Proof. We remind the reader that a family $\mathcal{F}$ of subsets of a topological space X contains all the Borel subsets of the space if it contains all the open sets and all the closed sets, and is closed under countable unions and countable intersections. To see this, let $\mathcal{A}_0 = \{U \subseteq X : U$ is open in $X\} \cup \{U \subseteq X : U$ is closed in $X\}$. For $\sigma < \omega_1$, if $\mathcal{A}_\sigma$ has been defined, let $\mathcal{A}_{\sigma+1} = \{\bigcup \mathcal{U} : \mathcal{U}$ is a nonempty countable subset of $\mathcal{A}_\sigma\} \cup \{\bigcap \mathcal{U} : \mathcal{U}$ is a nonempty countable subset of $\mathcal{A}_\sigma\}$. If σ is a nonzero limit ordinal, let $\mathcal{A}_\sigma = \bigcup_{\tau<\sigma} \mathcal{A}_\tau$. Then each $\mathcal{A}_\sigma \subseteq \mathcal{F}$ and, since each $\mathcal{A}_\sigma$ is closed under complementation, $\bigcup_{\sigma<\omega_1} \mathcal{A}_\sigma$ is the set of Borel subsets of X.

Let $\mathcal{F}$ denote the family of subsets of βS which are the union of 2^κ or fewer compact subsets of βS. Then $\mathcal{F}$ contains the open subsets of βS, because $\beta\mathbb{N}$ has a basis of 2^κ clopen sets, and $\mathcal{F}$ obviously contains the closed subsets of βS. It is also obvious that $\mathcal{F}$ is closed under countable unions. To see that $\mathcal{F}$ is closed under countable intersections let $\langle A_n \rangle_{n=1}^\infty$ be a sequence in $\mathcal{F}$ and for each $n \in \mathbb{N}$, pick a set $\mathcal{D}_n$ of at most 2^κ compact subsets of βS such that $A_n = \bigcup \mathcal{D}_n$. Then $\bigcap_{n=1}^\infty A_n = \bigcup \{\bigcap_{n=1}^\infty F(n) : F \in \times_{n=1}^\infty \mathcal{D}_n\}$ and $|\times_{n=1}^\infty \mathcal{D}_n| \leq (2^\kappa)^\omega = 2^\kappa$. □

Theorem 4.6. *Let A, B, U and S be defined as in Theorem 4.4. Assume that A and B are infinite sets of cardinality κ and that, for every $b \in B$, $|\{c \in B : c > b\}| = \kappa$. Then $K(\beta S)$ is not a Borel subset of βS.*

Proof. We note that $|\beta A| = |\beta U| = 2^{2^\kappa}$ by [2, Theorems 3.58 and 3.62]. Let $\langle p_i \rangle_{i<2^{2^\kappa}}$ and $\langle q_i \rangle_{i<2^{2^\kappa}}$ be enumerations of βA and U respectively as injective 2^{2^κ}-sequences. We claim that, for any compact subset C of $K(\beta S)$, $\{i < 2^{2^\kappa} : p_i \otimes q_i \in C\}$ is finite. Since $\{p_i \otimes q_i : i < 2^{2^\kappa}\} \subseteq K(\beta S)$, it will follow that $K(\beta S)$ cannot be covered by 2^κ compact subsets. So let C be a compact subset of $K(\beta S)$.

Assume that $D = \{i < 2^{2^\kappa} : p_i \otimes q_i \in C\}$ is infinite. Pick a countably infinite subset F of D such that $\{p_i : i \in F\}$ is strongly discrete. Pick an infinite subset E of F such that $\{q_i : i \in E\}$ is strongly discrete. We can choose pairwise disjoint families of subsets $\{P_i : i \in E\}$ and $\{Q_i : i \in E\}$ of A and B respectively, such that $P_i \in p_i$ and $Q_i \in q_i$ for every $i \in E$. By [2, Theorem 3.59], $|c\ell_{\beta S}(\{p_i \otimes q_i : i \in E\})| \geq 2^{\mathfrak{c}}$, and so there exists $x \in c\ell_{\beta S}(\{p_i \otimes q_i : i \in E\}) \setminus \{p_i \otimes q_i : i \in E\}$. Since $x \in K(\beta S)$, $x = p \otimes q$ for some $p \in \beta A$ and some $q \in U$. Then $p = \widetilde{\pi_1}(x) \in \widetilde{\pi_1}[c\ell_{\beta S}(\{p_i \otimes q_i : i \in E\})] = c\ell_{\beta A}\widetilde{\pi_1}[\{p_i \otimes q_i : i \in E\}] = c\ell_{\beta A}[\{p_i : i \in E\}$ and $q = \widetilde{\pi_2}(x) \in c\ell_U[\{q_i : i \in E\}]$.

Let $P = \bigcup_{i \in E} P_i$ and let $Q = \bigcup_{i \in E} Q_i$. Define $f : P \to E$ by $s \in P_{f(s)}$ and let $X = \bigcup_{s \in P} \big(\{s\} \times (Q \setminus Q_{f(s)})\big)$. Then $X \in x$ because $P \in p$ and, for each $s \in P$, $Q \setminus Q_{f(s)} \in q$. However, X is not a member of $q_i \otimes p_i$ for any $i \in E$, contradicting the assumption that $x \in c\ell_{\beta S}(\{p_i \otimes q_i : i \in E\})$. □

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