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TO THE BEST OF MY KNOWLEDGE, THIS IS THE FINAL VERSION AS IT WAS SUBMITTED TO THE PUBLISHER – NH.

ALGEBRA IN βS AND ITS APPLICATIONS TO RAMSEY THEORY

NEIL HINDMAN

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ABSTRACT. We present an elementary introduction to the algebraic structure of the Stone-Čech compactification βS of a discrete semigroup S, and survey recent results about this algebraic structure and its applications to the branch of combinatorial number theory known as *Ramsey Theory.*

1. INTRODUCTION

Given a semigroup $(S,\cdot)$, if one views S as a discrete topological space, one may extend the operation to the Stone-Čech compactification βS of S in such a way as to make $(\beta S,\cdot)$ into a right topological semigroup with S contained in its topological center. That is, given any $p\in\beta S$, the function $\rho_p:\beta S\longrightarrow\beta S$ defined by $\rho_p(q)=q\cdot p$ is continuous and, given any $x\in S$, the function $\lambda_x:\beta S\longrightarrow\beta S$ defined by $\lambda_x(q)=x\cdot q$ is continuous. (The fact that this can be done was implicitly established by Day [21] in 1957 using methods of Arens [1]. The first explicit statement seems to have been given by Civin and Yood [19].)

As a compact right topological semigroup, $(\beta S,\cdot)$ enjoys a significant amount of known structure. (See [16] for a detailed description of much that is known.) But many fascinating questions remain. For example, starting with possibly the simplest, and certainly the most familiar, infinite semigroup $(\mathbb{N},+)$ it is a famous, and still unsolved, problem to determine whether there is any infinte increasing chain of left ideals in $(\beta\mathbb{N},+)$. (See [44].) Further, until very recently, it was unknown whether $(\beta\mathbb{N},+)$ contains any nontrivial finite subgroups. (See below for the answer.)

My own introduction to this area came by way of its applications. In 1975 I had recently published a very complicated combinatorial proof of the Finite Sum Theorem.

1.1 Theorem (Finite Sum Theorem). *Let $r\in\mathbb{N}$ and let $\mathbb{N}=\bigcup_{i=1}^{r}A_i$. Then there exist some $i\in\{1,2,\ldots,r\}$ and some sequence $\langle x_n\rangle_{n=1}^{\infty}$ such that for every finite nonempty subset F of $\mathbb{N}$, $\Sigma_{n\in F}\,x_n\in A_i$.*

Then a letter arrived from Fred Galvin, presenting a proof of the Finite Sum Theorem which he and Steven Glazer had devised which was very simple and elegant and used a little of the algebraic structure of $(\beta\mathbb{N},+)$. (See Section 5 for a presentation of this proof

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as well as an even simpler proof of very recent origin.) Beginning with the Galvin-Glazer proof of the Finite Sum Theorem, the algebraic structure of βS has seen many significant applications to the branch of combinatorial number theory known as *Ramsey Theory*.

I have written four previous surveys of these topics [35,36,37,38], the most recent of which included only algebraic results. (The reader should be cautioned that in all of these papers I took βS to be left topological rather than right topological. I have changed sides to conform to majority usage, at least among my collaborators.)

In this paper we will begin with an elementary construction of βS and a derivation of its algebraic structure, as well as an elementary introduction to the combinatorial applications of this structure. The only background expected of the reader is that provided by the usual first year graduate courses in topology and algebra.

We take βS to be the set of ultrafilters on S. Section 2 consists of an introduction to ultrafilters and an elementary construction of the Stone-Čech compactification of a discrete space. In Section 3 we present several alternative characterizations of ultrafilters, based on the delightful article [17]. In Section 4 we show how to extend the operation of S to βS making it a right topological semigroup and present some of the basic facts about compact right topological semigroups. In Section 5 we present some of the easier applications of the algebra of βS to combinatorics.

We will then survey results on the algebraic structure of βS and its dynamical and combinatorial applications that have not been included in our earlier surveys.

Very recently, E. Zelenjuk [67] has answered what had been the outstanding unsolved problem regarding the algebra of $(\beta\mathbb{N},+)$, namely whether this semigroup contained any nontrivial finite groups. Zelenjuk in fact established that if G is any countable abelian group and G does not itself contain any nontrivial finite subgroups, then neither does βG. Since $\beta\mathbb{N} \subseteq \beta\mathbb{Z}$, this result answers the question about $\beta\mathbb{N}$ as well. We believe that this result is so significant that we present a proof of Zelenjuk's Theorem in its entirety as Section 6. In fact, we present a strengthening of Zelenjuk's conclusion which was observed by Dona Strauss. This strengthening allows one to omit the assumption of commutativity. (All material necessary to follow this version of the proof will have been presented in the earlier sections.)

For the rest of the paper we will follow the usual practice in survey articles, and cite most of the results that we present without proof. Exceptions will include results whose proofs are easily presented in their entirety as well as any results that are appearing here for the first time.

In Sections 7 and 8 we present algebraic results about βS that have been obtained since the publication of [38]. The remaining sections include dynamical and combinatorial applications (topics not covered in [38]) that have been obtained since the publication of [37]. It is not always clear from the statements of the combinatorial results how they involve βS, but we do restrict ourselves to results in Ramsey Theory that involve βS.

Section 7 consists of results involving natural orderings of idempotents in βS and Section 8 deals with results on ideals and cancellation in βS. Section 9 presents results involving connections with topology and topological dynamics. In Section 10 we present some Ramsey Theoretic applications and results involving equations in $\beta\mathbb{N}$. In Section 11 we deal with several results about partition regularity of matrices. In Section 12 we discuss the powerful notion of central sets.

Our set theoretic notation is more or less standard. We mention in particular that, given a function f we write $f[A] = \{f(x) : x \in A\}$. Given a set X we write $\mathcal{P}_f(X) = \{F : F$ is a finite nonempty subset of $X\}$. We write $\mathbb{N}$ for the set of positive integers and $\omega = \mathbb{N} \cup \{0\}$ for the set of nonnegative integers. (Also, ω is the cardinal number of countable infinity.) We will assume that all hypothesized topological spaces are Hausdorff.

I would like to thank my collaborators whose results are featured here. They are: John Baker, Vitaly Bergelson, John Berglund, Andreas Blass, Walter Deuber, David Gunderson, Bryna Kra, Jimmie Lawson, Imre Leader, Hanno Lefmann, Amha Lisan, Amir Maleki, John Pym, Dona Strauss, and Wen-jin Woan. Special thanks go to John Pym for translating [67] from Russian and to Dona Strauss for helping me to understand [67].

2. An Introduction to Ultrafilters

2.1 Definition. Let D be a set. Then $\mathcal{U}$ is a *filter* on D if and only if

(a) $\mathcal{U} \subseteq \mathcal{P}(D)$,
(b) $\emptyset \notin \mathcal{U}$,
(c) $D \in \mathcal{U}$,
(d) if $A, B \in \mathcal{U}$, then $A \cap B \in \mathcal{U}$, and
(e) if $A \in \mathcal{U}$ and $A \subseteq B \subseteq D$, then $B \in \mathcal{U}$.

Thus a *filter* on D is a nonempty set of nonempty subsets of D which is closed under finite intersections and supersets. A clasic example of a filter is the set of neighborhoods of a point in a topological space.

2.2 Definition. Let D be a set. An *ultrafilter* p on D is a maximal filter on D.

Even though an ultrafilter on D is a set of subsets of D, we usually denote ultrafilters by lower case letters because they are the points of the Stone-Čech compactification of D.

Recall that a set $\mathcal{A}$ of sets has the *finite intersection property* if and only if whenever $\mathcal{F}$ is a finite nonempty subset of $\mathcal{A}$, $\bigcap \mathcal{F} \neq \emptyset$.

2.3 Theorem. *Let D be a set and let $p \subseteq \mathcal{P}(D)$. If p is maximal with respect to the finite intersection property, then p is an ultrafilter on D.*

Proof. Assume that p is maximal with respect to the finite intersection property among subsets of $\mathcal{P}(D)$. Then conditions (a) and (b) of Definition 2.1 hold immediately. Since $p \cup \{D\}$ has the finite intersection property and $p \subseteq p \cup \{D\}$, one has $p = p \cup \{D\}$. That is, $D \in p$. Given $A, B \in p$, $p \cup \{A \cap B\}$ has the finite intersection property so $A \cap B \in p$. Given A and B with $A \in p$ and $A \subseteq B \subseteq D$, $p \cup \{B\}$ has the finite intersection property so $B \in p$. Thus p is a filter.

Suppose now that p is not a maximal filter and pick a filter $\mathcal{U}$ such that $p \subsetneq \mathcal{U}$. Pick $A \in \mathcal{U} \backslash p$. Then $p \subsetneq p \cup \{A\}$ so $p \cup \{A\}$ does not have the finite intersection property. Since we have already seen that p is a filter, this says that there is some $B \in p$ such that $B \cap A = \emptyset$. But since $B, A \in \mathcal{U}$, this is a contradiction. $\square$

The following simple result is of fundamental importance.

2.4 Theorem. *Let D be a set and let $\mathcal{A}$ be a subset of $\mathcal{P}(D)$ which has the finite intersection property. Then there is an ultrafilter p on D such that $\mathcal{A} \subseteq p$.*

Proof. Let $\Gamma = \{\mathcal{B} \subseteq \mathcal{P}(D) : \mathcal{A} \subseteq \mathcal{B}$ and $\mathcal{B}$ has the finite intersection property$\}$. Then $\mathcal{A} \in \Gamma$ so $\Gamma \neq \emptyset$. Given a chain $\mathcal{C}$ in Γ one has immediately that $\mathcal{A} \subseteq \bigcup \mathcal{C}$. Given $\mathcal{F} \in \mathcal{P}_f(\bigcup \mathcal{C})$ there is some $\mathcal{B} \in \mathcal{C}$ with $\mathcal{F} \subseteq \mathcal{B}$ so $\bigcap \mathcal{F} \neq \emptyset$. Thus by Zorn's Lemma we may pick a maximal member p of Γ. Trivially p is not only maximal in Γ, but in fact p is maximal with respect to the finite intersection property. By Theorem 2.3, p is an ultrafilter on D. $\square$

We observe that, given any $x \in D$, $\{A \subseteq D : x \in A\}$ is an ultrafilter on D.

2.5 Definition. Let D be a set and let $x \in D$. Then $e(x) = \{A \subseteq D : x \in A\}$ and $e(x)$ is the *principal ultrafilter generated by* x. An ultrafilter p on D is *principal* if and only if $p = e(x)$ for some $x \in D$. It is *nonprincipal* if and only if it is not principal.

It is customary to identify a principal ultrafilter $e(x)$ with the point x, and we will adopt that practice ourselves very shortly (as we implicitly did in the introduction). However, we will keep our virtue for a while and maintain the distinction between them.

2.6 Definition. Let D be a discrete topological space.
(a) $\beta D = \{p : p \text{ is an ultrafilter on } D\}$.
(b) Given $A \subseteq D$, $\overline{A} = \{p \in \beta D : A \in p\}$.

We caution the reader that $\overline{A}$ is not *defined* to be the closure of A in any topological space. (It will turn out later, as a consequence of Lemma 2.10, that once we start pretending that $D \subseteq \beta D$ we will have $\overline{A} = c\ell_{\beta D}\ A$.)

The proof of the following lemma is an easy exercise.

2.7 Lemma. *Let D be a discrete space.*
(a) For all $A, B \subseteq D$, $\overline{A \cap B} = \overline{A} \cap \overline{B}$.
(b) For all $A, B \subseteq D$, $\overline{A \cup B} = \overline{A} \cup \overline{B}$.
(c) For all $A \subseteq D$, $\overline{D \backslash A} = \beta D \backslash \overline{A}$.
(d) $\overline{D} = \beta D$.

2.8 Lemma. *Let D be a discrete space. Then $\{\overline{A} : A \subseteq D\}$ is a basis for (the open sets of) a Hausdorff topology on D.*

Proof. Let $\mathcal{B} = \{\overline{A} : A \subseteq D\}$. By Lemma 2.7(a), $\mathcal{B}$ is a basis for a topology on D. To see that this topology is Hausdorff, let p and q be distinct members of βD. Pick $A \in p \backslash q$. Then $D \backslash A \in q \backslash p$ so $p \in \overline{A}$ and $q \in \overline{D \backslash A}$ and $\overline{A} \cap \overline{D \backslash A} = \emptyset$. $\square$

From this point on we take βD to be the topological space whose topology has the basis $\{\overline{A} : A \subseteq D\}$. Notice that by Lemma 2.7(c) one has that each $\overline{A}$ is clopen (= "open and closed") and that $\{\overline{A} : A \subseteq D\}$ is also a basis for the closed sets of βD.

2.9 Lemma. *Let D be a discrete space. Then $e : D \longrightarrow \beta D$ is an embedding and $e[D]$ is an open subset of βD.*

Proof. Trivially e is one-to-one and any function from a discrete space is continuous. Given $x \in D$, $\overline{\{x\}} = \{e(x)\}$ so $\{e(x)\}$ is open. $\square$

The following lemma is the basis for the assertion that we can pretend that $\overline{A} = c\ell_{\beta D}\ A$.

2.10 Lemma. *Let D be a discrete space and let $A \subseteq D$. Then $\overline{A} = c\ell_{\beta D}\ e[A]$.*

Proof. Since $\overline{A} = \beta D \backslash (\overline{D \backslash A})$, $\overline{A}$ is closed. Given $x \in A$, $A \in e(x)$ so $e[A] \subseteq \overline{A}$ and consequently $c\ell_{\beta D}\ e[A] \subseteq \overline{A}$. For the reverse inclusion, let $p \in \overline{A}$ and let a basic neighborhood $\overline{B}$ of p be given. Then $A \cap B \in p$ so $A \cap B \neq \emptyset$. Pick $x \in A \cap B$. Then $e(x) \in \overline{B} \cap e[A]$ so $\overline{B} \cap e[A] \neq \emptyset$ as required. $\square$

2.11 Lemma. *Let D be a discrete space. Then βD is compact.*

Proof. Let $\mathcal{B}$ be a collection of closed subsets of βD with the finite intersection property. Let $\mathcal{A} = \{A \subseteq D : \text{there exists } B \in \mathcal{B} \text{ with } B \subseteq \overline{A}\}$. Then $\mathcal{A}$ has the finite intersection property. Pick by Theorem 2.4 some $p \in \beta D$ with $\mathcal{A} \subseteq p$. To see that $p \in \bigcap \mathcal{B}$, let $B \in \mathcal{B}$ and suppose that $p \notin B$. Since B is closed, pick a basic neighborhood $\overline{A}$ of p such that $\overline{A} \cap B = \emptyset$. Then $B \subseteq \beta D \backslash \overline{A} = \overline{D \backslash A}$ so $D \backslash A \in \mathcal{A}$, so $D \backslash A \in p$, a contradiction. $\square$

The following Theorem says that βD is the Stone-Čech compactification of D.

2.12 Theorem. *Let D be a discrete space. Then βD is a compact Hausdorff space and e is an embedding of D into βD. Further, given any compact Hausdorff space Y and any function $f : D \longrightarrow Y$, there is a continuous function $g : \beta D \longrightarrow Y$ such that $g \circ e = f$.*

Proof. The assertions in the second sentence follow from Lemmas 2.8, 2.9, and 2.11.

Let Y be a compact Hausdorff space and let $f : D \longrightarrow Y$. For each $p \in \beta D$ let $\mathcal{A}_p = \{c\ell_Y f[A] : A \in p\}$. Then for each $p \in \beta D$, $\mathcal{A}_p$ has the finite intersection property so has nonempty intersection so pick $g(p) \in \bigcap \mathcal{A}_p$. Then we have the following diagram.

$$\begin{tikzcd} & \beta D \arrow[dr, "g"] & \\ D \arrow[ur, "e"] \arrow[rr, "f"] & & Y \end{tikzcd}$$

We need to show that the diagram commutes and that g is continuous.

For the first assertion, let $x \in D$. Then $\{x\} \in e(x)$ so $g\big(e(x)\big) \in c\ell_Y f[\{x\}] = c\ell_Y[\{f(x)\}] = \{f(x)\}$ so $g \circ e = f$ as required.

To see that g is continuous, let $p \in \beta D$ and let U be a neighborhood of $g(p)$ in Y. Since Y is regular, pick a neighborhood V of $g(p)$ with $c\ell_Y V \subseteq U$ and let $A = f^{-1}[V]$. We claim that $A \in p$ so suppose instead that $D \backslash A \in p$. Then $g(p) \in c\ell_Y f[D \backslash A]$ and V is a neighborhood of $g(p)$ so $V \cap f[D \backslash A] \neq \emptyset$, contradicting the fact that $A = f^{-1}[V]$. Thus $\overline{A}$ is a neighborhood of p. We claim that $g[\overline{A}] \subseteq U$, so let $q \in \overline{A}$ and suppose that $g(q) \notin U$. Then $Y \backslash c\ell_Y V$ is a neighborhood of $g(q)$ and $g(q) \in c\ell_Y f[A]$ so $(Y \backslash c\ell_Y V) \cap f[A] \neq \emptyset$, again contradicting the fact that $A = f^{-1}[V]$. $\square$

From this point on we adopt the customary identification of $x \in D$ with $e(x)$. That is, we rely on the context to tell us, when we refer to x, whether we mean the point of D or the principal ultrafilter generated by that point. This seldom causes confusion, and when it does in this paper we will mention it.

Having made this identification, we have that $D \subseteq \beta D$ and Theorem 2.12 now reads as follows.

2.12 Theorem. *Let D be a discrete space. Then βD is a compact Hausdorff space and D is a discrete subspace of βD. Further, given any compact Hausdorff space Y and any function $f : D \longrightarrow Y$, there is a continuous function $g : \beta D \longrightarrow Y$ such that $g_{|D} = f$.*

The following notion will be useful in Section 6.

2.13 Definition. Let D be a discrete space and let φ be a filter on D. Then

$$\overline{\varphi} = \{p \in \beta D : \varphi \subseteq p\} \ .$$

2.14 Theorem. *Let D be a discrete space and let φ be a filter on D. Then $\overline{\varphi}$ is closed. In fact, given any subset B of βD, if $\varphi = \bigcap B$, then $\overline{\varphi} = c\ell B$.*

Proof. If $p \in \beta D \backslash \overline{\varphi}$, then there is some $A \in \varphi \backslash p$. Then $\overline{D \backslash A}$ is a neighborhood of p which misses $\overline{\varphi}$ and hence $\overline{\varphi}$ is closed.

Since trivially $B \subseteq \overline{\varphi}$ and $\overline{\varphi}$ is closed, we have that $c\ell B \subseteq \overline{\varphi}$. For the reverse inclusion, let $p \in \overline{\varphi}$ and let $A \in p$. Then $D \backslash A \notin \varphi$ so there is some $q \in B$ such that $D \backslash A \notin q$. That is, $q \in B \cap \overline{A}$. $\square$

We introduce now the notion of p-lim which was first used by Frolík in [28]. The notion is as versatile as the notion of nets, and has two significant advantages: (1) in a compact space a p-lim always converges and (2) it provides a "uniform" way of taking limits, as opposed to randomly choosing from among many possible limit points of a net.

2.15 Definition. Let D be a set, let p be an ultrafilter on D, let $\langle x_s\rangle_{s\in D}$ be an indexed family in a topological space X, and let $y \in X$. Then p-$\lim_{s\in D} x_s = p$-$\lim\langle x_s\rangle_{s\in D} = y$ if and only if for every neighborhood U of y, $\{s \in D : x_s \in U\} \in p$.

2.16 Theorem. *Let D be a set, let p be an ultrafilter on D, and let $\langle x_s\rangle_{s\in D}$ be an indexed family in a topological space X.*

(a) If X is a Hausdorff space and p-lim$\langle x_s\rangle_{s\in D}$ exists, then it is unique.

(b) If X is a compact space then p-lim$\langle x_s\rangle_{s\in D}$ exists.

Proof. (a) This is an easy exercise.

(b) Suppose that p-$\lim\langle x_s\rangle_{s\in D}$ does not exist and for each $y \in X$, pick an open neighborhood U_y of y such that $\{s \in D : x_s \in U_y\} \notin p$. Then $\{U_y : y \in X\}$ is an open cover of X so pick finite $F \subseteq X$ such that $X = \bigcup_{y\in F} U_y$. Then $D = \bigcup_{y\in F}\{s \in D : x_s \in U_y\}$ so pick $y \in F$ such that $\{s \in D : x_s \in U_y\} \in p$. This contradiction completes the proof. $\square$

As we shall see in the next section, Theorems 2.16 and 2.17 provide a characterization of ultrafilters.

2.17 Theorem. *Let D be a set, let p be an ultrafilter on D, let X and Y be Hausdorff topological spaces, let $\langle x_s\rangle_{s\in D}$ be an indexed family in X, and let $f : X \longrightarrow Y$. If f is continuous and p-lim$\langle x_s\rangle_{s\in D}$ exists, then p-lim$\langle f(x_s)\rangle_{s\in D} = f(p$-lim$\langle x_s\rangle_{s\in D})$.*

Proof. Let U be a neighborhood of $f(p$-$\lim\langle x_s\rangle_{s\in D})$ and pick a neighborhood V of p-$\lim\langle x_s\rangle_{s\in D}$ such that $f[V] \subseteq U$. Let $A = \{s \in D : x_s \in V\}$. Then $A \in p$ and $A \subseteq \{s \in D : f(x_s) \in U\}$. $\square$

The following fact will also be useful later.

2.18 Theorem. *Let D be a discrete space, let X be a compact Hausdorff space, and let $\langle x_s\rangle_{s\in D}$ be an indexed family in X. The function $f : \beta D \longrightarrow X$ defined by $f(p) = p$-lim$\langle x_s\rangle_{s\in D}$ is continuous.*

Proof. Notice that by Theorem 2.16, the equation $f(p) = p$-$\lim\langle x_s\rangle_{s\in D}$ does define a function. To see that this function is continuous, let $p \in \beta D$ and let U be a neighborhood of $f(p)$. Let $A = \{s \in D : x_s \in U\}$. Then $A \in p$ so $\overline{A}$ is a neighborhood of p. And $f[\overline{A}] \subseteq U$. $\square$

3. Alternate Characterizations of Ultrafilters

The material in the section is adapted from the paper [17] by A. Blass.

We have defined an ultrafilter on a set D as a set of subsets of D. There are other ways of viewing an ultrafilter. For example an ultrafilter is commonly viewed as a $\{0,1\}$-valued measure, so that it tells one which subsets of D are "large". This notion is quite convenient to keep in mind. For example, viewed this way one has that p-$\lim\langle x_s\rangle_{s\in D} = y$ if and only if x_s is p-almost always in every neighborhood of y. Another, less widely known, description is as a quantifier.

We have seen that ultrafilters can be used to construct the Stone-Čech compactification of a discrete space and to provide a uniform way of taking limits in a compact Hausdorff space. It turns out that each of these facts can also be used to characterize ultrafilters.

In this section we define five notions and describe why they can be viewed as essentially equivalent. The equivalence of four of these notions has been well known among the experts. The notion of an ultrafilter as a quantifier is an invention of Blass. Throughout this section we will take D to be a fixed infinite set.

3.1 Definition. $\mathrm{UF}_1 = \{p : p \text{ is an ultrafilter on } D\}$.

Thus if D is viewed as a discrete space, $\mathrm{UF}_1 = \beta D$. The notation is intended to represent "ultrafilters–version 1".

3.2 Definition. (a) A *uniform operator on functions from* D *to compact spaces* is an operator $\mathcal{O}$ which assigns to each indexed family $\langle x_s\rangle_{s\in D}$ in a compact Hausdorff space Y, some point of Y such that, whenever Y and Z are compact Hausdorff spaces, $\langle x_s\rangle_{s\in D}$ is an indexed family in Y, and f is a continuous function from Y to Z, one has $f\big(\mathcal{O}(\langle x_s\rangle_{s\in D})\big) = \mathcal{O}(\langle f(x_s)\rangle_{s\in D})$.

(b) $\mathrm{UF}_2 = \{\mathcal{O} : \mathcal{O} \text{ is a uniform operator on functions from } D \text{ to compact spaces}\}$.

For the third characterization, we fix a Stone-Čech compactification (φ, Z) of the discrete space D. That is

Z is a compact Hausdorff space and φ is an embedding of D into Z. Further, given any compact Hausdorff space Y and any function $f : D \longrightarrow Y$, there is a continuous function $g : Z \longrightarrow Y$ such that $g \circ \varphi = f$.

Of course one could take $(\varphi, Z) = (e, \beta D)$, but the point is that any choice will do.

3.3 Definition. Fix a Stone-Čech compactification (φ, Z) of the discrete space D. $\mathrm{UF}_3 = Z$.

3.4 Definition. $\mathrm{UF}_4 = \{\mu : \mu \text{ is a finitely additive } \{0,1\}\text{-valued measure on } \mathcal{P}(D) \text{ such that } \mu(D) = 1\}$.

Our final version of "ultrafilter" is as a "quantifier over D".

A *quantifier* $\mathcal{U}$ *over* D is an operation which applies to a formula $\gamma(s)$ with a free variable s ranging over D and produces a new formula $(\mathcal{U}s)\big(\gamma(s)\big)$ in which s is no longer free. It is required that, given any interpretation of the free variables other than s in $\gamma(s)$ (if any), $(\forall s \in D)\big(\gamma(s)\big)$ implies $(\mathcal{U}s)\big(\gamma(s)\big)$.

3.5 Definition. (a) Let $\mathcal{U}$ be a quantifier over D. Then $\mathcal{U}$ *respects propositional connectives* provided that whenever $\gamma(s)$ and $\psi(s)$ are statements about members s of D,

(1) $\neg(\mathcal{U}s)\big(\gamma(s)\big) \Leftrightarrow (\mathcal{U}s)\big(\neg\gamma(s)\big)$ and

(2) $(\mathcal{U}s)\big(\gamma(s)\big) \wedge (\mathcal{U}s)\big(\psi(s)\big) \Leftrightarrow (\mathcal{U}s)\big(\gamma(s) \wedge \psi(s)\big)$.

(b) $\mathrm{UF}_5 = \{\mathcal{U} : \mathcal{U} \text{ is a quantifier over } D \text{ that respects propositional connectives}\}$.

Note that, while we have defined "respects propositional connectives" only in terms of the connectives $\neg$ and $\wedge$, it follows for example that

$$(\mathcal{U}s)\big(\gamma(s)\big) \vee (\mathcal{U}s)\big(\psi(s)\big) \Leftrightarrow (\mathcal{U}s)\big(\gamma(s) \vee \psi(s)\big) \ .$$

To show that these notions are equivalent, we define how to get from one to another.

3.6 Definition. (a) Given $p \in \mathrm{UF}_1$, $\tau_1(p) = p\text{-lim}$.

(b) Given $\mathcal{O} \in \mathrm{UF}_2$, $\tau_2(\mathcal{O}) = \mathcal{O}(\langle\varphi(s)\rangle_{s\in D})$.

(c) Given $z \in \mathrm{UF}_3$ and $A \subseteq D$, $\tau_3(z)(A) = 1$ if and only if $z \in c\ell\varphi[A]$.

(d) Given $\mu \in \mathrm{UF}_4$, $\tau_4(\mu)$ is the quantifier over D such that for every statement $\gamma(s)$ about members s of D, $(\tau_4(\mu)x)\big(\gamma(s)\big)$ if and only if $\mu(\{s \in D : \gamma(s)\}) = 1$.

(e) Given $\mathcal{U} \in \mathrm{UF}_5$, $\tau_5(\mathcal{U}) = \{A \subseteq D : (\mathcal{U}s)(s \in A)\}$.

3.7 Lemma. *(a)* $\tau_1 : \mathrm{UF}_1 \longrightarrow \mathrm{UF}_2$.
(b) $\tau_2 : \mathrm{UF}_2 \longrightarrow \mathrm{UF}_3$.
(c) $\tau_3 : \mathrm{UF}_3 \longrightarrow \mathrm{UF}_4$.
(d) $\tau_4 : \mathrm{UF}_4 \longrightarrow \mathrm{UF}_5$.
(e) $\tau_5 : \mathrm{UF}_5 \longrightarrow \mathrm{UF}_1$.

Proof. (a) Theorems 2.16 and 2.17.

(b) φ is a function from D to a compact space.

(c) Let $z \in Z = \mathrm{UF}_3$ and let $\mu = \tau_3(z)$. Since $\varphi[D]$ is dense in Z, we have $\mu(D) = 1$. Thus we need to show that μ is finitely additive. Let A and B be disjoint subsets of D. If $\mu(A) = \mu(B) = 0$, then $z \notin c\ell\varphi[A] \cup c\ell\varphi[B] = c\ell\varphi[A \cup B]$ so $\mu(A \cup B) = 0$. If say $\mu[A] = 1$, then $z \in c\ell\varphi[A] \subseteq c\ell\varphi[A \cup B]$ so $\mu(A \cup B) = 1$ so it suffices to show that it is not possible to have $\mu(A) = \mu(B) = 1$. Indeed let χ_A be the characteriztic function of A and let $g : Z \longrightarrow \{0, 1\}$ be the continuous function with $g \circ \varphi = \chi_A$. Then $z \in c\ell\varphi[A] \subseteq g^{-1}[\{1\}]$ so $z \notin c\ell\varphi[B]$.

(d) Let $\mu \in \mathrm{UF}_4$ and let $\mathcal{U} = \tau_4(\mu)$. To see that $\mathcal{U}$ is a quantifier over D, let $\gamma(s)$ be given and assume that $(\forall s \in D)\big(\gamma(s)\big)$. Then $D = \{s \in D : \gamma(s)\}$, so $\mu(\{s \in D : \gamma(s)\}) = 1$, so $(\mathcal{U}s)\big(\gamma(s)\big)$. To see that $\mathcal{U}$ respects propositional connectives, let $\gamma(s)$ and $\psi(s)$ be statements about elements s of D. Then

$$\begin{aligned} \neg(\mathcal{U}s)\big(\gamma(s)\big) &\Leftrightarrow \neg\mu(\{s \in D : \gamma(s)\}) = 1 \\ &\Leftrightarrow \mu(\{s \in D : \gamma(s)\}) = 0 \\ &\Leftrightarrow \mu(D \backslash \{s \in D : \gamma(s)\}) = 1 \\ &\Leftrightarrow \mu(\{s \in D : \neg\gamma(s)\}) = 1 \\ &\Leftrightarrow (\mathcal{U}s)\big(\neg\gamma(s)\big) \ . \end{aligned}$$

Also

$$\begin{aligned} (\mathcal{U}s)\big(\gamma(s) \wedge \psi(x)\big) &\Leftrightarrow \mu(\{s \in D : \gamma(s) \wedge \psi(x)\}) = 1 \\ &\Leftrightarrow \mu(\{s \in D : \gamma(s)\} \cap \{s \in D : \psi(x)\}) = 1 \\ &\Leftrightarrow \mu(\{s \in D : \gamma(s)\}) = 1 \ \wedge \ \mu(\{s \in D : \psi(x)\}) = 1 \\ &\Leftrightarrow (\mathcal{U}s)\big(\gamma(s)\big) \wedge (\mathcal{U}s)\big(\psi(x)\big) \ . \end{aligned}$$

(e) Let $\mathcal{U} \in \mathrm{UF}_5$ and let $p = \tau_5(\mathcal{U})$. Since $\mathcal{U}$ is a quantifier over D and $(\forall s \in D)(s \in D)$, we have that $(\mathcal{U}s)(s \in D)$ and hence that $D \in p$. Since $(\forall s \in D)(\neg s \in \emptyset)$, we have $(\mathcal{U}s)(\neg s \in \emptyset)$ so, since $\mathcal{U}$ respects propositional connectives, $\neg(\mathcal{U}s)(s \in \emptyset)$ so $\emptyset \notin p$. Let $A, B \in p$. Then $(\mathcal{U}s)(s \in A) \wedge (\mathcal{U}s)(s \in B)$ so $(\mathcal{U}s)(s \in A \wedge s \in B)$ so $(\mathcal{U}s)(s \in A \cap B)$. That is $A \cap B \in p$. To complete the proof that p is a filter, assume that $A \in p$ and $A \subseteq B \subseteq D$. Then $(\mathcal{U}s)(s \in A)$. Also, $(\forall s \in D)(s \in A \Rightarrow s \in B)$ and hence $(\mathcal{U}s)(s \in A \Rightarrow s \in B)$. Since $\mathcal{U}$ respects propositional connectives, one has $(\mathcal{U}s)(s \in A) \Rightarrow (\mathcal{U}s)(s \in B)$. Since we know $(\mathcal{U}s)(s \in A)$, this says that $(\mathcal{U}s)(s \in B)$ so $B \in p$.

Finally to see that p is an ultrafilter, let $A \subseteq D$ be given. Then $(\forall s \in D)(s \in A \vee s \in D \backslash A)$ so that $(\mathcal{U}s)(s \in A \vee s \in D \backslash A)$ and hence, since $\mathcal{U}$ respects propositional connectives, $(\mathcal{U}s)(s \in A) \vee (\mathcal{U}s)(s \in D \backslash A)$. Thus $A \in p$ or $D \backslash A \in p$ as required. □

3.8 Theorem. *Each of τ_1, τ_2, τ_3, τ_4, and τ_5, is a bijection. In fact each of the following statements hold.*
(a) $\tau_5 \circ \tau_4 \circ \tau_3 \circ \tau_2 \circ \tau_1$ is the identity on UF_1.
(b) $\tau_1 \circ \tau_5 \circ \tau_4 \circ \tau_3 \circ \tau_2$ is the identity on UF_2.
(c) $\tau_2 \circ \tau_1 \circ \tau_5 \circ \tau_4 \circ \tau_3$ is the identity on UF_3.
(d) $\tau_3 \circ \tau_2 \circ \tau_1 \circ \tau_5 \circ \tau_4$ is the identity on UF_4.

(e) $\tau_4 \circ \tau_3 \circ \tau_2 \circ \tau_1 \circ \tau_5$ is the identity on UF_5.

Proof. The fact that each of τ_1, τ_2, τ_3, τ_4, and τ_5 is a bijection follows immediately from statements (a) through (e).

(a) Let $p \in \mathrm{UF}_1$ and let $q = \tau_5(\tau_4(\tau_3(\tau_2(\tau_1(p)))))$. Let $z = p\text{-}\lim\langle\varphi(s)\rangle_{s\in D} = \tau_2(\tau_1(p))$, let $\mu = \tau_3(z)$, and let $\mathcal{U} = \tau_4(\mu)$, so that $q = \tau_5(\mathcal{U})$.

Since p and q are both maximal filters, to see that $p = q$, it suffices to show that $q \subseteq p$, so let $A \in q$. Then $A \in \tau_5(\mathcal{U})$, so $(\mathcal{U}s)(s \in A)$. Since $\mathcal{U} = \tau_4(\mu)$, this says that $\mu(\{s \in D : s \in A\}) = 1$, i.e., $\mu(A) = 1$. Since $\mu = \tau_3(z)$, this says that $z \in c\ell\varphi[A]$ and hence that $p\text{-}\lim\langle\varphi(s)\rangle_{s\in D} \in c\ell\varphi[A]$. Now A and $D\backslash A$ are disjoint subsets of D and Z is a Stone-Čech compactification of D so $c\ell\varphi[A] \cap c\ell\varphi[D\backslash A] = \emptyset$ so $p\text{-}\lim\langle\varphi(s)\rangle_{s\in D} \in Z\backslash c\ell\varphi[D\backslash A]$, an open subset of Z. Consequently, $\{s \in D : \varphi(s) \in Z\backslash c\ell\varphi[D\backslash A]\} \in p$. But then $\{s \in D : \varphi(s) \in \varphi[A]\} \in p$. That is, $A \in p$ as required.

We leave the proofs of statements (b), (c), and (d) to the reader.

(e) Let $\mathcal{U} \in \mathrm{UF}_5$ and let $\mathcal{V} = \tau_4(\tau_3(\tau_2(\tau_1(\tau_5(\mathcal{U})))))$. Let $p = \tau_5(\mathcal{U})$, let $z = p\text{-}\lim\langle\varphi(s)\rangle_{s\in D} = \tau_2(\tau_1(p))$, and let $\mu = \tau_3(z)$.

Suppose that $\mathcal{U} \neq \mathcal{V}$ and choose a formula $\gamma(s)$ and an interpretation of the free variables other than s (if any) in $\gamma(s)$ such that $(\mathcal{U}s)(\gamma(s))$ and $\neg(\mathcal{V}s)(\gamma(s))$. (This choice is without loss of generality since $\gamma(s)$ may be replaced by $\neg\gamma(s)$.) With the same interpretation of the free variables let $A = \{s \in D : \gamma(s)\}$. Then $(\forall s \in D)(s \in A \Leftrightarrow \gamma(s))$ so $(\mathcal{U}s)(s \in A \Leftrightarrow \gamma(s))$ and $(\mathcal{V}s)(s \in A \Leftrightarrow \gamma(s))$. Since $\mathcal{U}$ and $\mathcal{V}$ respect propositional connectives, one has then that $(\mathcal{U}s)(s \in A) \Leftrightarrow (\mathcal{U}s)(\gamma(s))$ and $(\mathcal{V}s)(s \in A) \Leftrightarrow (\mathcal{V}s)(\gamma(s))$. Consequently $(\mathcal{U}s)(s \in A)$ and $\neg(\mathcal{V}s)(s \in A)$.

Since $(\mathcal{U}s)(s \in A)$ we have $A \in p$ and consequently $z \in c\ell\varphi[A]$. Then $\mu(A) = 1$, that is $\mu(\{s \in D : s \in A\}) = 1$. Thus $(\mathcal{V}s)(s \in A)$, a contradiction. □

4. Extending the Operation to βS

We show in this section that the operation $\cdot$ on S extends naturally to an operation on βS making $(\beta S, \cdot)$ into a compact right topological semigroup and we establish some of the basic properties that βS enjoys by virtue of that fact.

4.1 Definition. Let $(S, \cdot)$ be a semigroup. Define an operation $\cdot$ on βS, extending the operation $\cdot$ on S, in two stages as follows.

(a) Given $q \in \beta S\backslash S$ and $s \in S$,

$$s \cdot q = q\text{-}\lim_{t\in S}\ s \cdot t\ .$$

(b) Given $p \in \beta S\backslash S$ and $q \in \beta S$,

$$p \cdot q = p\text{-}\lim_{s\in S}\ s \cdot q\ .$$

Recall that we have identified any point of S with the principal ultrafilter generated by that point. Consequently, if $s, q \in S$, then $s \cdot q = q\text{-}\lim_{t\in S}\ s \cdot t$. That is statement (a) of Definition 4.1 holds for all $q \in \beta S$ and all $s \in S$. Similarly, statement (b) of Definition 4.1 holds for all $p, q \in \beta S$.

The following fact will be useful in establishing the associativity of the operation $\cdot$ on βS (and is worth knowing in any event). This theorem illustrates again the virtue of the uniformity of the process of passing to limits using p-lim . Observe that the hypotheses regarding the existence of limits can be dispensed with, by Theorem 2.16, if X is compact.

4.2 Theorem. *Let $(S,\cdot)$ be a semigroup, let X be a Hausdorff space, let $\langle x_s\rangle_{s\in S}$ be an indexed family in X, and let $p,q\in\beta S$. If all limits involved exist, then $(p\cdot q)$-$\lim_{v\in S} x_v = p$-$\lim_{s\in S}(q$-$\lim_{t\in S} x_{st})$.*

Proof. Let $z = (p\cdot q)$-$\lim_{v\in S} x_v$ and for each $s\in S$, let $y_s = q$-$\lim_{t\in S} x_{st}$. Suppose that p-$\lim_{s\in S} y_s \neq z$ and pick disjoint open neighborhoods U and V of p-$\lim_{s\in S} y_s$ and z respectively. Let $A=\{v\in S: x_v\in V\}$ and let $B=\{s\in S: y_s\in U\}$. Then $A\in p\cdot q$ and $B\in p$. Let $C=\{s\in S: s\cdot q\in \overline{A}\}$. Since $\overline{A}$ is a neighborhood of p-$\lim_{s\in S} s\cdot q$, $C\in p$. Then $B\cap C\in p$ so pick $s\in B\cap C$. Since $s\in B$, q-$\lim_{t\in S} x_{st}\in U$. Let $D=\{t\in S: x_{st}\in U\}$. Then $D\in q$. Since $s\in C$, $s\cdot q\in\overline{A}$ so $\overline{A}$ is a neighborhood of q-$\lim_{t\in S} s\cdot t$. Let $E=\{t\in S: s\cdot t\in\overline{A}\}$. Then $D\cap E\in q$ so pick $t\in D\cap E$. Since $t\in D$, $x_{st}\in U$. Since $t\in E$, $st\in A$ so $x_{st}\in V$ and hence $U\cap V\neq\emptyset$, a contradiction. □

4.3 Theorem. *Let $(S,\cdot)$ be a discrete semigroup. Then with the operation as defined by Definition 4.1, $(\beta S,\cdot)$ is a right topological semigroup. Further, for each $s\in S$, λ_s is continuous.*

Proof. As we have observed, given any $p,q\in\beta S$ one has that $p\cdot q = p$-$\lim_{s\in S} s\cdot q$. Consequently, by Theorem 2.18 one has that for each $q\in\beta S$, ρ_q is continuous. Likewise, since for all $s\in S$ and all $q\in\beta S$, $s\cdot q = q$-$\lim_{t\in S} s\cdot t$, one has that for each $s\in S$, λ_s is continuous.

It thus remains only to show that the operation $\cdot$ on βS is associative. To see this, let $p,q,r\in\beta S$. Then

$$\begin{aligned} p\cdot(q\cdot r) &= p\text{-}\lim_{s\in S}\big((q\cdot r)\text{-}\lim_{w\in S} sw\big) \\ &= p\text{-}\lim_{s\in S}\Big(q\text{-}\lim_{t\in S}\big(r\text{-}\lim_{v\in S} s(tv)\big)\Big) && \text{by Theorem 4.2} \\ &= p\text{-}\lim_{s\in S}\big(q\text{-}\lim_{t\in S}(r\text{-}\lim_{v\in S} (st)v)\big) \\ &= (p\cdot q)\text{-}\lim_{u\in S}(r\text{-}\lim_{v\in S} uv) && \text{by Theorem 4.2} \\ &= (p\cdot q)\cdot r\,. \end{aligned}$$

□

Given p and q in βS, we have that $p\cdot q$ is an ultrafilter, so it will be desirable to have a convenient characterization of its members. We have taken the notation $[A]_p$ from [67].

4.4 Definition. . Let $(S,\cdot)$ be a discrete semigroup and let $A\subseteq S$.

(a) For each $x\in S$, $x^{-1}A=\{y\in S: x\cdot y\in A\}$.

(b) For each $p\in\beta S$, $[A]_p=\{x\in S: x^{-1}A\in p\}$.

If S is a group, then $x^{-1}A=\{x^{-1}y: y\in A\}$, but in general this need not be true. In the first place, x^{-1} need not make any sense. Even when S is contained in a group, this equality may fail. For example if A is the set of odd positive integers, then in the semigroup $(\mathbb{N},\cdot)$, $2^{-1}A=\emptyset$.

4.5 Lemma. *Let $(S,\cdot)$ be a semigroup, let $p,q\in\beta S$, and let $A\subseteq S$.*

(a) $A\in p\cdot q$ if and only if $[A]_q\in p$.

(b) For each $x\in S$, $x^{-1}[A]_p=[x^{-1}A]_p$.

(c) $\big[[A]_q\big]_p=[A]_{p\cdot q}$.

Proof. (a) Necessity. Assume that $A\in p\cdot q$. Then $\overline{A}$ is a neighborhood of $\rho_q(p)$ so pick a member B of p such that $\rho_q[\overline{B}]\subseteq\overline{A}$. We claim that $B\subseteq[A]_q$ so that $[A]_q\in p$. To see this, let $x\in B$. Then $\lambda_x(q)\in\overline{A}$ so pick a member C of q such that $\lambda_x[\overline{C}]\subseteq\overline{A}$. Then $C\subseteq x^{-1}A$.

Sufficiency. Assume that $[A]_q\in p$ and suppose that $A\notin p\cdot q$. Then $S\backslash A\in p\cdot q$ so, by the necessity which has been established, $[S\backslash A]_q\in p$. Pick $x\in[A]_q\cap[S\backslash A]_q$. Then $x^{-1}A\in q$ and $x^{-1}(S\backslash A)\in q$ so pick $y\in x^{-1}A\cap x^{-1}(S\backslash A)$. Then $xy\in A\cap(S\backslash A)$, a contradiction.

(b) First observe that for any x and z in S, $(xz)^{-1}A = z^{-1}(x^{-1}A)$. Thus, given $x, z \in S$ one has

$$\begin{aligned} z \in x^{-1}[A]_q &\Leftrightarrow xz \in [A]_q \\ &\Leftrightarrow (xz)^{-1}A \in q \\ &\Leftrightarrow z^{-1}(x^{-1}A) \in q \\ &\Leftrightarrow z \in [x^{-1}A]_q \ . \end{aligned}$$

(c) Using part (b), we have that for any $x \in S$,

$$\begin{aligned} x \in \big[[A]_q\big]_p &\Leftrightarrow x^{-1}[A]_q \in p \\ &\Leftrightarrow [x^{-1}A]_q \in p \\ &\Leftrightarrow x^{-1}A \in p \cdot q \\ &\Leftrightarrow x \in [A]_{p \cdot q} \ . \end{aligned}$$

□

Of fundamental importance for the combinatorial applications of βS is the following fact due to R. Ellis [25, Corollary 2.10].

4.6 Theorem. *Let $(T, \cdot)$ be a compact Hausdorff right topological semigroup. Then T contains an idempotent.*

Proof. Let $\mathcal{A} = \{A \subseteq T : A \cdot A \subseteq A,\ A \neq \emptyset$, and A is compact$\}$. Notice that $T \in \mathcal{A}$ so $\mathcal{A} \neq \emptyset$. Given any chain $\mathcal{C}$ in $\mathcal{A}$, one has that $\bigcap \mathcal{C} \in \mathcal{A}$ so, by Zorn's Lemma, choose some minimal $A \in \mathcal{A}$ and pick some $x \in A$. (We will show that $xx = x$ so it will follow that in fact $A = \{x\}$. However, one does not need to notice this fact.)

Let $B = Ax$. Then $BB = (Ax)(Ax) \subseteq AAAx \subseteq Ax = B$, $xx \in B$, and $B = \rho_x[A]$ so (being the continuous image of a compact space) B is compact. Thus $B \in \mathcal{A}$. Since $B \subseteq A$ and A is minimal, we have $B = A$.

Let $C = \{y \in A : yx = x\}$. Since $B = A$, we have that $x \in Ax$ so $C \neq \emptyset$. Also $C = A \cap \rho_x^{-1}[\{x\}]$ so C is compact. And, given $y, z \in C$, $yz \in A$ and $yzx = yx = x$. Thus $C \in \mathcal{A}$ so $C = A$ and hence $x \in C$. That is, $xx = x$. □

As a consequence of Theorem 4.6, idempotents exist in βS for any semigroup S. Of special importance for the combinatorial applications are *minimal* idempotents. There are several natural interpretations of "minimal". Three of these involve partial orders, and a fourth the notion of minimal ideals. As we shall see, they are all equivalent.

4.7 Definition. Let $(T, \cdot)$ be any semigroup and let e and f be idempotents of T.

(a) $e \leq_L f$ if and only if e=ef.

(b) $e \leq_R f$ if and only if e=fe.

(c) $e \leq f$ if and only if e=fe=ef.

Notice that $\leq_L$, $\leq_R$, and $\leq$ are transitive and reflexive relations on the set of all idempotents in T. Observe also that $\leq$ is antisymmetric.

4.8 Theorem. *Let $(T, \cdot)$ be any semigroup and let e and f be idempotents of T. The following statements are equivalent.*

(a) The element e is minimal with respect to $\leq$.

(b) The element e is minimal with respect to $\leq_R$.

(c) The element e is minimal with respect to $\leq_L$.

Proof. (b) implies (a). Assume that e is minimal with respect to $\leq_R$ and let $f \leq e$. Then $f = ef$ so $f \leq_R e$ so $e \leq_R f$. Then $e = fe = f$.

We show that (a) implies (b). (Then the equivalence of (a) and (c) follows by a left-right switch.) Assume that e is minimal with respect to $\leq$ and let $f \leq_R e$. Let $g = fe$. Then

$gg = fefe = ffe = fe = g$ so g is an idempotent. Also, $g = fe = efe$ so $eg = eefe = efe = g = efee = ge$ so $g \le e$ so $g = e$ by the minimality of e. That is, $e = fe$ so $e \le_R f$ as required. $\square$

The fourth characterization of minimality involves the notion of minimal left and right ideals.

4.9 Definition. Let $(T, \cdot)$ be a semigroup and let $H \subseteq T$.
(a) H is a *left ideal* of T if and only if $H \neq \emptyset$ and $T \cdot H \subseteq H$.
(b) H is a *right ideal* of T if and only if $H \neq \emptyset$ and $H \cdot T \subseteq H$.
(c) H is an *ideal* of T if and only if H is both a left ideal and a right ideal of T.

4.10 Lemma. *Let $(T, \cdot)$ be a compact right topological semigroup. Every left ideal of T contains a minimal left ideal and minimal left ideals are closed.*

Proof. Let L be a left ideal of T. Let $\mathcal{A} = \{H : H$ is a compact left ideal of T and $H \subseteq L\}$. Then given $x \in L$, $Tx \in \mathcal{A}$ so $\mathcal{A} \neq \emptyset$. Since the intersection of a chain in $\mathcal{A}$ is again in $\mathcal{A}$, Zorn's Lemma yields the conclusion. $\square$

It is also true, but not so easy to show, that every right ideal contains a minimal right ideal. However, minimal right ideals in a compact right topological semigroup need not be closed. See [16] for proofs of these facts.

The smallest ideal of a compact right topological semigroup, whose existence we establish below, has an elaborate structure. For example, it is the union of pairwise isomorphic groups. The reader is again referred to [16] for the structure theorem.

4.11 Definition. Let $(T, \cdot)$ be a compact right topological semigroup. Then

$$K(T) = \bigcup\{L : L \text{ is a minimal left ideal of } T\}\ .$$

4.12 Theorem. *Let $(T, \cdot)$ be a compact right topological semigroup. Then $K(T)$ is an ideal of T. It is in fact the smallest two sided ideal of T.*

Proof. By Lemma 4.10, $K(T) \neq \emptyset$ and as the union of left ideals, $K(T)$ is trivially a left ideal. Given any two sided ideal I of T and any minimal left ideal L of T one has that $L \cap I \neq \emptyset$. (For if $a \in L$ and $b \in I$ then $ba \in L \cap I$.) Consequently $L \cap I$ is a left ideal which is contained in L and hence $L \cap I = L$. That is, $K(T) \subseteq I$.

Thus, to complete the proof, it suffices to show that $K(T)$ is a right ideal of T. To this end, let $a \in K(T)$ and let $b \in T$. Pick a minimal left ideal L of T such that $a \in L$. Then $ab \in Lb$, so it suffices to show that Lb is a minimal left ideal of T. It is trivially a left ideal, so suppose that H is a left ideal of T with $H \subseteq Lb$. Let $A = \{x \in L : xb \in H\}$. Then A is a left ideal of T which is contained in L so $A = L$ so $H = Lb$. $\square$

The fourth possible interpretation of a "minimal idempotent" is one which is a member of a minimal left ideal. We see now that this is equivalent to the other notions.

4.13 Theorem. *Let $(T, \cdot)$ be a compact right topological semigroup and let e be an idempotent in T. Then e is minimal with respect to any (and hence all) of the orders $\le_R$, $\le_L$, or $\le$ if and only if $e \in K(S)$.*

Proof. Necessity. Te is a left ideal of T so pick by Lemma 4.10 a minimal left ideal L of T with $L \subseteq Te$. Then L is closed (by Lemma 4.10) so pick by Theorem 4.6 an idempotent $f \in L$. Then $f \in Te$ so pick some $g \in T$ such that $f = ge$. Then $fe = gee = ge = f$ so $f \le_L e$. Since e is minimal with respect to $\le_L$, $e \le_L f$ so $e = ef \in L \subseteq K(T)$.

Sufficiency. Pick a minimal left ideal L of T such that $e \in L$. To see that e is minimal, let f be an idempotent with $f \leq e$. Now Te is a left ideal of T which is contained in L so $Te = L$. Since $f \leq_L e$, $f \in L$ so $Tf = L$. Thus $e \in Tf$ so pick $g \in T$ such that $e = gf$. Then $ef = gff = gf = e$. Since also $f \leq_R e$ we have $ef = f$. Thus $e = f$ as required. $\square$

Since we know by Lemma 4.10 that minimal left ideals are closed, we have by Theorem 4.6 that minimal idempotents exist. We see now that much more is true.

4.14 Theorem. *Let $(T, \cdot)$ be a compact right topological semigroup and let e be an idempotent in T. There is a minimal idempotent f such that $f \leq e$.*

Proof. As in the proof of the "necessity" of Theorem 4.13, pick a minimal left ideal L of T with $L \subseteq Te$ and pick an idempotent $f \in L$. As there, we see that $f = fe$. Now let $r = ef$. Then $r \in L$. Further $rr = efef = eff = ef = r$, so r is a minimal idempotent. Also, $er = eef = ef = r$ and $re = efe = ef = r$ so $r \leq e$. $\square$

We conclude this section with a result that will be needed in Section 6.

4.15 Theorem. *Let $(S, \cdot)$ be a discrete cancellative semigroup. Then $\beta G \backslash G$ is an ideal of βG.*

Proof. Let $p \in \beta G \backslash G$ and let $q \in \beta G$. Suppose first that $p \cdot q = a \in G$. Then $\{a\} \in p \cdot q$ so by Lemma 4.5(a), $[\{a\}]_q \in p$. Since p is nonprincipal, pick $x \neq y$ in $[\{a\}]_q$ and pick $z \in x^{-1}\{a\} \cap y^{-1}\{a\}$. Then $xz = yz$, a contradiction.

Now suppose that $q \cdot p = a \in G$. Then $\{a\} \in q \cdot p$ so by Lemma 4.5(a), $[\{a\}]_p \in q$. Pick $z \in [\{a\}]_p$. Then since p is nonprincipal, pick $x \neq y$ in $z^{-1}\{a\}$. Then $zx = zy$, a contradiction. $\square$

5. Easy Combinatorial Applications

We begin this section with the application which led us to the algebra of βS in the first place, namely the Galvin-Glazer proof of the Finite Sum Theorem. In fact, this proof is more general, applying to any semigroup, whether commutative or not. So we need to specify the order of our products. (We should point out that Corollary 5.3 can be derived from Theorem 1.1 by means of the equivalent Finite Union Theorem.)

5.1 Definition. Let $(S, \cdot)$ be a semigroup, let $\langle x_n \rangle_{n=1}^{\infty}$ be a sequence in S and let $F \in \mathcal{P}_f(\mathbb{N})$. Then $\Pi_{n \in F}\, x_n$ denotes the product in increasing order of indices and

$$FP(\langle x_n \rangle_{n=1}^{\infty}) = \{\Pi_{n \in F}\, x_n : F \in \mathcal{P}_f(\mathbb{N})\} .$$

In the event the operation in S is denoted by $+$, we write

$$FS(\langle x_n \rangle_{n=1}^{\infty}) = \{\Sigma_{n \in F}\, x_n : F \in \mathcal{P}_f(\mathbb{N})\} .$$

5.2 Theorem. *Let $(S, \cdot)$ be a discrete semigroup, let p be an idempotent in βS, and let $A \in p$. Then there is a sequence $\langle x_n \rangle_{n=1}^{\infty}$ in S such that $FP(\langle x_n \rangle_{n=1}^{\infty}) \subseteq A$.*

Proof. Notice that for each $B \in p$ we have that $[B]_p \in p$ by Lemma 4.5(a). Let $B_1 = A$ and choose $x_1 \in B_1 \cap [B_1]_p$. Let $B_2 = B_1 \cap {x_1}^{-1}B_1$. Inductively, assume we have $B_n \in p$, pick $x_n \in B_n \cap [B_n]_p$ and let $B_{n+1} = B_n \cap {x_n}^{-1}B_n$.

To see, for example, that $x_2x_5x_6 \in A$, we note that $x_6 \in B_6 \subseteq {x_5}^{-1}B_5$ so $x_5x_6 \in B_5 \subseteq B_4 \subseteq B_3 \subseteq {x_2}^{-1}B_2$ so that $x_2x_5x_6 \in B_2 \subseteq B_1 = A$.

More formally, one establishes by induction on $|F|$, that if $F \in \mathcal{P}_f(\mathbb{N})$ and $r = \min F$, then $\Pi_{n \in F}\, x_n \in B_r$. If $|F| = 1$, then $\Pi_{n \in F}\, x_n = x_r \in B_r$. So assume that $|F| = k > 1$ and assume that the statement is true for $k - 1$. Let $H = F \backslash \{r\}$ and let $s = \min H$. Then $\Pi_{n \in H}\, x_n \in B_s \subseteq B_{r+1} \subseteq {x_r}^{-1}B_r$ so $\Pi_{n \in F}\, x_n = x_r \cdot \Pi_{n \in H}\, x_n \in B_r$. $\square$

5.3 Corollary. *Let $(S,\cdot)$ be a semigroup, let $r \in \mathbb{N}$, and assume that $S = \bigcup_{i=1}^{r} A_i$. Then there exist $i \in \{1,2,\ldots,r\}$ and a sequence $\langle x_n \rangle_{n=1}^{\infty}$ in S such that $FP(\langle x_n \rangle_{n=1}^{\infty}) \subseteq A_i$.*

Proof. By Theorem 4.6 pick an idempotent $p \in \beta S$, pick some $i \in \{1,2,\ldots,r\}$ such that $A_i \in p$ and apply Theorem 5.2. $\square$

The proof that has just been presented is so short and pretty that I have been in love with it since first being introduced to it over 20 years ago. (Anyone with a very masochistic bent is invited to try to wade through the original combinatorial proof [33].)

It was thus with considerable surprise that I learned from Dona Strauss in 1995 that there is an even simpler version of the proof. In her simplification one is not forced to go backwards through a product to determine that it is in the desired set. This simplified proof depends on the following lemma. It is obvious that given $x \in B$ as defined below, one has $x^{-1}A \in p$. But more is true.

5.4 Lemma. *Let $(S,\cdot)$ be a discrete semigroup, let p be an idempotent in βS, and let $A \in p$. Let $B = A \cap [A]_p$. Then for each $x \in B$, $x^{-1}B \in p$.*

Proof. Let $x \in B$. Then $x^{-1}A \in p = p \cdot p$ so by Lemma 4.5(a), $[x^{-1}A]_p \in p$ while by Lemma 4.5(b), $[x^{-1}A]_p = x^{-1}[A]_p$. Thus $x^{-1}B = x^{-1}A \cap x^{-1}[A]_p \in p$. $\square$

Now let us examine the simplified proof of the Finite Product Theorem.

5.2 Theorem. *Let $(S,\cdot)$ be a discrete semigroup, let p be an idempotent in βS, and let $A \in p$. Then there is a sequence $\langle x_n \rangle_{n=1}^{\infty}$ in S such that $FP(\langle x_n \rangle_{n=1}^{\infty}) \subseteq A$.*

Proof. Let $B = A \cap [A]_p$ and choose $x_1 \in B$. Inductively, let $n \in \mathbb{N}$ and assume we have chosen $\langle x_t \rangle_{t=1}^{n}$ such that $FP(\langle x_t \rangle_{t=1}^{n}) \subseteq B$. Then by Lemma 5.4,

$$\bigcap\{a^{-1}B : a \in FP(\langle x_t \rangle_{t=1}^{n})\} \in p$$

so choose $x_{n+1} \in B \cap \bigcap\{a^{-1}B : a \in FP(\langle x_t \rangle_{t=1}^{n})\}$. Then $FP(\langle x_t \rangle_{t=1}^{n+1}) \subseteq B$. $\square$

We pause to note that a sort of converse to Theorem 5.2 is valid. (This is an old observation of Fred Galvin's).

5.5 Theorem. *Let $(S,\cdot)$ be a discrete semigroup and let $\langle x_n \rangle_{n=1}^{\infty}$ be a sequence in S. Then there is some idempotent $p \in \beta S$ such that $FP(\langle x_n \rangle_{n=1}^{\infty}) \in p$.*

Proof. Let $T = \bigcap_{m=1}^{\infty} \overline{FP(\langle x_n \rangle_{n=m}^{\infty})}$ (where the meaning of $FP(\langle x_n \rangle_{n=m}^{\infty})$ should be obvious). Then T is certainly compact and nonempty. We claim that T is a subsemigroup of βS. To see this, let $p, q \in T$ and let $m \in \mathbb{N}$. To see that $FP(\langle x_n \rangle_{n=m}^{\infty}) \in p \cdot q$, it suffices by Lemma 4.5 to prove that $[FP(\langle x_n \rangle_{n=m}^{\infty})]_q \in p$, for which it in turn suffices to show that $FP(\langle x_n \rangle_{n=m}^{\infty}) \subseteq [FP(\langle x_n \rangle_{n=m}^{\infty})]_q$. So let $a \in FP(\langle x_n \rangle_{n=m}^{\infty})$ and pick $F \in \mathcal{P}_f(\mathbb{N})$ such that $\min F \geq m$ and $a = \Pi_{n \in F}\, x_n$. Let $r = \max F$. Then $FP(\langle x_n \rangle_{n=r+1}^{\infty}) \in q$ and $FP(\langle x_n \rangle_{n=r+1}^{\infty}) \subseteq a^{-1}FP(\langle x_n \rangle_{n=m}^{\infty})$.

Since T is a compact right topological semigroup, by Theorem 4.6, there is an idempotent in T. $\square$

An old and easy application of Theorem 5.2 is the following theorem which was first published in 1979 [34]. It was not until 1994 that an elementary (though still more difficult) proof was obtained [12].

5.6 Theorem. *Let $\mathbb{N} = \bigcup_{i=1}^{r} A_i$. There exist $i \in \{1,2,\ldots,r\}$ and sequences $\langle x_n \rangle_{n=1}^{\infty}$ and $\langle y_n \rangle_{n=1}^{\infty}$ such that $FS(\langle x_n \rangle_{n=1}^{\infty}) \cup FP(\langle y_n \rangle_{n=1}^{\infty}) \subseteq A_i$.*

Proof. Let $\Gamma = \{p \in \beta\mathbb{N}$: for all $A \in p$, there is some sequence $\langle x_n \rangle_{n=1}^{\infty}$ such that $FS(\langle x_n \rangle_{n=1}^{\infty}) \subseteq A\}$. We claim that Γ is a compact subsemigroup of $(\beta\mathbb{N},\cdot)$, in fact a left ideal. To see that Γ is closed, let $p \in \beta\mathbb{N}\backslash\Gamma$. Then there is some $A \in p$ such that for any sequence $\langle x_n\rangle_{n=1}^{\infty}$ in $\mathbb{N}$, $FS(\langle x_n\rangle_{n=1}^{\infty})\backslash A \neq \emptyset$. Then $\overline{A}$ is a neighborhood of p which misses Γ.

To see that Γ is a left ideal of $(\beta\mathbb{N}, \cdot)$, let $p \in \beta\mathbb{N}$ and let $q \in \Gamma$. Let $A \in p \cdot q$. Then by Lemma 4.5, $[A]_q \in p$ so pick $z \in [A]_q$. Then $z^{-1}A \in q$ so pick $\langle x_n\rangle_{n=1}^{\infty}$ such that $FS(\langle x_n\rangle_{n=1}^{\infty}) \subseteq z^{-1}A$. Then $FS(\langle zx_n\rangle_{n=1}^{\infty}) \subseteq A$.

By Theorem 4.6 pick some $p \in \Gamma$ such that $p \cdot p = p$. Pick $i \in \{1, 2, \ldots, r\}$ such that $A_i \in p$. Since $p \in \Gamma$, pick a sequence $\langle x_n\rangle_{n=1}^{\infty}$ such that $FS(\langle x_n\rangle_{n=1}^{\infty}) \subseteq A_i$. Since $p = p \cdot p$, pick by Theorem 5.2, some sequence $\langle y_n\rangle_{n=1}^{\infty}$ such that $FP(\langle y_n\rangle_{n=1}^{\infty}) \subseteq A_i$. $\square$

6. Zelenjuk's Theorem

For some time, considerable effort has been invested in attempting to determine whether the semigroup $(\beta\mathbb{N}, +)$ contains any nontrivial finite subgroups. See for example [3] for some limited evidence that such groups need not exist.

We present in this section, with his kind permission, a version of E. G. Zelenjuk's proof that if G is a countable group and G has no nontrivial finite subgroups, then neither does βG. (Zelenjuk's original proof [67] included the assumption that G was abelian, but Dona Strauss observed in a personal communication that this assumption was not needed.)

The proof presented here is significantly different in organization than that in [67]. However, most of the notation and all of the ideas are Zelenjuk's (except for the observation of Strauss mentioned above).

We will assume throughout this section that G is a countable group with identity e, that we have a finite subgroup C of $\beta G\backslash G$, and that $\varphi = \bigcap C$, so that, by Theorem 2.14, $C = \overline{\varphi} = \{p \in \beta G : \varphi \subseteq p\}$. We will further assume that for some $\alpha \in \mathbb{N}\backslash\{1\}$, C is isomorphic to $\mathbb{Z}_\alpha$ and will write $C = \{p_0, p_1, \ldots, p_{\alpha-1}\}$ where $p_i \cdot p_j = p_{i+j}$ and the sum $i+j$ is computed in $\mathbb{Z}_\alpha$. (We may do this because C will in any event have such a subgroup.) We will also sometimes denote the identity p_0 of C by ϵ.

6.1 Definition. Let $p \in G$ and $A \subseteq G$. $[\varphi]$ is the filter generated by $\{\bigcup_{p\in\overline{\varphi}} [A]_p : A \in \varphi\}$.

Recall from Theorem 4.5 that given $p, q \in \beta G$ and $A \subseteq G$, $[A]_p \in q$ if and only if $A \in qp$ and that $\big[[A]_p\big]_q = [A]_{qp}$.

6.2 Lemma. $\overline{[\varphi]} = \{q \in \beta G : q\overline{\varphi} = \overline{\varphi}\}$.

Proof. Let $q \in \beta G$. Notice first that

$$\text{(a)} \qquad (\forall A \in \varphi)(\exists p \in \overline{\varphi})([A]_p \in q) \Leftrightarrow (\exists p \in \overline{\varphi})(\forall A \in \varphi)([A]_p \in q)$$

Indeed, the sufficiency is trivial. To establish the necessity, suppose instead that for each $p \in \overline{\varphi}$ we have some $A_p \in \varphi$ such that $[A_p]_p \notin q$ and let $A = \bigcap_{p\in\overline{\varphi}} A_p$. Then $A \in \varphi$ and for all $p \in \overline{\varphi}$, $[A]_p \subseteq [A_p]_p$, so $[A]_p \notin q$, a contradiction.

Second observe that

$$\text{(b)} \qquad (\exists p \in \overline{\varphi})(qp \in \overline{\varphi}) \Leftrightarrow q\overline{\varphi} = \overline{\varphi}$$

Again the sufficiency is trivial, while the necessity follows from the fact that $\overline{\varphi}$ is a group.

Thus we have

$$\begin{aligned} q \in \overline{[\varphi]} &\Leftrightarrow (\forall A \in \varphi)(\textstyle\bigcup_{p\in\overline{\varphi}} [A]_p \in q) \\ &\Leftrightarrow (\forall A \in \varphi)(\exists p \in \overline{\varphi})([A]_p \in q) \\ &\Leftrightarrow (\exists p \in \overline{\varphi})(\forall A \in \varphi)([A]_p \in q) \\ &\Leftrightarrow (\exists p \in \overline{\varphi})(\forall A \in \varphi)(A \in qp) \\ &\Leftrightarrow (\exists p \in \overline{\varphi})(qp \in \overline{\varphi}) \\ &\Leftrightarrow q\overline{\varphi} = \overline{\varphi}\ . \end{aligned}$$

$\square$

6.3 Lemma. *Let $A \in \varphi$ and let $W = \bigcup_{p\in\overline{\varphi}} [A]_p$. Then for all $q \in \overline{[\varphi]}$, $W = [W]_q$.*

Proof. Let $q \in \overline{[\varphi]}$. To see that $W \subseteq [W]_q$, let $x \in W$ and pick $p \in \overline{\varphi}$ such that $x \in [A]_p$. By Lemma 6.2 $q\overline{\varphi} = \overline{\varphi}$ so pick some $r \in \overline{\varphi}$ such that $qr = p$. Then $x \in [A]_{qr} = \big[[A]_r\big]_q$ so $x^{-1}[A]_r \in q$ so $x^{-1}W \in q$. That is, $x \in [W]_q$.

To see that $[W]_q \subseteq W$, let $x \in [W]_q$. Then $x^{-1}W \in q$ and $\overline{\varphi}$ is finite, so pick $p \in \overline{\varphi}$ such that $x^{-1}[A]_p \in q$. Then $x \in \big[[A]_p\big]_q = [A]_{qp} \subseteq W$ because $qp \in \overline{\varphi}$. □

6.4 Lemma. *Let $A \in \varphi$ and let $W = \bigcup_{p\in\overline{\varphi}} [A]_p$. Then*

(a) for all $a \in W$, there exists $V \in [\varphi]$ such that $aV \subseteq W$ and

(b) for all $a \in G\backslash W$, there exists $V \in [\varphi]$ such that $(aV) \cap W = \emptyset$.

Proof. (a). Let $a \in W$ and suppose that for all $V \in [\varphi]$, $aV\backslash W \neq \emptyset$. Then $\{V\backslash a^{-1}W : V \in [\varphi]\}$ has the finite intersection property, so choose $q \in \beta G$ such that $\{V\backslash a^{-1}W : V \in [\varphi]\} \subseteq q$, and note that $q \in \overline{[\varphi]}$. But then $a^{-1}W \notin q$ so $a \notin [W]_q$ and hence, by Lemma 6.3, $a \notin W$, a contradiction.

(b) Let $a \in G\backslash W$ and suppose that for all $V \in [\varphi]$, $aV \cap W \neq \emptyset$. Then $\{V \cap a^{-1}W : V \in [\varphi]\}$ has the finite intersection property, so choose $q \in \beta G$ such that $\{V \cap a^{-1}W : V \in [\varphi]\} \subseteq q$, and note that $q \in \overline{[\varphi]}$. But then $a^{-1}W \in q$ so $a \in [W]_q$ and hence, by Lemma 6.3, $a \in W$, a contradiction. □

6.5 Lemma. *There exists a choice of $A_p \in p$ for each $p \in \overline{\varphi}$ such that*

(1) $e \in A_\epsilon$,

(2) $A_p \cap A_q = \emptyset$ for $p \neq q$ in $\overline{\varphi}$,

and, letting $X = \bigcup_{p\in\overline{\varphi}} A_p$ and defining $f : X \longrightarrow \mathbb{Z}_\alpha$ by $f[A_{p_i}] = \{i\}$, one has

(3) for all $a \in X$ there is some $V \in [\varphi]$ such that for all $b \in V$, $ab \in X$ and $f(ab) = f(a) + f(b)$.

Proof. For each $p \in \overline{\varphi}$, choose $U_p \in p$ such that $U_p \cap U_q = \emptyset$ when $p \neq q$, and let

$$A_p = \bigcap\nolimits_{q\in\overline{\varphi}} [U_{pq}]_q = \bigcap\nolimits_{q\in\overline{\varphi}} \{a \in G : a^{-1}U_{pq} \in q\}\ .$$

Given $p, q \in \overline{\varphi}$, $U_{pq} \in pq$ so $[U_{pq}]_q \in p$ so $A_p \in p$. Also, given $q \in \overline{\varphi}$, $e \in [U_{\epsilon q}]_q$, so (1) holds.

To verify (2), suppose $a \in A_p \cap A_r$ where p and r are distinct members of $\overline{\varphi}$. Then $a \in [U_p]_\epsilon \cap [U_r]_\epsilon$ so that $a^{-1}U_p \cap a^{-1}U_r \in \epsilon$ so $U_p \cap U_r \neq \emptyset$, a contradiction.

To verify (3), let $a \in X$ and pick $p \in \overline{\varphi}$ such that $a \in A_p$. For each $q \in \overline{\varphi}$, let $V_q = A_q \cap a^{-1}A_{pq}$ and let $V = \bigcup_{q\in\overline{\varphi}} V_q$. We show first that for all $b \in V$, $ab \in X$ and $f(ab) = f(a) + f(b)$. So let $b \in V$ and pick $q \in \overline{\varphi}$ such that $b \in V_q$. Then $b \in a^{-1}A_{pq}$ so $ab \in A_{pq} \subseteq X$. Pick $i, j \in \mathbb{Z}_\alpha$ such that $p = p_i$ and $q = p_j$. Then $pq = p_{i+j}$ so $f(ab) = i + j = f(a) + f(b)$.

Thus it remains only to show that $V \in [\varphi]$. We show first that

$$\text{(i)} \qquad \text{for all } q \in \overline{\varphi},\ V_q \in q$$

So let $q \in \overline{\varphi}$. Since we know that $A_q \in q$ we need only show that $a^{-1}A_{pq} \in q$. Now $a \in A_p$ so for each $r \in \overline{\varphi}$, $a \in [U_{pqr}]_{qr} = \big[[U_{pqr}]_r\big]_q$ so $a^{-1}[U_{pqr}]_r \in q$. Thus $\bigcap_{r\in\overline{\varphi}} a^{-1}[U_{pqr}]_r \in q$. But $a^{-1}A_{pq} = a^{-1}\bigcap_{r\in\overline{\varphi}} [U_{pqr}]_r = \bigcap_{r\in\overline{\varphi}} a^{-1}[U_{pqr}]_r$ so $a^{-1}A_{pq} \in q$ as required.

Next we show that

$$\text{(ii)} \qquad (\forall b \in G)(\forall r \in \overline{\varphi})(b \in A_r \Leftrightarrow b^{-1}A_r \in \epsilon).$$

To see this let $b \in G$ and $r \in \overline{\varphi}$. Then

$$\begin{aligned} A_r &= \bigcap_{q\in\overline{\varphi}} [U_{rq}]_q \\ &= \bigcap_{q\in\overline{\varphi}} [U_{rq}]_{\epsilon q} \\ &= \bigcap_{q\in\overline{\varphi}} \big[[U_{rq}]_q\big]_\epsilon \end{aligned}$$

so

$$\begin{aligned} b \in A_r &\Leftrightarrow (\forall q \in \overline{\varphi})(b^{-1}[U_{rq}]_q \in \epsilon) \\ &\Leftrightarrow \bigcap_{q\in\overline{\varphi}} b^{-1}[U_{rq}]_q \in \epsilon \\ &\Leftrightarrow b^{-1}\bigcap_{q\in\overline{\varphi}} [U_{rq}]_q \in \epsilon \\ &\Leftrightarrow b^{-1}A_r \in \epsilon\ . \end{aligned}$$

Now to see that $V \in [\varphi]$, let $r \in \overline{[\varphi]}$. We show that $V \in r$. By Lemma 6.2 $r\epsilon \in \overline{\varphi}$ so by (i) we have $V_{r\epsilon} \in r\epsilon$ so $A_{r\epsilon} \in r\epsilon$ and $a^{-1}A_{pr\epsilon} \in r\epsilon$ so $\{b \in G : b^{-1}A_{r\epsilon} \in \epsilon\} \in r$ and $\{b \in G : b^{-1}(a^{-1}A_{pr\epsilon}) \in \epsilon\} \in r$. Thus, by (ii), $A_{r\epsilon} \in r$ and $a^{-1}A_{pr\epsilon} \in r$. That is, $V_{r\epsilon} \in r$ and thus, since $V_{r\epsilon} \subseteq V$, $V \in r$. $\square$

6.6 Lemma. *Let $\langle W_n\rangle_{n=1}^\infty$ be any sequence in $[\varphi]$. There is a sequence $\langle U_n\rangle_{n=1}^\infty$ in $[\varphi]$ such that*

(1) for all $n \in \mathbb{N}$, $U_{n+1} \subseteq U_n$,

(2) for all $n \in \mathbb{N}$, $U_n \subseteq W_n$,

(3) for all $n \in \mathbb{N}$ and all $a \in U_n\backslash U_{n+1}$ there is some $k \in \mathbb{N}$ such that $aU_k \subseteq U_n\backslash U_{n+1}$, and

(4) for all $a \in G\backslash U_1$ there is some $k \in \mathbb{N}$ such that $aU_k \cap U_1 = \emptyset$.

Proof. Enumerate G as $\{a_n : n \in \mathbb{N}\}$ with $a_1 = e$. We will choose $\langle U_n\rangle_{n=1}^\infty$ satisfying (1), (2), (3), (4) and

(5) for each $n \in \mathbb{N}$ there is some $D_n \in \varphi$ such that $U_n = \bigcup_{q\in\overline{\varphi}} [D_n]_q$.

Pick $D_1 \in \varphi$ such that $\bigcup_{q\in\overline{\varphi}} [D_1]_q \subseteq W_1$ and let $U_1 = \bigcup_{q\in\overline{\varphi}} [D_1]_q$. Now, let $n \in \mathbb{N}$ and assume that we have chosen $U_1, U_2, \ldots, U_n$ and $D_1, D_2, \ldots, D_n$.

For each $k \in \{1, 2, \ldots, n\}$, if $a_k \in U_n$, let $Z_k = G$. Otherwise, pick the first $l \in \{1, 2, \ldots, n\}$ such that $a_k \notin U_l$. If $l = 1$, pick by Lemma 6.4(b) some $Z_k \in [\varphi]$ such that $a_kZ_k \cap U_1 = \emptyset$. If $l > 1$, pick by Lemma 6.4(a) and (b) some $Z_k \in [\varphi]$ such that $a_kZ_k \cap U_l = \emptyset$ and $a_kZ_k \subseteq U_{l-1}$.

Let $V_{n+1} = U_n \cap W_{n+1} \cap \bigcap_{k=1}^n Z_k$. Then $V_{n+1} \in [\varphi]$ so pick $D_{n+1} \in \varphi$ such that $\bigcup_{q\in\overline{\varphi}}[D_{n+1}]_q \subseteq V_{n+1}$ and let $U_{n+1} = \bigcup_{q\in\overline{\varphi}}[D_{n+1}]_q$,

We claim that the sequence $\langle U_n\rangle_{n=1}^\infty$ is as required. Conclusions (1) and (2) are immediate. To verify (3) assume that $a_k \in U_l\backslash U_{l+1}$ and let $n = \max\{k, l+1\}$. Let Z_k be as chosen for a_k at stage $n+1$ in the induction. Then $a_kU_{n+1} \subseteq a_kZ_k \subseteq U_l\backslash U_{l+1}$.

Similarly, if $a_k \notin U_1$, then $a_kU_{k+1} \cap U_1 \subseteq a_kZ_k \cap U_1 = \emptyset$. $\square$

The following lemma summarizes our preliminary information.

6.7 Lemma. *Assume that for each $a \in G\backslash\{e\}$, $a\overline{\varphi} \neq \overline{\varphi}$. Then there exist a subset X of G, a function $f : X \longrightarrow \mathbb{Z}_\alpha$, a sequence $\langle U_n\rangle_{n=1}^\infty$, and a family $\mathcal{A}$ such that:*

(1) $\{U_n : n \in \mathbb{N}\} \subseteq \bigcap_{i=0}^{\alpha-1} p_i = \varphi$.

(2) For all $n \in \mathbb{N}$, $U_{n+1} \subseteq U_n \subseteq X$.

(3) For all $i, j \in \mathbb{Z}_\alpha$, $p_i \cdot p_j = p_{i+j}$ (where the addition is in $\mathbb{Z}_\alpha$).

(4) $\mathcal{A}$ is the algebra of subsets of X generated by $\{X \cap (x \cdot U_n) : x \in X$ and $n \in \mathbb{N}\}$.

(5) For each $A \in \mathcal{A}$ and each $x \in A$ there is some $k \in \mathbb{N}$ such that $x \cdot U_k \subseteq A$.

(6) $\bigcap_{n=1}^\infty U_n = \{e\}$.

(7) For each $x \in X$ there is some $n \in \mathbb{N}$ such that for every $y \in U_n$, $xy \in X$ and $f(xy) = f(x) + f(y)$.

(8) For each $i \in \mathbb{Z}_\alpha$, $f^{-1}[\{i\}] \in p_i$.

Proof. Let $\langle A_p \rangle_{p \in \overline{\varphi}}$ be as guaranteed by Lemma 6.5 and, as there, let $X = \bigcup_{p \in \overline{\varphi}} A_p$ and define $f : X \longrightarrow \mathbb{Z}_\alpha$ by $f[A_{p_i}] = \{i\}$. For each $a \in X$ choose some $V(a) \in [\varphi]$ such that for all $b \in V$, $ab \in X$ and $f(ab) = f(a) + f(b)$. For $a \in G \backslash X$, let $V(a) = G$.

Enumerate G as $\{a_n : n \in \mathbb{N}\}$ with $a_1 = e$. For each $n > 1$ we have by hypothesis that $a_n \overline{\varphi} \neq \overline{\varphi}$ so $a_n \notin \overline{[\varphi]}$ so the principal ultrafilter generated by a_n does not contain $[\varphi]$. That is, there is some $W \in [\varphi]$ such that $a_n \notin W$ and consequently $G \backslash \{a_n\} \in [\varphi]$.

Now let $W_1 = V(e)$ and for $n \in \mathbb{N} \backslash \{1\}$, let

$$W_n = V(e) \cap \bigcap_{k=2}^{n} \big((G \backslash \{a_k\} \cap V(a_k)\big) \ .$$

Then each $W_n \in [\varphi]$.

Let $\langle U_n \rangle_{n=1}^{\infty}$ be as guaranteed by Lemma 6.6 for $\langle W_n \rangle_{n=1}^{\infty}$ and let $\mathcal{A}$ be the algebra of subsets of X generated by $\{X \cap (x \cdot U_n) : x \in X \text{ and } n \in \mathbb{N}\}$. We verify that each of the conclusions hold, conclusion (5) being the only one that requires any effort.

Conclusion (1) holds because each $U_n \in [\varphi] \subseteq \varphi$. By Lemma 6.5, $e \in X$ and $eV(e) \subseteq X$ so $U_1 \subseteq X$ and thus (2) holds. Statement (3) holds by assumption and $\mathcal{A}$ was defined to satisfy statment (4). We know that $e \in \overline{[\varphi]}$ so $e \in \bigcap_{n=1}^{\infty} U_n$ and given $n > 1$, $a_n \notin U_n$ so conclusion (6) holds. To verify conclusion (7), let $x \in X$ and pick $n \in \mathbb{N}$ such that $x = a_n$. Then $U_n \subseteq W_n \subseteq V(a_n)$ so the conclusion holds by the choice of $V(a_n)$. For conclusion (8), one has for each $i \in \mathbb{Z}_\alpha$ that $f^{-1}[\{i\}] = A_{p_i} \in p_i$.

Finally, we verify conclusion (5). So let $A \in \mathcal{A}$ and let $x \in A$. Note that $x \in X$. Choose finite subsets L and M of X and functions $n : L \longrightarrow \mathbb{N}$ and $m : M \longrightarrow \mathbb{N}$ such that

$$x \in X \cap (\textstyle\bigcap_{y \in L} \ yU_{n(y)}) \backslash (\textstyle\bigcup_{y \in M} \ yU_{m(y)}) \subseteq A \ ,$$

(where $\bigcap_{y \in \emptyset} \ yU_{n(y)} = X$).

Given any $y \in L \backslash M$, one has $x \in yU_{n(y)}$. If $x = y$, let $k(y) = r(y) = n(y)$. Otherwise, $y^{-1}x \in U_{n(y)} \backslash \{e\}$ so pick $r(y) \geq n(y)$ such that $y^{-1}x \in U_{r(y)} \backslash U_{r(y)+1}$ and pick by Lemma 6.6 some $k(y)$ such that $y^{-1}xU_{k(y)} \subseteq U_{r(y)} \backslash U_{r(y)+1}$.

Given any $y \in L \cap M$, one has that $x \in (yU_{n(y)} \backslash (yU_{m(y)})$ so pick some

$$r(y) \in \{n(y), n(y)+1, \ldots, m(y)-1\} \text{ such that } y^{-1}x \in U_{r(y)} \backslash U_{r(y)+1}$$

and pick by Lemma 6.6 some $k(y)$ such that $y^{-1}xU_{k(y)} \subseteq U_{r(y)} \backslash U_{r(y)+1}$.

Given any $y \in M \backslash L$, one has that $x \notin yU_{m(y)}$. If $x \notin yU_1$, then pick by Lemma 6.6 some $k(y)$ such that $y^{-1}xU_{k(y)} \cap U_1 = \emptyset$. Otherwise pick $r(y) \in \{1, 2, \ldots, m(y)-1\}$ such that $y^{-1}x \in U_{r(y)} \backslash U_{r(y)+1}$ and pick by Lemma 6.6 some $k(y)$ such that $y^{-1}xU_{k(y)} \subseteq U_{r(y)} \backslash U_{r(y)+1}$.

Pick $l \in \mathbb{N}$ such that $x = a_l$ and let $k = \max(\{k(y) : y \in M \cup L\} \cup \{l\})$. Then $xU_k \subseteq xU_l \subseteq xV(a_l) = xV(x) \subseteq X$.

Given $y \in L$, $y^{-1}xU_k \subseteq U_{r(y)} \subseteq U_{n(y)}$ so $xU_k \subseteq yU_{n(y)}$.

Given $y \in M$, $(y^{-1}xU_k) \cap U_{m(y)} \subseteq (y^{-1}xU_k) \cap U_{r(y)+1} = \emptyset$ so $xU_k \cap yU_{m(y)} = \emptyset$. □

We are nearly ready to present Zelenjuk's Theorem, so we temporarily drop our standing assumptions about G, φ, and so on, so that we can state the theorem.

6.8 Definition. (a) $B = \bigoplus_{n=1}^{\infty}\mathbb{Z}_2$.

(b) For $n \in \mathbb{N}$, $V_n = \{s \in B : \text{for each } k \in \{1,2,\ldots,n\},\ s_k = 0\}$.

(c) $\tilde{B} = \bigcap_{n=1}^{\infty} c\ell V_n$, where the closure is taken in βB and B has the discrete topology.

(d) F is the free semigroup on the letters 0 and 1 with identity ϵ. Given $s \in F$, $\ell(s)$ is the length of s.

(e) The function $\gamma : F \longrightarrow B$ is defined by

$$\gamma(s)_i = \begin{cases} s_i & \text{if } i \le \ell(s) \\ 0 & \text{if } i > \ell(s)\ . \end{cases}$$

(f) The function $g : B \longrightarrow \mathbb{Z}_\alpha$ is defined by $g(s) \equiv |\{i \in \mathbb{N} : s_i \neq 0\}| \pmod{\alpha}$.

(g) The function $k : B \longrightarrow \mathbb{Z}_{\alpha^2}$ is defined by $k(s) \equiv |\{i \in \mathbb{N} : s_i \neq 0\}| \pmod{\alpha^2}$.

(h) The function $\nu : \mathbb{Z}_{\alpha^2} \longrightarrow \mathbb{Z}_\alpha$ is defined by $\nu(i) \equiv i \pmod{\alpha}$.

(i) For $m \in \omega$ and $i \in \{0,1,\ldots,m\}$, s_i^m is the member of F consisting of i 0's followed by $m - i$ 1's.

(j) Given s and t in F, we define $s+t$ to be the member of F with $\ell(s+t) = \max\{\ell(s),\ell(t)\}$ and $(s+t)_i = 1$ if and only if either $s_i = 1$ or $t_i = 1$.

For example, note that $s_0^0 = \epsilon$, $s_0^3 = 111$, $s_3^3 = 000$, and $s_1^3 + s_5^7 = 0110011$. Given any $t \in F$, t has a unique representation in the form $t = s_{i_0}^{m_0} + s_{i_1}^{m_1} + \ldots + s_{i_k}^{m_k}$ where $0 \le i_0 < m_0 < i_1 < m_1 < \ldots < i_k \le m_k$ (except, if $k = 0$, the requirement is $0 \le i_0 \le m_0$). We will call this the *canonical representation* of t. When we write $t = s_{i_0}^{m_0} + s_{i_1}^{m_1} + \ldots + s_{i_k}^{m_k}$ we will assume that this is the canonical representation.

6.9 Definition. Given $t \in F$, if $t = s_i^m$ for some $i, m \in \omega$, then $t' = \epsilon$ and $t^* = t$. Otherwise, if $t = s_{i_0}^{m_0} + s_{i_1}^{m_1} + \ldots + s_{i_{k+1}}^{m_{k+1}}$, then $t' = s_{i_0}^{m_0} + s_{i_1}^{m_1} + \ldots + s_{i_k}^{m_k}$ and $t^* = s_{i_{k+1}}^{m_{k+1}}$.

6.10 Theorem. *Let G be a countable group, let $\overline{\varphi}$ be a finite subgroup of $\beta G\backslash G$ and assume that for all $a \in G\backslash\{e\}$, $a\overline{\varphi} \neq \overline{\varphi}$. Then $\overline{\varphi}$ is trivial.*

Proof. Suppose that $\overline{\varphi}$ is not trivial. We may presume that $\overline{\varphi}$ is isomorphic to $\mathbb{Z}_\alpha$ for some $\alpha \in \mathbb{N}\backslash\{1\}$. Choose a subset X of G, a function $f : X \longrightarrow \mathbb{Z}_\alpha$, a sequence $\langle U_n\rangle_{n=1}^{\infty}$, elements $p_0, p_1, \ldots, p_{\alpha-1} \in G^*$, and a family $\mathcal{A}$ as guaranteed by Lemma 6.7. Define

$$\tilde{X} = \bigcap_{n=1}^{\infty} c\ell U_n$$

where the closure is taken in βG and G is discrete.

We observe that $\tilde{X}$ is a subsemigroup of βG. To see this, let $p, q \in \tilde{X}$ and let $n \in \mathbb{N}$. We show that $U_1 \subseteq [U_n]_q$, and hence that $U_n \in pq$. We have that $e \in X$ by conclusions (2) and (6) of Lemma 6.7, and consequently $U_n \in \mathcal{A}$. Thus, given $x \in U_1$ one has by conclusion (5) of Lemma 6.7 some $k \in \mathbb{N}$ such that $x \cdot U_k \subseteq U_n$ and hence $x^{-1}U_n \in q$.

Similarly $\tilde{B}$ is a subsemigroup of βB.

Next we observe that if $A \in \mathcal{A}$ and $A \neq \emptyset$, then for each $i \in \mathbb{Z}_\alpha$, $A \cap f^{-1}[\{i\}] \neq \emptyset$. To see this pick $a \in A$. Then by conclusion (5) of Lemma 6.7, pick $m \in \mathbb{N}$ such that $a \cdot U_m \subseteq A$. By conclusion (7) of Lemma 6.7 pick $r \in \mathbb{N}$ such that for all $y \in U_r$, $ay \in X$ and $f(ay) = f(a) + f(y)$. Let $j = i - f(a)$ (in $\mathbb{Z}_\alpha$). By conclusion (8) of Lemma 6.7, $f^{-1}[\{j\}] \in p_j$ and by conclusion (1), $U_m \cap U_r \in p_j$, so pick $y \in U_m \cap U_r \cap f^{-1}[\{j\}]$. Then $y \in U_m$ so $ay \in A$ and $y \in U_r$ so $f(ay) = f(a) + j = i$.

Enumerate X, placing e first. (When we refer to the first element of a subset of X, that reference is with respect to this order.) We define inductively on $\ell(s)$ for $s \in F$, $x(s)$ and $X(s)$ satisfying the following induction hypotheses for $n \in \mathbb{N}$.

0) If $s \in F$ and $\ell(s) = n$, then $x(s) \in X(s)$ and $X(s) \in \mathcal{A}$.

1) If $s \in F$ and $\ell(s) = n-1$, then
   (a) $X(s^\frown 0) \cap X(s^\frown 1) = \emptyset$,
   (b) $X(s^\frown 0) \cup X(s^\frown 1) = X(s)$, and
   (c) $x(s^\frown 0) = x(s)$.
2) (a) $X(s_n^n) \subseteq U_n$ and
   (b) for all $z \in X(s_n^n)$ and all $j \in \{0, 1, \ldots, n-1\}$, $x(s_j^{n-1}) \cdot z \in X$ and $f(x(s_j^{n-1}) \cdot z) = f\big(x(s_j^{n-1})\big) + f(z)$.
3) If $s \in F$ and $\ell(s) = n$, then
   (a) $X(s) = x(s') \cdot X(s^*)$ and
   (b) $x(s) = x(s') \cdot x(s^*)$.
4) For all $j \in \{0, 1, \ldots, n\}$, $f\big(x(s_j^n)\big) = g\big(\gamma(s_j^n)\big)$.
5) Let $a = \min(X \backslash \{x(s) : \ell(s) < n\})$. If $t \in F$, $\ell(t) = n-1$, and $a \in X(t)$, then $a \in X(t^\frown 1)$ and, if $f(a) = g\big(\gamma(t)\big) + 1$, then $a = x(t^\frown 1)$.

We begin by letting $x(\epsilon) = e$ and $X(\epsilon) = X$.

Now let $n = 1$. Let $a = \min(X \backslash \{e\})$. By conclusion (6) of Lemma 6.7, pick k such that $a \notin U_k$. If $f(a) = 1$, let $x(1) = a$. If $f(a) \neq 1$, then choose any $x(1) \in (X \backslash U_k) \cap f^{-1}[\{1\}]$. (We have already observed that $(X \backslash U_k) \cap f^{-1}[\{1\}] \neq \emptyset$.) Let $x(0) = e$, $X(0) = U_k$, and $X(1) = X \backslash U_k$. All induction hypotheses can be immediately checked.

Now let $n \in \mathbb{N}$ and assume we have chosen $x(s)$ and $X(s)$ satisfying the induction hypotheses for all $s \in F$ with $\ell(s) \leq n$.

We first establish 3 observations.

Observation 1. *If $t \in F$, the canonical representation of t is $t = s_{i_0}^{m_0} + s_{i_1}^{m_1} + \ldots + s_{i_{k+1}}^{m_{k+1}}$, and $m_{k+1} \leq n$, then $x(t) = x(s_{i_0}^{m_0}) \cdot x(s_{i_1}^{m_1}) \cdot \ldots \cdot x(s_{i_{k+1}}^{m_{k+1}})$.*

This observation follows from repeated applications of induction hypothesis 3)(b).

Observation 2. *If $t \in F$, the canonical representation of t is $t = s_{i_0}^{m_0} + s_{i_1}^{m_1} + \ldots + s_{i_{k+1}}^{m_{k+1}}$, $m_{k+1} \leq n$, and $c \in X(t^*)$, then $x(t') \cdot c \in X(s_{i_0}^{i_0})$.*

We establish Observation 2 by induction on k. If $k = 0$, we have

$$\begin{aligned} x(t') \cdot c &= x(s_{i_0}^{m_0}) \cdot c \\ &\in x(s_{i_0}^{m_0}) \cdot X(s_{i_1}^{m_1}) \\ &= X(s_{i_0}^{m_0} + s_{i_1}^{m_1}) && \text{by hypothesis 3)(a)} \\ &\subseteq X(s_{i_0}^{i_0}) && \text{by hypothesis 1)(b).} \end{aligned}$$

Now assume the observation is valid for k and let $t = s_{i_0}^{m_0} + s_{i_1}^{m_1} + \ldots + s_{i_{k+2}}^{m_{k+2}}$. Let $r = s_{i_1}^{m_1} + s_{i_2}^{m_2} + \ldots + s_{i_{k+2}}^{m_{k+2}}$. Then $r^* = t^*$ and

$$\begin{aligned} x(t') \cdot c &= x(s_{i_0}^{m_0} + s_{i_1}^{m_1} + \ldots + s_{i_{k+1}}^{m_{k+1}}) \cdot c \\ &= x(s_{i_0}^{m_0}) \cdot x(s_{i_1}^{m_1}) \cdot \ldots \cdot x(s_{i_{k+1}}^{m_{k+1}}) \cdot c && \text{by Observation 1} \\ &= x(s_{i_0}^{m_0}) \cdot \big(x(s_{i_1}^{m_1} + s_{i_2}^{m_2} + \ldots + s_{i_{k+1}}^{m_{k+1}})\big) \cdot c && \text{by Observation 1} \\ &= x(s_{i_0}^{m_0}) \cdot (x(r') \cdot c) \\ &\in x(s_{i_0}^{m_0}) \cdot X(s_{i_1}^{i_1}) && \text{by the induction hypothesis} \\ &\subseteq x(s_{i_0}^{m_0}) \cdot X(s_{m_0+1}^{m_0+1}) && \text{by hypothesis 1)(b)} \\ &= X(s_{i_0}^{m_0} + s_{m_0+1}^{m_0+1}) && \text{by hypothesis 3)(a)} \\ &\subseteq X(s_{i_0}^{i_0}) && \text{by hypothesis 1)(b).} \end{aligned}$$

Observation 3. *If $t \in F$ and $\ell(t) \le n$ then*
(a) $f\big(x(t)\big) = g\big(\gamma(t)\big)$ and
(b) for all $c \in X(t^*)$, $f(x(t') \cdot c) = g\big(\gamma(t')\big) + f(c)$.

If $t = s_i^m$ for some i and m, conclusion (a) is hypothesis 4) and conclusion b) is trivial. So assume the canonical representation of t is $t = s_{i_0}^{m_0} + s_{i_1}^{m_1} + \ldots + s_{i_{k+1}}^{m_{k+1}}$. We establish Observation 3 by induction on k.

Assume first that $k = 0$. Note that, since $x(s_{i_1}^{m_1}) \in X(s_{i_1}^{m_1}) \subseteq X(s_{m_0+1}^{m_0+1})$ we have by hypothesis 2)(b) that $f\big(x(s_{i_0}^{m_0}) \cdot x(s_{i_1}^{m_1})\big) = f\big(x(s_{i_0}^{m_0})\big) + f\big(x(s_{i_1}^{m_1})\big)$. Thus

$$\begin{aligned} f\big(x(t)\big) &= f\big(x(s_{i_0}^{m_0} + s_{i_1}^{m_1})\big) \\ &= f\big(x(s_{i_0}^{m_0}) \cdot x(s_{i_1}^{m_1})\big) && \text{by hypothesis 3)(b)} \\ &= f\big(x(s_{i_0}^{m_0})\big) + f\big(x(s_{i_1}^{m_1})\big) && \text{as noted above} \\ &= g\big(\gamma(s_{i_0}^{m_0})\big) + g\big(\gamma(s_{i_1}^{m_1})\big) && \text{by hypothesis 4)} \\ &\equiv m_0 - i_0 + m_1 - i_1 \ (\mod \alpha) \\ &= g\big(\gamma(s_{i_0}^{m_0} + s_{i_1}^{m_1})\big) \\ &= g\big(\gamma(t)\big) \end{aligned}$$

so conclusion (a) holds.

To establish conclusion (b), let $c \in X(t^*)$. Then $c \in X(s_{i_1}^{m_1}) \subseteq X(s_{m_0+1}^{m_0+1})$ so

$$\begin{aligned} f(x(t') \cdot c) &= f\big((x(s_{i_0}^{m_0})\big) + f(c) && \text{by hypothesis 2)(b)} \\ &= g\big(\gamma(s_{i_0}^{m_0})\big) + f(c) && \text{by hypothesis 4)} \\ &= g\big(\gamma(t')\big) + f(c)\ . \end{aligned}$$

Now assume that $k > 0$ and the observation is valid for $k - 1$. We verify conclusion (b) first. By Observation 1

$$\begin{aligned} x(t') &= x(s_{i_0}^{m_0}) \cdot x(s_{i_1}^{m_1}) \cdot \ldots \cdot x(s_{i_k}^{m_k}) \\ &= x(s_{i_0}^{m_0}) \cdot x(s_{i_1}^{m_1} + s_{i_2}^{m_2} + \ldots + s_{i_k}^{m_k})\ . \end{aligned}$$

Let $r = s_{i_1}^{m_1} + s_{i_2}^{m_2} + \ldots + s_{i_{k+1}}^{m_{k+1}}$. Then $r^* = t^*$ and $x(t') = x(s_{i_0}^{m_0}) \cdot x(r')$. By Observation 2, $x(r') \cdot c \in X(s_{i_1}^{i_1}) \subseteq X(s_{m_0+1}^{m_0+1})$ so

$$\begin{aligned} f(x(t') \cdot c) &= f(x(s_{i_0}^{m_0}) \cdot x(r') \cdot c) \\ &= f\big(x(s_{i_0}^{m_0})\big) + f(x(r') \cdot c) && \text{by hypothesis 2)(b)} \\ &= f\big(x(s_{i_0}^{m_0})\big) + g\big(\gamma(r')\big) + f(c) && \text{by the induction hypothesis} \\ &= g\big(\gamma(s_{i_0}^{m_0})\big) + g\big(\gamma(r')\big) + f(c) \\ &= g\big(\gamma(t')\big) + f(c)\ . \end{aligned}$$

To verify (a), note that $x(t^*) \in X(t^*)$ so

$$\begin{aligned} f\big(x(t)\big) &= f\big(x(t') \cdot x(t^*)\big) && \text{by Observation 1} \\ &= g\big(\gamma(t')\big) + f\big(x(t^*)\big) && \text{by conclusion (b)} \\ &= g\big(\gamma(t')\big) + g\big(\gamma(t^*)\big) && \text{by hypothesis 4)} \\ &= g\big(\gamma(t)\big)\ . \end{aligned}$$

The observations now being established, we proceed with the induction. By conclusion (7) of Lemma 6.7(applied $n+1$ times), pick $k(1) \in \mathbb{N}$ such that, for each $j \in \{0, 1, \ldots, n\}$ and each $z \in U_{k(1)}$, $x(s_j^n) \cdot z \in X$ and $f(x(s_j^n) \cdot z) = f\big(x(s_j^n)\big) + f(z)$.

Let $a = \min(X\backslash\{x(s) : s \in F \text{ and } \ell(s) \le n\})$ and, by hypothesis 1), pick $t \in F$ with $\ell(t) = n$ such that $a \in X(t)$. Since $\ell(t) = n$, pick $j_0 \in \{0, 1, \dots, n\}$ such that $t^* = s^n_{j_0}$.

Now $a \in X(t) = x(t') \cdot X(t^*)$ so pick $c \in X(t^*)$ such that $a = x(t') \cdot c$. Notice that $c \ne x(t^*)$ $\big($for then one would have $a = x(t') \cdot x(t^*) = x(t)\big)$. Choose by conclusion (6) of Lemma 6.7, some $k(2) \in \mathbb{N}$ such that $x(t^*)^{-1} \cdot c \notin U_{k(2)}$.

We observed early in the proof that if $A \in \mathcal{A}$ and $A \ne \emptyset$, then for each $i \in \mathbb{Z}_\alpha$, $A \cap f^{-1}[\{i\}] \ne \emptyset$. Using this fact, for each $j \in \{0, 1, \dots, n\}$ pick some

$$x(s^{n+1}_j) \in X(s^n_j) \cap f^{-1}[\{n - j + 1\}]\ .$$

If $f(c) = n - j_0 + 1$, then we demand that $x(s^{n+1}_{j_0}) = c$, which we can do because $c \in X(t^*) = X(s^n_{j_0})$.

Notice that for each $j \in \{0, 1, \dots, n\}$, $x(s^{n+1}_j) \ne x(s^n_j)$ because $f\big(x(s^n_j)\big) = g\big(\gamma(s^n_j)\big) = n - j$. So pick $k(3) \in \mathbb{N}$ such that for each $j \in \{0, 1, \dots, n\}$ $x(s^n_j)^{-1} \cdot x(s^{n+1}_j) \notin U_{k(3)}$.

Pick by conclusion (5) of Lemma 6.7 and hypothesis 0), some $k(4) \in \mathbb{N}$ such that for each $s \in F$ with $\ell(s) = n$, $x(s) \cdot U_{k(4)} \subseteq X(s)$.

Let $k = \max\{k(1), k(2), k(3), k(4), n\}$.

Define $x(s^{n+1}_{n+1}) = e$. For any $s \in F$ with $\ell(s) = n + 1$ such that s is not of the form s^{n+1}_j for any $j \in \{0, 1, \dots, n + 1\}$, define $x(s) = x(s') \cdot x(s^*)$.

Let $X(s^{n+1}_{n+1}) = U_k$. For $j \in \{0, 1, \dots, n\}$, let $X(s^{n+1}_j) = X(s^n_j)\backslash(x(s^n_j) \cdot U_k)$. For any $s \in F$ with $\ell(s) = n + 1$ such that $s \notin \{s^{n+1}_0, s^{n+1}_1, \dots, s^{n+1}_{n+1}\}$ define $X(s) = x(s') \cdot X(s^*)$.

We need to verify that each of hypotheses 0) – 5) holds.

To verify hypothesis 0), let $s \in F$ with $\ell(s) = n + 1$. If $s = s^{n+1}_{n+1}$, then $x(s) = e \in U_k = X(s)$ and $U_k \in \mathcal{A}$. If $j \in \{0, 1, \dots, n\}$ and $s = s^{n+1}_j$, then since $k \ge k(3)$, $x(s) \notin x(s^n_j) \cdot U_k$ so $x(s) \in X(s)$. Also $X(s^{n+1}_j) = X(s^n_j)\backslash(x(s^n_j) \cdot U_k) \in \mathcal{A}$. If s is not of the form s^{n+1}_j, then $x(s) = x(s') \cdot x(s^*) \in x(s') \cdot X(s^*) = X(s)$.

To complete the verification of hypothesis 0), assume that s is not of the form s^{n+1}_j. If $s^* = s^{n+1}_{n+1}$, then $X(s) = x(s') \cdot X(s^{n+1}_{n+1}) = x(s') \cdot U_k \in \mathcal{A}$. So assume that $s^* = s^{n+1}_j$ for some $j \in \{0, 1, \dots, n\}$. Let $r = s' + s^n_j$. Then $r^* = s^n_j$ and $r' = s'$. Consequently

$$\begin{aligned} X(s) &= x(s') \cdot X(s^{n+1}_j) \\ &= x(s') \cdot \big(X(s^n_j)\backslash(x(s^n_j) \cdot U_k)\big) \\ &= \big(x(s') \cdot X(s^n_j)\big)\backslash(x(s') \cdot x(s^n_j) \cdot U_k) \\ &= X(r)\backslash(x(r) \cdot U_k) \\ &\in \mathcal{A}\ . \end{aligned}$$

We verify hypothesis 1)(a) and 1)(b) together. Let $s \in F$ with $\ell(s) = n$. Assume first that $s = s^n_n$. Then $s^\frown 0 = s^{n+1}_{n+1}$ and $s^\frown 1 = s^{n+1}_n$. Also $X(s^{n+1}_{n+1}) = U_k$ and $U_k \subseteq X(s^n_n)$ because $k \ge k(4)$. And $X(s^{n+1}_n) = X(s^n_n)\backslash(x(s^n_n) \cdot U_k) = X(s^n_n)\backslash U_k$ because $x(s^n_n) = e$.

Next assume that $s = s^n_j$ for some $j \in \{0, 1, \dots, n-1\}$. Then $s^\frown 1 = s^{n+1}_j$ and $X(s^{n+1}_j) = X(s^n_j)\backslash(x(s^n_j) \cdot U_k)$. Also $(s^\frown 0)' = s^n_j$ and $(s^\frown 0)^* = s^{n+1}_{n+1}$ so $X(s^\frown 0) = x(s^n_j) \cdot X(s^{n+1}_{n+1}) = x(s^n_j) \cdot U_k$ and $x(s^n_j) \cdot U_k \subseteq X(s^n_j)$ because $k \ge k(4)$.

Now assume the rightmost letter of s is 0, but $s \ne s^n_n$ (in which case $s^* = s^n_n$). Then $(s^\frown 0)' = s'$ and $(s^\frown 0)^* = s^{n+1}_{n+1}$ so $X(s^\frown 0) = x(s') \cdot X(s^{n+1}_{n+1}) = x(s') \cdot U_k$. Also $U_k =$

$X(s^{n+1}_{n+1}) \subseteq X(s^n_n)$ so $x(s')\cdot U_k \subseteq x(s')\cdot X(s^n_n) = X(s)$. Now $(s^\frown 1)' = s'$ and $(s^\frown 1)^* = s^{n+1}_n$ so that

$$\begin{aligned} X(s^\frown 1) &= x(s')\cdot X(s^{n+1}_n) \\ &= x(s')\cdot (X(s^n_n)\backslash U_k) \\ &= \big(x(s')\cdot X(s^*)\big)\backslash (x(s')\cdot U_k) \\ &= X(s)\backslash X(s^\frown 0)\ . \end{aligned}$$

Finally assume the rightmost letter of s is 1, but $s \neq s^n_j$ for any $j \in \{0,1,\ldots,n-1\}$. Pick $j \in \{1,2,\ldots,n-1\}$ such that $s^* = s^n_j$. Then $(s^\frown 0)' = s$, $(s^\frown 0)^* = s^{n+1}_{n+1}$, $(s^\frown 1)' = s'$, and $(s^\frown 1)^* = s^{n+1}_j$. Thus $X(s^\frown 0) = x(s)\cdot X(s^{n+1}_{n+1}) = x(s)\cdot U_k \subseteq X(s)$ because $k \geq k(4)$. And

$$\begin{aligned} X(s^\frown 1) &= x(s')\cdot X(s^{n+1}_j) \\ &= x(s')\cdot \big(X(s^n_j)\backslash (x(s^n_j)\cdot U_k)\big) \\ &= \big(x(s')\cdot X(s^n_j)\big)\backslash (x(s')\cdot x(s^n_j)\cdot U_k) \\ &= X(s)\backslash (x(s)\cdot U_k) \\ &= X(s)\backslash X(s^\frown 0)\ . \end{aligned}$$

To verify 1)(c), let $s \in F$ with $\ell(s) = n$. If $s = s^n_n$, then $s^\frown 0 = s^{n+1}_{n+1}$ and $x(s^{n+1}_{n+1}) = e = x(s^n_n)$. For any other s, $(s^\frown 0)^* = s^{n+1}_{n+1}$. If the rightmost letter of s is 1, then $(s^\frown 0)' = s$ and $x(s^\frown 0) = x(s)\cdot x(s^{n+1}_{n+1}) = x(s)$. If the rightmost letter of s is 0, then $(s^\frown 0)' = s'$ so $x(s^\frown 0) = x(s')\cdot x(s^{n+1}_{n+1}) = x(s')$ and $x(s') = x(s)$ by induction because $s = s'^\frown 00\ldots 0$.

Hypothesis 2)(a) holds because $k \geq n$ and hypothesis 2)(b) holds because $k \geq k(1)$.

Hypothesis 3) holds by definition unless $s = s^{n+1}_j$ for some $j \in \{0,1,\ldots,n+1\}$, in which case $s' = \epsilon$ and $s^* = s$ so that the conclusion is trivial.

To verify hypothesis 4), note that $f\big(x(s^{n+1}_{n+1})\big) = f(e) = 0 = g\big(\gamma(s^{n+1}_{n+1})\big)$. (We know $f(e) = 0$ because of conclusion (7) of Lemma 6.7.) If $j \in \{0,1,\ldots,n\}$, then $f\big(x(s^{n+1}_j)\big) = n - j + 1 = g\big(\gamma(s^{n+1}_j)\big)$.

To verify hypothesis 5), recall that we chose $a = \min(X\backslash\{x(s) : s \in F \text{ and } \ell(s) \leq n\})$ and picked $t \in F$ with $\ell(t) = n$ such that $a \in X(t)$. We also picked $c \in X(t^*) = X(s^n_{j_0})$ such that $a = x(t')\cdot c$.

Since $k \geq k(2)$ we have $c \notin x(s^n_{j_0})\cdot U_k$ so that $c \in X(s^{n+1}_{j_0})$ and consequently $a \in x(t')\cdot X(s^{n+1}_{j_0})$. Now, if $t = s^n_{j_0}$, then $t' = \epsilon$ and $t^\frown 1 = s^{n+1}_{j_0}$ so $a = c \in X(t^\frown 1)$. If $f(a) = g\big(\gamma(t)\big) + 1 = n - j_0 + 1$, then $f(c) = n - j_0 + 1$ so $x(t^\frown 1) = x(s^{n+1}_{j_0}) = c = a$.

Now assume $t \neq s^n_{j_0}$. Then $(t^\frown 1)' = t'$ and $(t^\frown 1)^* = s^{n+1}_{j_0}$. Then $a = x(t')\cdot c \in x(t')\cdot X(s^{n+1}_{j_0}) = X(t^\frown 1)$. Assume $f(a) = g\big(\gamma(t)\big) + 1$. By Observation 3(b) we have that $f(a) = g\big(\gamma(t')\big) + f(c)$ so $f(c) = g\big(\gamma(t)\big) + 1 - g\big(\gamma(t')\big)$. Also

$$g\big(\gamma(t)\big) = g\big(\gamma(t')\big) + g\big(\gamma(t^*)\big) = g\big(\gamma(t')\big) + n - j_0$$

so $f(c) = n - j_0 + 1$. Consequently, $x(s^{n+1}_{j_0}) = c$ so that $a = x(t')\cdot x(s^{n+1}_{j_0}) = x(t^\frown 1)$.

We have thus completed the construction of $x(s)$ and $X(s)$ satisfying hypotheses 0) through 5).

Notice that, if s and t are in F and $x(s) = x(t)$, then $\gamma(s) = \gamma(t)$. To see this, suppose instead that $\gamma(s) \neq \gamma(t)$. Then we may assume without loss of generality that there are u, v, and w in F such that $s = u^\frown 1^\frown v$ and $t = u^\frown 0^\frown w$. Then $x(s) \in X(s) \subseteq X(u^\frown 1)$ and $x(t) \in X(t) \subseteq X(u^\frown 0)$ so $x(s) \neq x(t)$, a contradiction.

By hypothesis 5), for each $a \in X$ there is some $s \in F$ such that $a = x(s)$. Consequently, we may define

$$h : G \longrightarrow B \text{ by } h(a) = \begin{cases} \gamma(s) & \text{if } a = x(s) \in X \\ 0 & \text{if } a \in G\backslash X\ . \end{cases}$$

By Observation 3(a), we have that for each $s \in F$, $g\big(h(x(s))\big) = g(\gamma(s)) = f(x(s))$. That is, $g \circ h = f$. Consequently $g^\beta \circ h^\beta_{|\tilde{X}} = f^\beta_{|\tilde{X}}$ since these are continuous functions from βG to $\mathbb{Z}_\alpha$ which agree on X. Let $\tilde{h} = h^\beta_{|\tilde{X}}$, $\tilde{f} = f^\beta_{|\tilde{X}}$, $\tilde{k} = k^\beta_{|\tilde{B}}$, and $\tilde{g} = g^\beta_{|\tilde{B}}$. Once we have shown that $h^\beta[\tilde{X}] \subseteq \tilde{B}$, we will know that $\tilde{g} \circ \tilde{h} = \tilde{f}$.

To see that $h^\beta[\tilde{X}] \subseteq \tilde{B}$, let $p \in \tilde{X}$. To see that $h^\beta(p) \in \tilde{B}$, let $n \in \mathbb{N}$. We show that $V_n \in h^\beta(p)$. By conclusion (5) of Lemma 6.7, pick $k \in \mathbb{N}$ such that $U_k \subseteq X(s_n^n)$. Then $U_k \in p$. We claim that $h[U_k] \subseteq V_n$. So let $a \in U_k$. If $a = e$, then $h(a) = 0 \in V_n$ so assume that $a \neq e$ and pick $t \in F$ such that $a = x(t)$. Then $x(t) \in X(t) \cap X(s_n^n)$ so $t = s_n^n{}^\frown u$ for some $u \in F$. Thus $\gamma(t) \in V_n$. Since $\gamma(t) = h(a)$ we have $h(a) \in V_n$ as desired.

Now we show that $\tilde{h}$ is a homomorphism. Let $p, q \in \tilde{X}$. To see that $\tilde{h}(pq) = \tilde{h}(p) + \tilde{h}(q)$, that is, that $h^\beta(pq) = h^\beta(p) + h^\beta(q)$, it suffices to show that $h^\beta \circ \rho_q$ and $\rho_{h^\beta(q)} \circ h^\beta$ agree on X (and therefore agree at p). To see this, let $a \in X$. Then $(h^\beta \circ \rho_q)(a) = h^\beta(aq) = (h^\beta \circ \lambda_a)(q)$ and $(\rho_{h^\beta(q)} \circ h^\beta)(a) = h(a) + h^\beta(q) = (\lambda_{h(a)} \circ h^\beta)(q)$ so it suffices to show that $h^\beta \circ \lambda_a$ and $\lambda_{h(a)} \circ h^\beta$ agree on some member of q. Pick $t \in F$ such that $a = x(t)$ and let $n = \ell(t)$. Now $V_{n+1} \in h^\beta(q)$ so $h^{-1}[V_{n+1}] \in q$. Let $b \in h^{-1}[V_{n+1}]$ and pick $r \in F$ such that $b = x(r)$. Then $h(b) = \gamma(r) \in V_{n+1}$. Thus, considering the canonical representations of t and r we see that $\gamma(t+r) = \gamma(t) + \gamma(r)$ and, using Observation 1, $x(t+r) = x(t) \cdot x(r)$. Consequently

$$\begin{aligned}
(h^\beta \circ \lambda_a)(b) &= h(ab) \\
&= h\big(x(t) \cdot x(r)\big) \\
&= h\big(x(t+r)\big) \\
&= \gamma(t+r) \\
&= \gamma(t) + \gamma(r) \\
&= h\big(x(t)\big) + h\big(x(r)\big) \\
&= h(a) + h(b) \\
&= (\lambda_{h(a)} \circ h^\beta)(b) \ .
\end{aligned}$$

Since $\tilde{h}$ is a homomorphism and trivially $\tilde{g}$ is a homomorphism, we have that $\tilde{f}$ is a homomorphism. Define $\tau : \mathbb{Z}_\alpha \longrightarrow \tilde{X}$ by $\tau(i) = p_i$. (We know each $p_i \in \tilde{X}$ by conclusion (1) of Lemma 6.7.) By conclusion (3) of Lemma 6.7, τ is a homomorphism.

Thus we have established that the following diagram commutes and that each of the listed functions is a homomorphism.

$$\begin{tikzcd}
\tilde{X} \arrow[rr, "\tilde{h}"] \arrow[dr, shift left, "\tilde{f}"] & & \tilde{B} \arrow[dl, "\tilde{g}"'] \arrow[dr, "\tilde{k}"] & \\
& \mathbb{Z}_\alpha \arrow[ul, shift left, "\tau"] & & \mathbb{Z}_{\alpha^2} \arrow[ll, "\nu"']
\end{tikzcd}$$

Now consider the homomorphism $\tilde{k} \circ \tilde{h} \circ \tau$. Since the sum in $\mathbb{Z}_\alpha$ of 1 with itself α times is 0, the sum in $\mathbb{Z}_{\alpha^2}$ of $(\tilde{k} \circ \tilde{h} \circ \tau)(1)$ with itself α times is 0 so $(\tilde{k} \circ \tilde{h} \circ \tau)(1) = i\alpha$ for some $i \in \{0, 1, \ldots, \alpha - 1\}$ and consequently $(\nu \circ \tilde{k} \circ \tilde{h} \circ \tau)(1) = 0$. But

$$\begin{aligned}
(\nu \circ \tilde{k} \circ \tilde{h} \circ \tau)(1) &= (\tilde{g} \circ \tilde{h} \circ \tau)(1) \\
&= (\tilde{f} \circ \tau)(1) \\
&= f(p_1) \\
&= 1
\end{aligned}$$

by conclusion (8) of Lemma 6.7. This contradiction completes the proof. $\square$

As we have previously noted, in [67], Zelenjuk showed that in a countable commutative group with no nontrivial finite subgroups, one has for each finite subgroup $\overline{\varphi}$ of βG and each $a \in G$, $a\overline{\varphi} \neq \overline{\varphi}$ and hence by Theorem 6.10, $\overline{\varphi}$ must be a singleton. Dona Strauss observed in a personal communication that one does not need the commutativity assumption.

6.11 Lemma. *Let G be a group with no nontrivial finite subgroups and let C be a finite subgroup of βG. Then for all $a \in G\backslash\{e\}$, $aC \neq C$.*

Proof. Let $H = \{x \in G : xC = C\}$. Given $x \in H$, one has that $C = x^{-1}C$ so $x^{-1} \in H$. And, if $x, y \in H$, then $xyC = xC = C$. So H is a subgroup of G. We claim in fact that $H = \{e\}$. So suppose instead that $H \neq \{e\}$. Then, since G has no nontrivial finite subgroups, H must be infinite and consequently there must be some $x \neq y$ in H such that $x\epsilon = y\epsilon$, where ϵ is the identity of C. Let $a = y^{-1}x$. Then $a \in G\backslash\{e\}$ and $a\epsilon = \epsilon$.

Since $a \neq e$ we have for each $z \in G$ that $\lambda_a(z) \neq z$. Pick by [20, Lemma 9.1] sets A_0, A_1, and A_2 which partition G such that $A_i \cap \lambda_a[A_i] = \emptyset$ for each $i \in \{0,1,2\}$. Pick $i \in \{0,1,2\}$ such that $A_i \in \epsilon$. Then $a \cdot A_i \in a\epsilon = \epsilon$ so $A_i \cap (a \cdot A_i) \neq \emptyset$, a contradiction. □

Shortly after the above lemma was obtained, John Pym noted the following result (in a personal communication). It is somewhat stronger than Lemma 6.11 and its proof has the virtue of being self contained.

6.12 Lemma. *Let G be a group, let C be a finite subgroup of βG, and let $a \in G$ such that a generates an infinite subgroup of G. Then $aC \neq C$.*

Proof. Suppose instead that $aC = C$. Let $X = \{a^n : n \in \mathbb{Z}\}$ be the infinite subgroup of G generated by a and let A be a set consisting of one element of each right coset Xg of X. Then

$$\text{(1)} \qquad \text{for } m \neq n \text{ in } \mathbb{Z},\ a^n A \cap a^m A = \emptyset$$

and

$$\text{(2)} \qquad G = \textstyle\bigcup_{n\in\mathbb{Z}}\ a^n A\ .$$

Thus we may define $\tau : G \longrightarrow \mathbb{Z}$ by $g \in a^{\tau(g)}A$.

Let $m = |C|+1$ and define, for $i \in \{0,1,\ldots,m-1\}$, $D_i = \{g \in G : \tau(g) \equiv i \pmod{m}\}$. Then $\{\overline{D_i} : i \in \{0,1,\ldots,m-1\}\}$ partitions βG into m cells. Further, given any $p \in \beta G$, if $p \in \overline{D_i}$, the $ap \in \overline{D_{i+1}}$ (where the addition is in $\mathbb{Z}_m$).

Choose any $p \in C$. One thus has that $p, ap, a^2p, \ldots, a^{m-1}p$ are m distinct members of C, a contradiction. □

6.13 Corollary. *Let G be a countable group which has no nontrivial finite subgroups. Then βG has no nontrivial finite subgroups.*

Proof. Suppose that βG has a nontrivial finite subgroup $\overline{\varphi}$. Then $\overline{\varphi}$ is not contained in G. By Theorem 4.15 we have that $\beta G\backslash G$ is an ideal of βG so one has that $\overline{\varphi} \subseteq \beta G\backslash G$. Thus by Theorem 6.10, it suffices to show that for each $a \in G\backslash\{e\}$, $a\overline{\varphi} \neq \overline{\varphi}$. This follows from Lemma 6.11 or Lemma 6.12. □

Note that the requirement that G have no nontrivial finite subgroups is needed in Corollary 6.13 in the event that G is commutative if one wants to conclude that $\beta G\backslash G$ has no nontrivial finite subgroups. Indeed, given any subgroup H of G and any idempotent $\epsilon \in \beta G\backslash G$ one has, since H is contained in the center of βG, that $H \cdot \epsilon$ is a subgroup of $\beta G\backslash G$ which is isomorphic to H. We do not know whether there can be a countable noncommutative group G which has a nontrivial finite subgroup, but $\beta G\backslash G$ does not.

7. Idempotents and Order

We have seen in Theorem 4.6 that idempotents in βS exist, and in fact that there is a minimal idempotent below any idempotent (Theorem 4.14). But for all we know, all idempotents of βS may be minimal. In fact this is possible. For example, let S be an infinite (discrete) left zero semigroup (that is $xy = x$ for all x and y in S). Then βS is also a left zero semigroup. (Given $p, q \in \beta S$, $pq = p\text{-}\lim_{x\in S} q\text{-}\lim_{y\in S} xy = p\text{-}\lim_{x\in S} q\text{-}\lim_{y\in S} x = p\text{-}\lim_{x\in S} x = p$.) Consequently, $K(\beta S) = \beta S$, and all members of βS are minimal idempotents.

We shall see now some reasonable conditions which allow us to conclude first that non-minimal idempotents exist and second that they are plentiful. The following notion is borrowed from topological dynamics. The term *piecewise syndetic* originated in the context of $(\mathbb{N}, +)$. In $(\mathbb{N}, +)$, a set A is piecewise syndetic if and only if there exist a fixed bound b and arbitrarily long intervals in which the gaps of A are bounded by b.

7.1 Definition. Let S be a semigroup and let $A \subseteq S$. Then A is *piecewise syndetic* if and only if there is some $H \in \mathcal{P}_f(S)$ such that for every $F \in \mathcal{P}_f(S)$ there is some $x \in S$ with $F \cdot x \subseteq \bigcup_{t\in H} t^{-1}A$.

We should really call the notion defined above *right piecewise syndetic.* If we were taking βS to be left topological we would have replaced "$F \cdot x \subseteq \bigcup_{t\in H} t^{-1}A$" in the definition of piecewise syndetic by "$x \cdot F \subseteq \bigcup_{t\in H} At^{-1}$", where $At^{-1} = \{y \in S : yt \in A\}$.

7.2 Lemma. *Let S be a semigroup and let A and B be subsets of S. If $A \cup B$ is piecewise syndetic, then either A or B is piecewise syndetic.*

Proof. Pick $H \in \mathcal{P}_f(S)$ such that for every $F \in \mathcal{P}_f(S)$ there is some $x \in S$ with $F \cdot x \subseteq \bigcup_{t\in H} t^{-1}(A \cup B)$. Suppose that neither A nor B is piecewise syndetic.

Since A is not piecewise syndetic and $H \in \mathcal{P}_f(S)$, pick $F \in \mathcal{P}_f(S)$ such that for every $x \in S$, $F \cdot x \backslash \bigcup_{t\in H} t^{-1}A \neq \emptyset$.

Since B is not piecewise syndetic and $HF \in \mathcal{P}_f(S)$, pick $K \in \mathcal{P}_f(S)$ such that for every $x \in S$, $K \cdot x \backslash \bigcup_{s\in HF} s^{-1}B \neq \emptyset$.

Now $FK \in \mathcal{P}_f(S)$ so pick some $x \in S$ such that $FK \cdot x \subseteq \bigcup_{t\in H} t^{-1}(A \cup B)$. Pick $z \in K$ such that $zx \notin \bigcup_{s\in HF} s^{-1}B$. Then $F \cdot zx \backslash \bigcup_{t\in H} t^{-1}A \neq \emptyset$ so pick $y \in F$ such that $yzx \notin \bigcup_{t\in H} t^{-1}A$.

Now $yz \in FK$ so pick $t \in H$ such that $yzx \in t^{-1}(A \cup B)$. Since $yzx \notin t^{-1}A$, we must have that $tyzx \in B$. But then, $ty \in HF$ so $zx \in \bigcup_{s\in HF} s^{-1}B$, a contradiction. □

7.3 Theorem. *Let S be a discrete semigroup and let $I = \{p \in \beta S :$ for all $A \in p$, A is piecewise syndetic$\}$. Then I is an ideal of βS.*

Proof. Let $\mathcal{A} = \{A \subseteq S : S\backslash A$ is not piecewise syndetic$\}$. Since S is piecewise syndetic, $\emptyset \notin \mathcal{A}$ and by Lemma 7.2, $\mathcal{A}$ is closed under finite intersections. Thus, by Theorem 2.4 there is some $p \in \beta S$ such that $\mathcal{A} \subseteq p$. Then $p \in I$ and consequently, $I \neq \emptyset$.

Now let $p \in I$ and $q \in \beta S$. We need to show that $qp \in I$ and $pq \in I$. To see that $qp \in I$, let $A \in qp$ so that, by Theorem 4.5, $[A]_p \in q$ and hence $[A]_p \neq \emptyset$. Pick some $a \in [A]_p$. Then $a^{-1}A \in p$ so pick some $H \in \mathcal{P}_f(S)$ such that for every $F \in \mathcal{P}_f(S)$ there is some $x \in S$ with $F \cdot x \subseteq \bigcup_{t\in H} t^{-1}(a^{-1}A)$. Then $aH \in \mathcal{P}_f(S)$ and for every $F \in \mathcal{P}_f(S)$ there is some $x \in S$ with $F \cdot x \subseteq \bigcup_{t\in aH} t^{-1}A$.

To see that $pq \in I$, let $A \in pq$ so that $[A]_q \in p$. Pick $H \in \mathcal{P}_f(S)$ such that for every $F \in \mathcal{P}_f(S)$ there is some $x \in S$ with $F \cdot x \subseteq \bigcup_{t\in H} t^{-1}[A]_q$. We claim that for every $F \in \mathcal{P}_f(S)$ there is some $x \in S$ with $F \cdot x \subseteq \bigcup_{t\in H} t^{-1}A$, so let $F \in \mathcal{P}_f(S)$ be given. Pick

$x \in S$ such that $F \cdot x \subseteq \bigcup_{t \in H} t^{-1}[A]_q$. For each $y \in F$ pick $t_y \in H$ such that $t_y yx \in [A]_q$, that is, $(t_y yx)^{-1}A \in q$. Pick $z \in \bigcap_{y \in F} (t_y yx)^{-1}A$. Then $F \cdot xz \subseteq \bigcup_{t \in H} t^{-1}A$. □

In fact, the set I in Theorem 7.3 is the closure of the smallest ideal of βS. (See Corollary 9.14.)

7.4 Theorem. *Let S be a semigroup and assume there is a sequence $\langle x_n \rangle_{n=1}^{\infty}$ in S such that $FP(\langle x_n \rangle_{n=1}^{\infty})$ is not piecewise syndetic. Then there exists a nonminimal idempotent in βS.*

Proof. By Theorem 5.5, there is an idempotent $p \in \beta S$ with $FP(\langle x_n \rangle_{n=1}^{\infty}) \in p$. By Theorem 7.3, I is an ideal of βS and hence $K(\beta S) \subseteq I$. Thus $p \notin K(\beta S)$. □

In particular, as a consequence of Theorem 7.4 we have that there are nonminimal idempotents in $(\beta\mathbb{N}, +)$. (For instance, it is easy to see that $FS(\langle 2^{2n} \rangle_{n=1}^{\infty})$ is not piecewise syndetic.) The following theorem and its corollaries, obtained in collaboration with D. Strauss, show that in fact nonminimal idempotents in $(\beta\mathbb{N}, +)$ are plentiful. (Here $\mathfrak{c}$ is the cardinality of the continuum.)

7.5 Theorem. *Let p be any nonminimal idempotent in $(\beta\mathbb{N}, +)$. Then there exist $2^{\mathfrak{c}}$ nonminimal idempotents each lying immediately below p with respect to the order $\leq$.*

Proof. [47, Theorem 3.1]. □

7.6 Corollary. *Let p be any nonminimal idempotent in $(\beta\mathbb{N}, +)$. Then there exists a sequence $\langle p_n \rangle_{n=1}^{\infty}$ of idempotents such that $p_1 = p$ and for each $n \in \mathbb{N}$, $p_{n+1} < p_n$ and p_{n+1} is maximal among all idempotents less than p_n.*

Proof. [47, Corollary 3.2]. □

7.7 Corollary. *Let $K = K(\beta\mathbb{N}, +)$. There are $2^{\mathfrak{c}}$ idempotents in $c\ell K \backslash K$.*

Proof. [47, Corollary 3.3]. □

Going up in the order $\leq$ is much more difficult. For instance, while it seems obvious to me that no minimal idempotent in $(\beta\mathbb{N}, +)$ can be maximal with respect to $\leq$, we cannot prove that statement.

Theorem 7.5 was generalized by A. Maleki and D. Strauss.

7.8 Theorem. *Let S be any discrete, countably infinite, commutative, and cancellative semigroup. Let p be any nonminimal idempotent in βS. Then there exist $2^{\mathfrak{c}}$ nonminimal idempotents each lying immediately below p with respect to the order $\leq$.*

Proof. [55, Theorem 11]. □

Similarly, Maleki and Strauss obtained in the same more general setting results already known to be true in $(\beta\mathbb{N}, +)$.

7.9 Theorem. *Let S be any discrete, countably infinite, commutative, and cancellative semigroup. Then βS has $2^{\mathfrak{c}}$ minimal left ideals and $2^{\mathfrak{c}}$ minimal right ideals, each of which has $2^{\mathfrak{c}}$ idempotents.*

Proof. [55, Theorm 9]. □

The order relation among idempotents reflects a stronger ordering among *semi-principal left ideals* of $(\beta\mathbb{N}, +)$, as is shown by the following results of D. Strauss. We define $\mathbb{N}^* = \beta\mathbb{N} \backslash \mathbb{N}$.

7.10 Theorem. *Let $p \in \mathbb{N}^* \backslash K(\beta\mathbb{N}, +)$. There are $2^{\mathfrak{c}}$ left ideals of the form $\mathbb{N}^* + q \subsetneq \mathbb{N}^* + p$ such that there is no r with $\mathbb{N}^* + q \subsetneq \mathbb{N}^* + r \subsetneq \mathbb{N}^* + p$.*

Proof. [63, Theorem 1.4]. □

7.11 Theorem. *Let $\langle p_n \rangle_{n=1}^{\infty}$ be a sequence in $\mathbb{N}^*$ such that for each $n \in \mathbb{N}$, $\mathbb{N}^* + p_{n+1} \subsetneq \mathbb{N}^* + p_n$. Then there is some $q \in \mathbb{N}^* \backslash K(\beta\mathbb{N}, +)$ such that $\mathbb{N}^* + q \subseteq \bigcap_{n=1}^{\infty} \mathbb{N}^* + p_n$ and $\mathbb{N}^* + q$ is maximal with respect to this property.*

Proof. [63, Theorem 1.5]. □

Given an idempotent in βS, it is known that $p\beta Sp$ is a group if and only if p is minimal. In the same paper, Strauss established that if p is not minimal in $\beta\mathbb{N}$, then the center of $p + \beta\mathbb{N} + p$ is as small as possible.

7.12 Theorem. *Let p be a nonminimal idempotent in $(\beta\mathbb{N}, +)$. Then the center of $p + \beta\mathbb{N} + p$ is $p + \mathbb{Z}$.*

Proof. [63, Theorem 3.3]. □

It is well known that there are many idempotents in $(\beta\mathbb{N}, +)$ whose sums are again idempotents. For example, if e and f are idempotents in the same minimal left ideal, then $e = e + f$ and $f = f + e$ and a similar statement applies to members of the same minimal right ideal. It is not known whether two idempotents in $K(\beta\mathbb{N}, +)$ which do not belong to the same minimal left ideal or the same minimal right ideal can have an idempotent sum. The following result, however, was obtained in collaboration with J. Berglund.

7.13 Theorem. *There is a set C of $2^{\mathfrak{c}}$ idempotents in $K(\beta\mathbb{N}, +)$ such that whenever p and q are distinct members of C, neither $p + q$ nor $q + p$ is an idempotent.*

Proof. [15, Theorem 3.4]. □

From Theorem 7.13 we see that the idempotents do not form a subsemigroup of $(\beta\mathbb{N}, +)$, although they do of course generate a semigroup. Recall that in the proof of Theorem 5.6, we defined the set $\Gamma = \{p \in \beta\mathbb{N} : \text{for all } A \in p, \text{ there is some sequence } \langle x_n \rangle_{n=1}^{\infty} \text{ such that } FS(\langle x_n \rangle_{n=1}^{\infty}) \subseteq A\}$. It turned out that Γ, which is defined additively, is a left ideal of $(\beta\mathbb{N}, \cdot)$. However, it turns out that Γ is not even a subsemigroup of $(\beta\mathbb{N}, +)$. One reasonable candidate for the smallest compact subsemigroup of $(\beta\mathbb{N}, +)$ containing the idempotents would seem to be the intersection of all of the kernels of continuous homomorphisms from $(\beta\mathbb{N}, +)$ to compact topological groups. In collaboration with D. Strauss, we have seen that this is dramatically not true.

7.14 Theorem. *Let $C = \{p \in \beta\mathbb{N} :$ whenever φ is a continuous homomorphism from $(\beta\mathbb{N}, +)$ to a compact topological group, p is in the kernel of $\varphi\}$. For each $n \in \mathbb{N}\backslash\{1\}$, let $S_n = \{p \in \beta\mathbb{N} :$ for every $A \in p$ there is some sequence $\langle x_t \rangle_{t=1}^{n}$ such that $FS(\langle x_t \rangle_{t=1}^{n}) \subseteq A\}$. Then C and each S_n are compact subsemigroups of $(\beta\mathbb{N}, +)$ containing the idempotents and*

$$\ldots \subsetneq S_3 \subsetneq S_2 \subsetneq C \ .$$

Proof. [49, Lemma 2.3, Theorem 3.9, and Theorem 3.13]. □

8. Ideals and Cancellation

We saw in Theorem 7.3 that $\{p \in \beta S : \text{for all } A \in p, A \text{ is piecewise syndetic}\}$ is an ideal of βS. Generalizing a notion of density introduced by Polya in [57], H. Umoh introduced another notion of largeness and showed that it, too, determines an ideal.

8.1 Theorem. *Let S be any discrete, countably infinite, commutative, and cancellative semigroup and enumerate S as $\{a_n : n \in \mathbb{N}\}$. Given $A \subseteq S$, define $M(A) = \sup\{\alpha \geq 0 :$ for all $a \in S$ and all $n \in \mathbb{N}$, there exist $z \in aS$ and $m > n$ such that $|A \cap z \cdot \{a_1, a_2, \ldots, a_m\}| \geq \alpha m\}$. Let $\Delta = \{p \in \beta S :$ for all $A \in p$, $M(A) > 0\}$. Then Δ is an ideal of βS.*

Proof. [65, Theorem 3.3]. □

Recall that we have promised to present in Corollary 9.14 the result that the ideal I of Theorem 7.3 is in fact the closure of the smallest ideal of $(\beta\mathbb{N}, +)$, so that this closure is an ideal.

A question arises as to whether the ideals $K(\beta\mathbb{N}, +)$ and its closure are prime (i.e., if $p + q$ is an element then either p or q is) or semiprime (i.e., if $p + p$ is an element, then so is p). The following result, obtained in collaboration with D. Strauss, does not answer either question, but shows that the answers are the same.

8.2 Theorem. *Let $K = K(\beta\mathbb{N}, +)$. Then K is prime if and only if it is semiprime. Likewise $c\ell K$ is prime if and only if it is semiprime.*

Proof. [48, Theorem 6]. □

In a similar vein, it was shown (again in collaboration with D. Strauss) that it is consistent with ZFC that there are idempotents which are themselves "prime". That is, they can be written as sums in only trivial ways. In fact, such idempotents can be found very close to (that is sharing any fewer than $\mathfrak{c}$ members with) any given idempotent.

8.3 Theorem. *Assume Martin's Axiom, let p be an idempotent of $(\beta\mathbb{N}, +)$, and let $\mathcal{A} \subseteq p$ such that $|\mathcal{A}| < \mathfrak{c}$. Then there is an idempotent q such that $\mathcal{A} \subseteq q$ and whenever $q = r + s$, one has that $r, s \in q + \mathbb{Z}$.*

Proof. [45, Theorems 4.7 and 5.5]. □

Cancellation in βS has been extensively studied. (See the earlier survey [38].) Some new results have been obtained by M. Filali. One should notice the differences in the hypotheses. Weak p-points of $\beta S \backslash S$ (that is, points not in the closure of a countable subset of $\beta S \backslash S$) are known to exist in ZFC [53], while it is consistent that p-points of $\beta S \backslash S$ (that is points such that the intersection of any countable set of neighborhoods is again a neighborhood) do not exist.

8.4 Theorem. *Let S be a discrete infinite cancellative semigroup. The weak p-points of $\beta S \backslash S$ that are in the closure of a countable subset of S are right cancellable in βS.*

Proof. [26, Theorem 3]. □

8.5 Theorem. *Let S be a discrete infinite cancellative semigroup. The p-points of $\beta S \backslash S$ that are in the closure of a countable subset of S are left cancellable in βS.*

Proof. [27, Theorem 3]. □

9. Connections With Topological Dynamics

We have already defined the notion of "piecewise syndetic" (Definition 7.1).

9.1 Definition. Let S be a semigroup. A subset A of S is *syndetic* if and only if there is some $H \in \mathcal{P}_f(S)$ such that $S = \bigcup_{t \in H} t^{-1}A$.

Again, the definition of "syndetic" depends on the choice of continuity for βS. If we had chosen to make βS left topological, we would have required $S = \bigcup_{t \in H} At^{-1}$.

We extend the notion of "dynamical system", which is often defined only in the case that S is a group and X is a compact metric space. (We remind the reader that we are assuming that all hypothesized topological spaces are Hausdorff.)

9.2 Definition. A *topological dynamical system* is a pair $(X, \langle T_s \rangle_{s \in S})$ where
(1) X is a compact space,
(2) S is a semigroup,
(3) for each $s \in S$, T_s is a continuous function from X to X, and
(4) for each $s, t \in S$, $T_s \circ T_t = T_{st}$.

9.3 Definition. Given a topological dynamical system $(X, \langle T_s \rangle_{s \in S})$, a point x of X is a *uniformly recurrent point* of X if and only if for every neighborhood U of x, $\{s \in S : T_s(x) \in U\}$ is syndetic.

There is a standard notion of upper density for subsets of $\mathbb{N}$. That is, given $A \subseteq \mathbb{N}$, $\overline{d}(A) = \limsup_{n \to \infty} \frac{|A \cap \{1, 2, \ldots, n\}|}{n}$. In early research, reported in my first survey [35], the importance of the set $\{p \in \beta\mathbb{N} : \text{for all } A \in p,\ \overline{d}(A) > 0\}$ was established. It and its complement in $\mathbb{N}^*$ are both left ideals of both $(\beta\mathbb{N}, +)$ and of $(\beta\mathbb{N}, \cdot)$, and hence $\mathbb{N}$ is the center of each of these semigroups. M. Blumlinger has recently established a connection between this set and the Lévy group of permutations (which is the group $\mathcal{G}$ defined in Theorem 9.4). Given a permutation σ of $\mathbb{N}$ we will let T_σ be the continuous extension of σ from $\beta\mathbb{N}$ to $\beta\mathbb{N}$.

9.4 Theorem. *Let*

$$\mathcal{G} = \{\sigma : \sigma \text{ is a permutation of } \mathbb{N} \text{ and } \lim_{n \to \infty} \frac{|\{k \in \{1, 2, \ldots, n\} : \sigma(k) > n\}|}{n} = 0\}$$

and let $\Delta = \{p \in \beta\mathbb{N} : \text{for all } A \in p,\ \overline{d}(A) > 0\}$. *Then* $\Delta = \{p \in \mathbb{N}^* : p \text{ is a uniformly recurrent point for } \mathcal{G}\}$.

Proof. [18, Theorem 4]. □

It is well known that in any compact right topological semigroup, any two minimal left ideals are homeomorphic to each other. (See [16].) A topological characterization of minimal left ideals of $\beta\mathbb{N}$ that was discovered by B. Balcar and A. Blaszczyk is presented in Corollary 9.7.

9.5 Definition. Let X be a compact space. The *Gleason space* of X is the Stone space of the Boolean algebra of all regular open subsets of X.

9.6 Theorem. *Let* $f : \mathbb{N} \longrightarrow \mathbb{N}$ *be a one to one function which has at most finitely many finite orbits. Every closed nonempty subset of* $\beta\mathbb{N}\backslash\mathbb{N}$ *which is minimal invariant with respect to* f^β *is homeomorphic to the Gleason space of* $\{0, 1\}^{\mathfrak{c}}$.

Proof. [4, Theorem 2]. □

9.7 Corollary. *All minimal left ideals of* $(\beta\mathbb{N}, +)$ *are homeomorphic to the Gleason space of* $\{0, 1\}^{\mathfrak{c}}$.

Proof. [4, Corollary to Theorem 2]. □

Separate contrasts to Corollary 9.7 are given by the following two results, obtained in collaboration with D. Strauss.

9.8 Theorem. *Let S be a discrete, countably infinite, commutative, cancellative semigroup. Then there are* $2^{\mathfrak{c}}$ *homeomorphism classes of minimal right ideals of* βS.

Proof. [50, Theorem 1 and Theorem 5]. □

9.9 Theorem. *Let p and q be nonminimal idempotents in $(\beta\mathbb{N},+)$. Then there is no continuous homomorphism from $\beta\mathbb{N}+p$ onto $\beta\mathbb{N}+q$.*

Proof. [50, Theorem 2]. □

It is well known that in any compact right topological semigroup, maximal groups which are contained in the same minimal right ideal are topologically and algebraically equivalent. (See [16].) The following result, obtained in collaboration with D. Strauss, provides a significant contrast.

9.10 Theorem. *Let S be a commutative discrete semigroup and let L be a minimal left ideal of S. If S is infinite, then the maximal subgroups of L lie in at least $2^{\mathfrak{c}}$ homeomorphism classes.*

Proof. [50, Theorem 3]. □

Recall that, if $(T,\circ)$ is a semigroup of continuous functions from a topological space X to itself, then the *enveloping semigroup* of T is the closure of T in the product space of all functions from X to X. The following theorem was obtained in collaboration with J. Lawson and A. Lisan. (We do not know whether these statements hold in $(\beta\mathbb{N},+)$.)

9.11 Theorem. *Let S be discrete. The following statments are equivalent.*

(a) Whenever q and r are distinct points of βS, there is some $p\in K(\beta S)$ such that $qp\neq rp$.

(b) Whenever L is a minimal left ideal of βS, the enveloping semigroup of $\{\lambda_{s}{}_{|L} : s\in S\}$ is topologically isomorphic to βS.

Proof. [39, Theorem 2.5]. □

We extend the notion of "piecewise syndetic" to apply to a family of subsets of S.

9.12 Definition. Let $(S,\cdot)$ be a semigroup and let $\mathcal{A}\subseteq\mathcal{P}(S)$. Then $\mathcal{A}$ is *collectionwise piecewise syndetic* if and only if there exist functions $G:\mathcal{P}_f(\mathcal{A})\longrightarrow\mathcal{P}_f(S)$ and $x:\mathcal{P}_f(\mathcal{A})\times\mathcal{P}_f(S)\longrightarrow S$ such that for all $F\in\mathcal{P}_f(S)$ and all $\mathcal{F}$ and $\mathcal{H}$ in $\mathcal{P}_f(\mathcal{A})$ with $\mathcal{F}\subseteq\mathcal{H}$ one has $F\cdot x(\mathcal{H},F)\subseteq\bigcup_{t\in G(\mathcal{F})}t^{-1}(\bigcap\mathcal{F})$.

Note that a subset A is piecewise syndetic if and only if $\{A\}$ is collectionwise piecewise syndetic. An alternate characterisation is: $\mathcal{A}$ is collectionwise piecewise syndetic if and only if there exists a function $G:\mathcal{P}_f(\mathcal{A})\rightarrow\mathcal{P}_f(S)$ such that

$$\{y^{-1}\big(G(\mathcal{F})\big)^{-1}(\textstyle\bigcap\mathcal{F}) : y\in S \text{ and } \mathcal{F}\in\mathcal{P}_f(\mathcal{A})\}$$

has the finite intersection property. (Here $\{y^{-1}\big(G(\mathcal{F})\big)^{-1}(\bigcap\mathcal{F})=\bigcup_{t\in G(\mathcal{F})}y^{-1}(t^{-1}\bigcap\mathcal{F})$.)

The following characterization of sets contained in members of $K(\beta S)$ was obtained in collaboration with A. Lisan.

9.13 Theorem. *Let S be a semigroup and let $\mathcal{A}\subseteq\mathcal{P}(S)$. There exists some $p\in K(\beta S)$ such that $\mathcal{A}\subseteq p$ if and only if $\mathcal{A}$ is collectionwise piecewise syndetic.*

Proof. [42, Theorem 2.1]. □

9.14 Corollary. *Let S be a semigroup. Then $c\ell K(\beta S) = \{p \in \beta S :$ for all $A \in p$, A is piecewise syndetic$\}$.*

Proof. Let $I = \{p \in \beta S :$ for all $A \in p$, A is piecewise syndetic$\}$. Then we saw in Theorem 7.3 that I is an ideal of βS, and consequently $K(\beta S) \subseteq I$ and, since I is closed, $c\ell K(\beta S) \subseteq I$. To see that $I \subseteq c\ell K(\beta S)$, let $p \in I$ and let $A \in p$. Then A is piecewise syndetic so $\{A\}$ is collectionwise piecewise syndetic so pick, by Theorem 9.13, some $q \in \overline{A} \cap K(\beta S)$. □

Theorem 9.13 also allows us to characterize the closure of $K(\beta S)$ with respect to other finer topologies on βS.

9.15 Definition. Let X be a topological space and let κ be an infinite cardinal. The *κ-topology* on X is the topology with basis consisting of all intersections of at most κ members of the original topology on X.

It is known that there are members of $c\ell K(\beta\mathbb{N},+)$ which are not sums of two members of $\mathbb{N}^*$ and there are points of $c\ell K(\beta\mathbb{N},+)$ at which right cancellation holds. By contrast, we have the following, obtained in collaboration with A. Lisan.

9.16 Theorem. *Let M be the closure of $K(\beta\mathbb{N},+)$ in the ω-topology on $\beta\mathbb{N}$. Then $M \subseteq \mathbb{N}^* + \mathbb{N}^*$ and right cancellation fails at each point of M.*

Proof. [42, Theorem 3.3 and Corollary 3.4]. □

We have not been able to show that the set M of Theorem 9.16 is in fact distinct from $K(\beta\mathbb{N},+)$, but would be very surprised if it is not.

Points at which joint continuity holds in $\beta\mathbb{N}$ are known to be rare. (See the earlier survey [35].) Recent results of I. Protasov extend this knowledge considerably.

9.17 Theorem. *Let G be a discrete countable abelian group with a finite Boolean subgroup. If $(p,q) \in \beta G \times \beta G$ and $\cdot$ is jointly continuous at (p,q), then either p or q is in G.*

Proof. [58, Theorem 4]. □

9.18 Theorem. *Let G be a discrete abelian group. If $p \in \beta G$ and $\cdot$ is jointly continuous at (p,p), then $p \in G$.*

Proof. [58, Theorem 5]. □

A contrast to Theorems 9.17 and 9.18 is provided by the following result, obtained in collaboration with J. Baker and J. Pym.

9.19 Theorem. *The operation $+$ on $\beta\mathbb{N}$ is jointly continuous with respect to the ω-topology on $\beta\mathbb{N}$.*

Proof. [2, Corollary 3]. □

A major result of D. Strauss (featured in the survey [38]) was the fact that $\mathbb{N}^*$ does not contain a topological and algebraic copy of $\beta\mathbb{N}$ [62]. In collaboration with D. Strauss, this result has recently been extended to show that the only copies of $\mathbb{N}^*$ in $\mathbb{N}^*$ are the trivial ones.

9.20 Theorem. *If $\varphi : \mathbb{N}^* \longrightarrow \mathbb{N}^*$ is a continuous one to one homomorphism then there is some $k \in \mathbb{N}$ such that for all $p \in \mathbb{N}^*$, $\varphi(p) = k \cdot p$.*

Proof. [46, Theorem 3.3]. □

10. Arithmetic in βS and Ramsey Theory

The relationship between equations in βS and Ramsey Theory is illustrated by the proof of the Finite Product Theorem (Theorem 5.2), wherein the existence of a solution to the equation $p \cdot p = p$, yields the existence of $FP(\langle x_n \rangle_{n=1}^{\infty})$ contained in one cell of a partition. In other cases, combinatorial constructions show that certain equations cannot be solved. Such is the case with the following result of B. Balcar and P. Kalášek.

10.1 Theorem. *There do not exist $p, q \in \beta\mathbb{N}$ such that $q + p = p$ and $2q + p = p$.*

Proof. [5, the Proposition on page 527]. □

Notice that here, as before, by $2q$ one means the operation in $(\beta\mathbb{N}, \cdot)$, rather than $q + q$. In [5], Balcar and Kalášek remark without proof that for any $p \in \beta\mathbb{N}\backslash\mathbb{N}$, $p + p \neq 2p$. Since $\mathbb{N} \subseteq \mathbb{Z}$, this fact is a consequence of the following more general result of I. Protasov.

In an arbitrary semigroup $(S, +)$, one has the usual meaning of $2x = x + x$ and then can define for $p \in \beta S$, $2p = p\text{-}\lim_{x \in S} 2x$. (This then agrees with the product $2p$ in $(\beta\mathbb{N}, +)$.)

10.2 Theorem. *Let $(G, +)$ be a discrete abelian group and let $p \in \beta G\backslash G$. Then $p+p \neq 2p$.*

Proof. [58, Theorem 3]. □

Soon after the original proof of the Finite Sum Theorem, P. Erdős asked whether it is always possible, given a two cell partition of $\mathbb{N}$, to find one cell and an infinite subset all of whose "multilinear expressions...(where each variable occurs only once)" are in that cell. It has been known for some time that this is not possible. In fact, one can partition $\mathbb{N}$ into two cells in such a way that one cannot get a sequence $\langle x_n \rangle_{n=1}^{\infty}$ with $FS(\langle x_n \rangle_{n=1}^{\infty}) \cup FP(\langle x_n \rangle_{n=1}^{\infty})$ contained in one cell. Indeed, there is a partition of $\mathbb{N}$ into seven cells so that there is no sequence $\langle x_n \rangle_{n=1}^{\infty}$ with all sums $x_n + x_m$ and products $x_n \cdot x_m$ with $n \neq m$ in the same cell. (See the survey [36].)

However, recently G. Smith has shown that one can get sums of products which are taken in a restricted fashion in one cell, but that this cell can be made to depend on the number of terms in the sum.

10.3 Definition. Let $\langle x_t \rangle_{t=1}^{\infty}$ be a sequence in $\mathbb{N}$ and let $n \in \mathbb{N}$. Then

$$\begin{aligned} SP_n(\langle x_t \rangle_{t=1}^{\infty}) \quad = \quad & \{\Sigma_{k=1}^{n} \Pi_{t \in F_k}\, x_t : F_1, F_2, \ldots, F_n \\ & \text{are finite nonempty subsets of } \mathbb{N} \text{ and for each} \\ & k \in \{1, 2, \ldots, n-1\}, \max F_k < \min F_{k+1}\}\ . \end{aligned}$$

Thus, for example, $x_2 x_4 + x_6 + x_{10} x_{12} x_{13} \in SP_3(\langle x_t \rangle_{t=1}^{\infty})$.

10.4 Theorem. *Let $r, n \in \mathbb{N}$ and let $\mathbb{N} = \bigcup_{i=1}^{r} A_i$. Then there exist i and a sequence $\langle x_t \rangle_{t=1}^{\infty}$ such that $SP_n(\langle x_t \rangle_{t=1}^{\infty}) \subseteq A_i$.*

Proof. [61, Theorem 2.4]. □

10.5 Theorem. *Let $m_1, m_2, \ldots, m_r$ be r distinct members of $\mathbb{N}$. There is a partition $\{A_1, A_2, \ldots, A_r\}$ of $\mathbb{N}$ such that whenever $i, j \in \{1, 2, \ldots, r\}$ and $\langle x_t \rangle_{t=1}^{\infty}$ is a sequence in $\mathbb{N}$ with $SP_{m_i}(\langle x_t \rangle_{t=1}^{\infty}) \subseteq A_j$, one has $i = j$.*

Proof. [61, Theorem 3.17]. □

One may wonder what Theorems 10.4 and 10.5 have to do with arithmetic in $\beta\mathbb{N}$. Theorem 10.4 is proved by choosing some $p = p \cdot p$ in $\beta\mathbb{N}$ and picking some $A_i \in p + p + \ldots + p$, where the sum is taken n times. Similarly, Theorem 10.5 shows that, for example, if $p \cdot p = p$, then $p + p \neq p + p + p$.

Recently, in conversation with Erdős, A. Hajnal asked whether for each triangle free graph G with vertices in $\mathbb{N}$, there must exist a sequence $\langle x_n\rangle_{n=1}^{\infty}$ so that whenever F and H are distinct finite nonempty subsets of $\mathbb{N}$, $\{\Sigma_{n\in F}\, x_n, \Sigma_{n\in H}\, x_n\}$ is not an edge of G. (That is, $FS(\langle x_n\rangle_{n=1}^{\infty})$ is an independent set.)

The motivation for this question can be seen from the fact that it is somewhat stronger than the two cell version of the Finite Sum Theorem (which is known to imply the full Finite Sum Theorem). That is, given a partition $\{A_1, A_2\}$ of $\mathbb{N}$, define a graph G by $E(G) = \big\{\{x, y\} : x \in A_1 \text{ and } y \in A_2\big\}$. If one has a sequence $\langle x_n\rangle_{n=1}^{\infty}$ so that whenever F and H are distinct finite nonempty subsets of $\mathbb{N}$, $\{\Sigma_{n\in F}\, x_n, \Sigma_{n\in H}\, x_n\}$ is not an edge of G, then one must have $FS(\langle x_n\rangle_{n=1}^{\infty}) \subseteq A_1$ or $FS(\langle x_n\rangle_{n=1}^{\infty}) \subseteq A_2$.

T. Łuczak, V. Rödl, and T. Schoen [54, Theorem 3] have established a "semi-infinite" partial affirmative answer to Hajnal's question, while, in collaboration with W. Deuber, D. Gunderson, and D. Strauss, the original question has been answered in the negative [23, Theorem 2.3]. (The details of these results are not presented here because the proofs are purely combinatorial and hence outside of the bounds that we have set for this paper.)

Fortunately, the negative answer to Hajnal's question was somewhat slow in coming (since otherwise we might well have lost interest) and the following affirmative infinite version was obtained, using properties of an arbitrary idempotent in $(\beta\mathbb{N}, +)$. (Given $m \in \mathbb{N}$, a K_m is a complete graph on n vertices.)

10.6 Theorem. *Let $p + p = p \in \beta\mathbb{N}$ and let G be a graph with vertex set contained in $\mathbb{N}$. Assume there is some $m \in \mathbb{N}\backslash\{1, 2\}$ such that G contains no K_m and let $B \in p$. There is a sequence $\langle m_i\rangle_{i=1}^{\infty}$ with $FS(\langle m_i\rangle_{i=1}^{\infty}) \subseteq B$ such that whenever $F, H \in \mathcal{P}_f(\mathbb{N})$ with $F \cap H = \emptyset$,*

$$\{\Sigma_{i\in F}\, m_i, \Sigma_{i\in H}\, m_i\} \notin E(G)\ .$$

Proof. [23, Theorem 3.17]. □

Of course, except in the most trivial of instances, the semigroup $(\beta S, \cdot)$ is not commutative. However, if the semigroup S is not commutative, combinatorial results obtained from βS tend to be quite complicated, or else quite restricted in their validity. We see now some examples of this phenomenon in results obtained in collaboration with V. Bergelson.

Consider for example van der Waerden's Theorem. In case $(S, \cdot)$ is a commutative semigroup, this has a simple formulation (which is, for example, a corollary of Theorem 10.7 below). That is, given any sequence $\langle d_n\rangle_{n=1}^{\infty}$ in S, given $\ell, r \in \mathbb{N}$, and given that $S = \bigcup_{i=1}^{r} A_i$, there exist $a \in S$, $i \in \{1, 2, \ldots, r\}$, and $d \in FP(\langle d_n\rangle_{n=1}^{\infty})$ such that $\{a, ad, ad^2, \ldots, ad^{\ell}\} \subseteq A_i$. The general version is not so nice.

10.7 Theorem. *Let $(S, \cdot)$ be a semigroup, let $\ell, r \in \mathbb{N}$, let $\langle d_n\rangle_{n=1}^{\infty}$ be a sequence in S, and let $S = \bigcup_{i=1}^{r} A_i$. There exist $i \in \{1, 2, \ldots, r\}$, $n \in \mathbb{N}$, $a_1, a_2, \ldots, a_{n+1} \in S$, and $m(1) < m(2) < \ldots < m(n)$ in $\mathbb{N}$ with*

$$\begin{aligned}&\{a_1a_2\cdots a_{n+1}, a_1d_{m(1)}a_2d_{m(2)}\cdots d_{m(n)}a_{n+1},\\ &a_1d_{m(1)}^2a_2d_{m(2)}^2\cdots d_{m(n)}^2a_{n+1}, \ldots, a_1d_{m(1)}^{\ell}a_2d_{m(2)}^{\ell}\cdots d_{m(n)}^{\ell}a_{n+1}\} \subseteq A_i\ .\end{aligned}$$

Proof. [10, Corollary 3.1]. □

Notice that, in Theorem 10.7, the "increment" d has been separated by the a_t's. When one attempts to bring the increment back together one gets some very restricted results.

10.8 Theorem. *Let $(S,\cdot)$ be a semigroup, let $r \in \mathbb{N}$, let $\langle y_n\rangle_{n=1}^{\infty}$ be a sequence in S, and let $S = \bigcup_{i=1}^{r} A_i$. Then there exist $i \in \{1,2,\ldots,r\}$, and $d \in FP(\langle y_n\rangle_{n=1}^{\infty})$ such that $\{a, da, ad\} \subseteq A_i$. If S is a group, then there exist $i \in \{1,2,\ldots,r\}$, and $d \in FP(\langle y_n\rangle_{n=1}^{\infty})$ such that $\{da, ad, dad\} \subseteq A_i$.*

Proof. [11, Corollaries 3.3 and 3.6]. □

The conclusions of Theorem 10.8 are very weak, but as shown in the following theorem, they are as good as one can get in this generality.

10.9 Theorem. *Let G be the free group on the generators $\langle y_n\rangle_{n=1}^{\infty}$. There exist $r \in \mathbb{N}$ and sets $A_1, A_2, \ldots, A_r$ such that $G = \bigcup_{i=1}^{r} A_i$ and there do not exist $i \in \{1,2,\ldots,r\}$, $a \in G$ and $d \in FP(\langle y_n\rangle_{n=1}^{\infty})$ satisfying any of the following statements:*

(a) $\{a, ad, ad^2\} \subseteq A_i$.
(b) $\{a, da, d^2a\} \subseteq A_i$.
(c) $\{a, ad, dad\} \subseteq A_i$.
(d) $\{a, da, dad\} \subseteq A_i$.
(e) $\{a, d, ad, da\} \subseteq A_i$.
(f) $\{d, da, ad, dad\} \subseteq A_i$.

Proof. [11, Theorem 4.8]. □

We conclude this section with a result from [6], a paper produced in collaboration with V. Bergelson and A. Blass. The results of this paper were produced using idempotents that are algebraically related to each other in the Stone-Čech compactification of certain free semigroups. Most of the results involve the introduction of too much terminology to present here. The result that we do present is a generalization of a result of H. Furstenberg and Y. Katznelson [30, Theorem 3.1]. Even here, we must refer the reader to [6] for the precise definition of d-dimensional subspaces and strong infinite dimensional subspaces.

10.10 Theorem. *Let W be the free semigroup on a finite set of generators, let $d \in \mathbb{N}$, and let the collection of all d-dimensional subspaces be partitioned into finitely many pieces. There exists a strong infinite dimensional subspace of W all of whose d-dimensional subspaces lie in the same piece of the partition.*

Proof. [6, Theorem 7.1]. □

11. Partition Regularity of Matrices

In this section we deal with results related to two notions of partition regularity of matrices. These are "kernel partition regularity" and "image partition regularity". We discuss kernel partition regularity first in a rather general setting.

11.1 Definition. Let R be a commutative ring, let $u, v \in \mathbb{N}$, let C be a $u \times v$ matrix with entries from R, let M be a module over R, and let $B \subseteq M$. Then C is *kernel partition regular over B* if and only if whenever B is partitioned into finitely many classes, some one of these classes contains $x_1, x_2, \ldots, x_v$ with

$$C \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} .$$

It is an old and famous result of R. Rado's [60] characterizing those matrices with rational entries that are kernel partition regular over $\mathbb{N}$. These are precisely the matrices which satisfy the "columns condition" over $\mathbb{Q}$. We present a general definition of the columns condition that agrees with Rado's in the case $R = \mathbb{Q}$.

11.2 Definition. Let R be a commutative ring, let $u, v \in \mathbb{N}$, and let C be a $u \times v$ matrix with entries from R. Then C satisfies the *columns condition over* R if and only if the columns $\vec{c_1}, \vec{c_2}, \ldots, \vec{c_v}$ of C can be ordered so that there exist $m \in \mathbb{N}$ and $k_1, k_2, \ldots, k_m$ with $1 \leq k_1 < k_2 < \ldots < k_m = v$ and $d_1, d_2, \ldots, d_m \in R\backslash\{0\}$ such that

(1) $d_1 \cdot \Sigma_{i=1}^{k_1} \vec{c_i} = \vec{0}$,

(2) if $m > 1$ and $t \in \{2, 3, \ldots, m\}$, then there exist $\alpha_{1,t}, \alpha_{2,t}, \ldots, \alpha_{k_{t-1},t}$ in R with $\Sigma_{i=1}^{k_{t-1}} \alpha_{i,t} \vec{c_i} + d_t \cdot \Sigma_{i=k_{t-1}+1}^{k_t} \vec{c_i} = \vec{0}$, and

(3) if $m > 1$, then for each $n \in \mathbb{N}$, $R \cdot d_1 \cdot \Pi_{t=2}^{m} {d_t}^n$ is infinite.

Results obtained in collaboration with V. Bergelson, W. Deuber, and H. Lefmann have extended Rado's Theorem in two directions.

11.3 Theorem. *Let F be a finite field, let $u, v \in \mathbb{N}$, and let C be a $u \times v$ matrix with entries from F. The following statements are equivalent.*

(a) For each $r \in \mathbb{N}$ there is some $m \in \mathbb{N}$ such that whenever $n \geq m$, V is an n-dimensional vector space over F, and $V\backslash\{0\}$ is partitioned into r cells, there must exist $x_1, x_2, \ldots, x_v$ contained in one of these classes with

$$C \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} .$$

(b) C satisfies the columns condition over F.

Proof. [7, Theorem 3.4]. □

Notice that statement (a) in Theorem 11.3 is not quite the same as saying that C is kernel partition regular over $V\backslash\{0\}$ because the number of cells of the partition is restricted.

11.4 Theorem. *Let R be a commutative ring, let $u, v \in \mathbb{N}$, and let C be a $u \times v$ matrix with entries from R. If C satisfies the columns condition over R, then C is kernel partition regular over $R\backslash\{0\}$.*

Proof. [8, Theorem 2.4]. □

The applications that we present involving image partition regularity all deal with matrices with rational entries and with partitions of $\mathbb{N}$, so we only discuss the notion in this generality.

11.5 Definition. Let $u, v \in \mathbb{N}$ and let A be a $u \times v$ matrix with entries from $\mathbb{Q}$. Then A is *image partition regular over* $\mathbb{N}$ if and only if whenever $r \in \mathbb{N}$ and $\mathbb{N} = \bigcup_{i=1}^{r} E_i$, there exist $i \in \{1, 2, \ldots, r\}$ and $\vec{x} \in \mathbb{N}^v$ such that $A\vec{x} \in {E_i}^u$.

Notice that many of the classical theorems in Ramsey Theory are naturally stated in terms of the image partition regularity of certain matrices. For example, van der Waerden's Theorem says that given any $\ell, r \in \mathbb{N}$, whenever $\mathbb{N} = \bigcup_{i=1}^{r} E_i$, there exist $i \in \{1, 2, \ldots, r\}$ and $a, d \in \mathbb{N}$ such that $\{a, a+d, a+2d, \ldots, a+\ell d\} \subseteq E_i$, that is,

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ \vdots & \vdots \\ 1 & \ell \end{pmatrix} \begin{pmatrix} a \\ d \end{pmatrix} \in {E_i}^{\ell+1} .$$

In other words, van der Waerden's Theorem is the assertion that for each $\ell \in \mathbb{N}$, the matrix

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ \vdots & \vdots \\ 1 & \ell \end{pmatrix}$$

is image partition regular.

A particular class of matrices has long been known to be image partition regular.

11.6 Definition. Let $u, v \in \mathbb{N}$ and let A be a $u \times v$ matrix with entries from $\mathbb{Q}$. Then A satisfies the *first entries condition* if and only if no row of A is $\vec{0}$, the first (leftmost) nonzero entry of each row is positive, and whenever two rows have their first nonzero entry in the same column, that entry is the same on both rows.

In [22] W. Deuber showed that matrices satisfying the first entries condition are image partition regular. In fact he showed a much stronger statement which enabled him to prove a conjecture of Rado. That is he showed that, given two matrices A and B which satisfy the first entries condition and given $r \in \mathbb{N}$, there is some matrix C which satisfies the first entries condition and whenever $\vec{x} \in \mathbb{N}^v$ (where v is the number of columns of C) and the entries of $C\vec{x}$ are divided into r classes, there will be one of these classes E and vectors $\vec{y}$ and $\vec{z}$ (of the appropriate size) such that all entries of $A\vec{y}$ and all entries of $B\vec{z}$ are in E. (See [32] for a more detailed discussion of these results.)

In collaboration with H. Lefmann, Deuber's Theorem has been extended as follows.

11.7 Theorem. *Let $\langle A_n \rangle_{n=1}^{\infty}$ enumerate the matrices that satisfy the first entries condition and for each n, let $v(n)$ be the number of columns of A_n. Let $\mathcal{V} = \times_{n=1}^{\infty} \mathbb{N}^{v(n)}$ and for $\vec{x} \in \mathcal{V}$, let $\mathcal{S}(\vec{x}) = \{\Sigma_{n \in F}\, t_n : F \in \mathcal{P}_f(\mathbb{N})$ and for each $n \in \mathbb{N}$, t_n is an entry of $A_n\vec{x}(n)\}$. Given any $\vec{x} \in \mathcal{V}$ and any $r \in \mathbb{N}$, if $\mathcal{S}(\vec{x}) = \bigcup_{i=1}^{r} E_i$, then there exist $i \in \{1, 2, \ldots, r\}$ and $\vec{y} \in \mathcal{V}$ such that $\mathcal{S}(\vec{y}) \subseteq E_i$.*

Proof. [41, Theorem 2.7]. □

Given that many of the classical results of Ramsey Theory are naturally stated in terms of the image partition regularity of matrices and given that the problem of characterizing kernel partition regularity of matrices was solved by Rado in 1933 [60] it is surprising that the problem of characterizing image partition regularity of matrices has only recently been solved. The following theorem was obtained in collaboration with I. Leader.

11.8 Theorem. *Let $u, v \in \mathbb{N}$ and let A be a $u \times v$ matrix with entries from $\mathbb{Q}$. The following statements are equivalent.*

(a) The matrix A is image partition regular.

(b) Let $\vec{c_1}, \vec{c_2}, \ldots, \vec{c_v}$ be the columns of A. Then there exist positive rationals $t_1, t_2, \ldots, t_v$ such that the matrix

$$\begin{pmatrix} & & & & -1 & 0 & \ldots & 0 \\ & & & & 0 & -1 & \ldots & 0 \\ t_1\vec{c_1} & t_2\vec{c_2} & \ldots & t_v\vec{c_v} & \vdots & \vdots & \ddots & \vdots \\ & & & & 0 & 0 & \ldots & -1 \end{pmatrix}$$

satisfies the columns condition over $\mathbb{Q}$.

(c) For each $\vec{p} \in \omega^v \backslash \{\vec{0}\}$ there exists a positive rational b such that the matrix $\begin{pmatrix} A \\ b\vec{p}^T \end{pmatrix}$ is image partition regular.

(d) There exist $m \in \mathbb{N}$ and a $u \times m$ matrix B which satisfies the first entries condition such that for each $\vec{y} \in \mathbb{N}^m$ there is some $\vec{x} \in \mathbb{N}^v$ such that $A\vec{x} = B\vec{y}$.

Proof. [40, Theorem 3.1]. □

Notice that the condition given by statement (b) of Theorem 11.8 is effectively computable, so a finite computation suffices to determine whether a matrix is image partition regular. Notice also that statement (c) of Theorem 11.8 tells us that an image partition regular matrix can be expanded almost at will and remain image partition regular.

The situation with respect to image partition regularity of infinite matrices is considerably more complicated. Given that, as usual, one is asking questions about $\mathbb{N}$, it is reasonable to require that an infinite matrix have only finitely many nonzero entries in each row. With this restriction it is a natural problem to determine which $\omega \times \omega$ matrices are image partition regular. (Where the obvious modification to Definition 11.5 is made.)

For example, the Finite Sum Theorem tells us that the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 & \ldots \\ 0 & 1 & 0 & 0 & \ldots \\ 1 & 1 & 0 & 0 & \ldots \\ 0 & 0 & 1 & 0 & \ldots \\ 1 & 0 & 1 & 0 & \ldots \\ 0 & 1 & 1 & 0 & \ldots \\ 1 & 1 & 1 & 0 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

is image partition regular.

A natural conjecture is that any $\omega \times \omega$ matrix with rational entries and only finitely many nonzero entries on each row which satisfies the first entries condition should be image partition regular. Results obtained in collaboration with W. Deuber, I. Leader, and H. Lefmann produce a large class of infinite image partition regular matrices and also show that the above conjecture is wrong. These results ivolve the notion of a Milliken-Taylor system, named after the Milliken-Taylor Theorem [56, 64] which has a similar flavor.

11.9 Definition. Let $\vec{a} = \langle a_1, a_2, \ldots a_k \rangle$ be a finite sequence in $\mathbb{N}$ and let $\langle x_n \rangle_{n=1}^{\infty}$ be a sequence in $\mathbb{N}$. Then $MT(\vec{a}, \langle x_n \rangle_{n=1}^{\infty}) = \{\Sigma_{t=1}^{k} a_t \cdot (\Sigma_{n \in F_t}\, x_n) : F_1, F_2, \ldots, F_k$ are finite nonempty subsets of $\mathbb{N}$ and for $t \in \{1, 2, \ldots, k-1\}$, $\max F_t < \min F_{t+1}\}$.

Observe that any Milliken-Taylor system $MT(\vec{a}, \langle x_n \rangle_{n=1}^{\infty})$ is generated by a matrix. For example, if

$$A = \begin{pmatrix} 1 & 2 & 0 & 0 & 0 & \dots \\ 0 & 1 & 2 & 0 & 0 & \dots \\ 1 & 0 & 2 & 0 & 0 & \dots \\ 1 & 1 & 2 & 0 & 0 & \dots \\ 1 & 2 & 2 & 0 & 0 & \dots \\ 0 & 0 & 1 & 2 & 0 & \dots \\ 0 & 1 & 0 & 2 & 0 & \dots \\ 0 & 1 & 1 & 2 & 0 & \dots \\ 0 & 1 & 2 & 2 & 0 & \dots \\ 1 & 0 & 0 & 2 & 0 & \dots \\ 1 & 0 & 1 & 2 & 0 & \dots \\ 1 & 0 & 2 & 2 & 0 & \dots \\ 1 & 1 & 0 & 2 & 0 & \dots \\ 1 & 1 & 1 & 2 & 0 & \dots \\ 1 & 1 & 2 & 2 & 0 & \dots \\ 1 & 2 & 0 & 2 & 0 & \dots \\ 1 & 2 & 2 & 2 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \text{ and } \vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \end{pmatrix}$$

then $MT(\langle 1,2\rangle, \langle x_n\rangle_{n=1}^\infty)$ is the set of entries of $A\vec{x}$.

We define an equivalence relation $\approx$ on the set of finite sequences in $\mathbb{N}$.

11.10 Definition. Let S be the set of finite sequences in $\mathbb{N}$ and let $c : S \longrightarrow S$ be the function which deletes any consecutive repeated terms. (So $c(\langle 1,3,3,1,1,2,2,2\rangle) = \langle 1,3,1,2\rangle$.) Given $\vec{a}$ and $\vec{b}$ in S, $\vec{a} \approx \vec{b}$ if and only if there is a positive rational α such that $\alpha \cdot c(\vec{a}) = c(\vec{b})$.

Statement (a) in the following theorem tells us that the matrices associated with individual Milliken-Taylor systems are image partition regular, while statement (b) easily allows us to construct matrices with rational entries and only finitely many nonzero entries on each row which satisfy the first entries condition but are not image partition regular.

11.11 Theorem. *Let $\vec{a}$ and $\vec{b}$ be finite sequences in $\mathbb{N}$.*

(a) If $\vec{a} \approx \vec{b}$, then whenever $r \in \mathbb{N}$ and $\mathbb{N} = \bigcup_{i=1}^r A_i$, there exist $i \in \{1,2,\dots,r\}$ and sequences $\langle x_n\rangle_{n=1}^\infty$ and $\langle y_n\rangle_{n=1}^\infty$ in $\mathbb{N}$ such that $MT(\vec{a}, \langle x_n\rangle_{n=1}^\infty) \cup MT(\vec{b}, \langle y_n\rangle_{n=1}^\infty) \subseteq A_i$.

(b) If $\vec{a} \not\approx \vec{b}$, then there exist sets A_1 and A_2 such that $\mathbb{N} = A_1 \cup A_2$ and for any sequence $\langle x_n\rangle_{n=1}^\infty$ in $\mathbb{N}$ and any $i \in \{1,2\}$, if $MT(\vec{a}, \langle x_n\rangle_{n=1}^\infty) \subseteq A_i$, then $i = 1$ while if $MT(\vec{b}, \langle x_n\rangle_{n=1}^\infty) \subseteq A_i$, then $i = 2$.

Proof. [24, Theorems 3.2 and 3.14]. □

The relationship to arithmetic in $\beta\mathbb{N}$ in Theorem 11.11 is similar to that in Theorems 10.4 and 10.5. For example, if $\vec{a} = \langle 1,2,1,3\rangle$, to establish conclusion (a) of Theorem 11.11, one picks $p = p + p \in \beta\mathbb{N}$ and picks $A_i \in p + 2p + p + 3p$, while conclusion (b) of Theorem 11.11 tells us for example that if $p = p + p \in \beta\mathbb{N}$ and $q = q + q \in \beta\mathbb{N}$, then $p + 2p \neq q + 3q$ (because $\langle 1,2\rangle \not\approx \langle 1,3\rangle$).

12. Central Sets

The notion of a "central" subset of $\mathbb{N}$ was introduced by H. Furstenberg in [29, Chapter 8] based on dynamical systems. The definition depends on the notions of "syndetic"

(Definition 9.1), "uniform recurrence" (Definition 9.3) and "proximality". Furstenberg defined proximality in the context of a metric dynamical system as follows. (Recall that a topological dynamical system was defined in Definition 9.2.)

12.1 Definition. Let $(X, \langle T_s \rangle_{s\in S})$ be a topological dynamical system where (X,d) is a metric space and let $x, y \in X$. Then x and y are *proximal* if and only if there is a sequence $\langle s_k \rangle_{k=1}^{\infty}$ in S such that $\lim_{k\to\infty} d\big(T_{s_k}(x), T_{s_k}(y)\big) = 0$.

Furstenberg then stated that a subset $A \subseteq \mathbb{N}$ is *central* if and only if there exist a topological dynamical system $(X, \langle T_n \rangle_{n\in\mathbb{N}})$, where (X,d) is a metric space, a point x of X, a uniformly recurrent point y of X such that x and y are proximal, and a neighborhood U of y such that $A = \{n \in \mathbb{N} : T_n(x) \in U\}$.

Based on the fact that we could prove what we have come to call the Central Sets Theorem (Theorem 12.11 below) about elements of minimal idempotents while the same theorem had been proved in $(\mathbb{N}, +)$ by Furstenberg for his version of central sets, V. Bergelson and I defined the notion of central in an arbitrary discrete semigroup as follows.

12.2 Definition. Let S be a semigroup and let $A \subseteq S$. Then A is *central* if and only if there is some idempotent $p \in K(\beta S)$ such that $A \in p$.

With the assistance of B. Weiss, Bergelson and I then showed [9, Corollary 6.12] that if S is a countable semigroup, then $A \subseteq S$ is central (as we defined the notion) if and only if A satisfies Furstenberg's definition of central, where $\mathbb{N}$ is replaced by S.

Recently, S. Hong-ting and Y. Hong-wei showed that with an appropriate extension of Furstenberg's definition, the two notions agree in general. Key to this extension was devising an appropriate generalization of "proximal" to an arbitrary topological dynamical system.

12.3 Definition. Let $(X, \langle T_s \rangle_{s\in S})$ be a topological dynamical system and let $x, y \in X$. Then x and y are *proximal* if and only if there is some $p \in \beta S \backslash S$ such that $p\text{-}\lim_{s\in S} T_s(x) = p\text{-}\lim_{s\in S} T_s(y)$.

We need to check that Definitions 12.1 and 12.3 agree in the event that X is a metric space.

12.4 Lemma. *Let $(X, \langle T_s \rangle_{s\in S})$ be a topological dynamical system where (X,d) is a metric space and let $x, y \in X$. The following statements are equivalent.*

(a) There is some $p \in \beta S \backslash S$ such that $p\text{-}\lim_{s\in S} T_s(x) = p\text{-}\lim_{s\in S} T_s(y)$.

(b) There is a sequence $\langle s_k \rangle_{k=1}^{\infty}$ in S such that $\lim_{k\to\infty} d\big(T_{s_k}(x), T_{s_k}(y)\big) = 0$.

Proof. Necessity. Let $a = p\text{-}\lim_{s\in S} T_s(x)$ and for each $k \in \mathbb{N}$, let

$$U_k = \{z \in X : d(z,a) < 1/k\} .$$

Then each U_k is a neighborhood of a so $\{s \in S : T_s(x) \in U_k\} \in p$ and $\{s \in S : T_s(y) \in U_k\} \in p$, so pick $s_k \in \{s \in S : T_s(x) \in U_k\} \cap \{s \in S : T_s(y) \in U_k\}$.

Sufficiency. We may presume that for each $k \in \mathbb{N}$, $d\big(T_{s_k}(x), T_{s_k}(y)\big) < 1/k$. We may also presume that $\{s_k : k \in \mathbb{N}\}$ is infinite. Choose $p \in \beta S \backslash S$ such that $\big\{\{s_k : k > n\} : n \in \mathbb{N}\big\} \subseteq p$. Let $a = p\text{-}\lim_{s\in S} T_s(x)$ (which exists because X is compact). To see that $a = p\text{-}\lim_{s\in S} T_s(y)$, let U be a neighborhood of a and pick $n \in \mathbb{N}$ such that $\{z \in X : d(z,a) < 1/n\} \subseteq U$. Then $\{s_k : k > 2n\} \in p$ and $\{s \in S : d(T_s(x), a) < 1/(2n)\} \in p$ and $\{s_k : k > 2n\} \cap \{s \in S : d(T_s(x), a) < 1/(2n)\} \subseteq \{s \in S : T_s(y) \in U\}$. $\square$

12.5 Theorem. *Let S be a semigroup and let $A \subseteq S$. Then A is central if and only if there exist a topological dynamical system $(X, \langle T_s \rangle_{s\in S})$, a point x of X, a uniformly recurrent point y of X such that x and y are proximal, and a neighborhood U of y such that $A = \{n \in \mathbb{N} : T_n(x) \in U\}$.*

Proof. [52, Theorem 2.4]. □

Notice that as a consequence of Theorem 12.5 one has the nonobvious conclusion that the dynamical notion of "central" is closed under passage to supersets.

We now introduce some additional terminology due to Furstenberg [29].

12.6 Definition. Let S be a semigroup and let $A \subseteq S$.

(a) A is an *IP set* if and only if there is some sequence $\langle x_n \rangle_{n=1}^{\infty}$ in S such that $FP(\langle x_n \rangle_{n=1}^{\infty}) \subseteq A$.

(b) A is an *IP* set* if and only if for every IP set B, $A \cap B \neq \emptyset$.

(c) A is a *central* set* if and only if for every central set B, $A \cap B \neq \emptyset$.

Notice that, by Theorems 5.2 and 5.5, A is an IP set if and only if A is a member of some idempotent in βS. Thus A is an IP* set if and only if A is a member of every idempotent in βS. (If $p = pp$ and $A \notin p$, then $S \backslash A \in p$ and hence $S \backslash A$ is an IP set. Likewise, if A is a member of every idempotent and B is a member of some idempotent, then $A \cap B \neq \emptyset$.) Similarly, A is a central* set if and only if A is a member of every minimal idempotent.

In $(\mathbb{N}, +)$ central sets are very rich combinatorially. In particular, if A is a central subset of $\mathbb{N}$ and C is a $u \times v$ kernel partition regular matrix with entries in $\mathbb{Q}$, then there must exist some $\vec{x} \in A^v$ such that $C\vec{x} = \vec{0}$. Equivalently, if D is a $u \times v$ image partition regular matrix with entries in $\mathbb{Q}$, there must exist some $\vec{y} \in \mathbb{N}^v$ such that $D\vec{y} \in A^u$. (See [9, Section 5].)

However, this property of central sets does not extend to all semigroups. In fact, it does not even extend to $(\mathbb{N}, \cdot)$. To see this, consider the matrix $\begin{pmatrix} 2 & -2 & 1 \end{pmatrix}$. This matrix trivially satisfies the columns condition. But according to [9, Theorem 5.3], $\{x^2 : x \in \mathbb{N}\}$ is not central in $(\mathbb{N}, \cdot)$ and any solution to the equation $x^2 y^{-2} z = 1$ must have $z = (y/x)^2$.

In collaboration with W. Woan it was determined for a wide class of semigroups (a class that includes all abelian groups) precisely when central sets hold the same "central" combinatorial position that they hold in $(\mathbb{N}, +)$. In order to be able to use the customary matrix notation, we assume the operation is written additively in the statement of the next theorem.

12.7 Theorem. *Assume that $(S, +)$ is a commutative subsemigroup of a group and that for all a, b, and x in S and all $d \in \mathbb{N}$, if $x = da - db$, then $x \in dS$. Then the following statements are equivalent.*

(a) Whenever $u, v \in \mathbb{N}$, C is a $u \times v$ matrix with integer entries which satisfies the columns condition over $\mathbb{Q}$, and A is a central subset of S, there exists $\vec{x} \in A^v$ such that $C\vec{x} = \vec{0}$.

(b) For each $d \in \mathbb{N}$, dS is a central set.*

Proof. [51, Theorem 2.6]. □

We were also able to show that in $(\mathbb{N}, \cdot)$ central sets remain relatively rich combinatorially. In the statement of Theorem 12.8, the expression $\vec{x}^C = \vec{1}$ is the obvious multiplicative translation of the expression $C\vec{x} = \vec{0}$. For example, if $C = \begin{pmatrix} 1 & -1 & 2 \\ 0 & 3 & -2 \end{pmatrix}$, then the expression that $\vec{x}^C = \vec{1}$ means that $x_1 {x_2}^{-1} {x_3}^2 = 1$ and ${x_2}^3 {x_3}^{-2} = 1$.

12.8 Theorem. *Let $u, v \in \mathbb{N}$ and let C be a $u \times v$ matrix with entries from $\mathbb{Z}$. Then the following statements are equivalent.*

(a) Whenever A is a multiplicatively central subset of $\mathbb{N}$, there exists $\vec{x} \in A^v$ such that $\vec{x}^C = \vec{1}$.

(b) C satisfies the columns condition over $\mathbb{Z}$ with each $d_i = 1$ in Definition 11.2.

Proof. [51, Theorem 3.10]. □

We have already seen some interesting interactions among the operations in the algebraic system $(\beta\mathbb{N}, +, \cdot)$. (See for example Theorem 5.6.) In collaboration with V. Bergelson, it was shown that additively central sets in $\mathbb{N}$ need not have any multiplicative structure, while multiplicatively central sets must have some, but not too much, additive structure.

12.9 Theorem. *(a) There is an additively central set in $\mathbb{N}$ which does not contain any $\{x, y, xy\}$.*

(b) Given any multiplicatively central subset A of $\mathbb{N}$, for each $m \in \mathbb{N}$ there is a sequence $\langle x_n \rangle_{n=1}^m$ such that $FS(\langle x_n \rangle_{n=1}^m) \subseteq A$.

(c) There is a multiplicatively central set in $\mathbb{N}$ which does not contain any $FS(\langle x_n \rangle_{n=1}^\infty)$.

Proof. [13, Theorems 3.4, 3.5, and 3.6]. □

We have already introduced the notions of IP, IP*, central, and central* sets. By their very definitions, examples of IP sets in $(\mathbb{N}, +)$ are easy to come by; given any sequence $\langle x_n \rangle_{n=1}^\infty$ in $\mathbb{N}$ any set containing $FS(\langle x_n \rangle_{n=1}^\infty)$ is an IP set. And it is easy to see that for any $m \in \mathbb{N}$, $m\mathbb{N}$ is an IP* set, and hence a central* set, and hence a central set. (Given any sequence $\langle x_n \rangle_{n=1}^\infty$ in $\mathbb{N}$, choose a set $F \in \mathcal{P}_f(\mathbb{N})$ such that $|F| = m$ and for all $n, k \in F$, $x_n \equiv x_k \pmod{m}$. Then $\Sigma_{n \in F}\, x_n \in m\mathbb{N}$.) In collaboration with V. Bergelson and B. Kra, the following method of generating many explicit examples was obtained.

Sets of the form $\{\lfloor n\alpha + \gamma \rfloor : n \in \mathbb{N}\}$, where α and γ are real numbers and $\lfloor\ \rfloor$ is the greatest integer function, have been extensively studied and have some very interesting properties. See the introduction to [14] for a discussion of the history of these sets.

12.10 Theorem. *Let $\alpha > 0$ and let $0 \le \gamma < 1$ with $\gamma > 0$ if α is irrational. Define $g : \mathbb{N} \longrightarrow \mathbb{N}$ by $g(n) = \lfloor n\alpha + \gamma \rfloor$. Let $A \subseteq \mathbb{N}$.*

(a) If A is an IP set in $(\mathbb{N}, +)$, then so is $g[A]$.*

(b) If A is a central set in $(\mathbb{N}, +)$, then so is $g[A]$.*

(c) If A is a central set in $(\mathbb{N}, +)$, then so is $g[A]$.

(d) If A is an IP set in $(\mathbb{N}, +)$, then so is $g[A]$.

Proof. [14, Theorem 6.1]. □

Notice that one may iterate applications of g, with or without changes in the parameters α and γ, to obtain many examples of the kinds of sets mentioned.

We have already pointed out that central sets have a rich combinatorial structure. Most of this structure can be derived from the Central Sets Theorem, which we state here for a commutative semigroup. (There is a noncommutative version as well, but it is considerably more complicated to state–see [10, Theorem 2.8].)

12.11 Theorem. Central Sets Theorem. *Let S be a commutative semigroup and let Φ be the set of all functions $f : \mathbb{N} \to \mathbb{N}$ for which $f(n) \le n$ for all $n \in \mathbb{N}$. Let A be a central subset of S, and for each $\ell \in \mathbb{N}$, let $\langle y_{\ell,n} \rangle_{n=1}^\infty$ be a sequence in S. There exist a sequence $\langle a_n \rangle_{n=1}^\infty$ in S and a sequence $\langle H_n \rangle_{n=1}^\infty$ in $\mathcal{P}_f(\mathbb{N})$ such that $\max H_n < \min H_{n+1}$ for each $n \in \mathbb{N}$ and such that for each $f \in \Phi$, $FP(\langle a_n \cdot \prod_{t \in H_n} y_{f(n),t} \rangle_{n=1}^\infty) \subseteq A$.*

Proof. See [43, Theorem 2.8] for a proof of the Central Sets Theorem in this form. □

Recall that an IP set is defined in terms of finite products and characterized (via Theorems 5.2 and 5.5) by the property of belonging to some idempotent. Given that the known combinatorial conclusions about central sets are derivable from Theorem 12.11, it was thus a natural question as to whether this theorem in fact characterizes central sets. This question was recently answered in the negative in collaboration with A. Maleki and D. Strauss.

12.12 Theorem. *There is a subset A of $\mathbb{N}$ which satisfies the conclusion of the Central Sets Theorem in $(\mathbb{N},+)$ but is not a central set.*

Proof. [43, Theorem 4.4]. □

In the process of solving this problem, we did arrive at a combinatorial property which does characterize central sets. We conclude this paper with a presentation of this combinatorial characterization of central sets.

The characterization uses the notion of a "tree". In the definition we think of members of ω as ordinals, so that $0 = \emptyset$ and for $n \in \mathbb{N}$, $n = \{0, 1, \ldots, n-1\}$.

12.13 Definition. T is a *tree in A* if and only if T is a set of functions and for each $f \in T$, domain$(f) \in \omega$ and range$(f) \subseteq A$ and if domain$(f) = n > 0$, then $f_{|n-1} \in T$. T is a *tree* if and only if for some A, T is a tree in A.

The last requirement in the definition is not essential. We include the requirement in the definition for aesthetic reasons – it is not nice for branches at some late level to appear from nowhere.

12.14 Definition. (a) Let f be a function with domain$(f) = n \in \omega$ and let x be given. Then $f^\frown x = f \cup \{(n, x)\}$.

(b) Given a tree T and $f \in T$, $B_f = \{x : f^\frown x \in T\}$.

(c) Let S be a semigroup and let $A \subseteq S$. Then T is a *FP-tree in A* if and only if T is a tree in A and for all $f \in T$, $B_f = \{\Pi_{t \in F}\, g(t) : g \in T \text{ and } f \subsetneq g \text{ and } \emptyset \neq F \subseteq \mathrm{dom}(g) \backslash \mathrm{dom}(f)\}$.

The combinatorial characterization of central sets uses the notion of "collectionwise piecewise syndetic" (Definition 9.12).

12.15 Theorem. *Let S be an infinite semigroup and let $A \subseteq S$. Statements (1) and (2) are equivalent and are implied by statement (3). If S is countable, then all three statements are equivalent.*

(1) A is central.

(2) There is a FP-tree T in A such that $\{B_f : f \in T\}$ is collectionwise piecewise syndetic.

(3) There is a decreasing sequence $\langle C_n \rangle_{n=1}^{\infty}$ of subsets of A such that

(a) for each $n \in \mathbb{N}$ and each $x \in C_n$, there exists $m \in \mathbb{N}$ with $C_m \subseteq x^{-1} C_n$ and

(b) $\{C_n : n \in \mathbb{N}\}$ is collectionwise piecewise syndetic.

Proof. [43, Theorem 3.8]. □

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Department of Mathematics,Howard University Washington, DC 20059 USA
E-mail address: nhindman@aol.com