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Largeness of the Set of Finite Products in a Semigroup

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Abstract. We investigate when the set of finite products of distinct terms of a sequence $\langle x_n\rangle_{n=1}^{\infty}$ in a semigroup $(S,\cdot)$ is large in any of several standard notions of largeness. These include *piecewise syndetic*, *central*, *syndetic*, *central**, and *IP**. In the case of a "nice" sequence in $(S,\cdot)=(\mathbb{N},+)$ one has that $FS(\langle x_n\rangle_{n=1}^{\infty})$ has any or all of the first three properties if and only if $\{x_{n+1}-\sum_{t=1}^{n}x_t : n\in\mathbb{N}\}$ is bounded from above.

1. Introduction

Given a discrete semigroup $(S,\cdot)$, the operation can be extended to the Stone-Čech compactification βS of S so that $(\beta S,\cdot)$ is a right topological semigroup with S contained in its topological center. (That is, given any $p\in\beta S$, the function $\rho_p:\beta S\to\beta S$ defined by $\rho_p(q)=q\cdot p$ is continuous and, given any $x\in S$, the function $\lambda_x:\beta S\to\beta S$ defined by $\lambda_x(q)=x\cdot q$ is continuous.) Many powerful applications of this structure to Ramsey Theory have been obtained, beginning with the proof in 1975 by Fred Galvin and Steven Glazer of the Finite Sums Theorem. This theorem, which had been conjectured by Ron Graham and Bruce Rothschild [7], deals with the additive structure of the set $\mathbb{N}$ of positive integers. Given a sequence $\langle x_n\rangle_{n=1}^{\infty}$ in $\mathbb{N}$, $FS(\langle x_n\rangle_{n=1}^{\infty})=\{\sum_{n\in F}x_n : F\in\mathcal{P}_f(\mathbb{N})\}$, where for any set X, $\mathcal{P}_f(X)$ is the set of finite nonempty subsets of X.

1.1 Theorem (Finite Sums Theorem). *Let $r\in\mathbb{N}$ and let $\mathbb{N}=\bigcup_{i=1}^{r}A_i$. There exist $i\in\{1,2,\ldots,r\}$ and a sequence $\langle x_n\rangle_{n=1}^{\infty}$ such that $FS(\langle x_n\rangle_{n=1}^{\infty})\subseteq A_i$.*

The Galvin-Glazer proof of the Finite Sums Theorem used the fact, due to Robert Ellis [5, Corollary 2.10], that any compact right topological semigroup contains an idempotent. The Finite Sums Theorem follows immediately from the following more general fact about sequences in an arbitrary semigroup $(S,\cdot)$. When the operation is written multiplicatively, we write $FP(\langle x_n\rangle_{n=1}^{\infty})=\{\prod_{n\in F}x_n : F\in\mathcal{P}_f(\mathbb{N})\}$ where the products are taken in increasing order of indices.

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1.2 Theorem (Galvin). *Let* $(S, \cdot)$ *be a semigroup and let* $A \subseteq S$*. There exists an idempotent* p *of* βS *with* $p \in c\ell A$ *if and only if there is a sequence* $\langle x_n \rangle_{n=1}^{\infty}$ *in* S *with* $FP(\langle x_n \rangle_{n=1}^{\infty}) \subseteq A$*.*

Proof. [8, Theorem 5.12]. □

Theorem 1.1 is an immediate consequence of Theorem 1.2 because, if $p \in \beta S$, $r \in \mathbb{N}$, and $S = \bigcup_{i=1}^{r} A_i$, then there is some i with $p \in c\ell A_i$.

As a consequence of Theorem 1.2, sets which contain $FP(\langle x_n \rangle_{n=1}^{\infty})$ are interesting objects. The terminology in the following definition is due to Hillel Furstenberg [6], who viewed an IP-set as an "infinite dimensional parallelepiped".

1.3 Definition. Let $(S, \cdot)$ be a semigroup. A set $A \subseteq S$ is an *IP-set* if and only if there is some sequence $\langle x_n \rangle_{n=1}^{\infty}$ with $FP(\langle x_n \rangle_{n=1}^{\infty}) \subseteq A$.

Idempotents in βS are behind another very important notion of largeness, namely *central* sets. These sets are guaranteed to have substantial combinatorial structure. For example, a central subset of $\mathbb{N}$ must contain solutions to any partition regular system of homogenous linear equations with rational coefficients. See [8, Part III] for much more of the structure that must be present in central sets.

A subset J of a semigroup $(T, \cdot)$ is a left ideal if and only if $J \neq \emptyset$ and $T \cdot J \subseteq J$, a right ideal if and only if $J \neq \emptyset$ and $J \cdot T \subseteq J$, and a two sided ideal if and only if it is a left ideal and a right ideal. Any compact right topological semigroup $(T, \cdot)$ has a smallest two sided ideal denoted $K(T)$ and $K(T) = \bigcup\{L : L \text{ is a minimal left ideal of } T\} = \bigcup\{R : R \text{ is a minimal right ideal of } T\}$. Given a minimal left ideal L and a minimal right ideal R, $L \cap R$ is a group. An idempotent p in T is *minimal* if and only if $p \in K(T)$. Notice that if $p \in K(T)$, then Tp is a minimal left ideal of T and pT is a minimal right ideal. (See [4, Chapter 1] or [8, Chapter 2].)

1.4 Definition. Let $(S, \cdot)$ be a semigroup. A set $A \subseteq S$ is a *central* set if and only if there is an idempotent $p \in K(\beta S) \cap c\ell A$.

Thus a subset of S is central if and only if it is a member of a minimal idempotent.

Central sets were originally defined by Furstenberg [6] in terms of notions from topological dynamics. See [8, Section 19.3] for a derivation of the equivalence of the original definition and the one given above and see the notes to that chapter for the history of this derivation.

We introduce now a stronger notion.

1.5 Definition. Let $(S,\cdot)$ be a semigroup. A set $A \subseteq S$ is a *strongly central* set if and only if for every minimal left ideal L of βS, there is an idempotent $p \in L \cap c\ell A$.

The problem that originally caught our attention was the question of when in $(\mathbb{N},+)$ was $FS(\langle x_n\rangle_{n=1}^\infty)$ sufficiently large that its closure met the smallest ideal of $\beta\mathbb{N}$? And when was it even larger, that is when did the closure contain an idempotent in the smallest ideal? It turns out that for sufficiently civilized sequences the answers to those two questions are the same. We shall present these answers in Section 4.

We say that a sequence $\langle x_n\rangle_{n=1}^\infty$ in a semigroup $(S,\cdot)$ satisfies *uniqueness of finite products* provided that whenever $F, G \in \mathcal{P}_f(\mathbb{N})$ and $\prod_{t\in F} x_t = \prod_{t\in G} x_t$, one must have $F = G$. (If the operation is denoted by $+$, we call this property *uniqueness of finite sums*.)

We remark that there is a simple characterization of the abelian groups $(G,\cdot)$ with identity 1 for which $FS(\langle x_n\rangle_{n=1}^\infty)$ is as large as possible. Such a group G contains a sequence $\langle x_n\rangle_{n=1}^\infty$ satisfying uniqueness of finite products such that $FS(\langle x_n\rangle_{n=1}^\infty) = G\setminus\{1\}$ if and only if G has no elements of odd finite order [10, Corollary 4.8].

We shall also be interested in the following notions. Given $A \subseteq S$ and $x \in S$ we write $x^{-1}A = \{y \in S : xy \in A\}$.

1.6 Definition. Let $(S,\cdot)$ be a semigroup and let $A \subseteq S$.

(a) The set A is *thick* if and only if for all $F \in \mathcal{P}_f(S)$ there exists $x \in S$ such that $Fx \subseteq A$.

(b) The set A is *syndetic* if and only if there exists $G \in \mathcal{P}_f(S)$ such that $S = \bigcup_{t\in G} t^{-1}A$.

(c) The set A is *piecewise syndetic* if and only if there exists $G \in \mathcal{P}_f(S)$ such that $\bigcup_{t\in G} t^{-1}A$ is thick.

(d) The set A is *central** if and only if whenever B is a central subset of S, $A\cap B \neq \emptyset$.

(e) The set A is *IP** if and only if whenever B is an IP-set in S, $A\cap B \neq \emptyset$.

Notice that in $(\mathbb{N},+)$ a set A is thick precisely when it contains arbitrarily long blocks, syndetic precisely when there is a bound on the gaps of A, and piecewise syndetic precisely when there is a bound b and arbitrarily long blocks of $\mathbb{N}$ in which A has no gaps longer than b.

All of these notions have simple algebraic characterizations in terms of βS.

1.7 Lemma. *Let S be a semigroup and let $A \subseteq S$.*

(a) The set A is syndetic if and only if for every left ideal L of βS, $L\cap c\ell A \neq \emptyset$.

(b) The set A is thick if and only if there is some left ideal L of βS with $L \subseteq c\ell A$.

(c) The set A is piecewise syndetic if and only if $K(\beta S) \cap c\ell A \neq \emptyset$.

(d) The set A is IP if and only if $c\ell A$ contains all of the idempotents of βS.*

(e) The set A is central if and only if $c\ell A$ contains all of the idempotents of $K(\beta S)$.*

Proof. [2, Lemma 1.9]. □

As a consequence of Lemma 1.7 one sees easily that the following pattern of implications holds. Consult the table on page 24 of [2] to see that in $(\mathbb{N}, +)$ none of the missing implications is valid, except for the ones involving *strongly central*. The example given there which is central but neither thick nor syndetic is obviously also not strongly central. We will give an example of a subset of $\mathbb{N}$ which is strongly central but neither central* nor thick at the conclusion of this section.

$$\begin{tikzcd}
 & \text{IP*} \arrow[d] & \\
 & \text{central*} \arrow[d] & \\
 & \text{strongly central} \arrow[dl] \arrow[d] & \text{thick} \arrow[dl] \\
\text{syndetic} \arrow[d] & \text{central} \arrow[dl] \arrow[dr] & \\
\text{piecewise syndetic} & & \text{IP}
\end{tikzcd}$$

We shall show in this paper that under many circumstances, if an IP-set possesses one of these stronger properties, it must possess others as well. For example, we shall show in Section 2 that for any left cancellative semigroup S, an IP-set which is piecewise syndetic must in fact be central.

For most of our other results we restrict our attention to "nice" sequences. (To quote John Kelley [11, page 112] we are following the "time-honored custom of referring to a problem we cannot handle as abnormal, irregular, improper, degenerate, inadmissible, and otherwise undesirable.")

Section 2 will deal with arbitrary semigroups. In Section 3 we will show that nice sequences are relatively easy to come by. We shall restrict our attention in Section 4 to $(\mathbb{N}, +)$. Many of the results in this paper are based on results in the first author's dissertation [1].

The time has come to worry about what the points of βS are. We take these points to be the ultrafilters on S. We identify the principal ultrafilters with the points of S, and thus pretend that $S \subseteq \beta S$. Given a subset A of S, $\overline{A} = \{p \in \beta S : A \in p\}$. While it is true that $\overline{A} = c\ell A$ (so that $p \in c\ell A$ if and only if $A \in p$), the more important fact is that $\{\overline{A} : A \subseteq S\}$ is a basis for the open sets of βS. Given $p, q \in \beta S$ and $A \subseteq S$, $A \in p\cdot q$ if and only if $\{x \in S : x^{-1}A \in q\} \in p$. (If the operation is written $+$, we write that $A \in p+q$ if and only if $\{x \in S : -x+A \in q\} \in p$, where $-x+A = \{y \in S : x+y \in A\}$.) See [8] for any unfamiliar information about βS or its algebraic structure.

We shall frequently use the fact that if p is an idempotent in βS, then for all $q \in p\cdot\beta S$, $p\cdot q = q$. (To see this, pick $r \in \beta S$ such that $q = p\cdot r$. Then $p\cdot q = p\cdot p\cdot r = p\cdot r = q$.) Likewise, if $q \in \beta S \cdot p$, then $q \cdot p = q$.

Given $x \in \mathbb{N}$, we define $\mathrm{supp}(x) \in \mathcal{P}_f(\omega)$, where $\omega = \mathbb{N} \cup \{0\}$, in terms of the binary expansion of x, by $x = \sum_{t \in \mathrm{supp}(x)} 2^t$.

1.8 Theorem. *Let $A = \{x \in \mathbb{N} : \min \mathrm{supp}(x)$ is odd$\}$. Then A is strongly central but is neither central* nor thick.*

Proof. Let $\mathbb{H} = \bigcap_{n=1}^{\infty} \overline{2^n\mathbb{N}}$. By [8, Lemma 6.8] $\mathbb{H}$ is a compact subsemigroup of $(\beta\mathbb{N}, +)$ which contains all of the idempotents of $(\beta\mathbb{N}, +)$. We claim that $\overline{A} \cap \mathbb{H}$ is a right ideal of $\mathbb{H}$. To see this, let $p \in \overline{A} \cap \mathbb{H}$ and let $q \in \mathbb{H}$. We need to show that $A \in p + q$, which we do by showing that $A \subseteq \{x \in \mathbb{N} : -x + A \in q\}$. Given $x \in A$, let $k = \min \mathrm{supp}(x)$. Then $2^{k+1}\mathbb{N} \in q$ and $2^{k+1}\mathbb{N} \subseteq -x + A$. In an identical fashion one sees that $\overline{\mathbb{N} \setminus A} \cap \mathbb{H}$ is a right ideal of $\mathbb{H}$.

Now let L be a minimal left ideal of $(\beta\mathbb{N}, +)$. Then $L \cap \mathbb{H}$ is a left ideal of $\mathbb{H}$ and so $L \cap \overline{A}$ and $L \cap \overline{\mathbb{N} \setminus A}$ each contain groups, and therefore distinct idempotents. Consequently $\overline{A}$ is strongly central but not central*. Since A is contained in the set of even integers, it is not thick. □

2. Arbitrary Semigroups

In this section we establish some relationships that must hold between the various notions of size for an IP-set in an arbitrary semigroup. We begin with the most general of these, Theorem 2.2, wherein we only require that our specified sequence consist of left cancelable elements.

2.1 Lemma. *Let $(S, \cdot)$ be a semigroup and assume that $\langle x_n \rangle_{n=1}^{\infty}$ is a sequence of left cancelable elements of S and that $K(\beta S) \neq \beta S$. If L is a minimal left ideal of βS, and $L \cap \overline{FP(\langle x_n \rangle_{n=1}^{\infty})} \neq \emptyset$, then there is an idempotent in $L \cap \bigcap_{m=1}^{\infty} \overline{FP(\langle x_n \rangle_{n=m}^{\infty})}$.*

Proof. Since $K(\beta S) \neq \beta S$, no element of $K(\beta S)$ is left cancelable. Indeed, pick $q \in \beta S \setminus K(\beta S)$ and let $p \in K(\beta S)$. Then $R = p \cdot \beta S$ is a minimal right ideal of βS and $p \cdot R = R$ so there is some $r \in R \subseteq K(\beta S)$ such that $p \cdot q = p \cdot r$ and so one cannot cancel p on the left. We shall use the fact [8, Lemma 8.1] that any left cancelable element of S is also left cancelable in βS.

Let L be a minimal left ideal of βS and assume that we have some $q \in L \cap \overline{FP(\langle x_n \rangle_{n=1}^{\infty})}$. By [8, Lemma 5.11], $\bigcap_{m=1}^{\infty} \overline{FP(\langle x_n \rangle_{n=m}^{\infty})}$ is a subsemigroup of βS so it suffices to show that for each $m \in \mathbb{N}$, $L \cap \overline{FP(\langle x_n \rangle_{n=m}^{\infty})} \neq \emptyset$. (Minimal left ideals are closed, and so we then conclude that $L \cap \bigcap_{m=1}^{\infty} \overline{FP(\langle x_n \rangle_{n=m}^{\infty})} \neq \emptyset$ and so $L \cap \bigcap_{m=1}^{\infty} \overline{FP(\langle x_n \rangle_{n=m}^{\infty})}$ is a compact right topological semigroup so has an idempotent.) To this end, let $m \in \mathbb{N}$ with $m > 1$. Then

$$\begin{aligned} FP(\langle x_n \rangle_{n=1}^{\infty}) = {} & FP(\langle x_n \rangle_{n=m}^{\infty}) \cup FP(\langle x_n \rangle_{n=1}^{m-1}) \cup \\ & \bigcup \{ t \cdot FP(\langle x_n \rangle_{n=m}^{\infty}) : t \in FP(\langle x_n \rangle_{n=1}^{m-1}) \} \,. \end{aligned}$$

So we must have one of

(i) $FP(\langle x_n \rangle_{n=m}^{\infty}) \in q$,

(ii) $FP(\langle x_n \rangle_{n=1}^{m-1}) \in q$, or

(iii) $t \cdot FP(\langle x_n \rangle_{n=m}^{\infty}) \in q$ for some $t \in FP(\langle x_n \rangle_{n=1}^{m-1})$.

If $FP(\langle x_n \rangle_{n=m}^{\infty}) \in q$, then $q \in L \cap \overline{FP(\langle x_n \rangle_{n=m}^{\infty})}$.

Suppose now that one has $FP(\langle x_n \rangle_{n=1}^{m-1}) \in q$. Then $q = \prod_{n \in F} x_n$ for some F with $\emptyset \neq F \subseteq \{1, 2, \ldots, m-1\}$. (Recall that we are identifying the principal ultrafilters with the points of S.) But then q, being the product of left cancelable elements, is left cancelable, and $q \in L \subseteq K(\beta S)$, while there are no left cancelable elements of $K(\beta S)$, a contradiction.

Finally, assume that we have $t \in FP(\langle x_n \rangle_{n=1}^{m-1})$ such that $t \cdot FP(\langle x_n \rangle_{n=m}^{\infty}) \in q$. Since $t \in S$, λ_t is continuous so $\overline{t \cdot FP(\langle x_n \rangle_{n=m}^{\infty})} = t \cdot \overline{FP(\langle x_n \rangle_{n=m}^{\infty})}$ so pick $r \in \overline{FP(\langle x_n \rangle_{n=m}^{\infty})}$ such that $q = t \cdot r$. Pick an idempotent $p \in L$. Then $q \cdot p = q$ so $t \cdot r \cdot p = t \cdot r$. Now t is left cancelable and therefore $r \cdot p = r$. Since $p \in L$, $r \in L$ so $r \in L \cap \overline{FP(\langle x_n \rangle_{n=m}^{\infty})}$. □

2.2 Theorem. *Let $(S, \cdot)$ be a semigroup and assume that $\langle x_n \rangle_{n=1}^{\infty}$ is a sequence of left cancelable elements of S. Then the following statements are equivalent.*

(a) $FP(\langle x_n \rangle_{n=1}^{\infty})$ is piecewise syndetic.

(b) For all $m \in \mathbb{N}$, $FP(\langle x_n \rangle_{n=m}^{\infty})$ is central. In fact, there exists an idempotent in $K(\beta S) \cap \bigcap_{m=1}^{\infty} \overline{FP(\langle x_n \rangle_{n=m}^{\infty})}$.

Proof. That (b) implies (a) is trivial.

To see that (a) implies (b) assume first that $K(\beta S) = \beta S$. By [8, Lemma 5.11] there is an idempotent p of βS such that for all $m \in \mathbb{N}$, $FP(\langle x_n \rangle_{n=m}^{\infty}) \in p$. Since $p \in K(\beta S)$, we have that for all $m \in \mathbb{N}$, $FP(\langle x_n \rangle_{n=m}^{\infty})$ is central.

Now assume that $K(\beta S) \neq \beta S$. Since $FP(\langle x_n \rangle_{n=1}^{\infty})$ is piecewise syndetic and $K(\beta S) = \bigcup\{L : L \text{ is a minimal left ideal of } \beta S\}$, pick a minimal left ideal L of βS such that $L \cap \overline{FP(\langle x_n \rangle_{n=1}^{\infty})} \neq \emptyset$. Lemma 2.1 applies. □

We shall show in Theorem 2.4 that if a sequence $\langle x_n \rangle_{n=1}^{\infty}$ of left cancelable elements satisfies uniqueness of finite products and $FP(\langle x_n \rangle_{n=1}^{\infty})$ is piecewise syndetic, the sequence cannot be substantially thinned without losing piecewise syndeticity.

Recall that a *right zero* semigroup R is a semigroup such that $xy = y$ for all $x, y \in R$.

2.3 Lemma. *Let $(S, \cdot)$ be a semigroup. If there exists a sequence $\langle x_n \rangle_{n=1}^{\infty}$ of left cancelable elements of S which satisfies uniqueness of finite products, then $K(\beta S) \neq \beta S$.*

Proof. Suppose that $\langle x_n \rangle_{n=1}^{\infty}$ is a sequence of left cancelable elements which satisfies uniqueness of finite products and $K(\beta S) = \beta S$. Recall that by [8, Lemma 8.1], any left cancelable element of S is also left cancelable in βS, so βS has left cancelable elements. By [9, Theorem 5] there exist a finite group G and a right zero semigroup R such that S is isomorphic to $G \times R$, so we shall assume that $S = G \times R$. For each n, let $x_n = (a_n, b_n)$. Pick $n \neq s$ such that $a_n = a_s$ and pick $r \in \mathbb{N} \setminus \{n, s\}$. Then $x_n \cdot x_r = (a_n \cdot a_r, b_r) = (a_s \cdot a_r, b_r) = x_s \cdot x_r$. This contradicts the assumption that the sequence $\langle x_n \rangle_{n=1}^{\infty}$ satisfies uniqueness of finite products. □

2.4 Theorem. *Let $(S, \cdot)$ be a semigroup, let $\langle x_n \rangle_{n=1}^{\infty}$ be a sequence of left cancelable elements of S which satisfies uniqueness of finite products, and let $\langle y_n \rangle_{n=1}^{\infty}$ be a subsequence of $\langle x_n \rangle_{n=1}^{\infty}$. The following statements are equivalent.*

(a) $FP(\langle y_n \rangle_{n=1}^{\infty})$ is piecewise syndetic.

(b) $FP(\langle x_n \rangle_{n=1}^{\infty})$ is piecewise syndetic and $\{m \in \mathbb{N} : x_m \notin \{y_n : n \in \mathbb{N}\}\}$ is finite.

Proof. $(a) \Rightarrow (b)$. Trivially $FP(\langle x_n \rangle_{n=1}^{\infty})$ is piecewise syndetic. Suppose that $M = \{m \in \mathbb{N} : x_m \notin \{y_n : n \in \mathbb{N}\}\}$ is infinite and pick $q \in \overline{\{x_n : n \in \mathbb{N}\} \setminus \{y_n : n \in \mathbb{N}\}} \setminus S$. Now $K(\beta S) \cap \overline{FP(\langle y_n \rangle_{n=1}^{\infty})} \neq \emptyset$ so pick a minimal left ideal L of βS such that $L \cap \overline{FP(\langle y_n \rangle_{n=1}^{\infty})} \neq \emptyset$. By Lemma 2.3 $K(\beta S) \neq \beta S$ so by Lemma 2.1 pick an idempotent $p \in L \cap \bigcap_{m=1}^{\infty} \overline{FP(\langle y_n \rangle_{n=m}^{\infty})}$. Then $p \in K(\beta S) \cap \bigcap_{m=1}^{\infty} \overline{FP(\langle x_n \rangle_{n=m}^{\infty})}$. So by [8, Theorem 1.65], $p \in K\big(\bigcap_{m=1}^{\infty} \overline{FP(\langle x_n \rangle_{n=m}^{\infty})}\big)$. Also given $m \in \mathbb{N}$, $\{x_n : n \geq m\} \in q$ so

$q \in \bigcap_{m=1}^{\infty} \overline{FP(\langle x_n\rangle_{n=m}^{\infty})}$. By [8, Theorem 2.10] there is some $r \in \bigcap_{m=1}^{\infty} \overline{FP(\langle x_n\rangle_{n=m}^{\infty})}$ such that $p = rqp$.

Now $FP(\langle y_n\rangle_{n=1}^{\infty}) \in p$ so pick $B \in r$ such that $\overline{B}qp \subseteq \overline{FP(\langle y_n\rangle_{n=1}^{\infty})}$ and pick $z \in B \cap FP(\langle x_n\rangle_{n=1}^{\infty})$. Pick $F \in \mathcal{P}_f(\mathbb{N})$ such that $z = \prod_{n\in F} x_n$ and let $l = \max F$. Pick $C \in q$ such that $z\overline{C}p \subseteq \overline{FP(\langle y_n\rangle_{n=1}^{\infty})}$. Then $\{x_k : k \in M \text{ and } k > l\} \in q$ so pick $k \in M$ with $k > l$ such that $zx_kp \in \overline{FP(\langle y_n\rangle_{n=1}^{\infty})}$. Now $FP(\langle x_n\rangle_{n=k+1}^{\infty}) \in p$ so pick $G \in \mathcal{P}_f(\mathbb{N})$ with $\min G > k$ such that $zx_k \prod_{n\in G} x_n \in FP(\langle y_n\rangle_{n=1}^{\infty})$. Pick $H \in \mathcal{P}_f(\mathbb{N}\setminus M)$ such that $zx_k \prod_{n\in G} x_n = \prod_{n\in H} x_n$. Then $\prod_{n\in F\cup\{k\}\cup G} x_n = \prod_{n\in H} x_n$ so $F \cup \{k\} \cup G = H$. This is a contradiction since $k \in M$.

$(b) \Rightarrow (a)$. Pick a minimal left ideal L of βS such that $L \cap \overline{FP(\langle x_n\rangle_{n=1}^{\infty})} \neq \emptyset$ and pick $m \in \mathbb{N}$ such that $FP(\langle x_n\rangle_{n=m}^{\infty}) \subseteq FP(\langle y_n\rangle_{n=1}^{\infty})$. By Lemma 2.3 $K(\beta S) \neq \beta S$ so by Lemma 2.1 $L \cap \overline{FP(\langle x_n\rangle_{n=m}^{\infty})} \neq \emptyset$ so $L \cap \overline{FP(\langle y_n\rangle_{n=1}^{\infty})} \neq \emptyset$. □

For the remainder of our results, we need to assume that we are dealing with "nice" sequences. These sequences satisfy uniqueness of finite products and another technical condition. We shall see in the next section that there are many examples of nice sequences.

2.5 Definition. Let $(S, \cdot)$ be a semigroup, let I be the set of left identities of S, and let $\langle x_n\rangle_{n=1}^{\infty}$ be a sequence in S. Then $\langle x_n\rangle_{n=1}^{\infty}$ is *nice* if and only if

(a) $\langle x_n\rangle_{n=1}^{\infty}$ satisfies uniqueness of finite products and

(b) for all $s \in S \setminus (FP(\langle x_n\rangle_{n=1}^{\infty}) \cup I)$, there is some $k \in \mathbb{N}$ such that $FP(\langle x_n\rangle_{n=1}^{\infty}) \cap s \cdot FP(\langle x_n\rangle_{n=k}^{\infty}) = \emptyset$.

It will occasionally be useful to note that $FP(\langle x_n\rangle_{n=1}^{\infty}) \cap s \cdot FP(\langle x_n\rangle_{n=k}^{\infty}) = \emptyset$ if and only if $s^{-1}FP(\langle x_n\rangle_{n=1}^{\infty}) \cap FP(\langle x_n\rangle_{n=k}^{\infty}) = \emptyset$.

2.6 Lemma. *Let $(S, \cdot)$ be a semigroup and let $\langle x_n\rangle_{n=1}^{\infty}$ be a nice sequence in S. Any subsequence of $\langle x_n\rangle_{n=1}^{\infty}$ is also nice.*

Proof. Let $\langle y_n\rangle_{n=1}^{\infty}$ be a subsequence of $\langle x_n\rangle_{n=1}^{\infty}$ and pick an increasing $f : \mathbb{N} \to \mathbb{N}$ such that for each n, $y_n = x_{f(n)}$. Let I be the set of left identities of S. Let $s \in S\setminus(FP(\langle y_n\rangle_{n=1}^{\infty}) \cup I)$. If $s \notin FP(\langle x_n\rangle_{n=1}^{\infty})$, we may pick k as guaranteed by the definition of nice. Then $FP(\langle y_n\rangle_{n=1}^{\infty}) \cap s \cdot FP(\langle y_n\rangle_{n=k}^{\infty}) \subseteq FP(\langle x_n\rangle_{n=1}^{\infty}) \cap s \cdot FP(\langle x_n\rangle_{n=k}^{\infty}) = \emptyset$. So assume that $s = \prod_{n\in F} x_n$ for some $F \in \mathcal{P}_f(\mathbb{N})$. Let $k = \max F + 1$ and suppose that we have some $z \in s^{-1}FP(\langle y_n\rangle_{n=1}^{\infty}) \cap FP(\langle y_n\rangle_{n=k}^{\infty})$. Pick $G \in \mathcal{P}_f(\mathbb{N})$ with $\min G \geq k$ such that $z = \prod_{n\in G} y_n = \prod_{n\in f[G]} x_n$. Then $\min f[G] \geq k > \max F$, so $sz = \prod_{n\in F\cup f[G]} x_n$. Also, $sz \in FP(\langle y_n\rangle_{n=1}^{\infty})$ so pick $H \in \mathcal{P}_f(\mathbb{N})$ such that $sz = \prod_{n\in H} y_n = \prod_{n\in f[H]} x_n$.

Then by the uniqueness of finite products, we have that $f[H] = F \cup f[G]$. In particular, $F \subseteq f[H]$ and so $s \in FP(\langle y_n \rangle_{n=1}^{\infty})$, a contradiction. $\square$

We have need of the following algebraic lemma.

2.7 Lemma. *Let $(S, \cdot)$ be a semigroup, let $k \in \mathbb{N}$, and let $\langle x_n \rangle_{n=1}^{\infty}$ be a nice sequence in S. If I is the set of left identities of S, $p \in \bigcap_{m=1}^{\infty} \overline{FP(\langle x_n \rangle_{n=m}^{\infty})}$, $q \in \beta S \setminus \overline{I}$, and $q \cdot p \in \overline{FP(\langle x_n \rangle_{n=k}^{\infty})}$, then $q \in \overline{FP(\langle x_n \rangle_{n=k}^{\infty})}$.*

Proof. We need to show that $FP(\langle x_n \rangle_{n=k}^{\infty}) \in q$. We have that $I \notin q$ and that $\{s \in S : s^{-1}FP(\langle x_n \rangle_{n=k}^{\infty}) \in p\} \in q$ so it suffices to show that

$$\{s \in S \setminus I : s^{-1}FP(\langle x_n \rangle_{n=k}^{\infty}) \in p\} \subseteq FP(\langle x_n \rangle_{n=k}^{\infty}) .$$

So let $s \in S \setminus I$ such that $s^{-1}FP(\langle x_n \rangle_{n=k}^{\infty}) \in p$ and suppose that $s \notin FP(\langle x_n \rangle_{n=k}^{\infty})$. Since $\langle x_n \rangle_{n=k}^{\infty}$ is a subsequence of $\langle x_n \rangle_{n=1}^{\infty}$, we may pick by Lemma 2.6 some $r \geq k$ such that $s^{-1}FP(\langle x_n \rangle_{n=k}^{\infty}) \cap FP(\langle x_n \rangle_{n=r}^{\infty}) = \emptyset$. But this is a contradiction because $FP(\langle x_n \rangle_{n=r}^{\infty}) \in p$. $\square$

The following algebraic result is of some interest in its own right, and has as an immediate consequence that for IP-sets generated by nice sequences of left cancelable elements, piecewise syndetic implies strongly central.

2.8 Theorem. *Let $(S, \cdot)$ be a semigroup and let $\langle x_n \rangle_{n=1}^{\infty}$ be a nice sequence of left cancelable elements of S. If $FP(\langle x_n \rangle_{n=1}^{\infty})$ is piecewise syndetic, then there is a minimal right ideal R of βS such that every idempotent of R is in $\bigcap_{m=1}^{\infty} \overline{FP(\langle x_n \rangle_{n=m}^{\infty})}$.*

Proof. Pick by Theorem 2.2 some idempotent $p \in K(\beta S) \cap \bigcap_{m=1}^{\infty} \overline{FP(\langle x_n \rangle_{n=m}^{\infty})}$. Let $R = p \cdot \beta S$. Then R is a minimal right ideal of βS. Let q be an idempotent in R. Let I be the set of left identities of S. We claim that $q \in \beta S \setminus \overline{I}$. Indeed, suppose $q \in \overline{I}$. Given any $s \in S$, ρ_s is constantly equal to s on I and thus $qs = s$. Since $q \in K(\beta S)$, we then have that $S \subseteq K(\beta S)$. Pick $e \in I$. Then λ_e agrees with the identity function on S and thus on βS so that e is a left identity of βS. Thus $K(\beta S) = \beta S$. But this contradicts Lemma 2.3.

We have that $p \in R = q \cdot \beta S$ so $p = q \cdot p$. By Lemma 2.7, $q \in \bigcap_{m=1}^{\infty} \overline{FP(\langle x_n \rangle_{n=m}^{\infty})}$. $\square$

2.9 Corollary. *Let $(S, \cdot)$ be a semigroup and let $\langle x_n \rangle_{n=1}^{\infty}$ be a nice sequence of left cancelable elements of S. If $FP(\langle x_n \rangle_{n=1}^{\infty})$ is piecewise syndetic, then for each $m \in \mathbb{N}$, $FP(\langle x_n \rangle_{n=m}^{\infty})$ is strongly central.*

Proof. Let $m \in \mathbb{N}$. We need to show that for every minimal left ideal L of βS, there is an idempotent in $L \cap \overline{FP(\langle x_n \rangle_{n=m}^{\infty})}$. So let L be a minimal left ideal of βS. Pick R as guaranteed by Theorem 2.8. Then $L \cap R$ is a group so there is an idempotent $q \in L \cap R$. By Theorem 2.8, $q \in \overline{FP(\langle x_n \rangle_{n=m}^{\infty})}$. $\square$

We summarize what we have shown so far.

2.10 Theorem. *Let $(S, \cdot)$ be a semigroup and let $\langle x_n \rangle_{n=1}^{\infty}$ be a nice sequence of left cancelable elements of S. The following statements are equivalent.*

(a) $FP(\langle x_n \rangle_{n=1}^{\infty})$ *is piecewise syndetic.*
(b) For all $m \in \mathbb{N}$, $FP(\langle x_n \rangle_{n=m}^{\infty})$ *is piecewise syndetic.*
(c) $FP(\langle x_n \rangle_{n=1}^{\infty})$ *is strongly central.*
(d) For all $m \in \mathbb{N}$, $FP(\langle x_n \rangle_{n=m}^{\infty})$ *is strongly central.*
(e) $FP(\langle x_n \rangle_{n=1}^{\infty})$ *is syndetic.*
(f) For all $m \in \mathbb{N}$, $FP(\langle x_n \rangle_{n=m}^{\infty})$ *is syndetic.*
(g) $FP(\langle x_n \rangle_{n=1}^{\infty})$ *is central.*
(h) For all $m \in \mathbb{N}$, $FP(\langle x_n \rangle_{n=m}^{\infty})$ *is central.*

Proof. That $(a) \Rightarrow (d)$ is Corollary 2.9. The rest of the required implications are trivial. $\square$

Now we see that for IP-sets generated by nice sequences, if S has no left identities, then the notions of central* and IP* are equivalent. We shall see in Theorem 4.4 that there is a nice sequence $\langle d_n \rangle_{n=1}^{\infty}$ in $(\mathbb{N}, +)$ such that $FS(\langle d_n \rangle_{n=1}^{\infty})$ is syndetic (and thus central) but not central*.

2.11 Theorem. *Let $(S, \cdot)$ be a semigroup, let I be the set of left identities of S, and let $\langle x_n \rangle_{n=1}^{\infty}$ be a nice sequence of left cancelable elements of S. If $FP(\langle x_n \rangle_{n=1}^{\infty})$ is central*, then for every idempotent $q \in \beta S \setminus \overline{I}$, $q \in \overline{FP(\langle x_n \rangle_{n=1}^{\infty})}$.*

Proof. By Lemma 2.3, $K(\beta S) \neq \beta S$. Since each left ideal of βS contains an idempotent in $K(\beta S)$, we have by Lemma 2.1, that if L is a left ideal of βS, then there is an idempotent in $L \cap \bigcap_{m=1}^{\infty} \overline{FP(\langle x_n \rangle_{n=m}^{\infty})}$. Now let q be an idempotent in $\beta S \setminus \overline{I}$. Pick a minimal left ideal L of βS such that $L \subseteq \beta S \cdot q$ and pick an idempotent $p \in L \cap \bigcap_{m=1}^{\infty} \overline{FP(\langle x_n \rangle_{n=m}^{\infty})}$. Since $p \in \beta S \cdot q$, we have that $p = p \cdot q$. Therefore $q \cdot p \cdot q \cdot p = q \cdot p \cdot p = q \cdot p$. Then $q \cdot p$ is an idempotent in $L \subseteq K(\beta S)$. Therefore $q \cdot p \in \overline{FP(\langle x_n \rangle_{n=1}^{\infty})}$. By Lemma 2.7 we have that $q \in \overline{FP(\langle x_n \rangle_{n=1}^{\infty})}$. $\square$

2.12 Theorem. *There is a nice sequence* $\langle x_n\rangle_{n=1}^\infty$ *in* $(\mathbb{Z},+)$ *such that* $FS(\langle x_n\rangle_{n=1}^\infty)$ *is central* but not IP*.*

Proof. For each $n\in\mathbb{N}$, let $x_n=(-2)^n$. Then $\langle x_n\rangle_{n=1}^\infty$ satisfies uniqueness of finite sums, and $FS(\langle x_n\rangle_{n=1}^\infty)=2\mathbb{Z}\setminus\{0\}$. If $s\in S\setminus(FS(\langle x_n\rangle_{n=1}^\infty)\cup\{0\})$, then s is odd so $FS(\langle x_n\rangle_{n=1}^\infty)\cap\big(s+FS(\langle x_n\rangle_{n=1}^\infty)\big)=\emptyset$. $\square$

We now turn our attention to thickness. We see that for nice sequences $\langle x_n\rangle_{n=1}^\infty$, $FP(\langle x_n\rangle_{n=1}^\infty)$ is essentially never thick.

2.13 Theorem. *Let* $(S,\cdot)$ *be a semigroup, let* I *be the set of left identities for* S*, and let* $\langle x_n\rangle_{n=1}^\infty$ *be a nice sequence of left cancellable elements of* S*. If* $FP(\langle x_n\rangle_{n=1}^\infty)$ *is thick, then* $S\setminus I=FP(\langle x_n\rangle_{n=1}^\infty)$*.*

Proof. If a left identity is in $FP(\langle x_n\rangle_{n=1}^\infty)$ then $\langle x_n\rangle_{n=1}^\infty$ cannot satisfy uniqueness of finite products, so $FP(\langle x_n\rangle_{n=1}^\infty)\subseteq S\setminus I$. To verify the other inclusion, pick a minimal left ideal L of βS such that $L\subseteq\overline{FP(\langle x_n\rangle_{n=1}^\infty)}$. By Lemma 2.3 we have that $K(\beta S)\neq\beta S$ so by Lemma 2.1 there is an idempotent $p\in L\cap\bigcap_{m=1}^\infty\overline{FP(\langle x_n\rangle_{n=m}^\infty)}$.

Suppose that $FP(\langle x_n\rangle_{n=1}^\infty)\neq S\setminus I$ and pick $s\in S\setminus(FP(\langle x_n\rangle_{n=1}^\infty)\cup I)$. Pick $k\in\mathbb{N}$ such that $s^{-1}FP(\langle x_n\rangle_{n=1}^\infty)\cap FP(\langle x_n\rangle_{n=k}^\infty)=\emptyset$. Now $s\cdot p\in L\subseteq\overline{FP(\langle x_n\rangle_{n=1}^\infty)}$ so $s^{-1}FP(\langle x_n\rangle_{n=1}^\infty)\in p$ and $FP(\langle x_n\rangle_{n=k}^\infty)\in p$, a contradiction. $\square$

The next result shows that niceness is needed for the conclusion of Theorem 2.13.

2.14 Theorem. *There is a sequence* $\langle x_n\rangle_{n=1}^\infty$ *in* $(\mathbb{N},\cdot)$ *which satisfies uniqueness of finite products such that* $FP(\langle x_n\rangle_{n=1}^\infty)$ *is thick but not syndetic.*

Proof. Let $\langle x_n\rangle_{n=1}^\infty$ enumerate $\{2^{2^t}:t\in\omega\}\cup\{(2p)^{2^t}:p$ is an odd prime and $t\in\omega\}$. Then $FP(\langle x_n\rangle_{n=1}^\infty)$ is the set of positive integers x such that the number of factors of 2 in x is at least half of the length of the prime factorization of x. Given any $F\in\mathcal{P}_f(\mathbb{N}\setminus\{1\})$, let k be the maximum of the lengths of the prime factorizations of members of F. Then $F\cdot 2^k\subseteq FP(\langle x_n\rangle_{n=1}^\infty)$. So $FP(\langle x_n\rangle_{n=1}^\infty)$ is thick.

To see that $FP(\langle x_n\rangle_{n=1}^\infty)$ is not syndetic, suppose one has $G\in\mathcal{P}_f(\mathbb{N})$ such that $\mathbb{N}\subseteq\bigcup_{t\in G}t^{-1}FP(\langle x_n\rangle_{n=1}^\infty)$. Let k be the maximum of the lengths of the prime factorizations of members of G. Then $3^{k+1}\notin\bigcup_{t\in G}t^{-1}FP(\langle x_n\rangle_{n=1}^\infty)$. $\square$

3. Producing Nice Sequences

In this section we show that nice sequences are reasonably plentiful. We shall see as an immediate consequence of Theorem 3.2 that in $(\mathbb{N},+)$, any sequence $\langle x_n\rangle_{n=1}^\infty$ which has the property that $x_{n+1}>\sum_{t=1}^n x_t$ for all n is nice.

3.1 Lemma. *Let* $(S,\cdot)$ *be a semigroup and let* $\langle x_n\rangle_{n=1}^{\infty}$ *be a sequence of right cancelable elements of* S. *Let* $\varphi:(S,\cdot)\to(\mathbb{N},+)$ *be a homomorphism such that for all* n, $\sum_{t=1}^{n}\varphi(x_t)<\varphi(x_{n+1})$. *If* $s\in S$, $k\in\mathbb{N}$, $H,G\in\mathcal{P}_f(\mathbb{N})$, $\varphi(s)\leq\sum_{t=1}^{k-1}\varphi(x_t)$, $k\leq\min H$, *and* $s\cdot\prod_{t\in H}x_t=\prod_{t\in G}x_t$, *then* $H\subseteq G$ *and* $s=\prod_{t\in G\setminus H}x_t$.

Proof. We proceed by induction on $|H|$. Assume first that $H=\{l\}$ where $l\geq k$. Then $\sum_{t\in G}\varphi(x_t)=\varphi(s)+\varphi(x_l)\leq\sum_{t=1}^{l}\varphi(x_t)$ so $\max G\leq l$. If we had $\max G<l$, then we would have $\sum_{t\in G}\varphi(x_t)\leq\sum_{t=1}^{l-1}\varphi(x_t)<\varphi(x_l)<\varphi(s)+\varphi(x_l)$, a contradiction. Thus $\max G=l$. Since $\prod_{t\in G}x_t=s\cdot x_l$ and x_l is right cancelable, we have that $s=\prod_{t\in G\setminus\{l\}}x_t$.

Now assume that $|H|>1$ and the lemma is valid for smaller sets. Let $v=\max H$ and let $F=H\setminus\{v\}$. Then $\sum_{t\in G}\varphi(x_t)=\varphi(s)+\sum_{t\in H}\varphi(x_t)\leq\sum_{t=1}^{v}\varphi(x_t)$ so $\max G\leq v$. If $\max G<v$, then $\sum_{t\in G}\varphi(x_t)<\varphi(x_v)<\varphi(s)+\sum_{t\in H}\varphi(x_t)$, a contradiction. So $v=\max G$. Since x_v is right cancelable, we have that $s\cdot\prod_{t\in F}x_t=\prod_{t\in G\setminus\{v\}}x_t$ and so the induction hypothesis applies. □

3.2 Theorem. *Let* $(S,\cdot)$ *be a semigroup and let* $\varphi:(S,\cdot)\to(\mathbb{N},+)$ *be a homomorphism. Let* $\langle x_n\rangle_{n=1}^{\infty}$ *be a sequence in* S *with the property that for each* $m\in\mathbb{N}$, $\sum_{t=1}^{m}\varphi(x_t)<\varphi(x_{m+1})$. *If either* φ *is injective or* x_m *is right cancelable for each* $m\in\mathbb{N}$, *then* $\langle x_n\rangle_{n=1}^{\infty}$ *is a nice sequence.*

Proof. It is routine to establish that if $\langle y_n\rangle_{n=1}^{\infty}$ is a sequence in $\mathbb{N}$ with the property that for each $m\in\mathbb{N}$, $y_{m+1}>\sum_{t=1}^{m}y_t$, then $\langle y_n\rangle_{n=1}^{\infty}$ satisfies uniqueness of finite sums. Consequently, we have directly that $\langle x_n\rangle_{n=1}^{\infty}$ satisfies uniqueness of finite products in S.

Let $s\in S\setminus FP(\langle x_n\rangle_{n=1}^{\infty})$ and pick $k\in\mathbb{N}$ such that $\varphi(s)<\sum_{t=1}^{k-1}\varphi(x_t)$. Suppose that we have some $y\in s^{-1}FP(\langle x_n\rangle_{n=1}^{\infty})\cap FP(\langle x_n\rangle_{n=k}^{\infty})$. Pick $H\in\mathcal{P}_f(\mathbb{N})$ with $\min H\geq k$ such that $y=\prod_{t\in H}x_t$ and pick $G\in\mathcal{P}_f(\mathbb{N})$ such that $s\cdot y=\prod_{t\in G}x_t$. If each x_t is right cancelable we have directly from Lemma 3.1 that $s=\prod_{t\in G\setminus H}x_t$. So assume that φ is injective. Then since $\varphi(s)+\sum_{t\in H}\varphi(x_t)=\sum_{t\in G}\varphi(x_t)$ we have by Lemma 3.1 that $\varphi(s)=\sum_{t\in G\setminus H}\varphi(x_t)$. Since φ is injective, $s=\prod_{t\in G\setminus H}x_t$. Thus in either case, we have a contradiction. □

If the semigroup $(S,\cdot)$ is commutative, we may unambiguously write $\prod F$ for the product of the elements of the finite nonempty subset F of S. In this case, given $B\subseteq S$, $FP(B)=\{\prod F:F\in\mathcal{P}_f(B)\}$. The following lemma, whose routine proof we omit, says that in a commutative semigroup, niceness of a sequence depends only on its range (and the obvious fact that any nice sequence in any semigroup is injective).

3.3 Lemma. *Let* $(S,\cdot)$ *be a commutative semigroup and let* $\langle x_n\rangle_{n=1}^\infty$ *be a sequence in* S. *If* S *has an identity* e, *let* $I=\{e\}$ *and otherwise let* $I=\emptyset$. *Then* $\langle x_n\rangle_{n=1}^\infty$ *is nice if and only if* $A=\{x_n:n\in\mathbb{N}\}$ *has the following properties.*

(a) *For all* $F,G\in\mathcal{P}_f(A)$, *if* $\prod F=\prod G$, *then* $F=G$.

(b) *For all* $s\in S\setminus(FP(A)\cup I)$ *there exists* $G\in\mathcal{P}_f(A)$ *such that* $s^{-1}FP(A)\cap FP(A\setminus G)=\emptyset$.

We consider now the very noncommutative free semigroups on finite or countably infinite semigroups. (The free semigroup S on the alphabet A is the set of words over A with concatenation as the operation. We identify A with the length 1 words.)

3.4 Theorem. *Let* A *be a nonempty countable alphabet, let* $\emptyset\neq B\subseteq A$, *let* S *be the free semigroup on the alphabet* A, *and let* $T=B\cup\bigcup_{b\in B}bS=\{w\in S:$ *the leftmost letter of* w *is in* $B\}$. *There is a nice sequence* $\langle x_n\rangle_{n=1}^\infty$ *in* S *such that* $FP(\langle x_n\rangle_{n=1}^\infty)=T$. *For each* $k\in\mathbb{N}$, $FP(\langle x_n\rangle_{n=k}^\infty)$ *is syndetic.*

Proof. Let $\varphi:A\to\mathbb{N}$ be a finite-to-one function. Extend φ to all of S by $\varphi(w)=\sum_{i=1}^n\varphi(a_i)$ whenever $w=a_1a_2\cdots a_n$ with each $a_i\in A$. Notice that if w is a proper subword of u, then $\varphi(w)<\varphi(u)$. Notice also that if $k\in\mathbb{N}$, then $\{w\in S:\varphi(w)\le k\}$ is finite. Let $\gamma:\mathbb{N}\to T$ be an enumeration of T subject to the restriction that if $i<j$, then $\varphi\big(\gamma(i)\big)\le\varphi\big(\gamma(j)\big)$.

Define $\langle x_n\rangle_{n=1}^\infty$ inductively as follows. Let $x_1=\gamma(1)$. Having chosen $x_1,x_2,\ldots,x_{n-1}$, let $j=\min\{i\in\mathbb{N}:\gamma(i)\notin FP(\langle x_t\rangle_{t=1}^{n-1})\}$ and let $x_n=\gamma(j)$. Notice that if $x_n=\gamma(j)$ and $x_{n+1}=\gamma(k)$, then $j<k$ so that $\varphi(x_n)\le\varphi(x_{n+1})$. Trivially $FP(\langle x_n\rangle_{n=1}^\infty)=T$.

We next claim that if $m\in\mathbb{N}$, $F\in\mathcal{P}_f(\mathbb{N})$, and $x_m=\prod_{n\in F}x_n$, then $F=\{m\}$. If $|F|=1$, this is immediate, so suppose that $|F|>1$. Then for each $n\in F$, x_n is a proper subword of x_m and consequently $\varphi(x_n)<\varphi(x_m)$ so that $n<m$. Thus $x_m\in FP(\langle x_n\rangle_{n=1}^{m-1})$, a contradiction.

We now verify that $\langle x_n\rangle_{n=1}^\infty$ satisfies uniqueness of finite products. Suppose instead that we have $F\neq G$ in $\mathcal{P}_f(\mathbb{N})$ such that $\prod_{n\in F}x_n=\prod_{n\in G}x_n$. We may choose such F and G with $|F|+|G|$ as small as possible. As we have observed above, $|F|>1$ and $|G|>1$. Let $m=\max F$ and let $s=\max G$. Given $w\in S$, let $l(w)$ denote the length of w. We may assume that $l(x_m)\le l(x_s)$. If we had $l(x_m)=l(x_s)$ we would have $x_m=x_s$ and consequently $\prod_{n\in F\setminus\{m\}}x_n=\prod_{n\in G\setminus\{s\}}x_n$, contradicting the choice of F and G. Thus we have that $l(x_m)<l(x_s)$. Let r be the smallest member of F such that, if $H=\{n\in F:n\ge r\}$, then $\sum_{n\in H}l(x_n)<l(x_s)$. Let $k=\max F\setminus H$. Then $l(x_s)\le\sum_{n\in H\cup\{k\}}l(x_n)$. But if we had $l(x_s)=\sum_{n\in H\cup\{k\}}l(x_n)$, we would have $x_s=$

$\prod_{n\in H\cup\{k\}} x_n$, which we have seen is impossible. Therefore $l(x_s) < \sum_{n\in H\cup\{k\}} l(x_n)$. Let $v = l(x_s) - \sum_{n\in H} l(x_n)$. Then $v < l(x_k)$. Let w be the word consisting of the leftmost v letters of x_s, (which is the same as the rightmost v letters of x_k). Since the leftmost letter of w is the leftmost letter of x_s, $w \in T$. Now w is a proper subword of x_k, so $\varphi(w) < \varphi(x_k) \leq \varphi(x_r)$. Thus $w \in FP(\langle x_n\rangle_{n=1}^{r-1})$ so pick $L \subseteq \{1,2,\ldots,r-1\}$ such that $w = \prod_{n\in L} x_n$. Then $x_s = w\cdot\prod_{n\in H} x_n = \prod_{n\in L\cup H} x_n$, which we have seen is impossible.

To see that $\langle x_n\rangle_{n=1}^{\infty}$ is nice, let $s \in S\setminus T$. Then the leftmost letter of s is not in B, so $T\cap sT = \emptyset$.

Pick any $b\in B$. Then $S = b^{-1}T$, so $FP(\langle x_n\rangle_{n=1}^{\infty})$ is syndetic. By Theorem 2.10, for all $k\in\mathbb{N}$, $FP(\langle x_n\rangle_{n=k}^{\infty})$ is syndetic. □

We show now that under appropriate hypotheses, the existence of nice sequences is preserved under countable direct sums.

3.5 Theorem. *Let S_1 and S_2 be semigroups with two sided identities e_1 and e_2 and assume that for each $i\in\{1,2\}$, each $x\in S_i$ has at most finitely many right inverses. For $i\in\{1,2\}$ let $\langle y_{i,n}\rangle_{n=1}^{\infty}$ be a nice sequence in S_i. Let $f:\{1,2\}\times\mathbb{N}\xrightarrow[\text{onto}]{\text{1-1}}\mathbb{N}$ such that for all $(i,n)\in\{1,2\}\times\mathbb{N}$, $f(i,n)<f(i,n+1)$. Define a sequence $\langle z_n\rangle_{n=1}^{\infty}$ in $S=S_1\times S_2$ by, for $i\in\{1,2\}$ and $n\in\mathbb{N}$,*

$$(z_{f(i,n)})_j = \begin{cases} e_j & \text{if } j\neq i \\ y_{i,n} & \text{if } j = i\,. \end{cases}$$

Then $FP(\langle z_n\rangle_{n=1}^{\infty}) = \big((FP(\langle y_{1,n}\rangle_{n=1}^{\infty})\cup\{e_1\})\times(FP(\langle y_{1,n}\rangle_{n=1}^{\infty})\cup\{e_1\})\big)\setminus\{(e_1,e_2)\}$. The sequence $\langle z_n\rangle_{n=1}^{\infty}$ is nice. Also, $FP(\langle z_n\rangle_{n=1}^{\infty})$ is syndetic if and only if $FP(\langle y_{1,n}\rangle_{n=1}^{\infty})$ is syndetic and $FP(\langle y_{2,n}\rangle_{n=1}^{\infty})$ is syndetic.

Proof. This is not, at least not obviously, a corollary of Theorem 3.6 below. But the proof is essentially the same and simpler, so we leave the details as an exercise. □

Recall that, if for each $i\in\mathbb{N}$, S_i is a semigroup with an identified element e_i (typically a left or right or two sided identity), then the direct sum $S=\bigoplus_{i=1}^{\infty} S_i = \{x\in \times_{i=1}^{\infty} : \{i\in\mathbb{N} : x_i\neq e_i\}$ is finite$\}$.

3.6 Theorem. *For each $i\in\mathbb{N}$ let S_i be a semigroup with a two sided identity e_i such that each $x\in S_i$ has at most finitely many right inverses and let $\langle y_{i,n}\rangle_{n=1}^{\infty}$ be a nice sequence in S_i. Let $f:\mathbb{N}\times\mathbb{N}\xrightarrow[\text{onto}]{\text{1-1}}\mathbb{N}$ such that for all $(i,n)\in\mathbb{N}\times\mathbb{N}$, $f(i,n)<f(i,n+1)$.*

Define a sequence $\langle z_n\rangle_{n=1}^\infty$ *in* $S=\bigoplus_{i=1}^\infty S_i$ *by, for* $i,n\in\mathbb{N}$*,*

$$(z_{f(i,n)})_j=\begin{cases} e_j & \textit{if } j\neq i\\ y_{i,n} & \textit{if } j=i\,.\end{cases}$$

Then $FP(\langle z_n\rangle_{n=1}^\infty)=\big(\bigoplus_{i=1}^\infty(FP(\langle y_{i,n}\rangle_{n=1}^\infty)\cup\{e_i\})\big)\setminus\{e\}$*. The sequence* $\langle z_n\rangle_{n=1}^\infty$ *is nice. Also,* $FP(\langle z_n\rangle_{n=1}^\infty)$ *is syndetic if and only if each* $FP(\langle y_{i,n}\rangle_{n=1}^\infty)$ *is syndetic and* $\big\{i\in\mathbb{N}:FP(\langle y_{i,n}\rangle_{n=1}^\infty)\neq S_i\setminus\{e_i\}\big\}$ *is finite.*

Proof. To see that $FP(\langle z_n\rangle_{n=1}^\infty)\subseteq\big(\bigoplus_{i=1}^\infty(FP(\langle y_{i,n}\rangle_{n=1}^\infty)\cup\{e_i\})\big)\setminus\{e\}$, let $F\in\mathcal{P}_f(\mathbb{N})$ and let $x=\prod_{m\in F}z_m$. For each $i\in\mathbb{N}$, let $H_i=\{n\in\mathbb{N}:f(i,n)\in F\}$. Then, for each $i\in\mathbb{N}$, $x_i=\prod_{n\in H_i}y_{i,n}$, where we define $\prod_{n\in\emptyset}y_{i,n}=e_i$. Then $\{i\in\mathbb{N}:x_i\neq e_i\}$ is finite and nonempty. To see that $\big(\bigoplus_{i=1}^\infty(FP(\langle y_{i,n}\rangle_{n=1}^\infty)\cup\{e_i\})\big)\setminus\{e\}\subseteq FP(\langle z_n\rangle_{n=1}^\infty)$, let $x\in\big(\bigoplus_{i=1}^\infty(FP(\langle y_{i,n}\rangle_{n=1}^\infty)\cup\{e_i\})\big)\setminus\{e\}$ and let $M=\{i\in\mathbb{N}:x_i\neq e_i\}$ For each $i\in M$, pick $H_i\in\mathcal{P}_f(\mathbb{N})$ such that $x_i=\prod_{n\in H_i}y_{i,n}$ and let $F=f[\bigcup_{i\in M}(\{i\}\times H_i)]$. Then $x=\prod_{m\in F}z_m$.

To verify uniqueness of finite products, let $F,G\in\mathcal{P}_f(\mathbb{N})$ and assume that $\prod_{m\in F}z_m=\prod_{m\in G}z_m$. For each $i\in\mathbb{N}$, let $H_i=\{n\in\mathbb{N}:f(i,n)\in F\}$ and $K_i=\{n\in\mathbb{N}:f(i,n)\in G\}$. Notice that for each $i\in\mathbb{N}$, $e_i\notin FP(\langle y_{i,n}\rangle_{i=1}^\infty)$. (If, say, $e_i=\prod_{n\in L}y_{i,n}$ and $k=\max L$, then $\prod_{n\in L\cup\{k+1\}}y_{i,n}=y_{i,k+1}$.) Thus $\prod_{n\in H_i}y_{i,n}=e_i$ if and only if $H_i=\emptyset$. Thus, by the uniqueness of finite products in $FP(\langle y_{i,n}\rangle_{i=1}^\infty)$, we have that for each i, $H_i=K_i$, and thus $F=G$.

To complete the verification that $\langle z_n\rangle_{n=1}^\infty$ is nice, let $s\in S\setminus(FP(\langle z_n\rangle_{n=1}^\infty)\cup\{e\})$ and pick $i\in\mathbb{N}$ such that $s_i\notin FP(\langle y_{i,n}\rangle_{i=1}^\infty)\cup\{e_i\}$. Since s_i has only finitely many right inverses we may pick $r\in\mathbb{N}$ such that for all $w\in FP(\langle y_{i,n}\rangle_{n=r}^\infty)$, $s_iw\neq e_i$. Pick $k\geq r$ such that ${s_i}^{-1}FP(\langle y_{i,n}\rangle_{n=1}^\infty)\cap FP(\langle y_{i,n}\rangle_{n=k}^\infty)=\emptyset$. Let $m=f(i,n)$. We claim that $s^{-1}FP(\langle z_n\rangle_{n=1}^\infty)\cap FP(\langle z_n\rangle_{n=m}^\infty)=\emptyset$. Suppose instead we have $w\in s^{-1}FP(\langle z_n\rangle_{n=1}^\infty)\cap FP(\langle z_n\rangle_{n=m}^\infty)$. Pick $F\in\mathcal{P}_f(\mathbb{N})$ such that $\min F\geq m$ and $w=\prod_{n\in F}z_n$. Let $H=\{n\in\mathbb{N}:f(i,n)\in F\}$. Then $w_i=\prod_{n\in H}y_{i,n}$. Since $sw\in FP(\langle z_n\rangle_{n=1}^\infty)$ pick $G\in\mathcal{P}_f(\mathbb{N})$ such that $sw=\prod_{n\in G}z_n$. Let $K=\{n\in\mathbb{N}:f(i,n)\in G\}$. Then $s_iw_i=\prod_{n\in K}y_{i,n}$. If $H=\emptyset$, then $w_i=e_i$ so $s_i=s_iw_i=\prod_{n\in K}y_{i,n}$ so $s_i\in FP(\langle y_{i,n}\rangle_{n=1}^\infty)\cup\{e_i\}$, a contradiction. Thus $H\neq\emptyset$ and so $\min H\geq k\geq r$ so $s_iw_i\neq e_i$ so $K\neq\emptyset$ and thus $w_i\in {s_i}^{-1}FP(\langle y_{i,n}\rangle_{n=1}^\infty)\cap FP(\langle y_{i,n}\rangle_{n=k}^\infty)$, a contradiction.

Now assume that $FP(\langle z_n\rangle_{n=1}^\infty)$ is syndetic and pick $G\in\mathcal{P}_f(S)$ such that $S=\bigcup_{t\in G}t^{-1}FP(\langle z_n\rangle_{n=1}^\infty)$. To see that each $FP(\langle y_{i,n}\rangle_{n=1}^\infty)$ is syndetic, let $i\in\mathbb{N}$ and pick $x\in FP(\langle y_{i,n}\rangle_{n=1}^\infty)$. Let $G_i=\pi_i[G]\cup\{xw:w\in\pi_i[G]\}$. We claim that $S_i=$

$\bigcup_{t\in G_i} t^{-1}FP(\langle y_{i,n}\rangle_{n=1}^{\infty})$. Let $s \in S_i$ and define $u \in S$ by

$$u_j = \begin{cases} e_j & \text{if } j \neq i \\ s & \text{if } j = i\,. \end{cases}$$

Pick $g \in G$ such that $gu \in FP(\langle z_n\rangle_{n=1}^{\infty})$. Then $g_i s \in FP(\langle y_{i,n}\rangle_{n=1}^{\infty}) \cup \{e_i\}$. If $g_i s = e_i$, then $x g_i s = x \in FP(\langle y_{i,n}\rangle_{n=1}^{\infty})$.

Now let $M = \big\{i \in \mathbb{N} : FP(\langle y_{i,n}\rangle_{n=1}^{\infty}) \neq S_i \setminus \{e_i\}\big\}$ and suppose that M is infinite. Pick $i \in M$ such that for all $g \in G$, $g_i = e_i$ and pick $s \in (S_i \setminus \{e_i\}) \setminus FP(\langle y_{i,n}\rangle_{n=1}^{\infty})$. Define $u \in S$ by

$$u_j = \begin{cases} e_j & \text{if } j \neq i \\ s & \text{if } j = i\,. \end{cases}$$

Then for all $g \in G$, $g_i u_i = s \notin FP(\langle y_{i,n}\rangle_{n=1}^{\infty}) \cup \{e_i\}$ so $gu \notin FP(\langle z_n\rangle_{n=1}^{\infty})$, a contradiction.

Finally assume that each $FP(\langle y_{i,n}\rangle_{n=1}^{\infty})$ is syndetic and

$$M = \big\{i \in \mathbb{N} : FP(\langle y_{i,n}\rangle_{n=1}^{\infty}) \neq S_i \setminus \{e_i\}\big\}$$

is finite. If $M = \emptyset$, let $G = \{e, z_1\}$. Then $z_1 e \in FP(\langle z_n\rangle_{n=1}^{\infty})$ and for $s \in S \setminus \{e\}$, $es \in FP(\langle z_n\rangle_{n=1}^{\infty})$. So assume that $M \neq \emptyset$. For each $i \in M$, pick $G_i \in \mathcal{P}_f(S_i)$ such that $S_i = \bigcup_{t\in G_i} t^{-1}FP(\langle y_{i,n}\rangle_{n=1}^{\infty})$. Let

$$G = \{x \in S :\ \text{for all } i \in M\,,\ x_i \in G_i \text{ and for all } i \in \mathbb{N} \setminus M\,,\ x_i = e_i\}\,.$$

To see that $S = \bigcup_{t\in G} t^{-1}FP(\langle z_n\rangle_{n=1}^{\infty})$, let $v \in S$. For $i \in M$, pick $t_i \in G_i$ such that $t_i v_i \in FP(\langle y_{i,n}\rangle_{n=1}^{\infty})$. Define $u \in S$ by

$$u_i = \begin{cases} t_i & \text{if } i \in M \\ e_i & \text{if } i \notin M\,. \end{cases}$$

Then $u_i v_i \in FP(\langle y_{i,n}\rangle_{n=1}^{\infty}) \cup \{e_i\}$ for each i, and if $i \in M$, $u_i v_I \neq e_i$. $\square$

We note that it is reasonably simple to build semigroups S_i as required by Theorems 3.5 and 3.6.

3.7 Theorem. *Let S be a semigroup with no left identities, let $\langle y_n\rangle_{n=1}^{\infty}$ be a nice sequence in S, and adjoin a two sided identity $\{e\}$ to S. Then $\langle y_n\rangle_{n=1}^{\infty}$ is nice in $S \cup \{e\}$ and $FP(\langle y_{i,n}\rangle_{n=1}^{\infty})$ is syndetic in $S \cup \{e\}$ if and only if it is syndetic in S.*

Proof. That $\langle y_n\rangle_{n=1}^{\infty}$ is nice in $S \cup \{e\}$ is trivial. If $S = \bigcup_{t\in G} t^{-1}FP(\langle y_n\rangle_{n=1}^{\infty})$, then $S \cup \{e\} = \bigcup_{t\in G\cup\{y_1\}} t^{-1}FP(\langle y_n\rangle_{n=1}^{\infty})$. $\square$

Notice that the requirement in Theorem 3.7 that S have no left identities is needed because if f is a left identity of S, it is not a left identity of $S \cup \{e\}$. (Of course, if S

already had a two sided identity, there was no reason to "build" a new semigroup to use in Theorem 3.5 or Theorem 3.6.)

The following fact is easy to check.

3.8 Theorem. *For each* $n \in \omega$ *let* $y_n = (-2)^n$. *Then* $\langle y_n \rangle_{n=1}^{\infty}$ *is a nice sequence in* $(\mathbb{Z}, +)$ *and* $FS(\langle y_n \rangle_{n=1}^{\infty}) = \mathbb{Z} \setminus \{0\}$.

3.9 Corollary. *Let* $S = \bigoplus_{i=1}^{\infty} \mathbb{Z}$. *Then there is a nice sequence* $\langle z_n \rangle_{n=1}^{\infty}$ *in* S *with* $FS(\langle z_n \rangle_{n=1}^{\infty}) = S \setminus \{0\}$.

Proof. Theorems 3.6 and 3.8. $\square$

We write $\mathbb{Q}^+ = \{x \in \mathbb{Q} : x > 0\}$.

3.10 Corollary. *There is a nice sequence* $\langle x_n \rangle_{n=1}^{\infty}$ *in* $(\mathbb{Q}^+, \cdot)$ *such that* $FP(\langle x_n \rangle_{n=1}^{\infty}) = \mathbb{Q}^+ \setminus \{1\}$.

Proof. Let $\langle p_i \rangle_{i=1}^{\infty}$ enumerate the primes and define $f : \bigoplus_{i=1}^{\infty} \mathbb{Z} \to \mathbb{Q}$ by $f(x) = \prod_{i=1}^{\infty} {p_i}^{x_i}$. Then f is an isomorphism so Corollary 3.9 applies. $\square$

It is a consequence of [10, Theorem 2.3] that there is no nice sequence $\langle x_n \rangle_{n=1}^{\infty}$ in $(\mathbb{Q}^+, +)$ such that $\mathbb{Q}^+ = FS(\langle x_n \rangle_{n=1}^{\infty})$.

3.11 Question. *Is there a nice sequence* $\langle x_n \rangle_{n=1}^{\infty}$ *in* $(\mathbb{Q}^+, +)$ *such that* $FS(\langle x_n \rangle_{n=1}^{\infty})$ *is syndetic?*

3.12 Theorem. *There is a countably infinite group* $(S, +)$ *which has no sequence* $\langle x_n \rangle_{n=1}^{\infty}$ *satisfying uniqueness of finite sums with* $FS(\langle x_n \rangle_{n=1}^{\infty})$ *piecewise syndetic.*

Proof. Let $S = \bigoplus_{i=1}^{\infty} \mathbb{Z}_3$. Suppose we have a sequence $\langle x_n \rangle_{n=1}^{\infty}$ in S satisfying uniqueness of finite sums with $FS(\langle x_n \rangle_{n=1}^{\infty})$ piecewise syndetic. By Theorem 2.2 $FS(\langle x_n \rangle_{n=1}^{\infty})$ is central so by [8, Theorem 15.5*] there exist a and d in $S \setminus \{0\}$ such that $\{a, a+d, a+2d\} \subseteq FS(\langle x_n \rangle_{n=1}^{\infty})$. Pick F, G, and H in $\mathcal{P}_f(\mathbb{N})$ such that $a = \sum_{n \in F} x_n$, $a + d = \sum_{n \in G} x_n$, and $a + 2d = \sum_{n \in H} x_n$. For $i \in \{1, 2, 3\}$ let $K_i = \{n \in F \cup G \cup H : n$ is in exactly i of the sets F, G, and $H\}$. Then $0 = a + (a+d) + (a+2d) = \sum_{n \in K_1} x_n + \sum_{n \in K_2} 2x_n + \sum_{n \in K_1} 3x_n = \sum_{n \in K_1} x_n - \sum_{n \in K_2} x_n$ so $\sum_{n \in K_1} x_n = \sum_{n \in K_2} x_n$. By the uniqueness of finite sums, $K_1 = K_2$ so, since K_1 and K_2 are disjoint, $K_1 = K_2 = \emptyset$ and thus $F = G = H$. This is a contradiction because $d \neq 0$. $\square$

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*See the errata at `http://members.aol.com/nhindman/pdf/errata.pdf` for a correction to the first line of the proof of this theorem.

4. The Semigroup $(\mathbb{N},+)$

In this section we address the special case of the semigroup $(\mathbb{N},+)$. The first result requires no special assumptions about the sequence $\langle x_n\rangle_{n=1}^\infty$, except that terms be listed in their natural order.

4.1 Theorem. *Let $\langle x_n\rangle_{n=1}^\infty$ be a nondecreasing sequence in $\mathbb{N}$. Then $FS(\langle x_n\rangle_{n=1}^\infty)$ is syndetic if and only if the sequence $\langle x_{n+1}-\sum_{t=1}^n x_t\rangle_{n=1}^\infty$ is bounded from above.*

Proof. Necessity. Let b be a bound on the gaps of $FS(\langle x_n\rangle_{n=1}^\infty)$ and suppose we have n such that $x_{n+1}-\sum_{t=1}^n x_t > b$. Pick $F\in\mathcal{P}_f(\mathbb{N})$ such that $\sum_{t=1}^n x_t < \sum_{t\in F} x_t \leq \sum_{t=1}^n x_t + b$ and let $r=\max F$. If $r\geq n+1$, then $\sum_{t\in F} x_t \geq x_r \geq x_{n+1} > \sum_{t=1}^n x_t + b$, a contradiction. If $r\leq n$, then $\sum_{t\in F} x_t \leq \sum_{t=1}^n x_t$, a contradiction.

Sufficiency. Pick $b\in\mathbb{N}$ such that $x_1\leq b$ and for all $n\in\mathbb{N}$, $x_{n+1}-\sum_{t=1}^n x_t\leq b$. Let $a\in\mathbb{N}$. We shall show that $\{a+1,a+2,\ldots,a+b\}\cap FS(\langle x_n\rangle_{n=1}^\infty)\neq\emptyset$.

Define $\varphi:\mathbb{N}\xrightarrow{\text{onto}}FS(\langle x_n\rangle_{n=1}^\infty)$ by $\varphi(\sum_{t\in F}2^{t-1})=\sum_{t\in F}x_t$. Pick the first $m\in\mathbb{N}$ such that $\varphi(m)>a$. If $m=1$, then $\varphi(m)=x_1\leq b<a+b$, so assume that $m>1$. Let $F=1+\text{supp}(m)$ (so that $m=\sum_{t\in F}2^{t-1}$). If $1\in F$, then $m-1=\sum_{t\in F\setminus\{1\}}2^{t-1}$ so $\sum_{t\in F\setminus\{1\}}x_t=\varphi(m-1)\leq a<\varphi(m)=\sum_{t\in F\setminus\{1\}}x_t+x_1\leq a+b$. So assume that $1\notin F$ and let $s=\min F$. Let $G=(F\setminus\{s\})\cup\{1,2,\ldots,s-1\}$. Then $m-1=\sum_{t\in G}2^{t-1}$ so $\sum_{t\in G}x_t=\varphi(m-1)\leq a<\varphi(m)=\sum_{t\in G}x_t+x_s-\sum_{t=1}^{s-1}x_t\leq a+b$. □

4.2 Corollary. *Let $\langle x_n\rangle_{n=1}^\infty$ be a nice sequence in $\mathbb{N}$ written in increasing order. The following statements are equivalent.*

(a) $FS(\langle x_n\rangle_{n=1}^\infty)$ is piecewise syndetic.

(b) For all $m\in\mathbb{N}$, $FS(\langle x_n\rangle_{n=m}^\infty)$ is piecewise syndetic.

(c) $FS(\langle x_n\rangle_{n=1}^\infty)$ is strongly central.

(d) For all $m\in\mathbb{N}$, $FS(\langle x_n\rangle_{n=m}^\infty)$ is strongly central.

(e) $FS\langle x_n\rangle_{n=1}^\infty)$ is syndetic.

(f) For all $m\in\mathbb{N}$, $FS(\langle x_n\rangle_{n=m}^\infty)$ is syndetic.

(g) $FS(\langle x_n\rangle_{n=1}^\infty)$ is central.

(h) For all $m\in\mathbb{N}$, $FS(\langle x_n\rangle_{n=m}^\infty)$ is central.

(i) The sequence $\langle x_{n+1}-\sum_{t=1}^n x_t\rangle_{n=1}^\infty$ is bounded from above.

Proof. Theorems 2.10 and 4.1. □

In [3, Example 7.9] Vitaly Bergelson and Randall McCutcheon produced an example of a syndetic IP-set in $\mathbb{N}$ (defined by a nice sequence) which is not IP*. In view of

Theorem 2.11 this set is also not central*, since the set of left identities of $(\mathbb{N},+)$ is empty. We extend this example in Theorem 4.4. (The example produced by Bergelson and McCutcheon is the sequence of Lemma 4.3 determined by $r=2$.)

Recall that the *Banach density* of a set $A\subseteq\mathbb{N}$ is defined by

$$d^*(A)=\sup\Big\{\alpha\in\mathbb{R}: \text{for all } n\in\mathbb{N} \text{ there exist } m\ge n \text{ and } x\in\mathbb{N} \text{ such that } \frac{|A\cap\{x+1,x+2,\ldots,x+m\}|}{m}\ge\alpha\Big\}.$$

By [8, Theorems 20.5 and 20.6], $\Delta^*=\{p\in\beta\mathbb{N}: \text{for all } A\in p,\, d^*(A)>0\}$ is a two sided ideal of $\beta\mathbb{N}$ and consequently $K(\beta\mathbb{N})\subseteq\Delta^*$. So whenever A is a central subset of $\mathbb{N}$, one must have that $d^*(A)>0$. Consequently, if A is an IP-set which is not central*, one must have that $d^*(\mathbb{N}\setminus A)>0$. We shall see, however, that one can get such sets with Banach density as small as we please.

4.3 Lemma. *Let $r\in\mathbb{N}\setminus\{1\}$ and define for $n\in\omega$*

$$d_{rn}=\frac{2^{rn+r-1}+2^{r-1}-1}{2^r-1}$$

and for $n\in\omega$ and $j\in\{1,2,\ldots,r-1\}$

$$d_{rn+j}=\frac{2^{rn+r+j-1}-2^{j-1}}{2^r-1}\,.$$

(1) For $n\in\mathbb{N}$, $d_{rn}=2d_{rn-1}+1$, and for $n\in\omega$, $d_{rn+1}=2d_{rn}-1$.

(2) For $n\in\omega$ and $j\in\{2,3,\ldots,r-1\}$, $d_{rn+j}=2d_{rn+j-1}$.

(3) For $n\in\mathbb{N}$, $d_{rn}=\sum_{t=1}^{rn-1}d_t+2$.

(4) For $k\in\mathbb{N}$, if $k\not\equiv-1\pmod r$, then $d_{k+1}=\sum_{t=1}^{k}d_t+1$.

(5) If a and b are successive members of $\mathbb{N}\setminus FS(\langle d_n\rangle_{n=1}^\infty)$ with $a<b$, then $b-a=d_r=2^{r-1}+1$ or $b-a=d_{r+1}=2^r+1$.

Proof. The verification of conclusions (1) through (4) is a routine exercise. We verify conclusion (5). Notice that by conclusions (3) and (4), we have for all $F,G\in\mathcal{P}_f(\mathbb{N})$ that $\sum_{t\in F}d_t<\sum_{t\in G}d_t$ if and only if $\sum_{t\in F}2^t<\sum_{t\in G}2^t$. (This observation should help in the computation of b.)

By conclusions (3) and (4), all elements of $\mathbb{N}\setminus FS(\langle d_n\rangle_{n=1}^\infty)$ are of the form

$$\textstyle\sum_{t\in F}d_t+\sum_{t=1}^{rn-1}d_t+1$$

for some $n\in\mathbb{N}$ and some finite $F\subseteq\mathbb{N}$ such that either $F=\emptyset$ or $\min F\ge rn+1$. (We are using the convention that $\sum_{t\in\emptyset}d_t=0$.) So pick such n and F such that $a=\sum_{t\in F}d_t+\sum_{t=1}^{rn-1}d_t+1$.

Assume first that $n > 1$. Then $b = \sum_{t\in F} d_t + d_{rn} + \sum_{t=1}^{r-1} d_t + 1$ and so $b - a = d_{rn} - \sum_{t=1}^{rn-1} d_t + \sum_{t=1}^{r-1} d_t = 2 + d_r - 2 = d_r$.

Next assume that $n = 1$ and either $F = \emptyset$ or $\min F > r + 1$. Then $b = \sum_{t\in F} d_t + d_{r+1} + \sum_{t=1}^{r-1} d_t + 1$ so $b - a = d_{r+1}$.

Now assume that $n = 1$ and $\min F = r + 1$ and pick $l \geq r + 1$ such that $F = G\cup\{r+1, r+2, \ldots, l\}$ where either $G = \emptyset$ or $\min G \geq l+2$. Note that in this case, $a = \sum_{t\in G} d_t + \sum_{t=r+1}^{l} d_t + \sum_{t=1}^{r-1} d_t + 1$. If $l \equiv -1 \pmod{r}$, then $b = \sum_{t\in G} d_t + \sum_{t=1}^{l} d_t + 1$ and $b - a = d_r$. So assume that $l \not\equiv -1 \pmod{r}$. Then $b = \sum_{t\in G} d_t + d_{l+1} + \sum_{t=1}^{r-1} d_t + 1$ so $b - a = d_{l+1} - \sum_{t=r+1}^{l} d_t = d_{l+1} - \sum_{t=1}^{l} d_t + \sum_{t=1}^{r} d_t = 1 + \sum_{t=1}^{r} d_t = d_{r+1}$. $\square$

Notice that for the sequence $\langle d_n\rangle_{n=1}^{\infty}$ produced in the following theorem, even though in many respects $\mathbb{N}\setminus FS(\langle d_n\rangle_{n=1}^{\infty})$ is quite small, it is a central set and therefore contains all of the structure guaranteed to any central set.

4.4 Theorem. *Let $\epsilon > 0$. There is a nice sequence $\langle d_n\rangle_{n=1}^{\infty}$ in $\mathbb{N}$ such that $FS(\langle d_n\rangle_{n=1}^{\infty})$ is syndetic (and therefore central) but is not IP* (and therefore not central*) Also, $FS(\langle d_n\rangle_{n=1}^{\infty})$ has no gaps of length greater than 1 and $d^*\big(\mathbb{N}\setminus FS(\langle d_n\rangle_{n=1}^{\infty})\big) < \epsilon$. Furthermore, there is a sequence $\langle x_n\rangle_{n=1}^{\infty}$ in $\mathbb{N}$ such that $FS(\langle x_n\rangle_{n=1}^{\infty})\cap FS(\langle d_n\rangle_{n=1}^{\infty}) = \emptyset$ and for each $F\in\mathcal{P}_f(\mathbb{N})$, if $m = \max F$, then $2x_m + \sum_{k\in F\setminus\{m\}} x_k \notin FS(\langle d_n\rangle_{n=1}^{\infty})$.*

Proof. Pick $r\in\mathbb{N}$ such that $\dfrac{1}{2^{r-1}+1} < \epsilon$ and let $\langle d_n\rangle_{n=1}^{\infty}$ be as defined in Lemma 4.3. We then have by conclusions (3) and (4) of Lemma 4.3 that $\langle d_n\rangle_{n=1}^{\infty}$ is a nice sequence and $FS(\langle d_n\rangle_{n=1}^{\infty})$ has no gaps longer than 1. By conclusion (5), we have that $d^*\big(\mathbb{N}\setminus FS(\langle d_n\rangle_{n=1}^{\infty})\big) < \epsilon$.

To complete the proof we construct a sequence $\langle x_n\rangle_{n=1}^{\infty}$ satisfying the last sentence of the theorem. (The fact that $FS(\langle x_n\rangle_{n=1}^{\infty})\cap FS(\langle d_n\rangle_{n=1}^{\infty}) = \emptyset$ says that $FS(\langle d_n\rangle_{n=1}^{\infty})$ is not IP* and then Theorem 2.11 tells us that $FS(\langle d_n\rangle_{n=1}^{\infty})$ is not central*.)

Observe that by conclusion (1) of Lemma 4.3, for $H\in\mathcal{P}_f(\mathbb{N})$

$$(*)\qquad \text{if } |H| = s, \text{ then } 2\cdot\sum_{t\in H} d_{rt-1} + s = \sum_{t\in H}(d_{rt}-1) + s = \sum_{t\in H} d_{rt}\,.$$

We inductively define the sequence $\langle x_n\rangle_{n=1}^{\infty}$ and an auxiliary sequence $\langle H_n\rangle_{n=1}^{\infty}$ in $\mathcal{P}_f(\mathbb{N})$. Let $x_0 = 0$. Let $H_1 = \{2, 3, \ldots, d_r\}$ and let $x_1 = \sum_{t\in H_1} d_{rt-1} + \sum_{t=1}^{r-1} d_t + 1$. Given $k\in\mathbb{N}$ and x_k, let

$$H_{k+1} = \{d_r+1+x_0+x_1+\ldots+x_{k-1}, d_r+2+x_0+x_1+\ldots+x_{k-1}, \ldots, d_r+x_0+x_1+\ldots+x_k\}$$

and note that $|H_{k+1}| = x_k$. Let $x_{k+1} = \sum_{t\in H_{k+1}} d_{rt-1} + x_k$. Notice that for any $k\in\mathbb{N}$, any $t\in H_k$, and any $s\in H_{k+1}$, $rt < rs - 1$. Notice also that for any k,

$x_k = \sum_{l=1}^{k} \sum_{t\in H_l} d_{rt-1} + \sum_{t=1}^{r-1} d_t + 1$.

We now show that $FS(\langle x_n\rangle_{n=1}^{\infty}) \cap FS(\langle d_n\rangle_{n=1}^{\infty}) = \emptyset$ and for each $F \in \mathcal{P}_f(\mathbb{N})$, if $m = \max F$, then $2x_m + \sum_{k\in F\setminus\{m\}} x_k \notin FS(\langle d_n\rangle_{n=1}^{\infty})$. We do this by showing that each member of $FS(\langle x_n\rangle_{n=1}^{\infty})$ and each sum of the form $2x_m + \sum_{k\in F\setminus\{m\}} x_k$ can also be written as $\sum_{t\in L} d_t + \sum_{t=1}^{r-1} d_t + 1$ for some $L \in \mathcal{P}_f(\mathbb{N})$ with $\min L \geq r+1$. This will suffice because $\sum_{t\in L} d_t + d_r$ is the immediate successor of $\sum_{t\in L} d_t + \sum_{t=1}^{r-1} d_t$ in $FS(\langle d_n\rangle_{n=1}^{\infty})$ and $\sum_{t\in L} d_t + d_r = \sum_{t\in L} d_t + \sum_{t=1}^{r-1} d_t + 2$.

Notice that

$$\begin{aligned}\textstyle\sum_{t\in H_1} d_{rt-1} + x_1 &= \textstyle\sum_{t\in H_1} 2d_{rt-1} + \sum_{t=1}^{r-1} d_t + 1 \\ &= \textstyle\sum_{t\in H_1} d_{rt}\end{aligned}$$

by $(*)$ since $|H_1| = d_r - 1 = \sum_{t=1}^{r-1} d_t + 1$, and for $k > 1$,

$$\begin{aligned}\textstyle\sum_{t\in H_k} d_{rt-1} + x_k &= \textstyle\sum_{t\in H_k} d_{rt-1} + \sum_{t\in H_k} d_{rt-1} + x_{k-1} \\ &= \textstyle\sum_{t\in H_k} d_{rt} \text{ (by } (*)\text{, since } |H_k| = x_{k-1}).\end{aligned}$$

Using these facts, one easily establishes by induction on $|F|$ that for $F \in \mathcal{P}_f(\mathbb{N})$, if $m = \max F$, then

$$\begin{aligned}\textstyle\sum_{k\in F} x_k = &\textstyle\sum_{t\in H_m} d_{rt-1} + \sum_{k\in F\setminus\{m\}} \sum_{t\in H_k} d_{rt} + \\ &\textstyle\sum_{k\in\{1,2,\ldots,m\}\setminus F} \sum_{t\in H_k} d_{rt-1} + \sum_{t=1}^{r-1} d_t + 1\,.\end{aligned}$$

and

$$\begin{aligned}2x_m + \textstyle\sum_{k\in F\setminus\{m\}} x_k = &\textstyle\sum_{t\in H_m} d_{rt} + \sum_{k\in F\setminus\{m\}} \sum_{t\in H_k} d_{rt} + \\ &\textstyle\sum_{k\in\{1,2,\ldots,m\}\setminus F} \sum_{t\in H_k} d_{rt-1} + \sum_{t=1}^{r-1} d_t + 1\,.\end{aligned}$$

□

We know from Theorem 3.2 that any sequence $\langle x_n\rangle_{n=1}^{\infty}$ in $\mathbb{N}$ with the property that for all n, $x_{n+1} > \sum_{t=1}^{n} x_t$ is nice. We see in the next two theorems that if $\langle x_{n+1} - \sum_{t=1}^{n} x_t\rangle_{n=1}^{\infty}$ is bounded from below (which of course must hold if for sufficiently large n, $x_{n+1} > \sum_{t=1}^{n} x_t$), then such sequences nearly account for all nice sequences.

4.5 Theorem. *If $\langle x_n\rangle_{n=1}^{\infty}$ is an increasing sequence in $\mathbb{N}$ which satisfies uniqueness of finite sums and there is some $m \in \mathbb{N}$ such that for all $n \geq m$, $x_{n+1} > \sum_{t=1}^{n} x_t$, then $\langle x_n\rangle_{n=1}^{\infty}$ is nice.*

Proof. Let $s \in \mathbb{N} \setminus FS(\langle x_n\rangle_{n=1}^{\infty})$ and pick $l \in \mathbb{N}$ such that $l \geq m$ and $x_l > s$. Suppose that $\big(-s + FS(\langle x_n\rangle_{n=1}^{\infty})\big) \cap FS(\langle x_n\rangle_{n=l+1}^{\infty}) \neq \emptyset$. Pick $F \in \mathcal{P}_f(\{n \in \mathbb{N} : n > l\})$ with $\max F = r$ as small as possible such that $s + \sum_{t\in F} x_t \in FS(\langle x_n\rangle_{n=1}^{\infty})$ and pick $G \in \mathcal{P}_f(\mathbb{N})$ such that $s + \sum_{t\in F} x_t = \sum_{t\in G} x_t$.

If $r > \max G$, then $\sum_{t\in G} x_t \leq \sum_{t=1}^{r-1} x_t < x_r < s + \sum_{t\in F} x_t$, a contradiction. If $r < \max G$, then $s + \sum_{t\in F} x_t < x_l + \sum_{t\in F} x_t \leq \sum_{t=1}^{r} x_t < x_{r+1} \leq \sum_{t\in G} x_t$, again a contradiction. Thus $r = \max G$. Then $F = \{r\}$ since otherwise, by subtracting x_r from both sides one has a contradiction to the choice of F. So $s = \sum_{t\in G\setminus\{r\}} x_t$ as required. $\square$

4.6 Theorem. *If $\langle x_n\rangle_{n=1}^{\infty}$ is a nice sequence in $\mathbb{N}$ written in increasing order and $\langle x_{n+1} - \sum_{t=1}^{n} x_t\rangle_{n=1}^{\infty}$ is bounded from below, then there is some $m \in \mathbb{N}$ such that for all $n \geq m$, $x_{n+1} > \sum_{t=1}^{n} x_t$.*

Proof. Suppose instead that infinitely often $x_{n+1} < \sum_{t=1}^{n} x_t$. Using the fact that $\langle x_{n+1} - \sum_{t=1}^{n} x_t\rangle_{n=1}^{\infty}$ is bounded from below, choose s such that infinitely often $s = \sum_{t=1}^{n} x_t - x_{n+1}$. Pick $n \in \mathbb{N}$ such that $x_{n+1} > s$ and $s = \sum_{t=1}^{n} x_t - x_{n+1}$. Then $s \notin FS(\langle x_n\rangle_{n=1}^{\infty})$. (If we had $s = \sum_{t\in F} x_t$, then we would have $\max F < n+1$ and $\sum_{t\in F\cup\{n+1\}} x_t = s + x_{n+1} = \sum_{t=1}^{n} x_t$, contradicting the fact that $\langle x_n\rangle_{n=1}^{\infty}$ satisfies uniqueness of finite sums.) So pick $k \in \mathbb{N}$ such that $\bigl(-s + FS(\langle x_n\rangle_{n=1}^{\infty})\bigr) \cap FS(\langle x_n\rangle_{n=k}^{\infty}) = \emptyset$. Pick $n > k$ such that $s = \sum_{t=1}^{n} x_t - x_{n+1}$. Then $x_{n+1} \in \bigl(-s + FS(\langle x_n\rangle_{n=1}^{\infty})\bigr) \cap FS(\langle x_n\rangle_{n=k}^{\infty})$, a contradiction. $\square$

We would have liked that all nice sequences have the property that eventually $x_{n+1} > \sum_{t=1}^{n} x_t$. This is not the case as we shall see now.

4.7 Theorem. *There is a nice sequence in $\mathbb{N}$ written in increasing order such that $\langle x_{n+1} - \sum_{t=1}^{n} x_t\rangle_{n=1}^{\infty}$ is not bounded from below.*

Proof. For $t \in \omega$, let

$$x_{3t+1} = 12^t \cdot 2$$
$$x_{3t+2} = 12^t \cdot 4 \text{ and}$$
$$x_{3t+3} = 12^t \cdot 5\,.$$

Then $FS(\langle x_n\rangle_{n=1}^{\infty})$ consists of all those positive integers which, when written in base 12, use only the digits 0, 2, 4, 5, 6, 7, 9, and 11. By the uniqueness of base 12 expansions, $FS(\langle x_n\rangle_{n=1}^{\infty})$ satisfies uniqueness of finite sums.

Now let $s \in \mathbb{N} \setminus FS(\langle x_n\rangle_{n=1}^{\infty})$. Then somewhere the base 12 expansion of s uses a 1, 3, 8, or 10. Pick t such that $12^t > s$. If $y \in FS(\langle x_n\rangle_{n=3t+1}^{\infty})$, then $s + y$ uses that digit in the same position that s does.

Finally, given $t \in \omega$, $\sum_{i=1}^{3t+2} x_i - x_{3t+3} \geq 12^t \cdot 6 - 12^t \cdot 5 = 12^t$. $\square$

We strongly suspect that the answer to the following question is “no”.

4.8 Question. *Is there a nice sequence in* $\mathbb{N}$ *(written in increasing order) such that* $\langle x_{n+1} - \sum_{t=1}^{n} x_t \rangle_{n=1}^{\infty}$ *is not bounded from below and* $FS(\langle x_n \rangle_{n=1}^{\infty})$ *is syndetic?*

In view of the results of this section, it is natural to ask for a description of sequences for which $\langle x_{n+1} - \sum_{t=1}^{n} x_t \rangle_{n=1}^{\infty}$ is bounded. Given any sequence $\langle d_n \rangle_{n=1}^{\infty}$ in $\mathbb{N}$ for which $\langle d_{n+1} - \sum_{t=1}^{n} x_t \rangle_{n=1}^{\infty}$ is bounded and any $\alpha \in \mathbb{N}$, one may let $x_n = d_n + 2^n \cdot \alpha$, and obtain a sequence $\langle x_n \rangle_{n=1}^{\infty}$ with $\langle x_{n+1} - \sum_{t=1}^{n} x_t \rangle_{n=1}^{\infty}$ bounded. If one asks that x_n be reasonably approximated by $2^n \cdot \alpha$, one finds out that there is a unique choice.

4.9 Theorem. *Let* $\langle x_n \rangle_{n=1}^{\infty}$ *be a sequence in* $\mathbb{R}$ *and assume that* $\langle x_{n+1} - \sum_{t=1}^{n} x_t \rangle_{n=1}^{\infty}$ *is bounded. There exists a unique* $\alpha \in \mathbb{R}$ *such that, if for each* n, $d_n = x_n - 2^n \alpha$, *then the sequence* $\langle d_n \rangle_{n=1}^{\infty}$ *is bounded. If for all* $n \in \mathbb{N}$, $|x_{n+1} - \sum_{t=1}^{n} x_t| \leq b$, *then for all* $n \geq 2$, $|d_n| \leq b$. *(So* $\langle d_n \rangle_{n=1}^{\infty}$ *is bounded by* $\max\{b, |d_1|\}$*.) Further, for this sequence,* $\langle d_{n+1} - \sum_{t=1}^{n} d_t \rangle_{n=1}^{\infty}$ *is bounded.*

Proof. If $\langle d_n \rangle_{n=1}^{\infty}$ is bounded, then $\alpha = \lim_{n\to\infty} \frac{x_n}{2^n}$ so uniqueness is trivial, as is the assertion that $\langle d_{n+1} - \sum_{t=1}^{n} d_t \rangle_{n=1}^{\infty}$ is bounded.

For each $n \in \mathbb{N}$, let $c_n = x_{n+1} - \sum_{t=1}^{n} x_t$. Let $\alpha = \frac{1}{4}\left(x_2 - \frac{c_1}{2} + \sum_{t=2}^{\infty} \frac{c_t}{2^t}\right)$. Notice that for $n \geq 2$, $d_{n+1} - 2d_n = x_{n+1} - 2x_n = c_n - c_{n-1}$. Using this fact, one easily establishes by induction that for each $n \geq 2$, $d_n = \frac{c_{n-1}}{2} - \sum_{t=n}^{\infty} \frac{c_t}{2^{t-n+2}}$ so $|d_n| \leq b$. $\square$

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