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## IMAGE PARTITION REGULAR MATRICES AND CONCEPTS OF LARGENESS, II

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**ABSTRACT.** Let  $u$  and  $v$  be positive integers and let  $A$  be a  $u \times v$  matrix with rational entries. We determine several characterizations of the property that whenever  $B$  is a piecewise syndetic subset of the set  $\mathbb{N}$  of positive integers,  $\{\vec{x} \in \mathbb{N}^v : A\vec{x} \in B^u\}$  is piecewise syndetic in  $\mathbb{N}^v$  as well as the corresponding property with  $\mathbb{Z}$  replacing  $\mathbb{N}$ . We investigate related phenomena for several other notions of largeness in a semigroup.

### 1. INTRODUCTION

We are concerned throughout with the notion of *image partition regularity* of a matrix.

**Definition 1.1.** Let  $u, v \in \mathbb{N}$ , and let  $A$  be a  $u \times v$  matrix with rational entries.

- (1) The matrix  $A$  is *image partition regular over  $\mathbb{N}$*  (IPR/ $\mathbb{N}$ ) if and only if, whenever  $\mathbb{N}$  is finitely colored, there exists  $\vec{x} \in \mathbb{N}^v$  such that the entries of  $A\vec{x}$  are monochromatic.
- (2) The matrix  $A$  is *image partition regular over  $\mathbb{Z}$*  (IPR/ $\mathbb{Z}$ ) if and only if, whenever  $\mathbb{Z} \setminus \{0\}$  is finitely colored, there exists  $\vec{x} \in (\mathbb{Z} \setminus \{0\})^v$  such that the entries of  $A\vec{x}$  are monochromatic.

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In this paper we consider relationships involving matrices, image partition regularity, and several notions of largeness in a semigroup, continuing a study in [13]. We will present definitions of those notions in Section 2.

In 1998 Hillel Furstenberg and Eli Glasner proved the following theorem.

**Theorem 1.2.** *Let  $l \in \mathbb{N}$ , let  $AP_l$  be the set of all length  $l$  arithmetic progressions in  $\mathbb{Z}$ , and let  $B$  be a piecewise syndetic subset of  $\mathbb{Z}$ . Then  $AP_l \cap B^l$  is a piecewise syndetic subset of  $AP_l$ .*

*Proof.* [5, Theorem 1.1]. □

Note that, if  $A = \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ \vdots & \vdots \\ 1 & l \end{pmatrix}$ , then  $AP_l = \{A\vec{x} : \vec{x} \in \mathbb{Z}^2\}$ .

In 2002 we, with Imre Leader, proved the next theorem.

**Theorem 1.3.** *Let  $u, v \in \mathbb{N}$  and let  $A$  be a  $u \times v$  matrix with rational entries. Then  $A$  is IPR/ $\mathbb{N}$  if and only if, whenever  $B$  is a central subset of  $\mathbb{N}$ ,  $\{\vec{x} \in \mathbb{N}^v : A\vec{x} \in B^u\}$  is central in  $\mathbb{N}^v$ .*

*Proof.* [9, Theorem 2.10]. □

In [13] we proved theorems of the following form where  $\Psi$  is one of the notions of largeness being considered.

**Theorem 1.4.** *Let  $u$  and  $v$  be positive integers and let  $A$  be a  $u \times v$  matrix with entries from  $\underline{\text{(X)}}$  and assume that  $A$  has property  $\underline{\text{(Y)}}$ . If  $B$  is a  $\Psi$  set in  $\mathbb{N}$ , then  $\{\vec{x} \in \mathbb{N}^v : A\vec{x} \in B^u\}$  is a  $\Psi$  set in  $\mathbb{N}^v$ .*

The following table lists the version of Theorem 1.4 that apply to each of several of the notions of largeness. In the table  $\mathbb{Q}^+ = \{x \in \mathbb{Q} : x \geq 0\}$ .

Theorem	$\Psi$	$X$	$Y$
4.1	C, central, SC*	$\mathbb{Q}$	is IPR/ $\mathbb{N}$
4.2	D, QC	$\mathbb{Q}$	is IPR/ $\mathbb{N}$ and $\text{rank}(A) = u$
4.3	SC	$\mathbb{Q}^+$	is IPR/ $\mathbb{N}$ and $\text{rank}(A) = u$
4.4	thick	$\mathbb{Z}$	is IPR/ $\mathbb{N}$
4.5	PS*	$\omega$	is IPR/ $\mathbb{N}$
4.6	central*, QC*	$\mathbb{Q}^+$	is IPR/ $\mathbb{N}$
4.7	D*, C*, IP*, Q*	$\mathbb{Q}^+$	has no row equal to $\vec{0}$
4.8	B*, J*, P*	$\omega$	has no row equal to $\vec{0}$

Notice that the requirement that no row of  $A$  is identically 0 is strictly weaker than the requirement that  $A$  be IPR/ $\mathbb{N}$ .

**Corollary 1.5.** *Let  $u, v \in \mathbb{N}$ , let  $A$  be a  $u \times v$  matrix with entries from  $\omega$  such that  $A$  is IPR/ $\mathbb{N}$  and  $\text{rank}(A) = u$ . If  $\psi$  is any of the properties in Figure 1 that imply C and  $B$  is a  $\Psi$  set in  $\mathbb{N}$ , then  $\{\vec{x} \in \mathbb{N}^v : A\vec{x} \in B^u\}$  is a  $\Psi$  set in  $\mathbb{N}^v$ .*

As we pointed out in [13], if  $A = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$  and  $B = 2\mathbb{N} + 1$ , then  $A$  has entries from  $\omega$ ,  $A$  is IPR/ $\mathbb{N}$ ,  $\text{rank}(A) = 2$ , and  $B$  is syndetic, but  $\{\vec{x} \in \mathbb{N}^2 : A\vec{x} \in B^2\} = \emptyset$ . Thus none of the properties that are implied by syndetic satisfy Corollary 1.5.

We showed in [14, Theorem 4.2] that if  $A = \begin{pmatrix} 1 & 2 & 4 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}$  and  $B = FS(\langle 2^{4t} \rangle_{t=1}^\infty)$ , then  $A$  has entries from  $\omega$ ,  $A$  is IPR/ $\mathbb{N}$ ,  $\text{rank}(A) = 3$ , and  $B$  is an IP set, but  $\{\vec{x} \in \mathbb{N}^3 : A\vec{x} \in B^3\} = \emptyset$ . Thus neither IP nor Q satisfies Corollary 1.5. Consequently, if  $\Psi$  is one of the properties in Figure 1, then  $\Psi$  satisfies Corollary 1.5 if and only if  $\Psi$  implies C.

It was shown in [4, Corollary 4.2] that Theorem 1.2 holds with  $\mathbb{N}$  replacing  $\mathbb{Z}$ . So if  $l \in \mathbb{N}$ ,  $A = \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ \vdots & \vdots \\ 1 & l \end{pmatrix}$ , and  $B$  is piecewise syndetic in  $\mathbb{N}$ ,

then  $\{\vec{x} \in \mathbb{N}^2 : A\vec{x} \in B^l\} \neq \emptyset$ . Naturally, we wondered what there was about this matrix  $A$  which changed the conclusion.

In Section 3 of this paper we answer that question for the PS property in  $\mathbb{Z}$  and in Section 4 we answer that question for the PS property in  $\mathbb{N}$ . It turns out that several equivalent conditions involve other properties on our list, as well as ones related to those on our list.

In [4] Vitaly Bergelson and the first author of this paper proved results generalizing Theorem 1.2 for several of the notions in Figure 1 (as well as some not on that list). In Section 5 we extend this generalization to include  $SC^*$ ,  $QC$ ,  $QC^*$ , and  $P^*$  and derive combinatorial consequences that were not mentioned in that paper.

Section 6 will include some results applying to arbitrary commutative and cancellative semigroups as well as an investigation of properties of extensions of the maps  $\vec{x} \mapsto A\vec{x}$ .

## 2. DEFINITIONS

For all but two of the notions that we are studying, we will utilize a characterization in terms of the algebraic structure of the Stone-Ćech compactification of a discrete semigroup  $(S, +)$ . We give a very brief introduction to this structure now. For a detailed introduction see [11,

Part I]. We are writing our semigroups additively because we are primarily interested in the semigroups  $(\mathbb{N}^v, +)$  and  $(\mathbb{Z}^v, +)$ .

We let  $\beta S = \{p : p \text{ is an ultrafilter on } S\}$ , identifying the principal ultrafilters on  $S$  with the points of  $S$  so that we may assume that  $S \subseteq \beta S$ . Given  $A \subseteq S$ ,  $\bar{A} = \{p \in \beta S : A \in p\}$ . We choose  $\{\bar{A} : A \subseteq S\}$  as a basis for the topology of  $\beta S$ . Then  $\bar{A}$  is the closure of  $A$  in  $\beta S$ .

The operation  $+$  on  $S$  extends to an operation, also denoted  $+$ , on  $\beta S$  so that  $(\beta S, +)$  is a right topological semigroup with  $S$  contained in the topological center of  $\beta S$ . That is, for each  $p \in \beta S$ , the function  $\rho_p : \beta S \rightarrow \beta S$  defined by  $\rho_p(q) = q + p$  is continuous and for each  $x \in S$ , the function  $\lambda_x : \beta S \rightarrow \beta S$  defined by  $\lambda_x(q) = x + q$  is continuous. Despite the fact that it is denoted by  $+$ , the operation on  $\beta S$  is not likely to be commutative, even if  $S$  is commutative. In fact, if  $S = \mathbb{N}^v$  or  $S = \mathbb{Z}^v$ , the topological center of  $\beta S$  is equal to  $S$ ; that is, if  $p \in S^* = \beta S \setminus S$ , then  $\lambda_p$  is not continuous. Given  $p, q \in \beta S$  and  $A \subseteq S$ ,  $A \in p + q$  if and only if  $\{x \in S : -x + A \in q\} \in p$ , where  $-x + A = \{y \in S : x + y \in A\}$ .

As does any compact Hausdorff right topological semigroup,  $\beta S$  has idempotents and a smallest two sided ideal, denoted  $K(\beta S)$ , which is the union of all of the minimal left ideals of  $\beta S$  and also the union of all of the minimal right ideals of  $\beta S$ . An idempotent in  $\beta S$  is an element of  $K(\beta S)$  if and only if it is minimal with respect to the ordering of idempotents wherein  $p \leq q$  if and only if  $p + q = q + p = p$ . Such idempotents are simply said to be minimal. Minimal left ideals of  $\beta S$  are closed. The intersection of any minimal left ideal with any minimal right ideal is a group, and any two such groups are isomorphic. Given a subset  $X$  of  $\beta S$ , we let  $E(X) = \{p \in X : p + p = p\}$ .

Given a property  $\Psi$  of a subset of  $S$ , there is a dual property  $\Psi^*$  defined as follows. If  $A \subseteq S$ , then  $A$  has property  $\Psi^*$  if and only if  $A$  has nonempty intersection with any subset  $B$  of  $S$  which has property  $\Psi$ . All of the notions we will consider are closed under supersets and the empty set does not have any of these properties. In that situation,  $A$  has property  $\Psi^*$  if and only if  $S \setminus A$  does not have property  $\Psi$ . Further, under the same assumption, property  $\Psi$  implies property  $\theta$  if and only if property  $\theta^*$  implies property  $\Psi^*$  and property  $\Psi^{**}$  is the same as property  $\Psi$ .

As we define the notions, we will frequently give equivalent characterizations. For the proofs of the equivalences (or references to the proofs) see [7]. We write  $\mathcal{P}_f(X)$  for the set of finite nonempty subsets of  $X$ .

**Definition 2.1.** Let  $(S, +)$  be a commutative semigroup and let  $A \subseteq S$ .

- (1)  $A$  is a  $Q$  set if and only if there exists a sequence  $\langle x_n \rangle_{n=1}^\infty$  in  $S$  such that whenever  $m < n$ ,  $x_n \in x_m + A$ .

- (2)  $A$  is a *P set* if and only if for every  $k \in \mathbb{N}$ , there exist  $a, d \in S$  such that  $\{a, a + d, \dots, a + kd\} \subseteq A$ .
- (3)  $A$  is an *IP set* if and only if there exists a sequence  $\langle x_n \rangle_{n=1}^\infty$  in  $S$  such that  $FS(\langle x_n \rangle_{n=1}^\infty) \subseteq A$ , where  $FS(\langle x_n \rangle_{n=1}^\infty) = \{\sum_{n \in F} x_n : F \in \mathcal{P}_f(\mathbb{N})\}$ . Equivalently,  $A$  is an IP set if and only if there is an idempotent  $p \in \beta S$  such that  $A \in p$ .
- (4)  $A$  is a *J set* if and only if for every  $F \in \mathcal{P}_f(\mathbb{N}S)$ , there exist  $a \in S$  and  $H \in \mathcal{P}_f(\mathbb{N})$  such that for each  $f \in F$ ,  $a + \sum_{n \in H} f(n) \in A$ .
- (5)  $J(S) = \{p \in \beta S : (\forall A \in p)(A \text{ is a J set})\}$ .

Here  $\mathbb{N}S$  is the set of all sequences in  $S$ .

It is shown in [11, Section 14.4] that  $J(S)$  is a two sided ideal of  $\beta S$  and that a subset  $A$  of  $S$  is a C set if and only if there is an idempotent in  $\overline{A} \cap J(S)$ . (The proof of Theorem 14.14.4 should be moved to after Lemma 14.14.6, since one needs to know  $J(S) \neq \emptyset$ .)

**Definition 2.2.** Let  $(S, +)$  be a commutative semigroup and let  $A \subseteq S$ .

- (1)  $A$  is *piecewise syndetic* (that is a *PS set*) if and only if there exists  $G \in \mathcal{P}_f(S)$  such that for every  $F \in \mathcal{P}_f(S)$  there is some  $x \in S$  such that  $F + x \subseteq \bigcup_{t \in G} (-t + A)$ . Equivalently  $A$  is a PS set if and only if  $\overline{A} \cap K(\beta S) \neq \emptyset$ .
- (2)  $A$  is a *QC set* if and only if there is an idempotent in  $\overline{A} \cap \text{cl}K(\beta S)$ .
- (3)  $A$  is *central* if and only if there is an idempotent in  $\overline{A} \cap K(\beta S)$ .
- (4)  $A$  is *syndetic* if and only if there exists  $G \in \mathcal{P}_f(S)$  such that  $S = \bigcup_{t \in G} (-t + A)$ . Equivalently  $A$  is syndetic if and only if for every left ideal  $L$  of  $\beta S$ ,  $\overline{A} \cap L \neq \emptyset$ .
- (5)  $A$  is an *SC set* if and only if for every left ideal  $L$  of  $\beta S$ , there is an idempotent in  $\overline{A} \cap L$ .
- (6)  $A$  is *thick* if and only if for every  $F \in \mathcal{P}_f(S)$ , there exists  $x \in S$  such that  $F + x \subseteq A$ . Equivalently  $A$  is thick if and only if there exists a left ideal  $L$  of  $\beta S$  such that  $L \subseteq \overline{A}$ .

The last two of our notions involve the property of density of subsets of a commutative semigroup.

**Definition 2.3.** Let  $(S, +)$  be a commutative semigroup and let  $A \subseteq S$ . The *Følner density* of  $A$  is defined by

$$d(A) = \sup\{\alpha \in [0, 1] : (\forall H \in \mathcal{P}_f(S)) (\forall \epsilon > 0) (\exists K \in \mathcal{P}_f(S)) \\ (|A \cap K| \geq \alpha \cdot |K| \text{ and} \\ (\forall s \in H) (|K \triangle s + K| < \epsilon \cdot |K|))\}.$$

See [10] for information about the generality in which Følner density can be defined and its properties. If  $S = \mathbb{N}^v$  or  $S = \mathbb{Z}^v$ , a simpler notion was introduced by Vitaly Bergelson in [2].

**Definition 2.4.** Let  $v \in \mathbb{N}$ , let  $S = \mathbb{N}^v$  or  $S = \mathbb{Z}^v$ , and let  $A \subseteq S$ . Then the *Banach density* of  $A$  is defined by

$$d(A) = \sup\{\alpha \in [0, 1] : (\forall n \in \mathbb{N})(\exists k_1, k_2, \dots, k_v \in \{m \in \mathbb{N} : m > n\}) \\ (\exists \vec{a} \in S)(|A \cap (\vec{a} + \times_{i=1}^v \{0, 1, \dots, k_i - 1\})| \geq \alpha \cdot \prod_{i=1}^v k_i)\}.$$

It is a recent result of Vitaly Bergelson and Daniel Glasscock [3, Theorem 3.5 and Corollary 3.6] that for subsets of  $\mathbb{Z}^v$  or  $\mathbb{N}^v$ , the Banach density and Følner density are equal.

**Definition 2.5.** Let  $(S, +)$  be a commutative semigroup. Then  $\Delta(S) = \{p \in \beta S : (\forall A \in \mathcal{P}(S))(d(A) > 0)\}$ .

By [13, Theorem 3.1], if  $S = \mathbb{N}^v$  or  $S = \mathbb{Z}^v$ , then  $\Delta(S)$  is a closed two sided ideal of  $\beta S$ .

**Lemma 2.6.** *Let  $v \in \mathbb{N}$ , let  $S = \mathbb{N}^v$  or  $S = \mathbb{Z}^v$ , and let  $A \subseteq S$ . If  $d(A) > 0$ , then  $\overline{A} \cap \Delta(S) \neq \emptyset$ .*

*Proof.* It is a routine exercise to establish that if  $B$  and  $C$  are subsets of  $S$ , then  $d(B \cup C) \leq d(B) + d(C)$ . The conclusion is then an immediate consequence of [11, Theorem 3.11].  $\square$

**Definition 2.7.** Let  $(S, +)$  be a commutative semigroup and let  $A \subseteq S$ .

- (1)  $A$  is a *B set* if and only if  $d(A) > 0$ . Equivalently  $A$  is a B set if and only if  $\overline{A} \cap \Delta(S) \neq \emptyset$ .
- (2)  $A$  is a *D set* if and only if there is an idempotent in  $\overline{A} \cap \Delta(S)$ .

The names Q, P, IP, PS, QC, and SC come from quotient, progression, infinite dimensional parallelepiped, piecewise syndetic, quasi central, and strongly central respectively. (If quotient seems confusing, consider that when written multiplicatively,  $x_n \in x_m \cdot A$  says that  $\frac{x_n}{x_m} \in A$ .) The names C, J, B, and D, have no particular significance.

The implications in Figure 1 are established in [7] and examples are given in  $S = \mathbb{N}$  showing that the only implications that hold in general are those that follow from the diagram and transitivity.

### 3. MATRICES THAT PRESERVE PIECEWISE SYNDETICITY IN $\mathbb{Z}$

We single out  $\mathbb{Z}$  for separate attention because we are able to get stronger conclusions involving J sets and B sets in  $\mathbb{Z}$ .

**Lemma 3.1.** *Let  $v \in \mathbb{N}$ , Let  $I$  be a left ideal of  $\beta(\mathbb{N}^v)$ , let  $p$  be an idempotent in  $I$ , and let  $\vec{x} \in \mathbb{Z}^v$ . Then  $\vec{x} + p \in I$ .*

*Proof.* We first show that  $\vec{x} + p \in \beta(\mathbb{N}^v)$ . Let

$$E = \{\vec{y} \in \mathbb{N}^v : (\forall j \in \{1, 2, \dots, v\})(x_j + y_j > 0)\}.$$

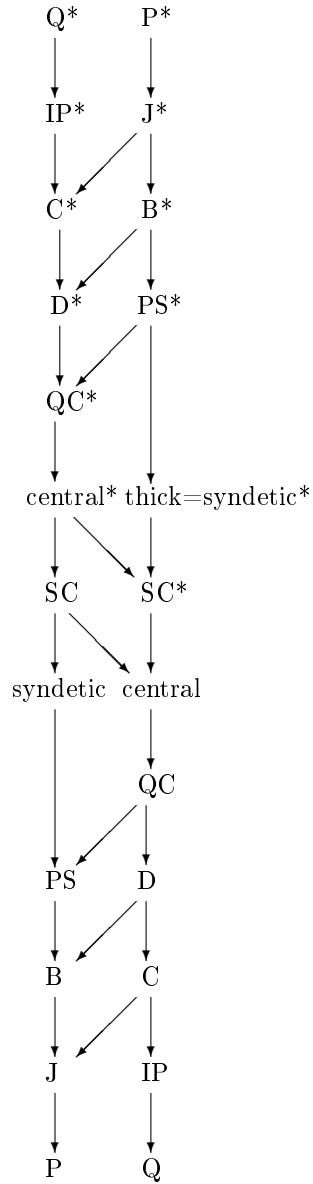


Figure 1: Implications for commutative  $S$ .

Then  $E$  is an IP\* set in  $\beta(\mathbb{N}^v)$ . (Given a sequence  $\langle \vec{w}_t \rangle_{t=1}^\infty$  in  $\mathbb{N}$ , if  $m \in \mathbb{N}$  and  $x_j > -m$  for every  $j \in \{1, 2, \dots, v\}$ , then  $\sum_{t=1}^m \vec{w}_t \in E$ .) Thus  $E \in p$ . Since  $E \subseteq -\vec{x} + \mathbb{N}^v$ , we have that  $-\vec{x} + \mathbb{N}^v \in p$  so  $\mathbb{N}^v \in \vec{x} + p$ . Since  $\vec{x} + p \in \beta(\mathbb{N}^v)$ ,  $\vec{x} + p = (\vec{x} + p) + p \in (\beta(\mathbb{N}^v) + p) \subseteq I$ .  $\square$

When we defined IPR/ $\mathbb{Z}$  in Definition 1.1 we required all of the entries of  $\vec{x}$  to be nonzero.

**Lemma 3.2.** *Let  $u, v \in \mathbb{N}$  and let  $A$  be a  $u \times v$  matrix with rational entries. Then  $A$  is IPR/ $\mathbb{Z}$  if and only if whenever  $\mathbb{Z} \setminus \{0\}$  is finitely colored, there exists  $\vec{x} \in \mathbb{Z}^v$  such that the entries of  $A\vec{x}$  are monochromatic.*

*Proof.* The necessity is trivial. Assume that whenever  $\mathbb{Z} \setminus \{0\}$  is finitely colored, there exists  $\vec{x} \in \mathbb{Z}^v$  such that the entries of  $A\vec{x}$  are monochromatic. By the fact that (a) implies (g) in [6, Theorem 4.13] we may pick  $b_1, b_2, \dots, b_v \in \mathbb{Q} \setminus \{0\}$  such that, if

$$C = \begin{pmatrix} & & A & & \\ b_1 & 0 & \dots & 0 & \\ 0 & b_2 & \dots & 0 & \\ \vdots & \vdots & \ddots & \vdots & \\ 0 & 0 & \dots & b_v & \end{pmatrix}$$

then whenever  $\mathbb{Z} \setminus \{0\}$  is finitely colored, there exists  $\vec{x} \in \mathbb{Z}^v$  such that the entries of  $C\vec{x}$  are monochromatic. If the entries of  $C\vec{x}$  are monochromatic, then for each  $j \in \{1, 2, \dots, v\}$ ,  $b_j x_j \neq 0$ .  $\square$

**Theorem 3.3.** *Let  $u, v \in \mathbb{N}$  and let  $A$  be a  $u \times v$  matrix with rational entries. Then  $A$  is IPR/ $\mathbb{Z}$  if and only if, whenever  $B$  is a central subset of  $\mathbb{Z}$ ,  $\{\vec{x} \in \mathbb{Z}^v : A\vec{x} \in B^u\}$  is central in  $\mathbb{Z}^v$ .*

*Proof.* For the sufficiency, let  $\mathbb{Z} \setminus \{0\}$  be finitely colored. Pick a color class  $B$  which is central in  $\mathbb{Z}$ . Then  $\{\vec{x} \in \mathbb{Z}^v : A\vec{x} \in B^u\}$  is central in  $\mathbb{Z}^v$  so is nonempty. By Lemma 3.2,  $A$  is IPR/ $\mathbb{Z}$ .

For the necessity, let  $B$  be central in  $\mathbb{Z}$  and pick an idempotent  $p$  in  $K(\beta\mathbb{Z})$  such that  $B \in p$ . By [11, Exercise 4.3.8]  $K(\beta\mathbb{Z}) = K(\beta\mathbb{N}) \cup -K(\beta\mathbb{N})$ . Assume first that  $p \in K(\beta\mathbb{N})$ . By the fact that (a) implies (j) in [6, Theorem 4.13]  $\{\vec{x} \in \mathbb{Z}^v : A\vec{x} \in (B \cap \mathbb{N})^u\}$  is central in  $\beta(\mathbb{Z}^v)$ .

Now assume that  $p \in -K(\beta\mathbb{N})$ . Then  $(-B) \cap \mathbb{N}$  is central in  $\mathbb{N}$  so  $\{\vec{x} \in \mathbb{Z}^v : A\vec{x} \in ((-B) \cap \mathbb{N})^u\}$  is central in  $\beta(\mathbb{Z}^v)$ . The mapping  $\vec{x} \mapsto -\vec{x}$  extends to an isomorphism and homeomorphism from  $\beta(\mathbb{Z}^v)$  to  $\beta(\mathbb{Z}^v)$  so  $\{\vec{x} \in \mathbb{Z}^v : A\vec{x} \in B^u\}$  is central in  $\beta(\mathbb{Z}^v)$ .  $\square$

Given  $n \in \mathbb{Z}$  we write  $\bar{n}$  for the vector of the appropriate size all of whose entries are equal to  $n$ .



**Theorem 3.4.** *Let  $u, v \in \mathbb{N}$ , let  $A$  be a  $u \times v$  matrix with rational entries, and assume that for each  $n \in \mathbb{Z}$ , there exists  $\vec{z} \in \mathbb{Z}^v$  such that  $A\vec{z} = \vec{n} \in \mathbb{Z}^u$ . If  $B$  is a piecewise syndetic subset of  $\mathbb{Z}$ , then  $\{\vec{x} \in \mathbb{Z}^v : A\vec{x} \in B^u\}$  is piecewise syndetic in  $\mathbb{Z}^v$ .*

*Proof.* Assume that  $B$  is a piecewise syndetic subset of  $\mathbb{Z}$ . By [11, Theorem 4.43] pick  $n \in \mathbb{Z}$  such that  $-n+B$  is central in  $\mathbb{Z}$  and let  $C = -n+B$ . Let  $D = \{\vec{y} \in \mathbb{Z}^v : A\vec{y} \in C^u\}$ . Note that the fact that there exists  $\vec{z} \in \mathbb{Z}^v$  such that  $A\vec{z} = \vec{1} \in \mathbb{N}^u$  implies that  $A$  is IPR/ $\mathbb{Z}$ . By Theorem 3.3,  $D$  is central in  $\mathbb{Z}^v$  so pick an idempotent  $p \in \overline{D} \cap K(\beta(S^v))$ . Pick  $\vec{z} \in \mathbb{Z}^v$  such that  $A\vec{z} = \vec{n}$ .

Let  $F = \{\vec{x} \in \mathbb{N}^v : A\vec{x} \in B^u\}$ . It suffices to show that  $D \subseteq -\vec{z} + F$  for then  $F \in \vec{z} + p$  so  $\overline{F} \cap K(\beta(\mathbb{Z}^v)) \neq \emptyset$ . So let  $\vec{y} \in D$ . Then  $\vec{z} + \vec{y} \in \mathbb{Z}^v$  and  $A\vec{y} \in C^u$ . We claim that  $A(\vec{z} + \vec{y}) \in B^u$ . To see this, let  $i \in \{1, 2, \dots, u\}$ . Then  $\sum_{j=1}^v a_{i,j}(z_j + y_j) = n + \sum_{j=1}^v a_{i,j}y_j \in n + C = B$ . (We are following the custom of denoting the entries of a matrix by the lower case letter whose upper case is the name of the matrix.)  $\square$

**Lemma 3.5.** *Let  $m, v \in \mathbb{N}$  and let  $D$  be an  $m \times v$  matrix with rational entries such that  $\text{rank}(D) = m$ . Let  $\vec{a} \in \mathbb{Z}^m$ . Assume that for each  $k \in \mathbb{N}$  there exists  $\vec{x} \in \mathbb{Z}^v$  such that  $D\vec{x} \in \vec{a} + (\mathbb{Z}k)^m$ . There exist  $d \in \mathbb{N}$  such that if  $k \in \mathbb{N}$ ,  $d$  divides  $k$ ,  $\vec{x} \in \mathbb{Z}^v$ , and  $D\vec{x} \in \vec{a} + (\mathbb{Z}k)^m$ , then there exist  $\vec{s}$  and  $\vec{z}$  in  $\mathbb{Z}^v$  such that  $\vec{x} = \vec{s} + \vec{z}$ ,  $D\vec{s} \in (\mathbb{Z}k)^m$  and  $D\vec{z} = \vec{a}$ .*

*Proof.* By adding a final column of  $\vec{0}$  if need be we may assume that  $v > m$ . We also assume that the first  $m$  columns of  $D$  are linearly independent. Let  $B$  consist of those first  $m$  columns, and let  $C$  consist of the other columns so that  $D = \begin{pmatrix} B & C \end{pmatrix}$ . Pick  $d \in \mathbb{N}$  such that all entries of  $dB^{-1}$  are integers. Choose  $k \in \mathbb{N}$  such that  $d$  divides  $k$  and  $\vec{x} \in \mathbb{Z}^v$  such that  $D\vec{x} \in \vec{a} + (\mathbb{Z}k)^m$ . Let  $\vec{w}$  be the first  $m$  entries of  $\vec{x}$  and let  $\vec{y}$  be the last  $v - m$  entries. Then  $D\vec{x} = \begin{pmatrix} B & C \end{pmatrix} \begin{pmatrix} \vec{w} \\ \vec{y} \end{pmatrix} = k\vec{f} + \vec{a}$  for some  $\vec{f} \in \mathbb{Z}^m$ .

Let  $\vec{e} = C\vec{y}$ . Pick  $\vec{g}$  and  $\vec{h}$  in  $\mathbb{Q}^m$  such that  $B\vec{g} = k\vec{f}$  and  $B\vec{h} = \vec{a} - \vec{e}$ . Then  $\vec{g} = kB^{-1}\vec{f}$  so  $\vec{g} \in \mathbb{Z}^m$ . Let  $\vec{s} = \begin{pmatrix} \vec{g} \\ \vec{0} \end{pmatrix}$  where  $\vec{0} \in \mathbb{Z}^{v-m}$ . Then  $\vec{s} \in \mathbb{Z}^v$  and  $D\vec{s} = B\vec{g} = k\vec{f} \in (\mathbb{Z}k)^m$ . Now  $B\vec{w} = k\vec{f} + \vec{a} - \vec{e} = B\vec{g} + B\vec{h}$  so  $\vec{w} = \vec{g} + \vec{h}$ . Since the entries of  $\vec{w}$  and  $\vec{g}$  are in  $\mathbb{Z}$ , so are the entries of  $\vec{h}$ .

Let  $\vec{z} = \begin{pmatrix} \vec{h} \\ \vec{y} \end{pmatrix}$ . Then  $\vec{z} \in \mathbb{Z}^v$  and  $D\vec{z} = B\vec{h} + C\vec{y} = \vec{a} - \vec{e} + \vec{e} = \vec{a}$ .  $\square$

**Theorem 3.6.** *Let  $u, v \in \mathbb{N}$  and let  $\vec{a} \in \mathbb{Z}^u$ . Let  $A$  be a  $u \times v$  matrix with rational entries which has the property that for any  $k \in \mathbb{N}$  there exists*

$\vec{x} \in \mathbb{Z}^v$  such that  $A\vec{x} \in \vec{a} + (\mathbb{Z}k)^u$ . Then there exists  $\vec{z} \in \mathbb{Z}^v$  such that  $A\vec{z} = \vec{a}$ .

*Proof.* Let the rows of  $A$  be  $\langle \vec{r}_i \rangle_{i=1}^u$ . Pick a maximal linearly independent set of rows of  $A$  and let  $m$  be the number of rows in this set. We may assume that these rows are the first  $m$  rows of  $A$ . Let  $D$  be the matrix consisting of those first  $m$  rows. If  $m = u$  we are done, by Lemma 3.5. So assume that  $m < u$ . For  $j \in \{m+1, m+2, \dots, u\}$  pick  $\langle q_{j,i} \rangle_{i=1}^m$  in  $\mathbb{Q}$  such that  $\vec{r}_j = \sum_{i=1}^m q_{j,i} \vec{r}_i$ .

Pick  $d$  as guaranteed by Lemma 3.5 and let  $k$  be a multiple of  $d$  such that  $k > |a_j|$  for each  $j \in \{m+1, m+2, \dots, u\}$  and  $kq_{j,i} \in \mathbb{Z}$  for every  $j \in \{m+1, \dots, u\}$  and every  $i \in \{1, 2, \dots, m\}$ . Observe that  $q_{j,i} \mathbb{Z}k^2 \subseteq \mathbb{Z}k$  for every  $j \in \{m+1, \dots, u\}$  and every  $i \in \{1, 2, \dots, m\}$ . Pick  $\vec{x} \in \mathbb{Z}^v$  such that  $A\vec{x} \in \vec{a} + (\mathbb{Z}k^2)^u$  and let  $\vec{b}$  consist of the first  $m$  entries of  $\vec{a}$ . Then  $D\vec{x} \in \vec{b} + (\mathbb{Z}k^2)^m$ . By Lemma 3.5 there exist  $\vec{s}$  and  $\vec{z}$  in  $\mathbb{Z}^v$  such that  $\vec{x} = \vec{s} + \vec{z}$ ,  $D\vec{s} \in (\mathbb{Z}k^2)^m$  and  $D\vec{z} = \vec{b}$ . So, for each  $i \in \{1, 2, \dots, m\}$ ,  $\vec{r}_i \cdot \vec{z} = a_i$ .

We claim that  $A\vec{z} = \vec{a}$ . That is, for each  $j \in \{m+1, m+2, \dots, u\}$ ,  $\vec{r}_j \cdot \vec{z} = a_j$ . So let  $j \in \{m+1, m+2, \dots, u\}$ . Note that  $\vec{r}_j \cdot \vec{x} \in a_j + \mathbb{Z}k^2$ . We also have  $\vec{r}_j \cdot \vec{s} = \sum_{i=1}^m q_{j,i} \vec{r}_i \cdot \vec{s} \equiv 0 \pmod{k}$  because  $\vec{r}_i \cdot \vec{s} \in \mathbb{Z}k^2$  for every  $i \in \{1, 2, \dots, m\}$ . So  $\vec{r}_j \cdot \vec{z} \equiv \vec{r}_j \cdot \vec{x} \equiv a_j \pmod{k}$ . Since  $k > |a_j|$ , it follows that  $\vec{r}_j \cdot \vec{z} = a_j$ .  $\square$

**Theorem 3.7.** *Let  $u, v \in \mathbb{N}$ , let  $A$  be a  $u \times v$  matrix with rational entries, and let  $B$  be a  $J$  set in  $\mathbb{Z}$ . Assume that for each  $a \in \mathbb{Z}$  there exists  $\vec{x} \in \mathbb{Z}^v$  such that  $A\vec{x} = \vec{a} \in \mathbb{Z}^u$ . Then  $\{\vec{y} \in \mathbb{Z}^v : A\vec{y} \in B^u\}$  is a  $J$  set in  $\mathbb{Z}^v$ .*

*Proof.* Let  $C = \{\vec{y} \in \mathbb{Z}^v : A\vec{y} \in B^u\}$ . Let  $F \in \mathcal{P}_f(\mathbb{N}(\mathbb{Z}^v))$ . Then for

$$\text{each } \vec{f} \in F \text{ and each } t \in \mathbb{N}, \vec{f}(t) = \begin{pmatrix} f_1(t) \\ \vdots \\ f_j(t) \end{pmatrix}.$$

Pick  $d \in \mathbb{N}$  such that all entries of  $dA$  are integers. Inductively choose a sequence  $\langle H_n \rangle_{n=1}^\infty$  in  $\mathcal{P}_f(\mathbb{N})$  such that  $\max H_n < \min H_{n+1}$  for each  $n$  and for each  $\vec{f} \in F$ , each  $j \in \{1, 2, \dots, v\}$ , and each  $n \in \mathbb{N}$ ,  $d$  divides  $\sum_{t \in H_n} f_j(t)$ .

For  $\vec{f} \in F$ ,  $i \in \{1, 2, \dots, u\}$ , and  $n \in \mathbb{N}$ , let

$$g_{\vec{f},i}(n) = \sum_{j=1}^v a_{i,j} \sum_{t \in H_n} f_j(t).$$

Then each  $g_{\vec{f},i}(n) \in \mathbb{N}\mathbb{Z}$ . Pick  $a \in \mathbb{Z}$  and  $G \in \mathcal{P}_f(\mathbb{N})$  such that for each  $\vec{f} \in F$  and each  $i \in \{1, 2, \dots, u\}$ ,  $a + \sum_{n \in G} g_{\vec{f},i}(n) \in B$ .

Pick  $\vec{x} \in \mathbb{Z}^v$  such that  $A\vec{x} = \vec{a} \in \mathbb{Z}^u$ . Let  $K = \bigcup_{n \in G} H_n$ . We claim that for  $\vec{f} \in F$ ,  $\vec{x} + \sum_{t \in K} \vec{f}(t) \in C$ . To see this we need to show that for  $i \in \{1, 2, \dots, u\}$ , entry  $i$  of  $A(\vec{x} + \sum_{t \in K} \vec{f}(t))$  is in  $B$ . That entry is  $a + \sum_{j=1}^v a_{i,j} \sum_{t \in K} f_j(t) = a + \sum_{j=1}^v a_{i,j} \sum_{n \in G} \sum_{t \in H_n} f_j(t) = a + \sum_{n \in G} \sum_{j=1}^v a_{i,j} \sum_{t \in H_n} f_j(t) = a + \sum_{n \in G} g_{\vec{f},i}(n) \in B$ .  $\square$

#### 4. MATRICES THAT PRESERVE DENSITY

We now turn our attention to B sets, where our results are more restrictive.

The following result is well known (as always with that term, by people who know it well). In this theorem,  $\mu$  being *left translation invariant* means that for every Borel  $B \subseteq \beta S$  and every  $x \in S$ ,  $\mu(\lambda_x^{-1}[B]) = \mu(B)$ .

**Theorem 4.1.** *Let  $v \in \mathbb{N}$ , let  $S = \mathbb{N}^v$  or  $S = \mathbb{Z}^v$ , and let  $B \subseteq S$ . Then  $d(B) > 0$  if and only if there exists a left translation invariant probability measure  $\mu$  on  $\beta S$  such that  $\mu(\overline{B}) > 0$ .*

*Proof.* By [10, Theorems 2.12 and 2.14]  $d(B) = \sup(\{\mu(\chi_B) : \mu \text{ is a left invariant mean on } S\})$ . Since the Banach space of bounded complex-valued functions defined on  $S$ , with the uniform norm, can be identified with the Banach space of continuous complex-valued functions defined on  $\beta S$ , with the uniform norm, the conclusion follows from the Riesz Representation Theorem.  $\square$

The following theorem is [13, Theorem 5.3].

**Theorem 4.2.** *Let  $(X, \langle Q_r \rangle_{r \in R})$  and  $(Y, \langle T_s \rangle_{s \in S})$  be dynamical systems, and assume that  $S$  is left amenable. Let  $\phi : Y \rightarrow X$  be a continuous surjection and let  $\mu$  denote a probability measure on  $X$  which is  $R$ -invariant. Assume that, for each  $s \in S$ , there exists  $r \in R$  such that  $\phi \circ T_s = Q_r \circ \phi$ . Then there is a probability measure  $\nu$  on  $Y$  which is  $S$ -invariant, such that  $\nu(f \circ \phi) = \mu(f)$  for every  $f \in C(X)$ .*

**Theorem 4.3.** *Let  $A$  be a  $u \times v$  matrix with integer entries and let  $V : \mathbb{Z}^v \rightarrow \mathbb{Z}^u$  be defined by  $V(\vec{z}) = A\vec{z}$ . Let  $\tilde{V} : \beta(\mathbb{Z}^v) \rightarrow \beta(\mathbb{Z}^u)$  be the continuous extension of  $V$ . Let  $\mu$  be a probability measure on  $\tilde{V}[\beta(\mathbb{Z}^v)]$  which is invariant under translations by elements of  $V[\mathbb{Z}^v]$ . There is a left translation invariant probability measure  $\nu$  on  $\beta(\mathbb{Z}^v)$  such that for each continuous function  $f : \tilde{V}[\beta(\mathbb{Z}^v)] \rightarrow \mathbb{R}$ ,  $\int (f \circ \tilde{V}) d\nu = \int f d\mu$ . In particular, for every clopen subset  $U$  of  $\tilde{V}[\beta(\mathbb{Z}^v)]$ ,  $\nu(\tilde{V}^{-1}[U]) = \mu(U)$ .*

*Proof.* After noting that for each  $\vec{x} \in \mathbb{Z}^v$ , if  $\vec{y} = V(\vec{x})$ , then  $V \circ \lambda_{\vec{x}} = \lambda_{\vec{y}} \circ V$ , this is the special case of Theorem 4.2, with  $Y = \beta(\mathbb{Z}^v)$ ,  $X = \tilde{V}[Y]$ ,  $\phi = \tilde{V}$ ,

$S = \mathbb{Z}^v$ ,  $R = V[\mathbb{Z}^v]$ ,  $T_{\vec{z}} = \lambda_{\vec{z}}$  for every  $\vec{z} \in \mathbb{Z}^v$ , and  $Q_{\vec{z}} = \lambda_{\vec{z}}$  for every  $\vec{z} \in V[\mathbb{Z}^v]$ .

Let  $U$  be a clopen subset of  $V[\beta(\mathbb{Z}^v)]$  and let  $f = 1_U$ , the characteristic function of  $U$ . Observe that, for every  $y \in Y$ ,  $(f \circ \tilde{V})(y) = 1$  if and only if  $\tilde{V}(y) \in U$ . So  $(f \circ \tilde{V})(y) = 1_{\tilde{V}^{-1}[U]}$ . Thus  $\nu(f \circ \tilde{V}^{-1}[U]) = \nu(\tilde{V}^{-1}[U])$ , while  $\mu(f) = \mu(U)$ .  $\square$

**Theorem 4.4.** *Let  $u, v \in \mathbb{N}$ . Let  $A$  be a  $u \times v$  matrix with entries from  $\mathbb{Z}$ . Let  $V$  and  $\tilde{V}$  be defined as in Theorem 4.3. Let  $B$  be a subset of  $\mathbb{Z}$ . If  $B^u \cap V[\mathbb{Z}^v]$  has positive density in  $\mathbb{Z}^u$ , then  $\{\vec{z} \in \mathbb{Z}^v : A\vec{z} \in B^u\}$  is a  $B$  set in  $\mathbb{Z}^v$ . The converse holds if  $A$  has rank  $u$ .*

*Proof.* Suppose that  $B^u \cap V[\mathbb{Z}^v]$  has positive density in  $\mathbb{Z}^u$ . By Theorem 4.1 there is a left invariant probability measure  $\mu$  on  $\beta(\mathbb{Z}^u)$  for which  $\mu(\overline{B^u} \cap \tilde{V}[\beta(\mathbb{Z}^v)]) > 0$ . By Theorem 4.3, applied to  $\mu[V[\beta(\mathbb{Z}^v)]]$ , there is an invariant probability measure  $\nu$  on  $\beta\mathbb{Z}^v$  for which  $\nu(\tilde{V}^{-1}[\overline{B^u} \cap \tilde{V}[\beta(\mathbb{Z}^v)]]) > 0$ . So  $\nu(\tilde{V}^{-1}[\overline{B^u}]) > 0$ . This implies that  $V^{-1}[B^u]$  has positive density in  $\mathbb{Z}^v$ .

Now assume that  $A$  has rank  $u$ . Suppose that  $B^u \cap V[\mathbb{Z}^v]$  has zero density in  $\mathbb{Z}^u$ , but that  $V^{-1}[B^u]$  has positive density in  $\mathbb{Z}^v$ . We can choose an invariant probability measure  $\nu$  on  $\beta(\mathbb{Z}^v)$  for which  $\nu(\tilde{V}^{-1}[\overline{B^u}]) > 0$ . We define a measure  $\mu$  on  $\tilde{V}[\beta(\mathbb{Z}^v)]$  by putting  $\mu(D) = \nu(\tilde{V}^{-1}[D])$  for every Borel subset  $D$  of  $\tilde{V}[\beta(\mathbb{Z}^v)]$ . We observe that  $\mu$  is invariant under translations by elements of  $V[\mathbb{Z}^v]$ . Furthermore,  $\mu(\overline{B^u} \cap \tilde{V}[\mathbb{Z}^v]) > 0$ , because  $\tilde{V}^{-1}[\overline{B^u} \cap \tilde{V}[\mathbb{Z}^v]] = \tilde{V}^{-1}[\overline{B^u}]$ . We shall obtain a contradiction by showing that we can extend  $\mu$  to an invariant measure on  $\beta(\mathbb{Z}^u)$ , which will imply that  $B^u \cap V[\mathbb{Z}^v]$  has positive density in  $\mathbb{Z}^u$ .

Let  $E = V[\mathbb{Z}^v]$ . Let  $\mathcal{C}$  denote the set of cosets of  $E$  in  $\mathbb{Z}^u$ . We claim that  $\mathcal{C}$  is finite. By [13, Lemma 3.14], we can choose  $k \in \mathbb{N}$  such that  $k\mathbb{Z}^u \subseteq E$ . Then  $\mathbb{Z}^u \subseteq \bigcup \{-\vec{x} + E : \vec{x} \in \{0, 1, \dots, k-1\}^u\}$ .

For each  $C \in \mathcal{C}$ , we choose  $\vec{w}_C \in C$ . We define  $\mu$  on  $\overline{C}$  by putting  $\mu(D) = \mu(-\vec{w}_C + D)$  for every Borel subset  $D$  of  $\overline{C}$ . We then define  $\mu$  on  $\beta(\mathbb{Z}^u)$  by putting  $\mu(D) = \sum_{C \in \mathcal{C}} \mu(D \cap \overline{C})$  for every Borel subset  $D$  of  $\beta(\mathbb{Z}^u)$ . To see that  $\mu$  is translation invariant, observe that we clearly have  $\mu(\vec{z} + D) = \mu(D)$  for every  $\vec{z} \in E$  and every Borel subset  $D$  of  $\beta\mathbb{Z}^u$ . Since  $\mathbb{Z}^u = \bigcup_{C \in \mathcal{C}} (\vec{w}_C + E)$ , it is sufficient to show that  $\mu$  is invariant under translations by the elements  $\vec{w}_C$ . Assume that  $C_1, C_2, C_3 \in \mathcal{C}$ , that  $D$  is a Borel subset of  $\overline{C_1}$  and that  $\vec{w}_{C_2} + D \subseteq \overline{C_3}$ . Then  $\mu(\vec{w}_{C_2} + D) = \mu(-\vec{w}_{C_3} + \vec{w}_{C_2} + D)$  and  $\mu(D) = \mu(-\vec{w}_{C_1} + D)$ . Now  $-\vec{w}_{C_1} + D$  and  $-\vec{w}_{C_3} + \vec{w}_{C_2} + D$  are subsets of  $\overline{E}$ , and  $\vec{w}_{C_2} + \vec{w}_{C_1} - \vec{w}_{C_3} \in E$ . Since  $-\vec{w}_{C_2} + D = (-\vec{w}_{C_1} - \vec{w}_{C_2} + \vec{w}_{C_3} + (-\vec{w}_{C_3}) + \vec{w}_{C_2} + D)$ , it follows that  $\mu(D) = \mu(\vec{w}_{C_2} + D)$ .  $\square$

We obtain an extension theorem as an immediate corollary of Theorem 4.4

**Corollary 4.5.** *Let  $u, v, w \in \mathbb{N}$ , let  $A$  be a  $u \times v$  matrix with entries in  $\mathbb{Z}$  such that  $\text{rank}(A) = u$  and let  $C$  be a  $u \times w$  matrix with entries in  $\mathbb{Z}$ . Let  $B$  be a subset of  $\mathbb{Z}$ . If  $\{\vec{z} \in \mathbb{Z}^v : A\vec{z} \in B^u\}$  has positive density in  $\mathbb{Z}^v$ , then  $\{\vec{z} \in \mathbb{Z}^{v+w} : (A \ C)\vec{z} \in B^u\}$  has positive density in  $\mathbb{Z}^{v+w}$ .*

*Proof.* This is immediate from Theorem 4.4 and the observation that  $\{A\vec{z} : \vec{z} \in \mathbb{Z}^v\} \subseteq \{(A \ C)\vec{z} : \vec{z} \in \mathbb{Z}^{v+w}\}$ .  $\square$

**Theorem 4.6.** *Let  $A$  be a  $u \times v$  matrix with entries from  $\mathbb{Z}$  and assume that  $\text{rank}(A) = u$  and  $(\exists \vec{x} \in \mathbb{Z}^v)(A\vec{x} = \bar{1} \in \mathbb{N}^u)$ . Whenever  $B$  is a B set in  $\mathbb{Z}$ ,  $\{\vec{x} \in \mathbb{Z}^v : A\vec{x} \in B^u\}$  is a B set in  $\mathbb{Z}^v$ .*

*Proof.* Define  $V$  and  $\tilde{V}$  as in Theorem 4.3.

Let  $B$  be a B set in  $\mathbb{Z}$  and pick  $p \in \overline{B} \cap \Delta(\mathbb{Z})$ . By [13, Lemma 3.14], pick  $k \in \mathbb{N}$  such that  $k\mathbb{Z}^u \subseteq V[\mathbb{Z}^v]$ . Pick  $n \in \{1, 2, \dots, k\}$  such that  $n + k\mathbb{Z} \in p$ . Pick  $\vec{z} \in \mathbb{Z}^v$  such that  $A\vec{z} = \bar{n} \in \mathbb{N}^u$ . Note that  $\bar{n} + k\mathbb{Z}^u \subseteq V[\mathbb{Z}^v]$ . (Given  $\vec{y} \in \mathbb{Z}^u$  pick  $\vec{x} \in \mathbb{Z}^v$  such that  $V(\vec{x}) = k\vec{y}$ . Then  $V(\vec{z} + \vec{x}) = \bar{n} + k\vec{y}$ .) Observe that  $(n + k\mathbb{Z})^u \subseteq \bar{n} + k\mathbb{Z}^u \subseteq V[\mathbb{Z}^v]$ . Since  $(n + k\mathbb{Z}) \cap B \in p$ ,  $(n + k\mathbb{Z}) \cap B$  has positive density in  $\mathbb{Z}$ . A routine application of Definition 2.4 that  $(n + k\mathbb{Z})^u \cap B^u$  has positive density in  $\mathbb{Z}^u$ . By Theorem 4.4,  $\{\vec{x} \in \mathbb{Z}^v : A\vec{x} \in B^u\}$  is a B set in  $\mathbb{Z}^v$ .  $\square$

We can now present the characterizations for  $\mathbb{Z}$ .

**Theorem 4.7.** *Let  $A$  be a  $u \times v$  matrix with rational entries. Statements (a) through (k) are equivalent and are implied by statement (l). If  $\text{rank}(A) = u$  and the entries of  $A$  are integers, then statement (l) is equivalent to the other statements.*

- (a) *Whenever  $B$  is a piecewise syndetic subset of  $\mathbb{Z}$ ,  $\{\vec{x} \in \mathbb{Z}^v : A\vec{x} \in B^u\}$  is piecewise syndetic in  $\mathbb{Z}^v$ .*
- (b) *Whenever  $B$  is a J subset of  $\mathbb{Z}$ ,  $\{\vec{x} \in \mathbb{Z}^v : A\vec{x} \in B^u\}$  is a J subset of  $\mathbb{Z}^v$ .*
- (c) *Whenever  $B$  is a J subset of  $\mathbb{Z}$ , there exists  $\vec{x} \in \mathbb{Z}^v$  such that  $A\vec{x} \in B^u$ .*
- (d) *Whenever  $B$  is a B subset of  $\mathbb{Z}$ , there exists  $\vec{x} \in \mathbb{Z}^v$  such that  $A\vec{x} \in B^u$ .*
- (e) *Whenever  $B$  is a piecewise syndetic subset of  $\mathbb{Z}$ , there exists  $\vec{x} \in \mathbb{Z}^v$  such that  $A\vec{x} \in B^u$ .*
- (f)  *$A$  is IPR/ $\mathbb{Z}$  and whenever  $B$  is a syndetic subset of  $\mathbb{Z}$ , there exists  $\vec{x} \in \mathbb{Z}^v$  such that  $A\vec{x} \in B^u$ .*
- (g) *Whenever  $B$  is a central subset of  $\mathbb{Z}$ , there exists  $\vec{x} \in \mathbb{Z}^v$  such that  $A\vec{x} \in (1 + B)^u$ .*

- (h)  $(\exists \vec{x} \in \mathbb{Z}^v)(A\vec{x} = \bar{1} \in \mathbb{N}^u)$ .
- (i)  $(\forall n \in \mathbb{Z})(\exists \vec{x} \in \mathbb{Z}^v)(A\vec{x} = \bar{n} \in \mathbb{Z}^u)$ .
- (j)  $(\forall n \in \mathbb{N})(\exists \vec{x} \in \mathbb{Z}^v)(A\vec{x} = \bar{n} \in \mathbb{N}^u)$ .
- (k)  $(\forall k \in \mathbb{N})(\exists \vec{x} \in \mathbb{Z}^v)(A\vec{x} \in (\mathbb{Z}k + 1)^u)$ .
- (l) *Whenever  $B$  is a  $B$  subset of  $\mathbb{Z}$ ,  $\{\vec{x} \in \mathbb{Z}^v : A\vec{x} \in B^u\}$  is a  $B$  set in  $\mathbb{Z}^v$ .*

*Proof.* That (i)  $\Rightarrow$  (a) follows from Theorem 3.4. That (i)  $\Rightarrow$  (b) follows from Theorem 3.7. That (k)  $\Rightarrow$  (h) follows from Theorem 3.6. If  $\text{rank}(A) = u$  and the entries of  $A$  are integers, then the fact that (i)  $\Rightarrow$  (l) follows from Theorem 4.6.

The following trivial implications then suffice to establish the claimed equivalences:

- (a)  $\Rightarrow$  (e), (b)  $\Rightarrow$  (c), (l)  $\Rightarrow$  (d), (c)  $\Rightarrow$  (d), (d)  $\Rightarrow$  (e), (e)  $\Rightarrow$  (f), (f)  $\Rightarrow$  (k), (e)  $\Rightarrow$  (g), (g)  $\Rightarrow$  (k), (h)  $\Rightarrow$  (i), (i)  $\Rightarrow$  (j), and (j)  $\Rightarrow$  (h).  $\square$

## 5. MATRICES THAT PRESERVE PIECEWISE SYNDETICITY IN $\mathbb{N}$

**Theorem 5.1.** *Let  $u, v \in \mathbb{N}$ , let  $A$  be a  $u \times v$  matrix with rational entries which is IPR/ $\mathbb{N}$ , and assume that for each  $n \in \mathbb{N}$ , there exists  $\vec{z} \in \mathbb{Z}^v$  such that  $A\vec{z} = \bar{n} \in \mathbb{N}^u$ . If  $B$  is a piecewise syndetic subset of  $\mathbb{N}$ , then  $\{\vec{x} \in \mathbb{N}^v : A\vec{x} \in B^u\}$  is piecewise syndetic in  $\mathbb{N}^v$ .*

*Proof.* Assume that  $B$  is a piecewise syndetic subset of  $\mathbb{N}$ . By [11, Theorem 4.43] pick  $n \in \mathbb{N}$  such that  $-n + B$  is central and let  $C = -n + B$ . Let  $D = \{\vec{y} \in \mathbb{N}^v : A\vec{y} \in C^u\}$ . By Theorem 1.3,  $D$  is central in  $\mathbb{N}^v$  so pick an idempotent  $p \in \overline{D} \cap K(\beta(\mathbb{N}^v))$ . Pick  $\vec{z} \in \mathbb{Z}^v$  such that  $A\vec{z} = \bar{n}$ . Let  $E = \{\vec{y} \in \mathbb{N}^v : (\forall j \in \{1, 2, \dots, v\})(z_j + y_j > 0)\}$ . Then  $E$  is an IP\* set in  $\mathbb{N}^v$  so  $E \in p$ .

Let  $F = \{\vec{x} \in \mathbb{N}^v : A\vec{x} \in B^u\}$ . It suffices to show that  $D \cap E \subseteq -\vec{z} + F$  for then  $F \in \vec{z} + p$  so by Lemma 3.1  $\overline{F} \cap K(\beta(\mathbb{N}^v)) \neq \emptyset$ . So let  $\vec{y} \in D \cap E$ . Then  $\vec{z} + \vec{y} \in \mathbb{N}^v$  and  $A\vec{y} \in C^u$ . We claim that  $A(\vec{z} + \vec{y}) \in B^u$ . To see this, let  $i \in \{1, 2, \dots, u\}$ . Then  $\sum_{j=1}^v a_{i,j}(z_j + y_j) = n + \sum_{j=1}^v a_{i,j}y_j \in n + C \subseteq B$ .  $\square$

What seems to be an essential part of the proof of Theorem 5.1 is the fact that if  $B$  is a piecewise syndetic subset of  $\mathbb{N}$ , that is if  $\overline{B} \cap K(\beta\mathbb{N}) \neq \emptyset$ , then there exist a idempotent  $p \in K(\beta\mathbb{N})$  and  $n \in \mathbb{N}$  such that  $-n + B \in p$ . The notion  $J$  stands in a similar relationship to  $C$  and the notion  $B$  stands in a similar relationship to  $D$ . That is, if  $v \in \mathbb{N}$ , then  $J(\mathbb{N})$  is a closed two sided ideal of  $\beta\mathbb{N}$ ,  $B$  is a  $J$  set in  $\mathbb{N}$  if and only if  $\overline{B} \cap J(\mathbb{N}) \neq \emptyset$ , and  $B$  is a  $C$  set in  $\mathbb{N}$  if and only if there is an idempotent  $p \in \overline{B} \cap J(\mathbb{N})$ . Similarly,  $\Delta(\mathbb{N})$  is a closed two sided ideal of  $\beta(\mathbb{N})$ ,  $B$  is a  $B$  set in  $\mathbb{N}$  if and only if

$\overline{B} \cap \Delta(\mathbb{N}^v) \neq \emptyset$  and  $B$  is a D set in  $\mathbb{N}^v$  if and only if there is an idempotent  $p \in \overline{B} \cap \Delta(\mathbb{N}^v)$ .

However, as we shall see now, as a consequence of a result of Ernst Straus, neither J nor B share the other crucial part of the relationship of piecewise syndetic to central.

For a subset  $B$  of  $\mathbb{N}$ , we are using  $d(B)$  for the Banach density. If the ordinary asymptotic density of  $B$  is  $\alpha$ , then  $d(B) \geq \alpha$ .

**Theorem 5.2.** *Let  $\epsilon > 0$ . There is a set  $B \subseteq \mathbb{N}$  such that the asymptotic density of  $B$  is greater than  $1 - \epsilon$  and there do not exist  $n \in \mathbb{Z}$  and a sequence  $\langle x_t \rangle_{t=1}^\infty$  in  $\mathbb{N}$  such that  $n + FS(\langle x_t \rangle_{t=1}^\infty) \subseteq B$ . This set  $B$  is a B set (and thus a J set) and there is no  $n \in \mathbb{N}$  such that  $-n + B$  is a member of an idempotent in  $J(\mathbb{N})$ .*

*Proof.* This is [1, Theorem 2.20]. (In the second line of the proof,  $\omega q_t$  should be  $\mathbb{N}q_t$ .)  $\square$

**Definition 5.3.** Let  $v \in \mathbb{N}$ .

- (1) A set  $B \subseteq \mathbb{N}^v$  is a *strong J set* if and only if there exist  $\vec{x} \in \mathbb{N}^v$  and an idempotent  $p \in J(\mathbb{N}^v)$  such that  $B \in -\vec{x} + p$ .
- (2) A set  $B \subseteq \mathbb{N}^v$  is a *strong B set* if and only if there exist  $\vec{x} \in \mathbb{N}^v$  and an idempotent  $p \in \Delta(\mathbb{N}^v)$  such that  $B \in -\vec{x} + p$ .

The proof for strong J sets is nearly identical to the proof for piecewise syndetic sets.

**Theorem 5.4.** *Let  $u, v \in \mathbb{N}$ , let  $A$  be a  $u \times v$  matrix with rational entries which is IPR/ $\mathbb{N}$ , and assume that for each  $n \in \mathbb{N}$ , there exists  $\vec{z} \in \mathbb{Z}^v$  such that  $A\vec{z} = \vec{n} \in \mathbb{N}^u$ . If  $B$  is a strong J subset of  $\mathbb{N}$ , then  $\{\vec{x} \in \mathbb{N}^v : A\vec{x} \in B^u\}$  is a strong J set in  $\mathbb{N}^v$ .*

*Proof.* Assume that  $B$  is a strong J subset of  $\mathbb{N}$ , pick  $n \in \mathbb{N}$  and an idempotent  $q \in J(\mathbb{N})$  such that  $-n + B \in q$  and let  $C = -n + B$ . Let  $D = \{\vec{y} \in \mathbb{N}^v : A\vec{y} \in C^u\}$ . By [13, Theorem 1.4], pick an idempotent  $p \in J(\mathbb{N}^v)$  such that  $D \in p$ . Pick  $\vec{z} \in \mathbb{Z}^v$  such that  $A\vec{z} = \vec{n}$ . Let  $E = \{\vec{y} \in \mathbb{N}^v : (\forall j \in \{1, 2, \dots, v\})(z_j + y_j > 0)\}$ . Then  $E$  is an IP\* set in  $\mathbb{N}^v$  so  $E \in p$ .

Let  $F = \{\vec{x} \in \mathbb{N}^v : A\vec{x} \in B^u\}$ . It suffices to show that  $D \cap E \subseteq -\vec{z} + F$  for then  $F \in \vec{z} + p$  so by Lemma 3.1  $\overline{F} \cap J(\mathbb{N}^v) \neq \emptyset$ . So let  $\vec{y} \in D \cap E$ . Then  $\vec{z} + \vec{y} \in \mathbb{N}^v$  and  $A\vec{y} \in C^u$ . We claim that  $A(\vec{z} + \vec{y}) \in B^u$ . To see this, let  $i \in \{1, 2, \dots, u\}$ . Then  $\sum_{j=1}^v a_{i,j}(z_j + y_j) = n + \sum_{j=1}^v a_{i,j}y_j \in n + C \subseteq B$ .  $\square$

**Theorem 5.5.** *Let  $u, v \in \mathbb{N}$  and let  $A$  be a  $u \times v$  matrix with rational entries which is IPR/ $\mathbb{N}$  and has  $\text{rank}(A) = u$ . Assume that for each*

$n \in \mathbb{N}$ , there exists  $\vec{z} \in \mathbb{Z}^v$  such that  $A\vec{z} = \bar{n} \in \mathbb{N}^u$ . If  $B$  is a strong  $B$  subset of  $\mathbb{N}$ , then  $\{\vec{x} \in \mathbb{N}^v : A\vec{x} \in B^u\}$  is a strong  $B$  set in  $\mathbb{N}^v$ .

*Proof.* This is essentially identical to the proof of Theorem 5.4. The reason we need to add the assumption that  $\text{rank}(A) = u$  is that that is an assumption of [13, Theorem 4.2] which establishes that if  $C$  is a  $D$  set in  $\mathbb{N}$ , then  $\{\vec{y} \in \mathbb{N}^v : A\vec{y} \in C^u\}$  is a  $D$  set in  $\mathbb{N}^v$ .  $\square$

We can now present the characterizations for  $\mathbb{N}$ .

**Theorem 5.6.** *Let  $A$  be a  $u \times v$  matrix with rational entries. Statements (a) through (k) are equivalent and are implied by statement (l). If  $\text{rank}(A) = u$ , then statement (l) is equivalent to the other statements.*

- (a) *Whenever  $B$  is a piecewise syndetic subset of  $\mathbb{N}$ ,  $\{\vec{x} \in \mathbb{N}^v : A\vec{x} \in B^u\}$  is piecewise syndetic in  $\mathbb{N}^v$ .*
- (b) *Whenever  $B$  is a strong  $J$  subset of  $\mathbb{N}$ ,  $\{\vec{x} \in \mathbb{N}^v : A\vec{x} \in B^u\}$  is a strong  $J$  subset of  $\mathbb{N}^v$ .*
- (c) *Whenever  $B$  is a strong  $J$  subset of  $\mathbb{N}$ , there exists  $\vec{x} \in \mathbb{N}^v$  such that  $A\vec{x} \in B^u$ .*
- (d) *Whenever  $B$  is a strong  $B$  subset of  $\mathbb{N}$ , there exists  $\vec{x} \in \mathbb{N}^v$  such that  $A\vec{x} \in B^u$ .*
- (e) *Whenever  $B$  is a piecewise syndetic subset of  $\mathbb{N}$ , there exists  $\vec{x} \in \mathbb{N}^v$  such that  $A\vec{x} \in B^u$ .*
- (f)  *$A$  is IPR/ $\mathbb{N}$  and whenever  $B$  is a syndetic subset of  $\mathbb{N}$ , there exists  $\vec{x} \in \mathbb{N}^v$  such that  $A\vec{x} \in B^u$ .*
- (g)  *$A$  is IPR/ $\mathbb{N}$  and whenever  $B$  is a central subset of  $\mathbb{N}$ , there exists  $\vec{x} \in \mathbb{N}^v$  such that  $A\vec{x} \in (1 + B)^u$ .*
- (h)  *$A$  is IPR/ $\mathbb{N}$  and  $(\exists \vec{x} \in \mathbb{Z}^v)(A\vec{x} = \bar{1} \in \mathbb{N}^u)$ .*
- (i)  *$A$  is IPR/ $\mathbb{N}$  and  $(\forall n \in \mathbb{Z})(\exists \vec{x} \in \mathbb{Z}^v)(A\vec{x} = \bar{n} \in \mathbb{Z}^u)$ .*
- (j)  *$A$  is IPR/ $\mathbb{N}$  and  $(\forall n \in \mathbb{N})(\exists \vec{x} \in \mathbb{Z}^v)(A\vec{x} = \bar{n} \in \mathbb{N}^u)$ .*
- (k)  *$A$  is IPR/ $\mathbb{N}$  and  $(\forall k \in \mathbb{N})(\exists \vec{x} \in \mathbb{Z}^v)(A\vec{x} \in (\mathbb{N}k + 1)^u)$ .*
- (l) *Whenever  $B$  is a strong  $B$  subset of  $\mathbb{N}$ ,  $\{\vec{x} \in \mathbb{N}^v : A\vec{x} \in B^u\}$  is a strong  $B$  subset of  $\mathbb{N}^v$ .*

*Proof.* That (j)  $\Rightarrow$  (a) follows from Theorem 5.1. That (j)  $\Rightarrow$  (b) follows from Theorem 5.4. That (k)  $\Rightarrow$  (h) follows from Theorem 3.6. If  $\text{rank}(A) = u$ , then the fact that (j)  $\Rightarrow$  (l) follows from Theorem 5.5.

The following trivial implications then suffice to establish the claimed equivalences:

- (a)  $\Rightarrow$  (e), (b)  $\Rightarrow$  (c), (l)  $\Rightarrow$  (d), (c)  $\Rightarrow$  (d), (d)  $\Rightarrow$  (e), (e)  $\Rightarrow$  (f), (f)  $\Rightarrow$  (k), (e)  $\Rightarrow$  (g), (g)  $\Rightarrow$  (k), (h)  $\Rightarrow$  (i), and (i)  $\Rightarrow$  (j).  $\square$

**Question 5.7.** *Can the analogues of statements (a) and (e) of Theorem 5.6 for  $J$  sets and  $B$  sets be added?*



We conclude this section by showing that the analogue of statement (a) for syndetic sets cannot be added to Theorem 5.6 and the analogue of statement (e) for P sets cannot be added to Theorem 5.6.

The example in the next theorem is based on [4, Theorem 3.9].

**Theorem 5.8.** *Let  $B = \mathbb{N} \setminus (\{2^{2n} + 2t - 1 : n \in \mathbb{N} \text{ and } t \in \{1, 2, \dots, n\}\} \cup \{2^{2n} + 2^{n+1} + 2t : n \in \mathbb{N} \text{ and } t \in \{1, 2, \dots, n\}\})$ , let  $A = \begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix}$ , and let  $D = \{\vec{x} \in \mathbb{N}^2 : A\vec{x} \in B^2\}$ . Then  $B$  is syndetic in  $\mathbb{N}$ ,  $A$  is IPR/ $\mathbb{N}$ ,  $\text{rank}(A) = 2$ , and there exists  $\vec{x} \in \mathbb{Z}^2$  such that  $A\vec{x} = \bar{1}$ , but  $D$  is not syndetic in  $\mathbb{N}^2$ .*

*Proof.* Since  $B$  has no gaps longer than 1,  $B$  is syndetic, and  $A$  is a first entries matrix so is IPR/ $\mathbb{N}$ . Suppose that  $D$  is syndetic in  $\mathbb{N}^2$  and pick  $F \in \mathcal{P}_f(\mathbb{N}^2)$  such that  $\mathbb{N}^2 = \bigcup_{\vec{x} \in F} -\vec{x} + D$ . Pick  $n \in \mathbb{N}$  such that for all  $\vec{x} \in F$ ,  $x_1 + x_2 < n$ . Now  $\begin{pmatrix} 2^{2n} \\ 2^n \end{pmatrix} \in \mathbb{N}^2$  so pick  $\vec{x} \in F$  such that  $\begin{pmatrix} 2^{2n} + x_1 \\ 2^n + x_2 \end{pmatrix} \in D$ .

Then  $2^{2n} + x_1 \in B$  and  $2^{2n} + 2^{n+1} + x_1 + 2x_2 \in B$ . Now  $2^{2n} < 2^{2n} + x_1 < 2^{2n} + n$  so  $x_1$  is even. Pick  $u \in \mathbb{N}$  such that  $x_1 = 2u$ . Then  $u + x_2 < x_1 + x_2 < n$  so  $2^{2n} + 2^{n+1} + 2u + 2x_2 \notin B$ , a contradiction.  $\square$

**Theorem 5.9.** *Let  $A = \begin{pmatrix} 1 & 2 & 4 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}$ . Then  $A$  is IPR/ $\mathbb{N}$ ,  $\text{rank}(A) = 3$ , and there exists  $\vec{x} \in \mathbb{Z}^3$  such that  $A\vec{x} = \bar{1}$ . There is a P set  $B \subseteq \mathbb{N}$  such that  $\{\vec{y} \in \mathbb{N}^3 : A\vec{y} \in B^3\} = \emptyset$ .*

*Proof.* Note that  $A^{-1} = \begin{pmatrix} 0 & 1 & 0 \\ -\frac{1}{2} & \frac{1}{2} & 2 \\ \frac{1}{2} & -\frac{1}{2} & -1 \end{pmatrix}$ .

Since  $A$  is a first entries matrix, it is IPR/ $\mathbb{N}$ . And  $A \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} = \bar{1}$ .

Let  $d_1 = 1$  and let  $a_1 = d_1 + 1$ . Let  $k \in \mathbb{N} \setminus \{1\}$  and assume we have chosen  $d_i$  and  $a_i$  for  $i \in \{1, 2, \dots, k-1\}$ . Let  $d_k = 8a_{k-1} + 8(k-1)a_{k-1}$  and let  $a_k = kd_k + 1$ . Let  $B = \{a_n + td_n : n \in \mathbb{N} \text{ and } t \in \{1, 2, \dots, n\}\}$ . Trivially  $B$  is a P set. Suppose we have  $\vec{y} \in \mathbb{N}^3$  such that  $A\vec{y} \in B^3$  and let  $\vec{\alpha} = A\vec{y}$ . Pick  $k, r, m \in \mathbb{N}$ ,  $b \in \{1, 2, \dots, k\}$ ,  $c \in \{1, 2, \dots, r\}$ , and  $e \in \{1, 2, \dots, m\}$  such that  $\alpha_1 = a_k + bd_k$ ,  $\alpha_2 = a_r + cd_r$ , and  $\alpha_3 = a_m + ed_m$ .

Then  $\vec{y} = A^{-1}\vec{\alpha}$  so  $0 < 2y_2 = -\alpha_1 + \alpha_2 + 4\alpha_3$  and  $0 < 2y_3 = \alpha_1 - \alpha_2 - 2\alpha_3$ . Therefore  $2\alpha_3 < \alpha_1 - \alpha_2 < 4\alpha_3$ . That is,  $2a_m + 2ed_m < a_k + bd_k - a_r - cd_r < 4a_m + 4ed_m$ . We note that  $\alpha_1 > \alpha_2$  so  $k \geq r$ .

We claim  $m < k$ . Suppose instead that  $k \leq m$ . Then  $2a_m < 2a_m + 2ed_m < a_k + bd_k - a_r - cd_r < a_k + bd_k \leq a_m + bd_m$  so  $a_m < bd_m \leq md_m = a_m - 1$ , a contradiction.

Now we claim that  $r = k$ . Suppose instead that  $r < k$ . Then  $a_r + cd_r \leq a_r + rd_r \leq a_{k-1} + (k-1)d_{k-1} < \frac{1}{2}d_k < \frac{1}{2}a_k$ . So  $4a_m + 4ed_m > a_k + bd_k - a_r - cd_r > a_k + bd_k - \frac{1}{2}a_k = \frac{1}{2}a_k + bd_k > \frac{1}{2}a_k$ . Therefore  $\frac{1}{2}a_k < 4a_m + 4ed_m \leq 4a_m + 4md_m \leq 4a_{k-1} + 4(k-1)d_{k-1} = \frac{1}{2}d_k < \frac{1}{2}a_k$ , a contradiction.

We thus have that  $m < k = r$ . Also  $(b-c)d_k = \alpha_1 - \alpha_2 > 0$  so  $b > c$ . We then have that  $d_k \leq (b-c)d_k < 4a_m + 4ed_m \leq 4a_m + 4md_m \leq 4a_{k-1} + 4(k-1)d_{k-1} = \frac{1}{2}d_k$ , a contradiction.  $\square$

## 6. LARGE IMAGE OF $\vec{x} \mapsto A\vec{x}$

We now turn our attention to direct analogues of Theorem 1.2 which

said that if  $l \in \mathbb{N}$ ,  $A = \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ \vdots & \vdots \\ 1 & l \end{pmatrix}$ , and  $B$  is a piecewise syndetic subset

of  $\mathbb{Z}$ , then  $B^l \cap \{A\vec{x} : \vec{x} \in \mathbb{N}^2\}$  is piecewise syndetic in  $\{A\vec{x} : \vec{x} \in \mathbb{N}^2\}$ .

Our results are consequences of an extension of [4, Theorem 3.7]. That theorem applied to arbitrary semigroups and we will prove the extension also for arbitrary semigroups (but will continue to write our semigroups additively).

In Section 2 the notions we are considering were defined for commutative semigroups. But for the notions in Theorem 6.3 all of the definitions except that of  $P^*$  are verbatim the same for an arbitrary semigroup.

**Definition 6.1.** Let  $(S, +)$  be an arbitrary semigroup and let  $B \subseteq S$ . Then  $B$  is a *P set* if and only if for every  $k \in \mathbb{N}$ , there exist  $m \in \mathbb{N}$ ,  $a \in S^{m+1}$ , and  $d \in S$  such that  $\{a(1) + td + a(2) + td + \dots + td + a(m+1) : t \in \{1, 2, \dots, k\}\} \subseteq B$ .

This property was called *wProg* in [8]. Note that if  $S$  is commutative, a subset  $A$  of  $S$  satisfies Definition 6.1 if and only if it satisfies Definition 2.1(2). (For the sufficiency, let  $k \in \mathbb{N}$ . Pick  $a, d \in S$  such that  $\{a, a + d, \dots, a + (k+1)d\} \subseteq A$ . Then with  $m = 1$ ,  $a(1) = a$ ,  $a(2) = d$ , (and  $d$  as itself), Definition 6.1 is satisfied.)

**Lemma 6.2.** *Let  $(S, +)$  and  $(T, +)$  be discrete semigroups, let  $\varphi : S \rightarrow T$  be a surjective homomorphism, and let  $\tilde{\varphi} : \beta S \rightarrow \beta T$  be its continuous extension. If  $B$  is a QC set in  $S$ , then  $\tilde{\varphi}[B]$  is a QC set in  $T$ .*

*Proof.* By [11, Exercise 1.7.3],  $\tilde{\varphi}[K(\beta S)] = K(\beta T)$ . Pick an idempotent  $p \in \overline{B} \cap \text{cl}K(\beta S)$ . Then  $\tilde{\varphi}(p)$  is an idempotent and  $\tilde{\varphi}(p) \in \tilde{\varphi}[\overline{B}] \cap \tilde{\varphi}[\text{cl}K(\beta S)] = \tilde{\varphi}[\overline{B}] \cap \text{cl}\tilde{\varphi}[K(\beta S)] = \tilde{\varphi}[\overline{B}] \cap \text{cl}K(\beta T)$ .  $\square$

**Theorem 6.3.** *Let  $(S, +)$  be a semigroup, let  $l \in \mathbb{N}$ , let  $E$  be a subsemigroup of  $S^l$  with  $\{\bar{a} \in S^l : a \in S\} \subseteq E$ , let  $I$  be an ideal of  $E$ , and let  $B \subseteq S$ . Let  $\Psi$  be any of the properties PS, central, central\*, thick, PS\*, IP\*, Q\*, SC\*, QC, QC\*, or P\*. If  $B$  is a  $\Psi$  set in  $S$ , then  $B^l \cap I$  is a  $\Psi$  set in  $I$ .*

*Proof.* The cases that  $\Psi$  is any of the first seven properties are established in [4, Theorem 3.7]. (The property Q\* was called there  $\Delta^*$ .)

Let  $Y = (\beta S)^l$ . Let  $\tilde{\tau} : \beta I \rightarrow \text{cl}_Y I$  be the continuous extension of the identity function.

We note that if  $p \in \beta S$  and  $\bar{p} \in (\beta S)^l$ , then  $\bar{p} \in \text{cl}_Y E$ . To see this let  $U$  be a neighborhood of  $\bar{p}$  and pick  $B \in p$  such that  $\overline{B}^l \subseteq U$ . Pick  $a \in B$ . Then  $\bar{a} \in U \cap E$ .

We also note that for  $i \in \{1, 2, \dots, l\}$ ,  $\pi_i[I]$  is thick in  $S$ . To see this, let  $i \in \{1, 2, \dots, l\}$  and let  $F \in \mathcal{P}_f(S)$ . Pick  $\vec{x} \in I$ . Given  $a \in F$ ,  $\bar{a} + \vec{x} \in I$  so  $a + x_i \in \pi_i[I]$ . Therefore  $F + x_i \subseteq \pi_i[I]$ .

Case  $\Psi$  is SC\*. Assume that  $B$  is an SC\* set in  $S$  and pick a minimal left ideal  $L$  of  $\beta S$  such that  $E(L) \subseteq \overline{B}$ . Pick  $p \in E(L)$ . By [4, Lemma 3.3],  $\bar{p} \in K(\text{cl}_Y I)$ . By [4, Lemma 3.5] pick  $r \in K(\beta I)$  such that  $\tilde{\tau}(r) = \bar{p}$ .

We claim that  $E((\beta I)r) \subseteq \overline{B}^l \cap I$  so that  $B^l \cap I$  is an SC\* set in  $I$ . So let  $q \in E((\beta I)r)$ . Pick  $v \in \beta I$  such that  $q = vr$ . Let  $\vec{s} = \tilde{\tau}(v)$ . Then  $\tilde{\tau}$  is a homomorphism so  $\tilde{\tau}(q) = \tilde{\tau}(v)\tilde{\tau}(r) = \vec{s}\bar{p}$  and therefore, for each  $i \in \{1, 2, \dots, l\}$ ,  $s_i p$  is an idempotent in  $L$  so  $s_i p \in \overline{B}$ . Since  $\tilde{\tau}(q) \in \overline{B}^l$ ,  $B^l \in q$ . Since  $q \in \beta I$ ,  $I \in q$ . Therefore  $q \in \overline{B}^l \cap I$  as required.

Case  $\Psi$  is QC. Assume  $B$  is a QC set in  $S$  and pick an idempotent  $p \in \text{cl}K(\beta S) \cap \overline{B}$ . We claim that  $\bar{p} \in \text{cl}K(\text{cl}_Y I)$ . To see this let  $U$  be a neighborhood of  $\bar{p}$  and pick  $A \in p$  such that  $\overline{A}^l \subseteq U$ . Pick  $q \in \overline{A} \cap K(\beta S)$ . Then  $\bar{q} \in U \cap ((K(\beta S))^l \cap \text{cl}_Y E)$ . By [4, Lemma 3.3]  $((K(\beta S))^l \cap \text{cl}_Y E) = K(\text{cl}_Y I)$  so  $\bar{p} \in \text{cl}K(\text{cl}_Y I)$  as claimed.

By [4, Lemma 3.5],  $\tilde{\tau}[K(\beta I)] = K(\text{cl}_Y I)$  so  $\tilde{\tau}[\text{cl}K(\beta I)] = \text{cl}\tilde{\tau}[K(\beta I)] = \text{cl}K(\text{cl}_Y I)$ . Thus  $T = \{r \in \text{cl}K(\beta I) : \tilde{\tau}(r) = \bar{p}\} \neq \emptyset$ . By [11, Theorem 4.44],  $\text{cl}K(\beta I)$  is an ideal of  $\beta I$  so  $T$  is a compact subsemigroup of  $\beta I$  so pick an idempotent  $r \in T$ . Then  $\tilde{\tau}(r) = \bar{p} \in \overline{B}^l$  so  $B^l \cap I \in r$  and thus  $B^l \cap I$  is a QC set in  $I$ .

Case  $\Psi$  is QC\*. Trivially QC is a partition regular notion, so by [4, Theorem 2.2], it suffices to let  $i \in \{1, 2, \dots, l\}$ , let  $B$  be a QC set in  $I$ , and show that  $\pi_i[B]$  is a QC set in  $S$ . So let  $i \in \{1, 2, \dots, l\}$  and let  $B$  be a QC set in  $I$ .

Now  $\pi_i[I]$  is thick in  $S$  so  $\overline{\pi_i[I]} \cap K(\beta S) \neq \emptyset$  so by [11, Theorem 1.65],  $K(\beta(\pi_i[I])) = K(\overline{\pi_i[I]}) = \overline{\pi_i[I]} \cap K(\beta S)$ .

By Lemma 6.2,  $\pi_i[B]$  is a QC set in  $\pi_i[I]$ . Pick an idempotent  $p \in \overline{\pi_i[B]} \cap \text{cl}K(\beta(\pi_i[I])) = \overline{\pi_i[B]} \cap \text{cl}(\overline{\pi_i[I]} \cap K(\beta S)) \subseteq \overline{\pi_i[B]} \cap \text{cl}K(\beta S)$ .

Case  $\Psi$  is P\*. If  $S$  is commutative, the fact that P is a partition regular notion is an immediate consequence of van der Waerden's Theorem. In the proof of [8, Theorem 2.6(9)] it was shown that for arbitrary  $S$ , P is a partition regular notion. (In [8] property P was called wProg.) So by [4, Theorem 2.2], it suffices to let  $i \in \{1, 2, \dots, l\}$ , let  $B$  be a P set in  $I$ , and show that  $\pi_i[B]$  is a P set in  $S$ . This is an immediate consequence of the fact that  $\pi_i$  is a homomorphism.  $\square$

**Corollary 6.4.** *Let  $A$  be a  $u \times v$  matrix with rational entries and let  $\Psi$  be any of the properties PS, central, central\*, thick, PS\*, IP\*, Q\*, SC\*, QC, QC\*, or P\*.*

- (1) *If there exists  $\vec{y} \in \omega^v$  such that  $A\vec{y} = \vec{1} \in \mathbb{N}^u$  and  $B$  is a  $\Psi$  set in  $\mathbb{N}$ , then  $B^u \cap \{A\vec{x} : \vec{x} \in \mathbb{N}^v\}$  is a  $\Psi$  set in  $\mathbb{N}^u \cap \{A\vec{x} : \vec{x} \in \mathbb{N}^v\}$ .*
- (2) *If there exists  $\vec{y} \in \mathbb{Z}^v$  such that  $A\vec{y} = \vec{1} \in \mathbb{N}^u$  and  $B$  is a  $\Psi$  set in  $\mathbb{Z}$ , then  $B^u \cap \{A\vec{x} : \vec{x} \in \mathbb{Z}^v\}$  is a  $\Psi$  set in  $\mathbb{Z}^u \cap \{A\vec{x} : \vec{x} \in \mathbb{Z}^v\}$ .*

*Proof.* We establish (1). The proof of conclusion (2) is similar and much simpler. Let  $S = \mathbb{N}$ , let  $I = \mathbb{N}^u \cap \{A\vec{x} : \vec{x} \in \mathbb{N}^v\}$ , let  $E = I \cup \{\vec{a} : a \in \mathbb{N}\}$ , and pick  $\vec{y} \in \omega^v$  such that  $A\vec{y} = \vec{1}$ .

We show first that  $I \neq \emptyset$ . Let  $T = \{j \in \{1, 2, \dots, v\} : y_j \neq 0\}$  and let  $R = \{1, 2, \dots, v\} \setminus T$ . Pick  $d \in \mathbb{N}$  such that all entries of  $dA$  are integers. Pick  $c \in \mathbb{N}$  such that for all  $i \in \{1, 2, \dots, u\}$ ,  $c + \sum_{j \in R} a_{i,j}d > 0$ . (So if  $R = \emptyset$ ,  $c = 1$  will do.) For  $j \in R$ , let  $x_j = d$  and for  $j \in T$ , let  $x_j = cy_j$ . Then for  $i \in \{1, 2, \dots, u\}$ , entry  $i$  of  $A\vec{x}$  is  $\sum_{j \in R} a_{i,j}d + \sum_{j \in T} a_{i,j}y_jc = \sum_{j \in R} a_{i,j}d + c \in \mathbb{N}$  so  $A\vec{x} \in I$ .

Trivially (now that we know it is nonempty)  $I$  is a semigroup. Also, if  $a, b \in \mathbb{N}$ , then  $\vec{a} + \vec{b} = \vec{a} + \vec{b}$ . So, to establish that  $E$  is a semigroup and  $I$  is an ideal of  $E$ , let  $\vec{x} \in \mathbb{N}^u$  such that  $A\vec{x} \in \mathbb{N}^u$  and let  $a \in \mathbb{N}$ . Then  $A\vec{x} + \vec{a} = A\vec{x} + \vec{1}a = A\vec{x} + A\vec{y}a = A(\vec{x} + \vec{y}a) \in I$ .

Now apply Theorem 6.3.  $\square$

Notice that Theorem 5.9 establishes that the hypothesis that there exists  $\vec{x} \in \mathbb{Z}^v$  such that  $A\vec{x} = \vec{1} \in \mathbb{N}^u$  is not sufficient for the conclusions of Corollary 6.4(1).

## 7. MORE GENERAL RESULTS

In this section we will occasionally be dealing with  $\beta\mathbb{Q}$ . We note that we are taking  $\mathbb{Q}$  to have the discrete topology. If  $\beta\mathbb{Q}$  is the Stone-Ćech compactification of  $\mathbb{Q}$  with its usual topology, then by [11, Theorem 21.47],  $\beta\mathbb{Q}$  cannot be made into a semigroup compactification of  $(\mathbb{Q}, +)$ .

**Lemma 7.1.** *Let  $(S, +)$  be a commutative and cancellative semigroup, let  $G$  be the group of differences of  $S$ , let  $v \in \mathbb{N}$ , and let  $p$  be a minimal idempotent in  $\beta(S^v)$ . Then  $\beta(G^v) + p \subseteq \beta(S^v) + p$  and  $p$  is minimal in  $\beta(G^v)$ .*

*Proof.* To see that  $\beta(G^v) + p \subseteq \beta(S^v) + p$  it suffices to show that  $G^v + p \subseteq \beta(S^v) + p$  so let  $\vec{x} \in G^v$  and let  $L = \beta(S^v) + p$ . Pick  $\vec{t}$  and  $\vec{s}$  in  $S^v$  such that  $\vec{x} = \vec{t} - \vec{s}$ . By [11, Theorem 6.54], the center of  $\beta(S^v)$  is  $S^v$  so  $\vec{s} + p = \vec{s} + p + p = p + \vec{s} + p \in p + \beta(S^v) + p$ , which is a group. Let  $w$  be the inverse of  $\vec{s} + p$  in  $p + \beta(S^v) + p$ . Then  $\vec{s} + w = \vec{s} + (p + w) = (\vec{s} + p) + w = p$ , so  $w = -\vec{s} + p$  (computed in  $\beta(G^v)$ ). Thus  $\vec{x} + p = (\vec{t} - \vec{s}) + p = \vec{t} + (-\vec{s} + p) = \vec{t} + w \in L$ .

Since  $\beta(G^v) + p \subseteq L$ ,  $K(\beta(G^v)) \cap \beta(S^v) \neq \emptyset$  so by [11, Theorem 1.65],  $K(\beta(S^v)) = K(\beta(G^v)) \cap \beta(S^v)$  and thus  $p \in K(\beta(G^v))$ .  $\square$

**Lemma 7.2.** *Let  $(S, +)$  be a commutative and cancellative semigroup, and let  $G$  be the group of differences of  $S$ . Let  $u, v \in \mathbb{N}$  and let  $A$  be a  $u \times v$  matrix with integer entries. If  $S = \mathbb{Q}$  or  $S = \mathbb{Q}^+$ , let the entries of  $A$  come from  $\mathbb{Q}$ . Define  $\varphi : S^v \rightarrow G^u$  by  $\varphi(\vec{x}) = A\vec{x}$  and let  $\tilde{\varphi} : \beta(S^v) \rightarrow (\beta G)^u$  be its continuous extension. Let  $p$  be a minimal idempotent in  $\beta S$  and assume that for every  $C \in p$  there exists  $\vec{x} \in S^v$  such that  $A\vec{x} \in C^u$ . There is a minimal idempotent  $q \in \beta(S^v)$  such that  $\tilde{\varphi}(q) = \bar{p} \in (\beta S)^u$ .*

*Proof.* Since for every  $C \in p$  we have some  $\vec{x} \in S^v$  such that  $\varphi(\vec{x}) \in C^u$ , we have that  $\bar{p} \in \tilde{\varphi}[\beta(S^v)]$ . Let  $M = \{q \in \beta(S^v) : \tilde{\varphi}(q) = \bar{p}\}$ . Then  $M$  is a compact subsemigroup of  $\beta(S^v)$  so pick an idempotent  $w \in M$ . Pick by [11, Theorem 1.60] a minimal idempotent  $q \in \beta S$  such that  $q \leq w$ . Then  $\tilde{\varphi}$  is a homomorphism so  $\tilde{\varphi}(q) \leq \tilde{\varphi}(w) = \bar{p}$ . By Lemma 7.1,  $p$  is minimal in  $\beta G$  so by [11, Theorem 2.23],  $\bar{p}$  is minimal in  $(\beta S)^u$  and so  $\tilde{\varphi}(q) = \bar{p}$ .  $\square$

**Theorem 7.3.** *Let  $(S, +)$  be a commutative and cancellative semigroup. Let  $A$  be a  $u \times v$  matrix with integer entries. If  $S = \mathbb{Q}$  or  $S = \mathbb{Q}^+$  let the entries of  $A$  come from  $\mathbb{Q}$ . Assume that for every central subset  $C$  of  $S$ , there exists  $\vec{x} \in S^v$  such that  $A\vec{x} \in C^u$ . Then for every central subset  $C$  of  $S$ ,  $\{\vec{x} \in S^v : A\vec{x} \in C^u\}$  is central in  $S^v$ .*

*Proof.* Let  $C$  be central in  $S$  and pick a minimal idempotent  $p$  in  $\overline{C} \cap \beta S$ . By Lemma 7.2 pick a minimal idempotent  $q \in \beta(S^v)$  such that  $\tilde{\varphi}(q) =$

$\bar{p} \in (\beta S)^u$ . Now  $\times_{i=1}^u \bar{C}$  is a neighborhood of  $\bar{p}$  so pick  $B \in q$  such that  $\bar{\varphi}[B] \subseteq \times_{i=1}^u \bar{C}$ . Then  $B \subseteq \{\bar{x} \in S^v : A\bar{x} \in C^u\}$  so  $\{\bar{x} \in S^v : A\bar{x} \in C^u\} \in q$ .  $\square$

**Theorem 7.4.** *Let  $(S, +)$  be a commutative and cancellative semigroup and let  $G$  be its group of differences. Let  $A$  be a  $u \times v$  matrix with integer entries. If*

- (1) *for every central subset  $C$  of  $S$ , there exists  $\bar{x} \in S^v$  such that  $A\bar{x} \in C^u$  and*
- (2) *for each  $s \in S$  there exists  $\bar{x} \in G^v$  such that  $A\bar{x} = \bar{s} \in S^u$ ,*

*then for every piecewise syndetic subset  $B$  of  $S$ ,  $\{\bar{x} \in S^v : A\bar{x} \in B^u\}$  is piecewise syndetic in  $S^v$ .*

*Proof.* Let  $B$  be a piecewise syndetic subset of  $S$  and by [11, Theorem 4.43] pick  $s \in S$  such that  $-s + B$  is central and let  $C = -s + B$ . Let  $D = \{\bar{y} \in S^v : A\bar{y} \in C^u\}$ . By assumption (1) and Theorem 7.3,  $D$  is central in  $S^v$ . Pick a minimal idempotent  $p$  in  $\beta(S^v)$  such that  $D \in p$ .

By assumption (2) pick  $\bar{x} \in G^v$  such that  $A\bar{x} = \bar{s} \in S^u$ . Let  $F = \{\bar{z} \in G^v : A\bar{z} \in B^u\}$ . We shall show that  $F \cap S^v$  is piecewise syndetic in  $S^v$ . For this we claim that it suffices to show that  $D \subseteq -\bar{x} + F$ . For then  $-\bar{x} + F \in p$  so  $F \in \bar{x} + p$  and by Lemma 7.1,  $\bar{x} + p \in K(\beta(S^v))$ . Since  $\bar{x} + p \in K(\beta(S^v))$  we have that  $S^v \in \bar{x} + p$  so  $F \cap S^v \in \bar{x} + p$  so  $F \cap S^v$  is piecewise syndetic in  $S^v$ .

To see that  $D \subseteq -\bar{x} + F$ , let  $\bar{y} \in D$ . Then we claim that  $A(\bar{x} + \bar{y}) \in B^u$ . So let  $i \in \{1, 2, \dots, u\}$ . We show that entry  $i$  of  $A(\bar{x} + \bar{y})$  is in  $B$ . Then  $\sum_{j=1}^v a_{i,j}(x_j + y_j) = s + \sum_{j=1}^v a_{i,j}y_j \in s + C \subseteq B$ .  $\square$

Because of Theorems 7.3 and 7.4 we are interested in conditions guaranteeing the existence of  $\bar{x}$  with entries of  $A\bar{x}$  in a given central set.

**Theorem 7.5.** *Let  $(S, +)$  be a commutative semigroup with identity 0. Let  $A$  be a  $u \times v$  matrix with integer entries. Assume that for all  $c \in \mathbb{N}$ ,  $cS$  is central\* in  $S$  and that  $A$  is IPR/ $\mathbb{N}$ . Then for every central subset  $C$  of  $S$ , there exists  $\bar{x} \in S^v$  such that  $A\bar{x} \in C^u$ .*

*Proof.* This follows from [12, Corollary 2.7(a)].  $\square$

**Theorem 7.6.** *Let  $(S, +)$  be a commutative and cancellative semigroup and let  $G$  be its group of differences. Let  $A$  be a  $u \times v$  matrix with integer entries. Assume that for all  $c \in \mathbb{N}$ ,  $cS$  is central\* in  $S$  and for every piecewise syndetic subset  $B$  of  $\mathbb{N}$ , there exists  $\bar{x} \in \mathbb{N}^v$  such that  $A\bar{x} \in B^u$ . Then for every piecewise syndetic subset  $B$  of  $S$ ,  $\{\bar{x} \in S^v : A\bar{x} \in B^u\}$  is piecewise syndetic in  $S^v$ .*

*Proof.* By Theorem 7.5, assumption (1) of Theorem 7.4 holds. Statement (e) of Theorem 5.6 holds so  $A$  is IPR/ $\mathbb{N}$  and we may pick  $\vec{x} \in \mathbb{Z}^v$  such that  $A\vec{x} = \bar{1} \in \mathbb{N}^u$ . Given  $s \in S$ , define  $\vec{y} \in G^v$  by for  $j \in \{1, 2, \dots, v\}$ ,  $y_j = x_j s$ . Then  $A\vec{y} = \bar{s}$  so assumption (2) of Theorem 7.4 holds.  $\square$

Let  $u, v \in \mathbb{N}$ . Let  $A$  be a  $u \times v$  matrix with entries in  $\mathbb{Q}$ . Then  $T_A : \beta(\mathbb{Q}^v) \rightarrow (\beta\mathbb{Q})^u$  and  $V_A : \beta(\mathbb{Q}^v) \rightarrow \beta(\mathbb{Q}^u)$  will denote the continuous homomorphisms extending the homomorphism from  $\mathbb{Q}^v$  to  $\mathbb{Q}^u$  defined by  $A$ . Let  $D$  denote the diagonal of  $(\beta\mathbb{Q})^u$ . I.e.  $D = \{\bar{x} \in (\beta\mathbb{Q})^u : x \in \beta\mathbb{Q}\}$ , which is obviously topologically isomorphic to  $\beta\mathbb{Q}$ .

We have previously considered questions equivalent to algebraic questions of the following form about a property  $\psi$  possessed by elements of certain semigroups: Whenever  $\bar{p} \in D$  has property  $\psi$ , does there exist  $q \in \beta(\mathbb{Q}^v)$  with this property for which  $T_A(q) = \bar{p}$ ? For example, if  $\psi$  is the property of being a minimal idempotent, we claim that the answer to this question is “yes” if and only if, for every central subset  $C$  of  $\mathbb{Q}$ , there exists  $\vec{x} \in \mathbb{Q}^v$  such that  $A\vec{x} \in C^u$ . The sufficiency follows from Lemma 7.2. The necessity follows from the fact that if  $C$  is a central subset of  $\mathbb{Q}$ , then there is a minimal idempotent  $p \in \beta\mathbb{Q}$  such that  $C \in p$  so  $\times_{i=1}^u \bar{C}$  is a neighborhood of  $\bar{p}$ .

It seems natural to ask analogous questions about elements  $p$  in the spaces  $(\beta\mathbb{Q})^u$  and  $\beta(\mathbb{Q}^u)$  rather than in  $D$ . We shall discuss the following properties of a  $u \times v$  matrix  $A$  with entries in  $\mathbb{Q}$ . We remark that every continuous homomorphism from  $\beta(\mathbb{Q}^v)$  to  $(\beta\mathbb{Q})^u$  which maps  $\mathbb{Q}^v$  to  $\mathbb{Q}^u$ , has the form  $T_A$ , and every continuous homomorphism from  $\beta(\mathbb{Q}^v)$  to  $\beta(\mathbb{Q}^u)$  which maps  $\mathbb{Q}^v$  to  $\mathbb{Q}^u$ , has the form  $V_A$ . So the properties of these maps have some interest in the study of the relationships between these semigroups.

- (1) For every minimal idempotent  $\bar{p}$  in  $(\beta\mathbb{N})^u$ , there exists a minimal idempotent  $q$  in  $\beta(\mathbb{N}^v)$  for which  $T_A(q) = \bar{p}$ .
- (2) For every minimal idempotent  $p$  in  $\beta(\mathbb{N}^u)$ , there exists a minimal idempotent  $q$  in  $\beta(\mathbb{N}^v)$  for which  $V_A(q) = p$ .
- (3) For every  $\bar{p} \in K((\beta\mathbb{N})^u)$ , there exists  $q \in K(\beta(\mathbb{N}^v))$  for which  $T_A(q) = \bar{p}$ .
- (4) For every  $p \in K(\beta(\mathbb{N}^u))$ , there exists  $q \in K(\beta(\mathbb{N}^v))$  for which  $V_A(q) = p$ .

Since these properties involve  $K(\beta\mathbb{Q})$  and  $K(\beta\mathbb{N})$  we feel it is worth noting that an idempotent in  $\beta\mathbb{N}$  cannot be minimal in  $\beta\mathbb{Q}$  (and similarly an idempotent in  $\beta(\mathbb{N}^u)$  cannot be minimal in  $\beta(\mathbb{Q}^u)$ ).

**Theorem 7.7.**  $\mathbb{Z}$  is not piecewise syndetic in  $\mathbb{Q}$  and  $\mathbb{N}$  is not piecewise syndetic in  $\mathbb{Q}^+$ . Consequently  $\beta\mathbb{Z} \cap K(\beta\mathbb{Q}) = \emptyset$  and  $\beta\mathbb{N} \cap K(\beta(\mathbb{Q}^+)) = \emptyset$ .

*Proof.* We establish the first assertion. Suppose  $\mathbb{Z}$  is piecewise syndetic in  $\mathbb{Q}$ . Pick  $G \in \mathcal{P}_f(\mathbb{Q})$  such that for every  $F \in \mathcal{P}_f(\mathbb{Q})$  there exists  $x \in \mathbb{Q}$  such that  $F + x \subseteq \bigcup_{t \in G} (-t + \mathbb{Z})$ . Pick  $k \in \mathbb{N}$  such that for all  $t \in G$ ,  $kt \in \mathbb{Z}$ . Let  $F = \{\frac{1}{k}, \frac{1}{2k}\}$  and pick  $x \in \mathbb{Q}$  such that  $F + x \subseteq \bigcup_{t \in G} (-t + \mathbb{Z})$ . Pick  $s$  and  $t$  in  $G$  such that  $s + \frac{1}{k} + x \in \mathbb{Z}$  and  $t + \frac{1}{2k} + x \in \mathbb{Z}$ . Then  $s - t + \frac{1}{k} - \frac{1}{2k} \in \mathbb{Z}$ . But then  $sk - tk + 1 - \frac{1}{2} \in \mathbb{Z}$ , a contradiction.  $\square$

In the following theorem, we explore the relationships among these four properties and between these properties and properties that we considered previously.

**Theorem 7.8.** (a) (2) implies (1), and (4) implies (3).  
 (b) (3) holds if and only if (1) holds and, for every  $\vec{a} \in \mathbb{Z}^u$ , there exists  $\vec{x} \in \mathbb{Z}^v$  for which  $A\vec{x} = \vec{a}$ .  
 (c) (4) holds if and only if (2) holds, and, for every  $\vec{a} \in \mathbb{Z}^u$ , there exists  $\vec{x} \in \mathbb{Z}^v$  for which  $A\vec{x} = \vec{a}$ .  
 (d) (1) implies that  $A$  is IPR/ $\mathbb{N}$

*Proof.* (a) Let  $\pi : \beta(\mathbb{Q}^u) \rightarrow (\beta\mathbb{Q})^u$  denote the continuous surjective homomorphism which extends the identity map of  $\mathbb{Q}^u$ . We observe that  $\pi \circ V_A = T_A$  because these two functions coincide on the dense subspace  $\mathbb{Q}^u$  of  $\beta(\mathbb{Q}^u)$ . First assume that (2) holds and that  $\vec{p}$  is a minimal idempotent in  $(\beta\mathbb{N})^u$ . The restriction of  $\pi$  to  $\beta(\mathbb{N}^u)$  takes  $\beta(\mathbb{N}^u)$  onto  $(\beta\mathbb{N})^u$  so by [11, Exercise 1.7.3], there exists a minimal idempotent  $p'$  in  $\beta(\mathbb{N}^u)$  for which  $\pi(p') = \vec{p}$ . By assumption, there is a minimal idempotent  $q \in \beta(\mathbb{N}^v)$  for which  $V_A(q) = p'$ . This implies that  $T_A(q) = \vec{p}$ , and so (1) holds. Thus (2) implies (1), and a similar proof shows that (4) implies (3).

(b) Assume that (1) holds and that, for every  $\vec{a} \in \mathbb{Z}^u$ , there exists  $\vec{x} \in \mathbb{Z}^v$  for which  $A\vec{x} = \vec{a}$ . Let  $\vec{p} \in K((\beta\mathbb{N})^u)$ . Then  $\vec{p} = \vec{p} + \vec{r}$  for some minimal idempotent  $\vec{r} \in (\beta\mathbb{N})^u$ . By (1), there exists a minimal idempotent  $s$  in  $\beta(\mathbb{N}^v)$  for which  $T_A(s) = \vec{r}$ . Now  $\mathbb{Z}^u \subseteq T_A[\mathbb{Z}^v]$  so  $\beta(\mathbb{Z}^u) \subseteq T_A[\beta(\mathbb{Z}^v)]$  so pick  $w \in \beta(\mathbb{Z}^v)$  for which  $T_A(w) = \vec{p}$ . Put  $q = w + s$ . We shall show that  $q \in K(\beta(\mathbb{N}^v))$ . For each  $i \in \{1, 2, \dots, v\}$ , let  $\pi_i$  denote the  $i$ 'th projection map of  $\mathbb{Z}^v$  and denote by  $\tilde{\pi}_i : \beta(\mathbb{Z}^v) \rightarrow \beta\mathbb{Z}$  its continuous extension. Since  $\tilde{\pi}_i[\beta(\mathbb{N}^v)] = \beta\mathbb{N}$ , by [11, Exercise 4.3.5],  $\tilde{\pi}_i(s)$  is a minimal idempotent in  $\beta\mathbb{N}$  and so  $\tilde{\pi}_i(s) \in \mathbb{N}^*$ . Now  $\mathbb{N}^*$  is a left ideal in  $\beta\mathbb{Z}$ , by [11, Exercise 4.3.5]. It follows that  $w + s \in \beta(\mathbb{N}^v)$ , because  $\bigcap_{i=1}^v \pi_i^{-1}[\mathbb{N}] \in w + s$ . Since  $s \in K(\beta(\mathbb{N}^v))$ ,  $q = w + s = (w + s) + s \in K(\beta(\mathbb{N}^v))$  and  $T_A(q) = T_A(w) + T_A(s) = \vec{p} + \vec{r} = \vec{p}$ . We have proved that (3) holds.

To prove the converse, assume that (3) holds. To verify (1), let  $\vec{p}$  be a minimal idempotent in  $(\beta\mathbb{N})^u$ . Pick  $q \in K(\beta(\mathbb{N}^v))$  such that  $T_A(q) = \vec{p}$  and pick a minimal left ideal  $L$  of  $\beta(\mathbb{N}^v)$  such that  $q \in L$ . Then  $L \cap$



$T_A^{-1}[\{\bar{p}\}]$  is a compact subsemigroup of  $\beta(\mathbb{N}^v)$  which therefore has an idempotent.

Now let  $\bar{a} \in \mathbb{Z}^u$  be given. Pick a minimal idempotent  $p \in \beta\mathbb{N}$ . By Lemma 3.1 (with  $v = 1$ ),  $a_i + p \in K(\beta\mathbb{N})$  for each  $i \in \{1, 2, \dots, u\}$  so by [11, Theorem 2.23],  $\bar{a} + \bar{p} \in K((\beta\mathbb{N})^u)$ . Pick  $q \in K(\beta(\mathbb{N}^v))$  such that  $T_A(q) = \bar{a} + \bar{p}$ . By Theorem 3.6, it suffices to show that for each  $k \in \mathbb{N}$  there is some  $\bar{x} \in \mathbb{Z}^v$  such that  $A\bar{x} \in \bar{a} + (\mathbb{N}k)^u$  so let  $k \in \mathbb{N}$  be given. Then  $\mathbb{N}k \in p$  so  $\times_{i=1}^u \overline{a_i + \mathbb{N}k}$  is a neighborhood of  $\bar{a} + \bar{p}$ . Pick  $B \in q$  such that  $T_A[\overline{B}] \subseteq \times_{i=1}^u \overline{a_i + \mathbb{N}k}$ . Pick  $\bar{x} \in B$ .

(c) The proof of (c) is essentially the same as the proof of (b) except for the proof that (4) implies that for every  $\bar{a} \in \mathbb{Z}^u$ , there exists  $\bar{x} \in \mathbb{Z}^v$  for which  $A\bar{x} = \bar{a}$ . So assume that (4) holds and let  $\bar{a} \in \mathbb{Z}^u$  be given. Pick a minimal idempotent  $p \in \beta(\mathbb{N}^u)$ . Then by Lemma 3.1,  $\bar{a} + p \in K(\beta(\mathbb{N}^u))$ . Pick  $q \in K(\beta(\mathbb{N}^v))$  such that  $V_A(q) = \bar{a} + p$ . By Theorem 3.6, it suffices to show that for each  $k \in \mathbb{N}$  there is some  $\bar{x} \in \mathbb{Z}^v$  such that  $A\bar{x} \in \bar{a} + (\mathbb{N}k)^u$  so let  $k \in \mathbb{N}$  be given. Since  $p$  is an idempotent,  $(\mathbb{N}k)^u \in p$  so  $\bar{a} + (\mathbb{N}k)^u \in \bar{a} + p$  so pick  $B \in q$  such that  $V_A[\overline{B}] \subseteq \bar{a} + (\mathbb{N}k)^u$ .

(d) Assume that (1) holds. By Theorem 1.3 it suffices to show that if  $B$  is a central subset of  $\mathbb{N}$ , then  $\{\bar{x} \in \mathbb{N}^v : A\bar{x} \in B^u\}$  is central in  $\mathbb{N}^v$ . So let  $B$  be a central subset of  $\mathbb{N}$  and pick a minimal idempotent  $p$  in  $\beta\mathbb{N}$  such that  $B \in p$ . Then by [11, Theorem 2.23],  $\bar{p} \in K((\beta\mathbb{N})^u)$ . Pick a minimal idempotent  $q \in \beta(\mathbb{N}^v)$  such that  $T_A(q) = \bar{p}$ . Then  $\overline{B^u}$  is a neighborhood of  $\bar{p}$  so  $\{\bar{x} \in \mathbb{N}^v : A\bar{x} \in B^u\} \in p$ .  $\square$

We shall now discuss the matrices which have property (2). The following example suggests that this is a very strong property.

In the proof of the following theorem we will write  $(x, y)$  instead of the technically correct  $\begin{pmatrix} x \\ y \end{pmatrix}$ .

**Theorem 7.9.** *Let  $A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ . Then  $A$  does not have property (2), but does have property (1).*

*Proof.* We claim that we can choose a minimal idempotent  $p \in \beta(\mathbb{N}^2)$  for which  $\{(m, n) \in \mathbb{N}^2 : m \leq n\} \in p$ . To see this, observe that the map  $\sigma : \mathbb{N}^2 \rightarrow \mathbb{N}^2$ , defined by  $\sigma(m, n) = (n, m)$ , is an isomorphism. So its continuous extension  $\tilde{\sigma} : \beta(\mathbb{N}^2) \rightarrow \beta(\mathbb{N}^2)$  is an isomorphism and thus takes  $K(\beta(\mathbb{N}^2))$  to itself. Let  $q$  be any minimal idempotent in  $\beta(\mathbb{N}^2)$ . Since  $\{(m, n) \in \mathbb{N}^2 : m \leq n\} \in q$  or  $\{(m, n) \in \mathbb{N}^2 : m \leq n\} \in \tilde{\sigma}(q)$ , it follows that  $\{(m, n) \in \mathbb{N}^2 : m \leq n\}$  is a member of  $q$  or of  $\tilde{\sigma}(q)$ .

Thus there is an idempotent  $p \in K(\beta(\mathbb{N}^2))$  such that  $\{(m, n) \in \mathbb{N}^2 : m \leq n\} \in p$ . If (2) held there would have to exist  $p' \in \beta(\mathbb{N}^2)$  such that

$V_A(p') = p$  and hence some  $\vec{x} \in \mathbb{N}^2$  such that  $A\vec{x} \in \{(m, n) \in \mathbb{N}^2 : m \leq n\}$ . This is impossible.

To see that  $A$  has property (1), note that  $T_A[\mathbb{N}^2] = \{\vec{x} \in \mathbb{N}^2 : x_1 > x_2\}$ , and so the restriction of  $T_A$  to  $\beta(\mathbb{N}^2)$  is a surjective homomorphism onto the semigroup  $\mathcal{cl}_{(\beta\mathbb{N})^2}\{\vec{x} \in \mathbb{N}^2 : x_1 > x_2\}$ . If  $\vec{p}$  is a minimal idempotent in  $(\beta\mathbb{N})^2$  then  $\vec{p} \in \mathcal{cl}_{(\beta\mathbb{N})^2}\{\vec{x} \in \mathbb{N}^2 : x_1 > x_2\}$ . Thus by [11, Exercise 1.7.3] we can pick a minimal idempotent  $q \in \beta(\mathbb{N}^2)$  such that  $T_A(q) = p$ .  $\square$

Since the very simple matrix in the preceding example does not have property (2), one might expect that there are no non-trivial examples of matrices which do have property (2). However, we shall see that there is quite a rich set of matrices which do have this property, and that there are many which have property (4), which is even stronger. (A trivial example of a matrix with property (2), which does not have property (4), is the  $1 \times 1$  matrix (2). This does not have (4), because, if  $q$  is a minimal idempotent in  $\beta\mathbb{N}$ ,  $1 + q \in K(\beta\mathbb{N})$  and  $1 + 2\mathbb{N} \in 1 + q$ . It has property (2) by [11, Lemma 5.19.2].)

**Definition 7.10.** Let  $u, v \in \mathbb{N}$ .  $\mathcal{A}_{u,v}$  will denote the set of  $u \times v$  matrices with entries in  $\mathbb{Q}$  which have property (2), and  $\mathcal{B}_{u,v}$  will denote the set of these matrices which have property (4).

**Lemma 7.11.** *Let  $u, v \in \mathbb{N}$  and let  $A$  be a  $u \times v$  matrix with rational entries. If for every  $\vec{x} \in \mathbb{N}^u$  there exists  $\vec{y} \in \mathbb{N}^v$  such that  $A\vec{y} = \vec{x}$ , then  $A$  has property (2).*

*Proof.* We have that  $\mathbb{N}^u \subseteq V_A[\mathbb{N}^v]$ , and so  $\beta(\mathbb{N}^u) \subseteq V_A[\beta(\mathbb{N}^v)]$ . Let  $p$  be a minimal idempotent in  $\beta(\mathbb{N}^u)$ . Then  $\{q \in \beta(\mathbb{N}^v) : V_A(q) = p\}$  is nonempty so is a compact semigroup so one may pick an idempotent  $q \in \beta(\mathbb{N}^v)$  such that  $V_A(q) = p$ . Pick by [11, Theorem 1.60] a minimal idempotent  $r \in \beta(\mathbb{N}^v)$  such that  $r \leq q$ . Then  $r = r + q = q + r$  so  $V_A(r) = V_A(r) + p = p + V_A(r)$ . Therefore  $V_A(r) \leq p$  so  $V_A(r) = p$ .  $\square$

**Theorem 7.12.** (a) *Let  $u \in \mathbb{N}$ . If  $A$  is an invertible  $u \times u$  matrix with entries in  $\omega$ , then  $A^{-1} \in \mathcal{B}_{u,u}$ .*  
 (b) *Let  $u, v, w \in \mathbb{N}$ . If  $A \in \mathcal{A}_{u,v}$  and  $B \in \mathcal{A}_{v,w}$ , then  $AB \in \mathcal{A}_{u,w}$ .*  
 (c) *Let  $u, v, w \in \mathbb{N}$ . If  $A \in \mathcal{B}_{u,v}$  and  $B \in \mathcal{B}_{v,w}$ , then  $AB \in \mathcal{B}_{u,w}$ .*  
 (d) *Let  $u, v \in \mathbb{N}$ . If  $A \in \mathcal{A}_{u,v}$ , then  $rA \in \mathcal{A}_{u,v}$  for every positive rational number  $r$ .*  
 (e) *Let  $u, v, w \in \mathbb{N}$ . If  $A \in \mathcal{A}_{u,v}$  and if  $B$  is an arbitrary  $u \times w$  matrix with entries in  $\mathbb{Q}$ , then  $(AB) \in \mathcal{A}_{u,v+w}$ .*  
 (f) *Let  $u, v, w \in \mathbb{N}$ . If  $A \in \mathcal{B}_{u,v}$  and if  $B$  is an arbitrary  $u \times w$  matrix with entries in  $\mathbb{Q}$ , then  $(AB) \in \mathcal{B}_{u,v+w}$ .*

*Proof.* (a) Let  $A$  be an invertible  $u \times u$  matrix with entries in  $\omega$ . For every  $\vec{x} \in \mathbb{N}^u$ ,  $A^{-1}(A\vec{x}) = \vec{x}$  and  $A\vec{x} \in \mathbb{N}^u$ . So  $A^{-1} \in \mathcal{A}_{u,u}$  by Lemma 7.11. For every  $\vec{a} \in \mathbb{Z}^u$ ,  $A^{-1}(A\vec{a}) = \vec{a}$  and  $A\vec{a} \in \mathbb{Z}^u$ . So  $A^{-1} \in \mathcal{B}_{u,u}$  by Theorem 7.8(c).

The proofs of (b) and (c) are obvious.

(d) Let  $u, v \in \mathbb{N}$ , let  $p$  be a minimal idempotent in  $\beta(\mathbb{N}^u)$ , let  $A \in \mathcal{A}_{u,v}$ , and let  $r$  be a positive rational number. Define  $l : \mathbb{N}^u \rightarrow \mathbb{Q}^u$  by  $l(\vec{x}) = \frac{1}{r} \cdot \vec{x}$  and let  $\tilde{l} : \beta(\mathbb{N}^u) \rightarrow \beta(\mathbb{Q}^u)$  be its continuous extension. As in the proof of [11, Lemma 5.19.2] one sees that  $\tilde{l}(p)$  is a minimal idempotent in  $\beta(\mathbb{N}^u)$ , and so we may pick a minimal idempotent  $q \in \beta(\mathbb{N}^v)$  such that  $V_A(q) = \tilde{l}(p)$ . Then one easily checks that  $V_{rA}(q) = p$ .

(e) Let  $I = \{A\vec{x} + B\vec{y} : \vec{x} \in \mathbb{N}^u \text{ and } \vec{y} \in \mathbb{N}^w\}$  and let  $E = I \cup \{A\vec{x} : \vec{x} \in \mathbb{N}^u\}$ . Then  $E \cap \mathbb{N}^u$  is a subsemigroup of  $\mathbb{N}^u$ , and  $I \cap \mathbb{N}^u$  is an ideal of  $E$ . Let  $\bar{I} = \text{cl}_{\beta(\mathbb{N}^u)}(I \cap \mathbb{N}^u)$  and let  $\bar{E} = \text{cl}_{\beta(\mathbb{N}^u)}(E \cap \mathbb{N}^u)$ . By [11, Theorem 4.17],  $\bar{E}$  is a subsemigroup of  $\beta(\mathbb{N}^u)$  and  $\bar{I}$  is an ideal of  $\bar{E}$ . Let  $p$  be an idempotent in  $K(\beta(\mathbb{N}^u))$ . We can choose an idempotent  $q$  in  $K(\beta(\mathbb{N}^v))$  for which  $V_A(q) = p$  and so  $p \in \bar{E}$ . Hence, by [11, Theorem 1.65],  $p \in K(\bar{E})$  and it follows that  $p \in \bar{I}$ . So  $p \in \text{cl}_{\beta(\mathbb{N}^u)}\{(A B)(\vec{s}) : \vec{s} \in \mathbb{N}^{v+w}\}$ . Thus there exists  $r \in \beta(\mathbb{N}^{v+w})$  for which  $V_{(A B)}(r) = p$ . As in the proof of Lemma 7.11,  $r$  can be chosen to be an idempotent in  $K(\beta(\mathbb{N}^{v+w}))$ .

(f) Let  $p \in K(\beta(\mathbb{N}^u))$ . Pick a minimal idempotent  $s \in \beta(\mathbb{N}^u)$  such that  $p = p + s$ . By (e),  $(A B) \in \mathcal{A}_{u,v+w}$  so pick a minimal idempotent  $q \in \beta(\mathbb{N}^{v+w})$  such that  $V_{(A B)}(q) = s$ . As in the proof of (e) we get  $r \in \beta(\mathbb{N}^{v+w})$  such that  $V_{(A B)}(r) = p$ . Then  $r + q \in K(\beta(\mathbb{N}^{v+w}))$  and  $V_{(A B)}(r + q) = p + s = p$ .  $\square$

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