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Partition Theorems for Layered Partial Semigroups

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Abstract. We introduce the notions of *layered* semigroups and partial semigroups, and prove some Ramsey type partition results about them. These results generalize previous results of Gowers [5, Theorem 1], of Furstenberg [3, Proposition 8.21], and of Bergelson, Blass, and Hindman [1, Theorem 4.1]. We give some applications of these results (see e.g., Theorem 1.1), and present examples suggesting that our results are rather optimal.

1. Introduction

Ramsey Theory studies the existence of large homogeneous structures. For example, Ramsey's Theorem itself [10] (or see [6, Theorem 1.5]) says that whenever $k \in \mathbb{N}$ and the set $[\mathbb{N}]^k$ of k element subsets of $\mathbb{N}$ is partitioned into finitely many classes (or *finitely colored*) there must exist an infinite set $X \subseteq \mathbb{N}$ such that $[X]^k$ is contained in one class (or is *monochrome*). (Throughout, we take $\mathbb{N}$ to be the set of positive integers, while the first infinite ordinal $\omega = \mathbb{N} \cup \{0\}$.) Another typical example of a Ramseyan principle says that for every finite coloring of the set $\mathbb{N}$ of all natural numbers there is an infinite $A \subseteq \mathbb{N}$ such that the set of all finite sums of elements of A without repetitions is monochrome (see [6, Theorem 3.16] or [9, Corollary 5.17]). In this paper we study a specific form of Ramsey theory. Our Ramseyan spaces will consist of finitely many layers, and besides the associative operation on the space they will be equipped with a set of homomorphisms sending higher layers to lower ones. A typical result will say that, under certain conditions, for every partition of the space into finitely many

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colors there is an infinite sequence included in the top layer such that the 'subspace' it generates has a monochromatic intersection with each layer.

Let us now describe a simple corollary to one of our results. For a finite alphabet Σ let $W(\Sigma)$ denote the free semigroup (with identity e) on the alphabet Σ. That is, $W(\Sigma)$ is the set of all words (including the empty word) with letters from Σ and the operation is concatenation.

Note that every endomorphism f of $W(\Sigma)$ is uniquely determined by its restriction to Σ. If $\Sigma = \{a,b,c\}$ and $x,y,z \in \{a,b,c,e\}$, then let f_{xyz} be the endomorphism of $W(\{a,b,c\})$ uniquely determined by $f(a) = x$, $f(b) = y$, and $f(c) = z$. Given a set X, $\mathcal{P}_f(X)$ is the set of finite nonempty subsets of X. By $\prod_{n\in F} x_n$ we mean the product in increasing order of indices.

1.1 Theorem. *For every $r \in \mathbb{N}$ and every partition $W(\{a,b,c\}) = \bigcup_{j=1}^{r} C_j$ there exist an infinite $\langle x_n\rangle_{n=1}^{\infty}$ in $W(\{a,b,c\}) \setminus W(\{a,b\})$ and $\gamma\colon \{a,b,c\} \to \{1,2,\ldots,r\}$ such that if $\sigma \in \{f_{eab}, f_{aeb}, f_{aab}\}$ and $\mathcal{F} = \{f_{abc}, f_{abb}, f_{aba}, f_{abe}, \sigma\} \cup \big\{f_{xyz} | x,y,z \in \{a,e\}\big\}$, then we have*

$$\{\textstyle\prod_{n\in F} g_n(x_n) : F \in \mathcal{P}_f(\mathbb{N})\,,\ \text{and for each } n \in F\,,\ g_n \in \mathcal{F}\} \cap \big(W(\{a,b,c\}) \setminus W(\{a,b\})\big) \subseteq C_{\gamma(a)}$$

$$\{\textstyle\prod_{n\in F} g_n(x_n) : F \in \mathcal{P}_f(\mathbb{N})\,,\ \text{and for each } n \in F\,,\ g_n \in \mathcal{F}\} \cap \big(W(\{a,b\}) \setminus W(\{a\})\big) \subseteq C_{\gamma(b)}$$

$$\{\textstyle\prod_{n\in F} g_n(x_n) : F \in \mathcal{P}_f(\mathbb{N})\,,\ \text{and for each } n \in F\,,\ g_n \in \mathcal{F}\} \cap \big(W(\{a\}) \setminus \{e\}\big) \subseteq C_{\gamma(c)}$$

Proof. This is Corollary 3.14. □

Note that Theorem 1.1 is saying that the set generated by $\langle x_n\rangle_{n=1}^{\infty}$ and $\mathcal{F}$ is at most three-chromatic, i.e., it has a nonempty intersection only with $C_{\gamma(a)}, C_{\gamma(b)}$ and $C_{\gamma(c)}$. This number clearly cannot improved to two, as long as we require that $\langle x_n\rangle_{n=1}^{\infty}$ is included in $W(\{a,b,c\}) \setminus W(\{a,b\})$.

We do not know whether any or all of the three choices for $\mathcal{F}$ in Theorem 1.1 above is a maximal set of functions of the form f_{xyz} for which the conclusion of this theorem holds. However, if one colors $W(\{a,b,c\}) \setminus W(\{a,b\})$ by six colors according to the order of the first occurrences of a, b, and c and colors $W(\{a,b\}) \setminus W(\{a\})$ by two colors according to whether a or b occurs first, one can (rather laboriously) prove that $\mathcal{F} \cup \{f_{xyz}\}$ does not satisfy the conclusion of Theorem 1.1 unless $f_{xyz} \in \mathcal{F} \cup \{f_{eab}, f_{aeb}, f_{aab}\}$.

In [5], W. T. Gowers proved (as a tool for attacking a problem in the theory of

Banach spaces) a remarkable Ramsey Theoretic result which serves as the inspiration for this paper. While it was not stated this way by Gowers, his theorem can be naturally stated in terms of the notion of a "partial semigroup" introduced in [1].

1.2 Definition. A *partial semigroup* is a pair $(S,*)$, where S is a nonempty set and $*$ maps a subset D of $S\times S$ to S so that for all $x,y,z\in S$,

(a) if $(x,y)\in D$ and $(x*y,z)\in D$, then $(y,z)\in D$, $(x,y*z)\in D$, and $(x*y)*z = x*(y*z)$ and

(b) if $(y,z)\in D$ and $(x,y*z)\in D$, then $(x,y)\in D$, $(x*y,z)\in D$, and $(x*y)*z = x*(y*z)$.

If $(S,*)$ is a partial semigroup and $(x,y)\in\text{domain}(*)$, we say that "$x*y$ is defined". The requirements of Definition 1.2(a) and (b), can then be more succinctly stated as "$(x*y)*z = x*(y*z)$ in the sense that, whenever either side is defined, so is the other and they are equal." We shall develop some machinery for dealing with partial semigroups in Section 2.

Let $k\in\mathbb{N}$, the set of positive integers, and let

$$Y=\{f:\ f:\mathbb{N}\to\{0,1,\ldots,k\}\text{ and }\{x\in\mathbb{N}:f(x)\neq 0\}\text{ is finite}\}\,.$$

Given $f\in Y$, let $\text{supp}(f)=\{x\in\mathbb{N}:f(x)\neq 0\}$ and for $f,g\in Y$, define $f+g$ pointwise, but only when $\text{supp}(f)\cap\text{supp}(g)=\emptyset$. Then $(Y,+)$ is a partial semigroup. Let $Y_k=\{f\in Y:\max(f[\mathbb{N}])=k\}$. Define $\sigma:Y\to Y$ by

$$\sigma(f)(x)=\begin{cases} f(x)-1 & \text{if } f(x)>0\\ 0 & \text{if } f(x)=0\,.\end{cases}$$

Notice that σ is a partial semigroup homomorphism in the sense that $\sigma(f+g)=\sigma(f)+\sigma(g)$ whenever $f+g$ is defined. (If we did not have the disjointness of support requirement, this need not be true.)

We can now state Gowers' result.

1.3 Theorem. *Let k, Y, Y_k and σ be as defined above, let $r\in\mathbb{N}$, and let $Y=\bigcup_{i=1}^r C_i$. Then there exist $i\in\{1,2,\ldots,r\}$ and a sequence $\langle f_n\rangle_{n=1}^\infty$ in Y_k such that* $\text{supp}(f_n)\cap\text{supp}(f_m)=\emptyset$ *for all distinct m and n in $\mathbb{N}$ and*

$$\{\textstyle\sum_{n\in F}\sigma^{t(n)}(f_n):F\in\mathcal{P}_f(\mathbb{N})\,,\ t:F\to\{0,1,\ldots,k-1\}\,,\ \textit{and } t^{-1}[\{0\}]\neq\emptyset\}\subseteq C_i\,.$$

Proof. [5, Theorem 1]. □

Notice that the requirement that $t^{-1}[\{0\}]\neq\emptyset$ is clearly needed, since otherwise, one could have $C_1=\{f\in Y:k\in f[\mathbb{N}]\}$ and $C_2=Y\backslash C_1$.

Notice also that this result already generalizes several other Ramsey Theoretic results, including the Finite Unions Theorem (see [6, Theorem 3.16] or [9, Corollary 5.17]), which is trivially equivalent to the $k = 1$ instance of Theorem 1.3.

Theorem 1.3 translates into a statement about Lipshitz functions on the positive part of the unit sphere of the classical Banach space c_0: Every such function is 'approximately constant' on some infinite-dimensional slice of the unit sphere. The corresponding statement about Lipshitz functions on the whole unit sphere of c_0 is also proved in [5], but it does not correspond to a Ramsey-type result.

Another result naturally stated in terms of partial semigroups is Theorem 4.1 of [1]. In this case, one can again let $k \in \mathbb{N}$ (now assuming that $k > 1$) and work with the same set Y defined above. The operation $\oplus$ is defined pointwise, but in this case $f \oplus g$ is defined only when $\max\ \mathrm{supp}(f) < \min\ \mathrm{supp}(g)$. For $t \in \{1, 2, \ldots, k-1\}$, define $\mu_t : Y \to Y$ by

$$\mu_t(f)(x) = \begin{cases} t & \text{if } f(x) = k \\ f(x) & \text{if } f(x) \neq k\,. \end{cases}$$

One may think of k as a "variable", so that $\mu_t(f)$ is obtained by "substituting" t for occurrences of the variable k. Let $\mathcal{F} = \big\{\mu_t : t \in \{1, 2, \ldots, k-1\}\big\}$. We denote the identity function (on an appropriate set) by ι.

1.4 Theorem. *Let k, Y, Y_k and $\mathcal{F}$ be as defined above, let $r \in \mathbb{N}$, and let $Y = \bigcup_{i=1}^{r} C_i$. Then there exist $\gamma(1)$ and $\gamma(2)$ in $\{1, 2, \ldots, r\}$ and a sequence $\langle f_n \rangle_{n=1}^{\infty}$ in Y_k such that*

(a) $\max\ \mathrm{supp}(f_n) < \min\ \mathrm{supp}(f_{n+1})$ *for each* $n \in \mathbb{N}$,

(b) $\{\bigoplus_{n \in F} \tau_n(f_n) : F \in \mathcal{P}_f(\mathbb{N})$ *and* $\tau_n \in \mathcal{F}$ *for each* $n \in F\} \subseteq C_{\gamma(1)}$,

(c) $\{\bigoplus_{n \in F} \tau_n(f_n) : F \in \mathcal{P}_f(\mathbb{N})\,, \tau_n \in \mathcal{F} \cup \{\iota\}$ *for each* $n \in F$, *and some* $\tau_n = \iota\} \subseteq C_{\gamma(2)}$.

Proof. [1, Theorem 4.1]. □

In Section 3 we shall present Theorem 3.13 which is a common generalization of Theorems 1.3 and 1.4, in terms of what we call a "layered partial semigroup".

In Section 4 we give examples showing that Theorem 3.13 cannot be strengthened in certain directions. We also give two variants of this result, one of which (Theorem 4.5) is the optimal result in the case of a semigroup with only two nontrivial layers.

A notion that has become quite important in Ramsey Theory is that of "central sets". This concept was introduced by Furstenberg [3] and defined in terms of notions of topological dynamics. Central sets have a nice characterization in terms of the algebraic structure of βS, the Stone-Čech compactification of the semigroup S. We shall present this characterization below, after introducing the necessary background information.

Let $(S, \cdot)$ be an infinite discrete semigroup. We take the points of βS to be the ultrafilters on S, the principal ultrafilters being identified with the points of S. By this identification, we pretend that $S \subseteq \beta S$. In a similar fashion, if $S \subseteq T$, we pretend that $\beta S \subseteq \beta T$ by identifying the ultrafilter p on S with the ultrafilter $\{A \subseteq T : A \cap S \in p\}$ on T. Given a set $A \subseteq S$, $\overline{A} = \{p \in \beta S : A \in p\}$. The set $\{\overline{A} : A \subseteq S\}$ is a basis for the open sets (as well as a basis for the closed sets) of βS.

There is a natural extension of the operation $\cdot$ of S to βS making βS a compact right topological semigroup with S contained in its topological center. This says that for each $p \in \beta S$ the function $\rho_p : \beta S \to \beta S$, defined by $\rho_p(q) = q \cdot p$, is continuous and for each $x \in S$, the function $\lambda_x : \beta S \to \beta S$, defined by $\lambda_x(q) = x \cdot q$ is continuous. See [9] for an elementary introduction to the semigroup βS as well as for any unfamiliar algebraic terminology enountered here. (We shall frequently cite [9] for basic results that we need. This is not to be construed as a claim of originality for those results. Original sources can usually be found by consulting the chapter notes in [9].)

Any compact Hausdorff right topological semigroup $(T, \cdot)$ has a smallest two sided ideal $K(T)$ which is the union of all of the minimal left ideals of T, each of which is closed [9, Theorem 2.8 and Corollary 2.6], and any compact right topological semigroup contains idempotents [9, Theorem 2.5]. Since the minimal left ideals are themselves compact right topological semigroups, this says in particular that there are idempotents in the smallest ideal. There is a partial ordering of the idempotents of T determined by $p \leq q$ if and only if $p = p \cdot q = q \cdot p$. An idempotent p is minimal with respect to this order if and only if $p \in K(T)$ [9, Theorem 1.59]. Such an idempotent is called simply "minimal."

1.5 Definition. Let $(S, \cdot)$ be an infinite discrete semigroup and let $A \subseteq S$. Then A is *central* if and only if there is some minimal idempotent p in βS such that $A \in p$. Also, A is *central** if and only if $A \cap B \neq \emptyset$ whenever B is a central subset of S.

See [9, Theorem 19.27] for a proof of the equivalence of the definition above with the original dynamical definition.

The following theorem is the "Central Sets Theorem" for commutative semigroups. (We shall actually be concerned with a generalization of the Central Sets Theorem for arbitrary semigroups, but it is more complicated to state.) Given a sequence $\langle x_n \rangle_{n=1}^{\infty}$ in a semigroup $(S, \cdot)$, we write $FP(\langle x_n \rangle_{n=1}^{\infty}) = \{\prod_{n \in F} x_n : F \in \mathcal{P}_f(\mathbb{N})\}$, the set of finite products from $\langle x_n\rangle_{n=1}^{\infty}$. Recall that, if $(S,\cdot)$ is not commutative, we specify that the product $\prod_{n\in F} x_n$ is taken in increasing order of indices.

1.6 Theorem. *Let $(S,\cdot)$ be a commutative semigroup, let A be a central subset of S, and for each $l\in\mathbb{N}$, let $\langle y_{l,n}\rangle_{n=1}^{\infty}$ be a sequence in S. There exist a sequence $\langle a_n\rangle_{n=1}^{\infty}$ in S and a sequence $\langle H_n\rangle_{n=1}^{\infty}$ in $\mathcal{P}_f(\mathbb{N})$ such that $\max H_n < \min H_{n+1}$ for each $n\in\mathbb{N}$ and such that for every $f:\mathbb{N}\to\mathbb{N}$ satisfying $f(n)\leq n$ for each n, $FP(\langle a_n\cdot\prod_{t\in H_n} y_{f(n),t}\rangle_{n=1}^{\infty})\subseteq A$.*

Proof. [9, Theorem 14.11]. (Or see [3, Proposition 8.21] where the original Central Sets Theorem for the semigroup $(\mathbb{N},+)$ was proved.) $\square$

In Section 5 we shall prove an extension of the Central Sets Theorem, valid for layered partial semigroups. Some of the results of Section 3 will in fact be corollaries of this extension. However, we feel justified in beginning with simpler versions of the more general construction.

In Section 6 we state several open problems and give remarks puting the subject of the present paper in a somewhat broader context.

2. Partial Semigroups

In this section we present some basic results about an arbitrary partial semigroup and an associated subspace of its Stone-Čech compactification. Some of this material overlaps that in [8].

We saw in the introduction two examples of partial semigroups. Another natural example is the set $FP(\langle x_n\rangle_{n=1}^{\infty})$ where $\langle x_n\rangle_{n=1}^{\infty}$ is a sequence in a semigroup. In this case, $FP(\langle x_n\rangle_{n=1}^{\infty})$ is not likely to be closed under the restriction of the operation of the entire semigroup. However, if one only defines $(\prod_{n\in F} x_n)\cdot(\prod_{n\in G} x_n)$ when $F\cap G=\emptyset$ (if the original semigroup is commutative) or when $\max F<\min G$ (otherwise), then one does have a well behaved partial semigroup.

2.1 Definition. Let $(S,\cdot)$ be a partial semigroup.

(a) For $x\in S$, $\varphi(x)=\varphi_S(x)=\{y\in S: x\cdot y \text{ is defined}\}$.

(b) The semigroup S is *adequate* if and only if for every $F\in\mathcal{P}_f(S)$, $\bigcap_{x\in F}\varphi(x)\neq\emptyset$.

(c) $\delta S=\bigcap_{x\in S} c\ell_{\beta S}\big(\varphi(x)\big)$.

All of the partial semigroups that we have mentioned have been adequate. Notice that the assertion that S is adequate is exactly the assertion that $\delta S\neq\emptyset$. An important fact is that, for an adequate partial semigroup S, δS is in a natural way a compact right topological *semigroup*. This fact is part of the next theorem.

Note that the operation of a partial semigroup S is defined precisely on $\bigcup_{x\in S}\big(\{x\}\times\varphi(x)\big)$.

2.2 Theorem. *Let $(S,\cdot)$ be an adequate partial semigroup. Let*

$$D=\big(\bigcup_{x\in S}(\{x\}\times\overline{\varphi(x)}\big)\cup(\beta S\times\delta S)\,.$$

Then the operation $\cdot$ can be extended uniquely to D so that

(a) for each $x\in S$, the function $\lambda_x:\overline{\varphi(x)}\to\beta S$, defined by $\lambda_x(q)=x\cdot q$, is continuous, and

(b) for each $p\in\delta S$, the function $\rho_p:\beta S\to\beta S$, defined by $\rho_p(q)=q\cdot p$ is continuous.

Proof. For each $x\in S$, define $l_x:\varphi(x)\to S$ by $l_x(y)=x\cdot y$. Then l_x has a unique continuous extension $\widetilde{l_x}:\overline{\varphi(x)}\to\beta S$. For $q\in\beta S$, define $x\cdot q=\widetilde{l_x}(q)$ whenever $x\cdot q$ has not already been defined. Then λ_x is continuous.

Now, for each $p\in\delta S$, $x\cdot p$ is defined for all $x\in S$. Define $r_p(x)=x\cdot p$ and let $\widetilde{r_p}:\beta S\to\beta S$ be the unique continuous extension of r_p. For $q\in\beta S$, define $q\cdot p=\widetilde{r_p}(q)$ whenever $q\cdot p$ has not already been defined. $\square$

The points of δS are ultrafilters, so we are interested in describing the members of $p\cdot q$ in terms of the members of p and q.

2.3 Definition. Let S be a partial semigroup, let $x\in S$, and let $A\subseteq S$. Then $x^{-1}A=\{y\in\varphi(x):x\cdot y\in A\}$.

Notice that there is no suggestion, even in the event that S has an identity, that any or all elements of S have inverses. Also, if the operation in S is denoted by $+$, then we write $-x+A$ for $\{y\in\varphi(x):x+y\in A\}$.

2.4 Lemma. *Let S be an adequate partial semigroup.*

(a) Let $x\in S$, let $q\in\overline{\varphi(x)}$, and let $A\subseteq S$. Then $A\in x\cdot q$ if and only if $x^{-1}A\in q$.

(b) Let $p\in\beta S$, let $q\in\delta S$, and let $A\subseteq S$. Then $A\in p\cdot q$ if and only if $\{x\in S:x^{-1}A\in q\}\in p$.

Proof. (a) Necessity. Pick $B\in q$ such that $\lambda_x\big[\,\overline{B\cap\varphi(x)}\,\big]\subseteq\overline{A}$. Then $\varphi(x)\cap B\subseteq x^{-1}A$.

Sufficiency. Suppose that $A\notin x\cdot q$. Then $S\backslash A\in x\cdot q$ so that, by the already established necessity, $x^{-1}(S\backslash A)\in q$ while $x^{-1}A\cap x^{-1}(S\backslash A)=\emptyset$, a contradiction.

(b) Necessity. Pick $B \in p$ such that $\rho_q\big[\,\overline{B}\,\big] \subseteq \overline{A}$. Then by ($a$), $B \subseteq \{x \in S : x^{-1}A \in q\}$.

Sufficiency. Suppose that $A \notin p \cdot q$. Then $S \backslash A \in p \cdot q$ so that, by the already established necessity, $\{x \in S : x^{-1}(S\backslash A) \in q\} \in p$ while $\{x \in S : x^{-1}(S\backslash A) \in q\} \cap \{x \in S : x^{-1}A \in q\} = \emptyset$, a contradiction. □

2.5 Lemma. *Let S be an adequate partial semigroup, let $p \in \beta S$, $q \in \delta S$, and $a \in S$. Then $\varphi(a) \in p \cdot q$ if and only if $\varphi(a) \in p$.*

Proof. Necessity. Assume that $\varphi(a) \in p \cdot q$ so that $\{b \in S : b^{-1}\varphi(a) \in q\} \in p$. We show that $\{b \in S : b^{-1}\varphi(a) \in q\} \subseteq \varphi(a)$. So let $b^{-1}\varphi(a) \in q$. Pick $c \in b^{-1}\varphi(a)$. Then $c \in \varphi(b)$ and $b \cdot c \in \varphi(a)$ so $a \cdot (b \cdot c)$ is defined and thus $a \cdot (b \cdot c) = (a \cdot b) \cdot c$ and in particular $b \in \varphi(a)$.

Sufficiency. Assume that $\varphi(a) \in p$. We claim that $\varphi(a) \subseteq \{b \in S : b^{-1}\varphi(a) \in q\}$ so that $\varphi(a) \in p \cdot q$. Let $b \in \varphi(a)$. Since $q \in \delta S$, $\varphi(a \cdot b) \in q$. Therefore it suffices to show that $\varphi(a \cdot b) \subseteq b^{-1}\varphi(a)$. Let $c \in \varphi(a \cdot b)$. Then $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ so $c \in \varphi(b)$ and $b \cdot c \in \varphi(a)$. That is, $c \in b^{-1}\varphi(a)$ as required. □

2.6 Theorem. *Let S be an adequate partial semigroup. Then with the restriction of the operation given in Theorem* 2.2, *δS is a compact right topological semigroup.*

Proof. We have by Lemma 2.5 that if $p, q \in \delta S$, then $p \cdot q \in \delta S$. Since δS is a closed subset of βS we have that δS is compact. By Theorem 2.2, we have that ρ_q is continuous for each $q \in \delta S$. It thus suffices to show that the operation is associative on δS.

To this end, let $p, q, r \in \delta S$. Suppose that $p \cdot (q \cdot r) \neq (p \cdot q) \cdot r$ and pick

$$A \in p \cdot (q \cdot r) \backslash (p \cdot q) \cdot r. \text{ Let } B = \{a \in S : a^{-1}(S\backslash A) \in r\}.$$

Then $B \in p \cdot q$ so $\{b \in S : b^{-1}B \in q\} \in p$. Also, $\{b \in S : b^{-1}A \in q \cdot r\} \in p$ so pick $b \in S$ such that $b^{-1}B \in q$ and $b^{-1}A \in q \cdot r$. Then $\{c \in S : c^{-1}(b^{-1}A) \in r\} \in q$ so pick $c \in b^{-1}B$ such that $c^{-1}(b^{-1}A) \in r$. Then $c \in \varphi(b)$ and $b \cdot c \in B$ so $(b \cdot c)^{-1}(S\backslash A) \in r$. Pick $a \in c^{-1}(b^{-1}A) \cap (b \cdot c)^{-1}(S\backslash A)$. Then $a \in \varphi(c)$ and $c \cdot a \in b^{-1}A$ so $c \cdot a \in \varphi(b)$ and $b \cdot (c \cdot a) \in A$. On the other hand, $a \in \varphi(b \cdot c)$ and $(b \cdot c) \cdot a \in S\backslash A$, a contradiction. □

The fact that δS is a compact right topological semigroup provides a natural context for the notion of "central" in an adequate partial semigroup.

2.7 Definition. Let S be an adequate partial semigroup and let $A \subseteq S$. Then A is *central* if and only if there is some minimal idempotent $p \in \delta S$ such that $A \in p$. Also A is *central** if and only if $A \cap B \neq \emptyset$ whenever B is a central subset of S.

Notice that A is central* if and only if A is a member of every minimal idempotent in δS.

We shall be concerned extensively with more than one partial semigroup at a time.

2.8 Definition. Let S and T be partial semigroups and let $f : S \to T$. Then f is a *partial semigroup homomorphism* if and only if whenever $x \in S$ and $y \in \varphi_S(x)$, one has that $f(y) \in \varphi_T\big(f(x)\big)$ and $f(x \cdot y) = f(x) \cdot f(y)$.

It would be natural to define an "adequate partial subsemigroup" S of an adequate semigroup T to be a subset which is an adequate partial semigroup under the inherited operation. We see now that this is not enough to guarantee that $\delta S \subseteq \delta T$.

2.9 Lemma. *Let T be an adequate partial semigroup and let S be a subset of T which is an adequate partial semigroup under the inherited operation. Then $\delta S \subseteq \delta T$ if and only if for all $y \in T$ there exists $H \in \mathcal{P}_f(S)$ such that $\bigcap_{x\in H} \varphi_S(x) \subseteq \varphi_T(y)$.*

Proof. The sufficiency is immediate. For the necessity, let $y \in T$ and suppose that for all $H \in \mathcal{P}_f(S)$, $\bigcap_{x\in H} \varphi_S(x)\setminus\varphi_T(y) \neq \emptyset$. Then

$$\{\textstyle\bigcap_{x\in H} \varphi_S(x)\setminus\varphi_T(y) : H \in \mathcal{P}_f(S)\}$$

has the finite intersection property so pick $p \in \beta S$ such that

$$\{\textstyle\bigcap_{x\in H} \varphi_S(x)\setminus\varphi_T(y) : H \in \mathcal{P}_f(S)\} \subseteq p\,.$$

Then $p \in \delta S\setminus\delta T$, a contradiction. □

We shall see in Theorem 2.12 that the condition of Lemma 2.9 does not have to hold.

2.10 Definition. Let T be a partial semigroup. Then S is an *adequate partial subsemigroup* of T if and only if $S \subseteq T$, S is an adequate partial semigroup under the inherited operation, and for all $F \in \mathcal{P}_f(T)$ there exists $H \in \mathcal{P}_f(S)$ such that $\bigcap_{x\in H} \varphi_S(x) \subseteq \bigcap_{x\in F} \varphi_T(x)$.

2.11 Remark. *Notice that:*

(1) By Lemma 2.9 if S is an adequate partial subsemigroup of T, then $\delta S \subseteq \delta T$.

(2) If T is a partial semigroup which has an adequate partial subsemigroup, then necessarily T is an adequate partial semigroup.

(3) If S is a subset of T which is a partial semigroup under the inherited operation, then every adequate partial subsemigroup of T included in S is an adequate partial subsemigroup of S.

Notice that "is an adequate partial subsemigroup of" is a transitive relation. However, the following result establishes that the notion is not as well behaved as one might like.

2.12 Theorem. *There exist an adequate partial semigroup T and adequate partial subsemigroups R and S of T such that $R \cap S$ is an adequate partial semigroup with the inherited operation, but $R \cap S$ is not an adequate partial subsemigroup of T.*

Proof. Let $T = \mathcal{P}_f(\omega + \omega)$, where $\omega + \omega$ is the ordinal sum. For $\alpha, \beta \in T$, define $\alpha * \beta = \alpha \cup \beta$ exactly when $\max \alpha < \min \beta$. It is easy to see that T is an adequate partial semigroup.

Let $A = \omega \cup \{\omega + 2n : n \in \omega\}$ and $B = \omega \cup \{\omega + 2n + 1 : n \in \omega\}$. Let $R = \mathcal{P}_f(A)$ and let $S = \mathcal{P}_f(B)$. It is routine to verify that both R and S are adequate partial subsemigroups of T. Now $R \cap S = \mathcal{P}_f(\omega)$. To see that $R \cap S$ is not an adequate partial subsemigroup of T, let $F = \big\{\{\omega\}\big\}$. Then there is no $H \in \mathcal{P}_f(R \cap S)$ such that $\bigcap_{\alpha \in H} \varphi_{R \cap S}(\alpha) \subseteq \bigcap_{\alpha \in F} \varphi_T(\alpha)$ (which is $\{\alpha \in T : \min \alpha > \omega\}$). $\square$

Theorem 2.12 shows in particular that one may have adequate partial semigroups S and T such that $S \subseteq T$ (and S has the inherited operation) but $\delta S \setminus \delta T \neq \emptyset$. If $q \in \delta S \setminus \delta T$ and $p \in \beta S \setminus S$, then $p \cdot q$ is defined in βS, but is not defined in βT. This fact raises the possibility of some ambiguity concerning what is meant by $p \cdot q$. The following result shows that, if it is defined, $p \cdot q$ can mean only one thing.

2.13 Lemma. *Let T be an adequate partial semigroup and let R and S be subsets of T which are both adequate partial semigroups under the inherited operation. Let $p, q \in \beta(R \cap S)$. If $p \cdot q$ is defined in S and $p \cdot q$ is defined in R, then it is the same object under both definitions.*

Proof. Let $A \subseteq R \cap S$ and assume that $A \in p \cdot q$ as that object is defined in R. We show that $A \in p \cdot q$ as that object is defined in S. Assume first that $p \in R \cap S$ so that (because $p \cdot q$ is defined), $\varphi_R(p) \in q$ and $\varphi_S(p) \in q$. Then by Lemma 2.4(a) $\{y \in \varphi_R(p) : p \cdot y \in A\} \in q$ and $\{y \in \varphi_R(p) : p \cdot y \in A\} \cap \varphi_S(p) \subseteq \{y \in \varphi_S(p) : p \cdot y \in A\}$ and hence $\{y \in \varphi_S(p) : p \cdot y \in A\} \in q$.

Now assume that $p \in \beta(R \cap S) \setminus (R \cap S)$ and hence (because $p \cdot q$ is defined), $q \in (\delta R \cap \delta S)$. Then by Lemma 2.4(b) $\{x \in R : \{y \in \varphi_R(x) : x \cdot y \in A\} \in q\} \in p$. Also $S \in p$. We claim that

$$\{x \in R : \{y \in \varphi_R(x) : x \cdot y \in A\} \in q\} \cap S \subseteq \{x \in S : \{y \in \varphi_S(x) : x \cdot y \in A\} \in q\}$$

so that $\{x \in S : \{y \in \varphi_S(x) : x \cdot y \in A\} \in q\} \in p$ as required. To this end, let $x \in S \cap R$ such that $\{y \in \varphi_R(x) : x \cdot y \in A\} \in q$. Since also $\varphi_S(x) \in q$ and

$$\{y \in \varphi_R(x) : x \cdot y \in A\} \cap \varphi_S(x) \subseteq \{y \in \varphi_S(x) : x \cdot y \in A\}$$

we have that $\{y \in \varphi_S(x) : x \cdot y \in A\} \in q$ as required. □

We now establish conditions guaranteeing that the continuous extension of a partial semigroup homomorphism is a homomorphism.

2.14 Lemma. *Let S and T be adequate partial semigroups, let $f : S \to T$ be a partial semigroup homomorphism, and let $\widetilde{f} : \beta S \to \beta T$ be the continuous extension of f. If $p \in \beta S$, $q \in \delta S$, and $\widetilde{f}(q) \in \delta T$, then $\widetilde{f}(p \cdot q) = \widetilde{f}(p) \cdot \widetilde{f}(q)$. If $f[S]$ is an adequate partial subsemigroup of T, then $\widetilde{f}[\delta S] \subseteq \delta T$ and $\widetilde{f}_{|\delta S}$ is a (semigroup) homomorphism.*

Proof. Assume first that $p \in \beta S$, $q \in \delta S$, and $\widetilde{f}(q) \in \delta T$ and suppose that $\widetilde{f}(p \cdot q) \neq \widetilde{f}(p) \cdot \widetilde{f}(q)$. Pick disjoint open neighborhoods U and V of $\widetilde{f}(p \cdot q)$ and $\widetilde{f}(p) \cdot \widetilde{f}(q)$ respectively. Pick $A \in p$ such that $\widetilde{f} \circ \rho_q\big[\,\overline{A}\,\big] \subseteq U$ and $\rho_{\widetilde{f}(q)} \circ \widetilde{f}\big[\,\overline{A}\,\big] \subseteq V$ and pick $x \in A$. Then $\widetilde{f}(x \cdot q) \in U$ and $f(x) \cdot \widetilde{f}(q) \in V$. Since $\lambda_{f(x)}\big(\widetilde{f}(q)\big) \in V$, pick $B \in \widetilde{f}(q)$ such that $\lambda_{f(x)}\big[\,\overline{B \cap \varphi_T\big(f(x)\big)}\,\big] \subseteq V$. Pick $C \in q$ such that $\widetilde{f} \circ \lambda_x\big[\,\overline{C \cap \varphi_S(x)}\,\big] \subseteq U$ and $\widetilde{f}\big[\,\overline{C}\,\big] \subseteq \overline{B \cap \varphi_T\big(f(x)\big)}$. Pick $y \in C \cap \varphi_S(x)$. Then $f(x \cdot y) \in U$ and $f(x) \cdot f(y) \in V$, a contradiction.

Now assume that $f[S]$ is an adequate partial subsemigroup of T. Let $p \in \delta S$ and let $x \in T$. Suppose that $\varphi_T(x) \notin \widetilde{f}(p)$. Pick $A \in p$ such that $\widetilde{f}\big[\,\overline{A}\,\big] \cap \overline{\varphi_T(x)} = \emptyset$ and pick $F \in \mathcal{P}_f(f[S])$ such that $\bigcap_{y \in F} \varphi_{f[S]}(y) \subseteq \varphi_T(x)$. Let $G \in \mathcal{P}_f(S)$ be such that $f[G] = F$, and pick $b \in A \cap \bigcap_{a \in G} \varphi_S(a)$. Then $f(b) \in f[A] \cap \bigcap_{y \in F} \varphi_{f[S]}(y) \subseteq f[A] \cap \varphi_T(x)$, a contradiction. □

We now extend the notions of "right ideal", "left ideal", and "ideal" to partial semigroups.

2.15 Definition. Let S be a partial semigroup.

(a) A subset I of S is a *left ideal* of S if and only if $x \cdot y \in I$ whenever $x \in S$ and $y \in I \cap \varphi(x)$.

(b) A subset I of S is a *right ideal* of S if and only if $x \cdot y \in I$ whenever $x \in I$ and $y \in \varphi(x)$.

(c) A subset I of S is an *ideal* of S if and only if I is both a left ideal and a right ideal of S.

2.16 Lemma. *Let T be a partial semigroup, let S be an adequate partial subsemigroup of T and assume that S is an ideal of T. Then δS is an ideal of δT. In particular, $K(\delta S) = K(\delta T)$.*

Proof. By Lemma 2.9, $\delta S \subseteq \delta T$. Let $p \in \delta S$ and $q \in \delta T$. To see that $q \cdot p \in \delta S$, let $x \in S$. We need to show that $\varphi_S(x) \in q \cdot p$. Since $q \in \delta T$, $\varphi_T(x) \in q$. We claim that $\varphi_T(x) \subseteq \{y \in T : y^{-1}\varphi_S(x) \in p\}$. (Here $y^{-1}\varphi_S(x)$ is interpreted in T, so $y^{-1}\varphi_S(x) = \{z \in \varphi_T(y) : y \cdot z \in \varphi_S(x)\}$.) So let $y \in \varphi_T(x)$ and pick $H \in \mathcal{P}_f(S)$ such that $\bigcap_{z \in H} \varphi_S(z) \subseteq \varphi_T(x \cdot y)$. We claim that $\bigcap_{z \in H} \varphi_S(z) \subseteq y^{-1}\varphi_S(x)$, and thus that $y^{-1}\varphi_S(x) \in p$ as required. So let $w \in \bigcap_{z \in H} \varphi_S(z)$. Then $w \in \varphi_T(x \cdot y)$. So $(x \cdot y) \cdot w$ is defined in T and so $x \cdot (y \cdot w)$ is defined in T. In particular, $y \cdot w \in \varphi_T(x)$. Since $w \in S$, and S is an ideal of T, $y \cdot w \in \varphi_T(x) \cap S = \varphi_S(x)$. Also, $w \in \varphi_T(y)$. Thus, $w \in y^{-1}\varphi_S(x)$.

To see that $p \cdot q \in \delta S$, let $x \in S$. We need to show that $\varphi_S(x) \in p \cdot q$. We claim that $\varphi_S(x) \subseteq \{y \in T : y^{-1}\varphi_S(x) \in q\}$. (Again $y^{-1}\varphi_S(x)$ is interpreted in T.) Let $y \in \varphi_S(x)$. We claim that $\varphi_T(x \cdot y) \subseteq y^{-1}\varphi_S(x)$, so that $y^{-1} \cdot \varphi_S(x) \in q$. Let $z \in \varphi_T(x \cdot y)$. Then $(x \cdot y) \cdot z$ is defined in T and so $x \cdot (y \cdot z)$ is defined in T. Also since S is an ideal of T, $y \cdot z \in S$, so $y \cdot z \in \varphi_S(x)$.

Finally, since δS is an ideal of δT, we have that $K(\delta T) \subseteq \delta S$ and in particular $K(\delta T) \cap \delta S \neq \emptyset$. Consequently by [9, Theorem 1.65], $K(\delta S) = K(\delta T) \cap \delta S = K(\delta T)$. □

We conclude this section with three technical lemmas that are of interest in terms of our descriptions of "layered partial semigroups" in the next section.

2.17 Lemma. *Let T and S be adequate partial semigroups, let $\sigma : T \to S$, and let $\widetilde{\sigma} : \beta T \to \beta S$ be the continuous extension of σ. Then $\delta S \subseteq \widetilde{\sigma}[\delta T]$ if and only if for every $F \in \mathcal{P}_f(T)$ there exists $H \in \mathcal{P}_f(S)$ such that $\bigcap_{x \in H} \varphi_S(x) \subseteq \sigma\left[\bigcap_{x \in F} \varphi_T(x)\right]$.*

Proof. Necessity. Let $F \in \mathcal{P}_f(T)$ and let $B = \bigcap_{x \in F} \varphi_T(x)$. Suppose that for all $H \in \mathcal{P}_f(S)$, $\bigcap_{x \in H} \varphi_S(x) \backslash \sigma[B] \neq \emptyset$. Then $\{\varphi_S(x) \backslash \sigma[B] : x \in S\}$ has the finite intersection property so pick $p \in \beta S$ such that $\{\varphi_S(x) \backslash \sigma[B] : x \in S\} \subseteq p$. Then $p \in \delta S$ so pick $q \in \delta T$ such that $\widetilde{\sigma}(q) = p$. Now $B \in q$ so $\sigma[B] \in p$, a contradiction.

Sufficiency. Let $p \in \delta S$. It suffices to show that $\{\sigma^{-1}[A] : A \in p\} \cup \{\varphi_T(x) : x \in T\}$ has the finite intersection property. (For then, picking $q \in \beta T$ such that $\{\sigma^{-1}[A] : A \in p\} \cup \{\varphi_T(x) : x \in T\} \subseteq q$ we have that $q \in \delta T$ and $\widetilde{\sigma}(q) = p$.) Let $A \in p$ and $F \in \mathcal{P}_f(T)$. Pick $H \in \mathcal{P}_f(S)$ such that $\bigcap_{x \in H} \varphi_S(x) \subseteq \sigma\left[\bigcap_{x \in F} \varphi_T(x)\right]$. Pick $y \in A \cap \bigcap_{x \in H} \varphi_S(x)$ and pick $z \in \bigcap_{x \in F} \varphi_T(x)$ such that $y = \sigma(z)$. Then $z \in \bigcap_{x \in F} \varphi_T(x) \cap \sigma^{-1}[A]$. □

2.18 Lemma. *Let T and S be adequate partial semigroups, let $\sigma : T \to S$ be a partial semigroup homomorphism such that $\sigma[T] = S$, and let $\widetilde{\sigma} : \beta T \to \beta S$ be the continuous extension of σ. The following are equivalent.*

(a) $\delta S \subseteq \widetilde{\sigma}[\delta T]$.

(b) $\delta S = \widetilde{\sigma}[\delta T]$.

(c) *For every $F \in \mathcal{P}_f(T)$ there exists $H \in \mathcal{P}_f(S)$ such that*

$$\bigcap_{x\in H} \varphi_S(x) \subseteq \sigma\left[\bigcap_{x\in F} \varphi_T(x)\right] .$$

Proof. Statements (a) and (c) are equivalent by Lemma 2.17, and trivially (b) implies (a). That (a) implies (b) follows from Lemma 2.14. □

2.19 Lemma. *Let T and S be adequate partial semigroups, let $\sigma : T \to S$, and let $\widetilde{\sigma} : \beta T \to \beta S$ be the continuous extension of σ. Then $\{p \in K(\delta S) : p \cdot p = p\} \subseteq \widetilde{\sigma}[\delta T]$ if and only if for every $F \in \mathcal{P}_f(T)$, $\sigma\left[\bigcap_{x\in F} \varphi_T(x)\right]$ is central* in S.*

Proof. Sufficiency. Let $p \in K(\delta S)$ such that $p \cdot p = p$. Then for every $F \in \mathcal{P}_f(T)$, $\sigma\left[\bigcap_{x\in F} \varphi_T(x)\right] \in p$. Thus $\{\varphi_T(x) : x \in T\} \cup \{\sigma^{-1}[A] : A \in p\}$ has the finite intersection property and so we may pick $q \in \beta S$ such that

$$\{\varphi_T(x) : x \in T\} \cup \{\sigma^{-1}[A] : A \in p\} \subseteq q\,.$$

Since $\{\varphi_T(x) : x \in T\} \subseteq q$, $q \in \delta T$. Since $\{\sigma^{-1}[A] : A \in p\} \subseteq q$, $\widetilde{\sigma}(q) = p$.

Necessity. Let $F \in \mathcal{P}_f(T)$. To see that $\sigma\left[\bigcap_{x\in F} \varphi_T(x)\right]$ is central* in S, let $p = p \cdot p \in K(\delta S)$. Pick $q \in \delta T$ such that $\widetilde{\sigma}(q) = p$. Then $\bigcap_{x\in F} \varphi_T(x) \in q$ so $\sigma\left[\bigcap_{x\in F} \varphi_T(x)\right] \in p$. □

3. Layered Partial Semigroups

In this section, we introduce our main objects of study and prove a common generalization of Theorems 1.3 and 1.4. Notice that any semigroup is also a partial semigroup and if S is a semigroup, then $\delta S = \beta S$.

3.1 Definition. The set S is a *layered partial semigroup* (with k layers) if and only if there exist $k \in \mathbb{N}\backslash\{1\}$ and $S_0, S_1, \ldots, S_k$, such that

(1) $\{S_0, S_1, \ldots, S_k\}$ is a partition of S;

(2) $S_0 = \{e\}$ where e is a two sided identity for S with $\varphi_S(e) = S$ and $e \in \bigcap_{x\in S} \varphi_S(x)$;

(3) for $n \in \{1, 2, \ldots, k\}$, $\bigcup_{i=0}^{n} S_i$ is an adequate partial semigroup; and

(4) for $n \in \{1, 2, \ldots, k\}$, S_n is an adequate partial subsemigroup of S and an ideal of $\bigcup_{i=0}^{n} S_i$.

Notice that if S is a layered partial semigroup, then by requirement (3) of the definition, $S = \bigcup_{i=0}^{k} S_i$ is an adequate partial semigroup.

If S is a layered partial semigroup which is in fact a semigroup, we shall say that S is a *layered semigroup*.

We shall not be concerned with layered partial semigroups by themselves, but rather in conjunction with certain functions acting on all or part of these semigroups.

3.2 Definition. Let S be a layered partial semigroup with k layers, let $S_0, S_1, \ldots, S_k$ be as in Definition 3.1, and let $n \in \{2, 3, \ldots, k\}$. A function σ is a *shift* on $\bigcup_{i=0}^{n} S_i$ if and only if

(1) σ is a partial semigroup homomorphism from $\bigcup_{i=0}^{n} S_i$ to $\bigcup_{i=0}^{n-1} S_i$;
(2) $\sigma[S_n]$ is an adequate partial subsemigroup of S_{n-1}; and
(3) for every $F \in \mathcal{P}_f(S_n)$, $\sigma\left[\bigcap_{x \in F} \varphi_{S_n}(x)\right]$ is central* in S_{n-1}.

It is not in general easy to tell whether a given subset of a partial semigroup is central*. In practice it is often convenient to establish that for every $F \in \mathcal{P}_f(S_n)$ there exists $H \in \mathcal{P}_f(S_{n-1})$ such that $\bigcap_{x \in H} \varphi_{S_{n-1}}(x) \subseteq \sigma\left[\bigcap_{x \in F} \varphi_{S_n}(x)\right]$. Then, by Lemma 2.17, $\delta S_{n-1} \subseteq \widetilde{\sigma}[\delta S_n]$ and in particular $\{p \in K(\delta S_{n-1}) : p \cdot p = p\} \subseteq \widetilde{\sigma}[\delta S_n]$ so that by Lemma 2.19, $\sigma\left[\bigcap_{x \in F} \varphi_{S_n}(x)\right]$ is central* in S_{n-1} for every $F \in \mathcal{P}_f(S)$.

Notice that requirement (2) of Definition 3.2 holds automatically in the event that $\sigma[S_n] = S_{n-1}$. Notice also that, in the event that S is a semigroup, requirement (2) is equivalent to the assertion that $\sigma[S_n] \subseteq S_{n-1}$ and requirement (3) is equivalent to the assertion that $\sigma[S_n]$ is central* in S_{n-1}.

3.3 Definition. Let S be a layered partial semigroup with k layers and let $S_0, S_1, \ldots, S_k$ be as in Definition 3.1. Then $\langle \mathcal{F}_n \rangle_{n=2}^{k}$ is a *layered action on S* if and only if for every $n \in \{2, 3, \ldots, k\}$, $\mathcal{F}_n$ is a nonempty finite set of partial semigroup homomorphisms from $\bigcup_{i=0}^{n} S_i$ to $\bigcup_{i=0}^{n-1} S_i$ such that

(1) for each $f \in \mathcal{F}_n$, either
  (a) the restriction of f to $\bigcup_{i=0}^{n-1} S_i$ is the identity or
  (b) f is a shift on $\bigcup_{i=0}^{n} S_i$ and either
    (i) $n > 2$ and the restriction of f to $\bigcup_{i=0}^{n-1} S_i$ is a member of $\mathcal{F}_{n-1}$ or
    (ii) $f[\bigcup_{i=0}^{n-1} S_i] = \{e\}$; and
(2) for all but at most one member of $\mathcal{F}_n$, condition (1)(a) holds.

The following simple lemma will be useful later.

3.4 Lemma. *Let S be a layered partial semigroup with k layers, let $S_0, S_1, \ldots, S_k$ be as in Definition* 3.1*, and let $\langle \mathcal{F}_n \rangle_{n=2}^k$ be a layered action on S. For $n \in \{1, 2, \ldots, k\}$, let $T_n = \bigcup_{i=0}^n S_i$. If $n \in \{2, 3, \ldots, k\}$ and $f \in \mathcal{F}_n$, then there is some $v \in \{2, 3, \ldots, n\}$ such that for all $s \in \{v, v+1, \ldots, n\}$ $f_{|T_s} \in \mathcal{F}_s$, and either $f_{|T_{v-1}} = \iota_{T_{v-1}}$ or $f[T_{v-1}] = \{e\}$.*

Proof. This is a routine induction (in the usual upwards direction) on n. □

We pause to note that we already have examples of layered partial semigroups.

3.5 Lemma. *Let $k \in \mathbb{N}$ and let Y, $+$, and σ be as defined before Theorem* 1.3*. For each $n \in \{2, 3, \ldots, k\}$, let $\mathcal{F}_n = \{\sigma_{|\cup_{i=0}^n S_i}\}$. Then Y is a layered partial semigroup with k layers and $\langle \mathcal{F}_i \rangle_{i=2}^k$ is a layered action on Y.*

Proof. For $n \in \{0, 1, \ldots, k\}$, let $S_n = \{f \in Y : \max(f[\mathbb{N}]) = n\}$. Requirements (1) through (3) of Definition 3.1 are easily verified. Requirement (4) holds because, given any $n \in \{1, 2, \ldots, k\}$ and any $f \in Y \backslash \{\overline{0}\}$ there exists $g \in S_n$ such that $\mathrm{supp}(g) = \mathrm{supp}(f)$ and therefore $\varphi_{S_n}(g) = \varphi_Y(f)$.

To complete the proof, we need to show that for each $n \in \{2, 3, \ldots, k\}$, $\sigma_{|\cup_{i=0}^n S_i}$ is a shift on $\bigcup_{i=0}^n S_i$, since then condition (1)(b)(ii) of Definition 3.3 holds for $n = 2$, while condition (1)(b)(i) holds for $n > 2$.

Requirement (1) of Definition 3.2 is immediate and requirement (2) holds because $\sigma[S_n] = S_{n-1}$ for each $n \in \{1, 2, \ldots, k\}$. To verify requirement (3), let $n \in \{2, 3, \ldots, k\}$ and let $F \in \mathcal{P}_f(S_n)$. We show that there exists $H \in \mathcal{P}_f(S_{n-1})$ such that $\bigcap_{x \in H} \varphi_{S_{n-1}}(x) \subseteq \sigma\left[\bigcap_{x \in F} \varphi_{S_n}(x)\right]$, which suffices, as we remarked, by Lemmas 2.17 and 2.19. For each $f \in F$, define $h_f \in S_{n-1}$ by

$$h_f(t) = \begin{cases} n-1 & \text{if } f(t) \neq 0 \\ 0 & \text{if } f(t) = 0 \end{cases}$$

and let $H = \{h_f : f \in F\}$. Now let $g \in \bigcap_{x \in H} \varphi_{S_{n-1}}(x)$. Define $r \in S_n$ by

$$r(t) = \begin{cases} g(t)+1 & \text{if } g(t) \neq 0 \\ 0 & \text{if } g(t) = 0\,. \end{cases}$$

Then, $r \in \bigcap_{x \in F} \varphi_{S_n}(x)$ since for each $f \in F$, $\mathrm{supp}(h_f) = \mathrm{supp}(f)$. And $\sigma(r) = g$. □

3.6 Lemma. *Let $k \in \mathbb{N} \backslash \{1\}$ and let Y, $\oplus$, and $\mathcal{F}_2 = \mathcal{F}$ be as defined before Theorem* 1.4*. Then Y is a layered partial semigroup with* 2 *layers and $\langle \mathcal{F}_i \rangle_{i=2}^2$ is a layered action on Y.*

Proof. Let $S_0 = \{\overline{0}\}$, let $S_1 = \{f \in Y : 0 < \max(f[\mathbb{N}]) < k\}$, let $S_2 = \{f \in Y : \max(f[\mathbb{N}]) = k\}$, and let $\mathcal{F}_2 = \mathcal{F}$. It is easy to verify that condition (1)(a) of Definition 3.3 applies to each $f \in \mathcal{F}_2$. $\square$

Lemmas 3.5 and 3.6 raise the natural question of whether we can have a layered partial semigroup S and a layered action $\langle \mathcal{F}_n \rangle_{n=2}^k$ on S for which conditions (1)(a) and (1)(b) each apply to members of $\mathcal{F}_n$. We see in fact that a very familiar semigroup (not just partial semigroup) satisfies these requirements.

3.7 Lemma. *Let $k \in \mathbb{N}\backslash\{1\}$ and let S be the free semigroup on k letters with identity e. Then S is a layered semigroup with k layers. Further for any $m \in \mathbb{N}$, there is a layered action $\langle \mathcal{F}_n \rangle_{n=2}^k$ on S such that for each n, $|\mathcal{F}_n| \geq m$ and condition (1)(b) of Definition* 3.3 *applies to one member of $\mathcal{F}_n$.*

Proof. Let the k letters be $a_1, a_2, \ldots, a_k$. Let $S_0 = \{e\}$, and for $n \in \{1, 2, \ldots, k\}$, let $S_n = \{w \in S : \max\{t : a_t \text{ occurs in } w\} = n\}$.

Recall that a homomorphism on a free semigroup is completely determined by its values at the letters. Define a homomorphism $\sigma : S \to S$ by $\sigma(a_1) = e$ and $\sigma(a_n) = a_{n-1}$ for $n \in \{2, 3, \ldots, k\}$. Let $m \in \mathbb{N}$ be given and for each $n \in \{2, 3, \ldots, k\}$, choose a finite $F_n \subseteq \bigcup_{i=0}^{n-1} S_i$ such that $|F_n| \geq m$. For each $n \in \{2, 3, \ldots, k\}$ and each $w \in F_n$, define a homomorphism $f_{n,w} : \bigcup_{i=0}^{n} S_i \to \bigcup_{i=0}^{n-1} S_i$ by

$$f_{n,w}(a_t) = \begin{cases} w & \text{if } t = n \\ a_t & \text{if } t < n \end{cases}$$

and let $\mathcal{F}_n = \{f_{n,w} : w \in F_n\} \cup \{\sigma_{|\cup_{i=0}^{n} S_i}\}$. All requirements can be easily verified. $\square$

The following is our major algebraic tool. The proof combines ideas from the proofs of [5, Theorem 1] and [1, Theorem 4.1].

3.8 Theorem. *Let S be a layered partial semigroup and let k and $S_0, S_1, \ldots, S_k$ be as in Definition* 3.1. *Let $\langle \mathcal{F}_n \rangle_{n=2}^k$ be a layered action on S. Let p be any minimal idempotent in δS_1. For $n \in \{2, 3, \ldots, k\}$ and $f \in \mathcal{F}_n$, let $\widetilde{f} : \beta(\bigcup_{i=0}^{n} S_i) \to \beta(\bigcup_{i=0}^{n-1} S_i)$ be the continuous extension of f. Then for each $i \in \{1, 2, \ldots, k\}$, there is an idempotent p_i, minimal in δS_i, such that*

(1) $p_1 = p$;

(2) if $i \in \{2, 3, \ldots, k\}$ and $f \in \mathcal{F}_i$, then $\widetilde{f}(p_i) = p_{i-1}$ and

(3) if $i, j \in \{1, 2, \ldots, k\}$ and $i \leq j$, then $p_j \leq p_i$.

Proof. For each $n \in \{1, 2, \ldots, k\}$, let $T_n = \bigcup_{i=0}^{n} S_i$.

Now let $n \in \{2, 3, \ldots, k\}$ and assume that we have chosen $p_0, p_1, \ldots, p_{n-1}$ as required. We claim that it suffices to produce an idempotent p_n minimal in δS_n such that $p_n \leq p_{n-1}$ and for $\sigma \in \mathcal{F}_n$ satisfying condition (1)(b) of Definition 3.3, if any, $\widetilde{\sigma}(p_n) = p_{n-1}$.

Assume that we have done this. Then conclusions (1) and (3) hold directly. Let $f \in \mathcal{F}_n$ such that f satisfies condition (1)(a) of Definition 3.3. Since $f[T_{n-1}] = T_{n-1}$, f is surjective so by Lemma 2.14, $\widetilde{f}_{|\delta T_n} : \delta T_n \to \delta T_{n-1}$ is a homomorphism. Therefore, $\widetilde{f}(p_n) \leq \widetilde{f}(p_{n-1})$. Since f equals the identity on T_{n-1}, $\widetilde{f}(p_{n-1}) = p_{n-1}$. Therefore, $\widetilde{f}(p_n) \leq p_{n-1}$. By Lemma 2.16 and Remark 2.11(3), $p_{n-1} \in K(\delta T_{n-1})$ and so p_{n-1} is minimal in δT_{n-1} and so $\widetilde{f}(p_n) = p_{n-1}$, and conclusion (2) holds.

Notice that, by Remark 2.11(1) and Remark 2.11(3), $\delta S_{n-1} \subseteq \delta T_n$ and $\delta S_n \subseteq \delta T_n$. Notice also that if σ is a shift on T_n, then by Lemma 2.14, $\widetilde{\sigma}_{|\delta S_n}$ is a homomorphism for each $n \in \{2, 3, \ldots, k\}$.

If all $f \in \mathcal{F}_n$ satisfy condition (1)(a) of Definition 3.3, we simply note that $p_{n-1} \in \delta S_{n-1} \subseteq \delta T_n$. Thus we may pick by [9, Theorem 1.60] an idempotent $p_n \in K(\delta T_n)$ such that $p_n \leq p_{n-1}$. By Lemma 2.16, $K(\delta S_n) = K(\delta T_n)$ and so p_n is minimal in δS_n.

So we assume that $\sigma \in \mathcal{F}_n$ satisfies condition (1)(b) of Definition 3.3. Let $M = \{q \in \delta S_n : \widetilde{\sigma}(q) = p_{n-1}\}$. By requirement (3) of Definition 3.2 and Lemma 2.19, we have that $p_{n-1} \in \widetilde{\sigma}[\delta S_n]$ so that $M \neq \emptyset$. Trivially M is a compact subsemigroup of δS_n (since $\widetilde{\sigma}$ is a homomorphism on δS_n). Now $p_{n-1} \in \delta S_{n-1} \subseteq \delta T_n$ and $M \subseteq \delta S_n \subseteq \delta T_n$ so $M \cdot p_{n-1}$ is a compact subset of δT_n. Since δS_n is an ideal of δT_n by Lemma 2.16 and Remark 2.11(3), we have $M \cdot p_{n-1} \subseteq \delta S_n$.

We claim that $M \cdot p_{n-1} \subseteq M$. To see this, let $q \in M$. We have just seen that $q \cdot p_{n-1} \in \delta S_n$. Now either $n > 2$ and $\sigma_{|T_{n-1}} \in \mathcal{F}_{n-1}$ or $\sigma[T_{n-1}] = \{e\}$. In the first case $\widetilde{\sigma}(p_{n-1}) = p_{n-2}$ by the induction hypothesis and so $\widetilde{\sigma}(p_{n-1}) \in \delta S_{n-2} \subseteq T_{n-1}$. In the second case $\widetilde{\sigma}(p_{n-1}) = e \in T_{n-1}$. Thus in either case we have by Lemma 2.14 that $\widetilde{\sigma}(q \cdot p_{n-1}) = \widetilde{\sigma}(q) \cdot \widetilde{\sigma}(p_{n-1}) = p_{n-1} \cdot \widetilde{\sigma}(p_{n-1}) = p_{n-1}$.

Since $M \cdot p_{n-1} \subseteq M$ we have that $M \cdot p_{n-1} \cdot M \cdot p_{n-1} \subseteq M \cdot p_{n-1}$. That is, $M \cdot p_{n-1}$ is a subsemigroup of δS_n. Pick an idempotent q minimal in $M \cdot p_{n-1}$. Then $q = r \cdot p_{n-1}$ for some $r \in M$ so that $q \cdot p_{n-1} = r \cdot p_{n-1} \cdot p_{n-1} = r \cdot p_{n-1} = q$. Also $q \in M \cdot p_{n-1} \subseteq M$. Let $p_n = p_{n-1} \cdot q$ and note that $p_n \cdot p_{n-1} = p_{n-1} \cdot p_n = p_n$. Also note that $p_n \cdot p_n = p_n \cdot p_{n-1} \cdot q = p_n \cdot q = p_{n-1} \cdot q \cdot q = p_{n-1} \cdot q = p_n$. Thus p_n is an idempotent with $p_n \leq p_{n-1}$.

We claim first that $p_n \in M$ (so, since $p_n = p_n \cdot p_{n-1}$, $p_n \in M \cdot p_{n-1}$). We have that $p_n \in \delta S_n$ because δS_n is an ideal of δT_n and $q \in \delta S_n$. Also $q \in \delta S_n \subseteq \delta T_n$ and

$\widetilde{\sigma}(q) \in \delta S_{n-1} \subseteq \delta T_{n-1}$ so by Lemma 2.14, $\widetilde{\sigma}(p_n) = \widetilde{\sigma}(p_{n-1} \cdot q) = \widetilde{\sigma}(p_{n-1}) \cdot \widetilde{\sigma}(q) = \widetilde{\sigma}(p_{n-1}) \cdot p_{n-1} = p_{n-1}$, using again the fact that either $\widetilde{\sigma}(p_{n-1}) = p_{n-2}$ or $\widetilde{\sigma}(p_{n-1}) = e$. It remains only to show that p_n is minimal in δS_n which we shall do in three steps.

Next we claim that p_n is minimal in $M \cdot p_{n-1}$. Indeed, $p_n = p_{n-1} \cdot q = p_{n-1} \cdot q \cdot q = p_n \cdot q \in (M \cdot p_{n-1}) \cdot K(M \cdot p_{n-1}) \subseteq K(M \cdot p_{n-1})$.

Now we show that p_n is minimal in M. So let s be an idempotent in M with $s \leq p_n$. Then $s = s \cdot p_n = s \cdot p_n \cdot p_{n-1} \in M \cdot p_{n-1}$ and so $s = p_n$. (We used here the fact noted earlier that M is a semigroup.)

Finally we show that p_n is minimal in δS_n. So let s be an idempotent in δS_n with $s \leq p_n$. We need to show that $s \in M$. To see that $\widetilde{\sigma}(s) = p_{n-1}$ it suffices to show that $\widetilde{\sigma}(s) \leq p_{n-1}$ since $\widetilde{\sigma}(s)$ is an idempotent in δS_{n-1}. Now $\widetilde{\sigma}(s) \cdot p_{n-1} = \widetilde{\sigma}(s) \cdot \widetilde{\sigma}(p_n) = \widetilde{\sigma}(s \cdot p_n) = \widetilde{\sigma}(s)$ and $p_{n-1} \cdot \widetilde{\sigma}(s) = \widetilde{\sigma}(p_n) \cdot \widetilde{\sigma}(s) = \widetilde{\sigma}(p_n \cdot s) = \widetilde{\sigma}(s)$. Thus $\widetilde{\sigma}(s) \leq p_{n-1}$ as required. □

We now introduce some notation that will be used to describe the structures which we can guarantee to lie in one cell of a partition of a layered partial semigroup. The notation does not reflect its dependence on the choice of semigroup or the choice of layered action.

3.9 Definition. Let S be a layered partial semigroup with k layers, let $S_0, S_1, \ldots, S_k$ be as in Definition 3.1, and let $\langle \mathcal{F}_n \rangle_{n=2}^k$ be a layered action on S. Let $\mathcal{G}_k = \{\iota_S\}$. For $l \in \{1, 2, \ldots, k-1\}$, given that $\mathcal{G}_{l+1}$ has been defined, let $\mathcal{G}_l = \{f \circ g : g \in \mathcal{G}_{l+1}$ and $f \in \mathcal{F}_{l+1}\}$.

The following lemma will be needed in Section 6.

3.10 Lemma. *Let S be a layered partial semigroup with k layers, let $S_0, S_1, \ldots, S_k$ be as in Definition* 3.1*, and let $\langle \mathcal{F}_n \rangle_{n=2}^k$ be a layered action on S. For $l \in \{1, 2, \ldots, k\}$ let $\mathcal{G}_l$ be as in Definition* 3.9*. Let $l, m \in \{1, 2, \ldots, k\}$, let $f \in \mathcal{G}_l$, and let $g \in \mathcal{G}_m$. Then there is some $t \in \{1, 2, \ldots, k\}$, with $t \leq \min\{l, m\}$, such that $f \circ g \in \mathcal{G}_t \cup \{\overline{e}\}$, where $\overline{e}$ is the function constantly equal to e.*

Proof. For $n \in \{1, 2, \ldots, k\}$, let $T_n = \bigcup_{i=0}^n S_i$. We proceed by downward induction on l. If $l = k$, then $f \circ g = g \in \mathcal{G}_m$, so assume that $l < k$ and the lemma is valid for $l+1$ and m. Pick $r \in \mathcal{G}_{l+1}$ and $h \in \mathcal{F}_{l+1}$ such that $f = h \circ r$. Pick $t \in \{1, 2, \ldots, k\}$, with $t \leq \min\{l+1, m\}$, such that $r \circ g \in \mathcal{G}_t \cup \{\overline{e}\}$.

If $r \circ g = \overline{e}$, then $f \circ g = h \circ r \circ g = \overline{e}$ because h is a partial semigroup homomorphism. So assume that $r \circ g \in \mathcal{G}_t$. Now $h \in \mathcal{F}_{l+1}$ so pick by Lemma 3.4 some $v \in \{2, 3, \ldots, l+1\}$

such that for all $s \in \{v, v+1, \ldots, l+1\}$ $h_{|T_s} \in \mathcal{F}_s$, and either $h_{|T_{v-1}} = \iota_{T_{v-1}}$ or $h[T_{v-1}] = \{e\}$.

We have that $t \leq l+1$. Assume first that $t \geq v$. Then $h_{|T_t} \in \mathcal{F}_t$ and $r \circ g \in \mathcal{G}_t$ so $f \circ g = h \circ r \circ g \in \mathcal{G}_{t-1}$ and $t - 1 \leq \min\{l, m\}$.

Thus we may assume that $t < v$. Then $r \circ g[S] \subseteq T_t \subseteq T_{v-1}$ and either $h_{|T_{v-1}} = \iota_{T_{v-1}}$ or $h[T_{v-1}] = \{e\}$. If $h_{|T_{v-1}} = \iota_{T_{v-1}}$, then $h \circ r \circ g = r \circ g \in \mathcal{G}_t$ and $t \leq v-1 \leq l$ so that $t \leq \min\{l, m\}$. If $h[T_{v-1}] = \{e\}$, then $h \circ r \circ g = \overline{e}$. □

Notice that Lemma 3.10 says in particular that $\{\overline{e}\} \cup \bigcup_{l=1}^{k} \mathcal{G}_l$ is a semigroup under composition.

3.11 Lemma. *Let S be a layered partial semigroup with k layers and let $S_0, S_1, \ldots, S_k$ be as in Definition 3.1. For $n \in \{1, 2, \ldots, k\}$, let $T_n = \bigcup_{i=0}^{n} S_i$. For each $n \in \{2, 3, \ldots, k\}$, let $\mathcal{F}_n$ be a finite set of partial semigroup homomorphisms from T_n into T_{n-1}. Let $\mathcal{G}_k = \{\iota_S\}$ and for $l \in \{1, 2, \ldots, k-1\}$, let $\mathcal{G}_l = \{f \circ g : g \in \mathcal{G}_{l+1}$ and $f \in \mathcal{F}_{l+1}\}$. For $n \in \{1, 2, \ldots, k\}$ let p_n be an idempotent in δS_n such that, if $n \geq 2$, then $p_n \leq p_{n-1}$ and $\widetilde{f}(p_n) = p_{n-1}$ for every $f \in \mathcal{F}_n$.*

- *(a) For each $l \in \{1, 2, \ldots, k\}$, $\mathcal{G}_l$ is a finite set of partial semigroup homomorphisms from S into T_l.*
- *(b) For each $l \in \{1, 2, \ldots, k\}$ and each $h \in \mathcal{G}_l$, $\widetilde{h}(p_k) = p_l$, where $\widetilde{h} : \beta S \to \beta T_l$ is the continuous extension of h.*
- *(c) For $i \in \{1, 2, \ldots, k\}$, let $A_i \in p_i$. Given any $i, j \in \{1, 2, \ldots, k\}$ and any $g \in \mathcal{G}_j$, $\{w \in S_k : g(w)^{-1} A_{\max\{i,j\}} \in p_i\} \in p_k$.*

Proof. The first two conclusions are immediate. To verify conclusion (c), let $i, j \in \{1, 2, \ldots, k\}$.

Assume first that $j \leq i$, in which case $p_i \leq p_j$. In particular, $p_i = p_j \cdot p_i$. Since $A_i \in p_i$, we have that $\{w \in S : w^{-1} A_i \in p_i\} \in p_j$. Since $p_j = \widetilde{g}(p_k)$ by conclusion (b), $\widetilde{g}(p_k) \in \overline{\{w \in S : w^{-1} A_i \in p_i\}}$, so there exists $B \in p_k$ such that $\widetilde{g}[\,\overline{B}\,] \subseteq \overline{\{w \in S : w^{-1} A_i \in p_i\}}$. Then $B \subseteq \{w \in S_k : g(w)^{-1} A_i \in p_i\}$ and so $\{w \in S_k : g(w)^{-1} A_i \in p_i\} \in p_k$ as required.

Now assume that $i < j$, so that $p_j \leq p_i$ and in particular, $p_j = p_j \cdot p_i$. Since $A_j \in p_j$, we have that $\{w \in S : w^{-1} A_j \in p_i\} \in p_j = \widetilde{g}(p_k)$ and thus $\{w \in S_k : g(w)^{-1} A_j \in p_i\} \in p_k$ as required. □

3.12 Lemma. *Let S be a layered partial semigroup with k layers and let $S_0, S_1, \ldots, S_k$ be as in Definition 3.1. For $n \in \{1, 2, \ldots, k\}$, let $T_n = \bigcup_{i=0}^{n} S_i$. For each $n \in \{2, 3, \ldots, k\}$, let $\mathcal{F}_n$ be a finite set of partial semigroup homomorphisms from T_n into T_{n-1}.*

Let $\mathcal{G}_k = \{\iota_S\}$ and for $l \in \{1, 2, \ldots, k-1\}$, let $\mathcal{G}_l = \{f \circ g : g \in \mathcal{G}_{l+1}$ and $f \in \mathcal{F}_{l+1}\}$. For $n \in \{1, 2, \ldots, k\}$ let p_n be an idempotent in δS_n such that, if $n \geq 2$, then $p_n \leq p_{n-1}$ and $\widetilde{f}(p_n) = p_{n-1}$ for every $f \in \mathcal{F}_n$. Let $r \in \mathbb{N}$ and assume that $S = \bigcup_{i=1}^r C_i$. For each $l \in \{1, 2, \ldots, k\}$, pick $\gamma(l) \in \{1, 2, \ldots, r\}$ such that $C_{\gamma(l)} \cap S_l \in p_l$. For each $l \in \{1, 2, \ldots, k\}$ let $\langle B_{l,m}\rangle_{m=1}^\infty$ be a sequence of elements of p_l. Then there exists a sequence $\langle w_n\rangle_{n=1}^\infty$ in S_k such that $\prod_{n\in F} g_n(w_n)$ is defined for each $F \in \mathcal{P}_f(\mathbb{N})$ and each choice of $g_n \in \bigcup_{i=1}^k \mathcal{G}_i$, and for each $l \in \{1, 2, \ldots, k\}$ and each $m \in \mathbb{N}$,

$$\begin{aligned}\{\textstyle\prod_{n\in F} g_n(w_n) : \quad & F \in \mathcal{P}_f(\mathbb{N})\,,\, g_n \in \textstyle\bigcup_{i=1}^l \mathcal{G}_i \text{ for each } n \in F\,, \min F \geq m\,,\\ & \text{and there exists } n \in F \text{ such that } g_n \in \mathcal{G}_l\} \subseteq C_{\gamma(l)} \cap B_{l,m}\,.\end{aligned}$$

Proof. For $i \in \{1, 2, \ldots, k\}$, let $A_{i,1} = C_{\gamma(i)} \cap B_{i,1}$ and note that $A_{i,1} \in p_i$. We inductively construct a sequence $\langle w_n\rangle_{n=1}^\infty$ in S_k and, for each $i \in \{1, 2, \ldots, k\}$, a sequence $\langle A_{i,n}\rangle_{n=1}^\infty$ in p_i such that

(1) for $j \in \{1, 2, \ldots, k\}$, $n \in \mathbb{N}$, and $g \in \mathcal{G}_j$, $g(w_n) \in A_{j,n}$;

(2) for $i, j \in \{1, 2, \ldots, k\}$, $n \in \mathbb{N}$, and $g \in \mathcal{G}_j$, $g(w_n)^{-1}A_{\max\{i,j\},n} \in p_i$; and

(3) for $i \in \{1, 2, \ldots, k\}$ and $n \in \mathbb{N}$,

$$A_{i,n+1} = B_{i,n+1} \cap A_{i,n} \cap \left(\textstyle\bigcap_{j=1}^i \bigcap_{g\in\mathcal{G}_j} g(w_n)^{-1}A_{i,n}\right) \cap \left(\textstyle\bigcap_{j=i}^k \bigcap_{g\in\mathcal{G}_j} g(w_n)^{-1}A_{j,n}\right)\,.$$

So let $n \in \mathbb{N}$ and assume that we have $A_{i,n} \in p_i$ for each $i \in \{1, 2, \ldots, k\}$. By Lemma 3.11($c$), for any $i, j \in \{1, 2, \ldots, k\}$ and any $g \in \mathcal{G}_j$,

$$\{w \in S_k : g(w)^{-1}A_{\max\{i,j\},n} \in p_i\} \in p_k\,.$$

Also, by Lemma 3.11(b), for any $j \in \{1, 2, \ldots, k\}$ and any $g \in \mathcal{G}_j$, $\widetilde{g}(p_k) = p_j$ and so $g^{-1}[A_{j,n}] \in p_k$. Thus we may pick

$$w_n \in \left(\textstyle\bigcap_{i=1}^k \bigcap_{j=1}^k \bigcap_{g\in\mathcal{G}_j} \{w \in S_k : g(w)^{-1}A_{\max\{i,j\},n} \in p_i\}\right) \cap \left(\textstyle\bigcap_{j=1}^k \bigcap_{g\in\mathcal{G}_j} g^{-1}[A_{j,n}]\right)\,.$$

Hypotheses (1) and (2) are satisfied directly. For $i \in \{1, 2, \ldots, k\}$, let $A_{i,n+1}$ be as required by hypothesis (3). By hypothesis (2), $A_{i,n+1} \in p_i$.

The construction being complete, we show by induction on $|F|$ that if $F \in \mathcal{P}_f(\mathbb{N})$, $g : F \to \bigcup_{t=1}^k \mathcal{G}_t$, $a = \min F$, and $l = \max\{t : g[F] \cap \mathcal{G}_t \neq \emptyset\}$, then $\prod_{n\in F} g(n)(w_n) \in A_{l,a}$. Since $A_{l,a} \subseteq A_{l,1} = C_{\gamma(l)}$ and $A_{l,a} \subseteq B_{l,m}$ for all $m \in \{1, 2, \ldots, a\}$, this will suffice.

If $F = \{a\}$, we have by hypothesis (1) that $g(a)(w_a) \in A_{l,a}$ as required. So assume that $|F| > 1$ and the assertion is true for all smaller sets. Let $G = F \backslash \{a\}$, let $b = \min G$, and let $m = \max\{t : g[G] \cap \mathcal{G}_t \neq \emptyset\}$. By assumption $\prod_{n\in G} g(n)(w_n) \in A_{m,b}$. Pick $j \in \{1, 2, \ldots, k\}$ such that $g(a) \in \mathcal{G}_j$, and note that $l = \max\{m, j\}$.

Assume first that $l = m$ (so that $m \geq j$). Then

$$\prod_{n\in G} g(n)(w_n) \in A_{m,a+1} \subseteq \big(g(a)(w_a)\big)^{-1} A_{m,a}$$

and thus $\prod_{n\in F} g(n)(w_n) \in A_{m,a} = A_{l,a}$.

Now assume that $l = j$ (so that $m \leq j$). Then

$$\prod_{n\in G} g(n)(w_n) \in A_{m,a+1} \subseteq \big(g(a)(w_a)\big)^{-1} A_{j,a}$$

and thus $\prod_{n\in F} g(n)(w_n) \in A_{j,a} = A_{l,a}$. □

The following is the main result of this section.

3.13 Theorem. *Let S be a layered partial semigroup with k layers, let $S_0, S_1, \ldots, S_k$ be as in Definition 3.1, and let $\langle \mathcal{F}_n\rangle_{n=2}^k$ be a layered action on S. Let $\mathcal{G}_1, \mathcal{G}_2, \ldots, \mathcal{G}_k$ be as in Definition 3.9. Let $r \in \mathbb{N}$ and assume that $S = \bigcup_{i=1}^r C_i$. Then there exists $\gamma : \{1, 2, \ldots, k\} \to \{1, 2, \ldots, r\}$ and a sequence $\langle w_n\rangle_{n=1}^\infty$ in S_k such that $\prod_{n\in F} g_n(w_n)$ is defined for each $F \in \mathcal{P}_f(\mathbb{N})$ and each choice of $g_n \in \bigcup_{i=1}^k \mathcal{G}_k$ and for each $l \in \{1, 2, \ldots, k\}$,*

$$\begin{aligned}\{\textstyle\prod_{n\in F} g_n(w_n) : \quad & F \in \mathcal{P}_f(\mathbb{N}),\ g_n \in \textstyle\bigcup_{i=1}^l \mathcal{G}_i \text{ for each } n \in F, \\ & \text{and there exists } n \in F \text{ such that } g_n \in \mathcal{G}_l\} \subseteq C_{\gamma(l)}.\end{aligned}$$

Further, for each $i \in \{1, 2, \ldots, k\}$, $C_{\gamma(i)} \cap S_i$ is central in S_i and $\gamma(1)$ can be any $i \in \{1, 2, \ldots, r\}$ such that $C_i \cap S_1$ is central in S_1.

Proof. Pick $\gamma(1) \in \{1, 2, \ldots, r\}$ such that $C_{\gamma(1)} \cap S_1$ is central in S_1 and pick an idempotent p minimal in δS_1 such that $C_{\gamma(1)} \in p$. For $i \in \{1, 2, \ldots, k\}$ pick p_i as guaranteed by Theorem 3.8. The result now follows from Lemma 3.12. □

We illustrate an application by proving Theorem 1.1 from the introduction.

3.14 Corollary. *For every $r \in \mathbb{N}$ and every partition $W(\{a, b, c\}) = \bigcup_{j=1}^r C_j$ there exist an infinite $\langle x_n\rangle_{n=1}^\infty$ in $W(\{a, b, c\}) \setminus W(\{a, b\})$ and $\gamma\colon \{a, b, c\} \to \{1, 2, \ldots, r\}$ such that if $\sigma \in \{f_{eab}, f_{aeb}, f_{aab}\}$ and $\mathcal{F} = \{f_{abc}, f_{abb}, f_{aba}, f_{abe}, \sigma\} \cup \big\{f_{xyz} | x, y, z \in \{a, e\}\big\}$, then we have*

$$\begin{aligned}\{\textstyle\prod_{n\in F} g_n(x_n) : F \in \mathcal{P}_f(\mathbb{N}),\ \text{and for each } n \in F,\ g_n \in \mathcal{F}\} & \\ \cap \big(W(\{a, b, c\}) \setminus W(\{a, b\})\big) & \subseteq C_{\gamma(a)}\end{aligned}$$

$$\begin{aligned}\{\textstyle\prod_{n\in F} g_n(x_n) : F \in \mathcal{P}_f(\mathbb{N}),\ \text{and for each } n \in F,\ g_n \in \mathcal{F}\} & \\ \cap \big(W(\{a, b\}) \setminus W(\{a\})\big) & \subseteq C_{\gamma(b)}\end{aligned}$$

$$\begin{aligned}\{\textstyle\prod_{n\in F} g_n(x_n) : F \in \mathcal{P}_f(\mathbb{N}),\ \text{and for each } n \in F,\ g_n \in \mathcal{F}\} & \\ \cap \big(W(\{a\}) \setminus \{e\}\big) & \subseteq C_{\gamma(c)}\end{aligned}$$

Proof. Let $S = W(\{a,b,c\})$, $S_0 = \{e\}$, $S_1 = W(\{a\})\backslash\{e\}$, $S_2 = W(\{a,b\}) \setminus W(\{a\})$, and $S_3 = S\backslash W(\{a,b\})$. Then S is a layered semigroup. Let $\mathcal{F}_2 = \{f_{aae|W(\{a,b\})}, f_{aee|W(\{a,b\})}, f_{eab|W(\{a,b\})}\}$ and let $\mathcal{F}_3 = \{f_{abb}, f_{aba}, f_{abe}, \sigma\}$.

We claim that $\langle \mathcal{F}_n \rangle_{n=2}^3$ is a layered action on S. Trivially $f_{aae|W(\{a\})}$ and $f_{aee|W(\{a\})}$ are equal to the identity on $W(\{a\})$ and $f_{abb|W(\{a,b\})}$, $f_{aba|W(\{a,b\})}$, and $f_{abe|W(\{a,b\})}$ are equal to the identity on $W(\{a,b\})$. Also, f_{eab} is a shift on S, $f_{eab|W(\{a,b\})}$ is a shift on $W(\{a,b\})$, and $f_{eab}[W(\{a\})] = \{e\}$. Finally, f_{aeb} and f_{aab} are shifts on S, $f_{aeb|W(\{a,b\})} = f_{aee|W(\{a,b\})} \in \mathcal{F}_2$, and $f_{aab|W(\{a,b\})} = f_{aae|W(\{a,b\})} \in \mathcal{F}_2$.

It is easily checked that $\mathcal{G}_1$, $\mathcal{G}_2$ and $\mathcal{G}_3$ defined in Definition 3.9 satisfy $\mathcal{F}\backslash\{f_{eee}\} = \mathcal{G}_1 \cup \mathcal{G}_2 \cup \mathcal{G}_3$. Thus the conclusion follows by Theorem 3.13 and the fact that for any $w \in S$, $f_{eee}(w) = e$. $\square$

Notice that by Lemma 3.5, Theorem 1.3 is a corollary to Theorem 3.13, and by Lemma 3.6, Theorem 1.4 is a corollary to Theorem 3.13. The reader is invited to amuse herself by seeing what sorts of configurations can be guaranteed to be monochromatic in the free semigroup on k letters. As an illustration of the process, we derive the Hales–Jewett Theorem [7].

Recall that, given an alphabet Σ, a variable word over Σ is a word over the alphabet $\Sigma \cup \{v\}$ in which v actually occurs, where v is a "variable" which is not a member of Σ. Given a variable word w and $a \in \Sigma$, $w(a)$ is the result of substituting a for each occurrence of v.

3.15 Corollary (Hales–Jewett). *Let Σ be a finite nonempty alphabet, let R be the free semigroup over Σ, let $r \in \mathbb{N}$, and let $R = \bigcup_{i=1}^r C_i$. Then there exist $i \in \{1,2,\ldots,r\}$ and a variable word u over Σ such that $\{u(a) : a \in \Sigma\} \subseteq C_i$.*

Proof. We may presume that we have $k \in \mathbb{N}\backslash\{1\}$ such that $\Sigma = \{a_1, a_2, \ldots, a_{k-1}\}$. Let S be the free semigroup with identity e over $\{a_1, a_2, \ldots, a_k\}$. Let $S_0 = \{e\}$, let $S_1 = R$, and let $S_2 = \{w \in S : a_k \text{ occurs in } w\}$. Let $T_2 = S = S_0 \cup S_1 \cup S_2$ and let $T_1 = S_0 \cup S_1$. For $i \in \{1,2,\ldots,k-1\}$ define a homomorphism $f_i : T_2 \to T_1$ by

$$f_i(a_t) = \begin{cases} a_i & \text{if } t = k \\ a_t & \text{if } t < k\,. \end{cases}$$

Let $\mathcal{F}_2 = \big\{f_i : i \in \{1,2,\ldots,k-1\}\big\}$.

Then S is a layered semigroup with two layers. Let $C_0 = S_0$ and let $C_{r+1} = S_2$ (or divide S_2 up any way you please). Choose $\gamma : \{1,2\} \to \{0,1,\ldots,r+1\}$ and a sequence $\langle w_n \rangle_{n=1}^\infty$ in S_2 as guaranteed by Theorem 3.13. Define a variable word u

over Σ by replacing all occurrences of a_k in w_1 by v. Then $\big\{u(a_t) : t \in \{1, 2, \ldots, k-1\}\big\} = \{f(w_1) : f \in \mathcal{F}_2\} \subseteq C_{\gamma(1)}$. □

In fact, we also get a significant strengthening of the Hales–Jewett Theorem as a corollary to Theorem 3.13. (This result has probably not been previously stated, although it is derivable as a consequence of the noncommutative Central Sets Theorem [2, Theorem 2.8].)

3.16 Corollary. *Let Σ be a finite nonempty alphabet, let R be the free semigroup over Σ, let $r \in \mathbb{N}$, and let $R = \bigcup_{i=1}^{r} C_i$. Then there exist $i \in \{1, 2, \ldots, r\}$ and a sequence of variable words $\langle u_n \rangle_{n=1}^{\infty}$ over Σ such that for every $F \in \mathcal{P}_f(\mathbb{N})$ and every $h : F \to \Sigma$, $\prod_{n \in F} u_n\big(h(n)\big) \in C_i$.*

Proof. Let k, S, S_0, S_1, S_2, T_1, T_2, $f_1, f_2, \ldots, f_{k-1}$, $\mathcal{F}_2$, C_0, C_{r+1}, γ, and $\langle w_n \rangle_{n=1}^{\infty}$ be as in the proof of Corollary 3.15. For each $n \in \mathbb{N}$, define a variable word u_n by replacing each occurrence of a_k in w_n by the variable v.

Now let $F \in \mathcal{P}_f(\mathbb{N})$ and let $h : F \to \Sigma$. For $n \in F$, let $g_n = f_j$, where $h(n) = a_j$. Then $\prod_{n \in F} u_n\big(h(n)\big) = \prod_{n \in F} g_n(w_n) \in C_{\gamma(1)}$. □

4. Restrictions on Shifts

Requirement (3) of Definition 3.2 is of a more esoteric character than the other requirements in Definitions 3.1 and 3.2, and it would be nice if it could be eliminated. We see now that it cannot, given that we want the conclusion of Theorem 3.13 to hold, or even the weakened version which does not require that the chosen cells be central.

4.1 Theorem. *There exist a layered semigroup S with 2 layers, and a set $\mathcal{F}_2$ such that $\langle \mathcal{F}_n \rangle_{n=2}^{2}$ would be a layered action on S if requirement (3) of Defintition 3.2 were eliminated, for which the conclusion of Theorem 3.13 fails.*

Proof. Let $S = W(\{1, 2, 3\})$, $S_0 = \{e\}$, $S_1 = W(\{1, 2\}) \backslash \{e\}$, and $S_2 = S \backslash W(\{1, 2\})$ Define homomorphisms σ and f on S by $\sigma(1) = \sigma(2) = e$, $\sigma(3) = 1$, $f(1) = 1$, and $f(2) = f(3) = 2$. Let $\mathcal{F}_2 = \{f, \sigma\}$. It is routine to verify that S is a layered partial semigroup with 2 layers and that $\langle \mathcal{F}_n \rangle_{n=2}^{2}$ would be a layered action on S if requirement (3) of Definition 3.2 were eliminated.

Every word in $f[S_2]$ has at least one letter equal to 2, while all words in $\sigma[S_2]$ consist only of 1's. Thus the sets $f[S_2]$ and $\sigma[S_2]$ are disjoint, and this clearly implies that the conclusion of Theorem 3.13 fails. □

In our results about layered partial semigroups, it is striking how differently conditions (1)(a) and (1)(b) of Definition 3.3 are treated. It would seem far more natural to simply require that each $f \in \mathcal{F}_n$ satisfy either condition (1)(a) or condition (1)(b). However, given that our goal is Theorem 3.13, this is not possible. In fact not only cannot one allow two of the functions in $\mathcal{F}_n$ to satisfy only condition (1)(b), but indeed one cannot use the same choice of colors for the semigroup layered via two such choices. (See also Question 6.6 and the paragraph following it.)

4.2 Theorem. *Let $k \in \mathbb{N}\setminus\{1\}$. There exist a layered semigroup $(S,+)$ with k layers, sets C_1 and C_2, and functions $\sigma : S \to S$ and $\sigma' : S \to S$ such that*

(1) if for each $n \in \{2,3,\ldots,k\}$, $\mathcal{F}_n = \{\sigma_{|\cup_{i=1}^n S_i}\}$, then $\langle \mathcal{F}_n\rangle_{n=2}^k$ is a layered action on S;

(2) if for each $n \in \{2,3,\ldots,k\}$, $\mathcal{F}'_n = \{\sigma'_{|\cup_{i=1}^n S_i}\}$, then $\langle \mathcal{F}'_n\rangle_{n=2}^k$ is a layered action on S;

(3) $S = C_1 \cup C_2$;

(4) if $\mathcal{G}_1, \mathcal{G}_2, \ldots, \mathcal{G}_k$ are as in Definition 3.9 for the layered action $\mathcal{F}_n = \{\sigma_{|\cup_{i=1}^n S_i}\}$, $\mathcal{G}'_1, \mathcal{G}'_2, \ldots, \mathcal{G}'_k$ are as in Definition 3.9 for the layered action $\mathcal{F}'_n = \{\sigma_{|\cup_{i=1}^n S_i}\}$, $\gamma : \{1,2,\ldots,k\} \to \{1,2\}$, $\gamma' : \{1,2,\ldots,k\} \to \{1,2\}$, and $\langle w_n\rangle_{n=1}^\infty$ is a sequence in S_k such that

(a) for each $l \in \{1,2,\ldots,k\}$,

$$\{\textstyle\sum_{n\in F} g_n(w_n) : \ F \in \mathcal{P}_f(\mathbb{N}),\ g_n \in \bigcup_{i=1}^l \mathcal{G}_i \text{ for each } n \in F, \text{ and there exists } n \in F \text{ such that } g_n \in \mathcal{G}_l\} \subseteq C_{\gamma(l)}\,,$$

and

(b) for each $l \in \{1,2,\ldots,k\}$,

$$\{\textstyle\sum_{n\in F} g_n(w_n) : \ F \in \mathcal{P}_f(\mathbb{N}),\ g_n \in \bigcup_{i=1}^l \mathcal{G}'_i \text{ for each } n \in F, \text{ and there exists } n \in F \text{ such that } g_n \in \mathcal{G}'_l\} \subseteq C_{\gamma'(l)}\,;$$

then for all $t, l \in \{1,2,\ldots,k-1\}$, $\gamma(t) = \gamma(l) \neq \gamma'(t) = \gamma'(l)$.

Proof. Let $S = W(\{1,2,\ldots,2k\})$. Let $S_0 = \{e\}$, $S_1 = W(\{1,2\})\setminus\{e\}$, and for $n \in \{2,3,\ldots,k\}$, let $S_n = W(\{1,2,\ldots,2n\})\setminus W(\{1,2,\ldots,2n-2\})$. Define homomorphisms $\sigma : S \to S$ and $\sigma' : S \to S$ by agreeing for $i \in \{1,2,\ldots,2k\}$ that

$$\sigma(i) = \begin{cases} i-2 & \text{if } i > 2 \\ e & \text{if } i \in \{1,2\} \end{cases} \text{ and}$$

$$\sigma'(i) = \begin{cases} 2k-3 & \text{if } i = 2k \\ 2k-2 & \text{if } i = 2k-1 \\ i-2 & \text{if } 2 < i < 2k-1 \\ e & \text{if } i \in \{1,2\} \end{cases}.$$

For each $w \in S\backslash\{e\}$, let $\mu(w) = n$ where $w \in S_n$. Let $C_1 = \{w \in S : 2\mu(w) - 1$ occurs in w before any occurrence of $2\mu(w)\}$ and let $C_2 = S\backslash C_1$.

It is routine to verify that $\langle \mathcal{F}_n \rangle_{n=2}^k$ and $\langle \mathcal{F}'_n \rangle_{n=2}^k$ are layered actions on S (via in each case condition (1)(b) of Definition 3.3). Let γ, γ', and $\langle w_n \rangle_{n=1}^\infty$ be as in conclusion (4). Assume without loss of generality that each $w_n \in C_1$. Then for any $g \in \bigcup_{i=1}^{k-1} \mathcal{G}_i$ one has $g(w_n) \in C_1$ and thus $\gamma(l) = 1$ for each $l \in \{1, 2, \ldots, k-1\}$. Also for any $g \in \bigcup_{i=1}^{k-1} \mathcal{G}'_i$ one has $g(w_n) \in C_2$ and thus $\gamma(l) = 2$ for each $l \in \{1, 2, \ldots, k-1\}$. □

As we have just seen, Theorem 3.13 cannot be extended by adding another shift (Theorem 4.2) or by relaxing the requirements on σ (Theorem 4.1). Theorem 4.4 characterizes exactly when $\mathcal{F}_2$ may be taken to be a specified set of homomorphisms in the case when $k = 2$ and S is countable while Theorem 4.5 provides a simpler description and removes the countability assumption in the event that S is a semigroup. The proof of the following lemma repeats a portion of the proof of Theorem 3.8 which was extracted from [5].

4.3 Lemma. *Let S be an adequate partial semigroup, and let $\mathcal{H}$ be a finite set of partial semigroup homomorphisms from S into itself such that $f[S]$ is an adequate partial subsemigroup of S for each $f \in \mathcal{H}$. Let p be an idempotent in δS such that $\widetilde{f}(p) \cdot p = p$ for every $f \in \mathcal{H}$, where $\widetilde{f} : \beta S \to \beta S$ is the continuous extension of f.*

(a) If there exists $q \in \delta S$ such that $\widetilde{f}(q) = p$ for every $f \in \mathcal{H}$, then there exists an idempotent $r \in \delta S$ such that $r \leq p$ and $\widetilde{f}(r) = p$ for every $f \in \mathcal{H}$.

(b) Assume that X is a compact subsemigroup of δS such that both $p \cdot X$ and $X \cdot p$ are included in X. If there exists $q \in X$ such that $\widetilde{f}(q) = p$ for every $f \in \mathcal{H}$, then there exists an idempotent $r \in X$ such that $r \leq p$ and $\widetilde{f}(r) = p$ for every $f \in \mathcal{H}$.

Proof. Notice that for each $f \in \mathcal{H}$, $\widetilde{f}_{|\delta S}$ is a homomorphism by Lemma 2.14 and the assumption that $f[S]$ is an adequate partial subsemigroup of S. Since (a) is a special case of (b), when $X = \delta S$, we shall prove only (b). Let $M = \{q \in \delta S : \widetilde{f}(q) = p$ for every $f \in \mathcal{H}\}$. By assumption $M \cap X \neq \emptyset$. Since $\widetilde{f}_{|\delta S}$ is a homomorphism for each $f \in \mathcal{H}$, we have that M is a compact semigroup. Thus $M \cap X$ is also a compact semigroup.

We claim that $(M \cap X) \cdot p$ is a subsemigroup of δS. Fix q' and r' in $(M \cap X) \cdot p$, and let $q, r \in M \cap X$ be such that $q' = q \cdot p$ and $r' = r \cdot p$. We need to prove that $q \cdot p \cdot r \in M \cap X$. Since $X \cdot p \subseteq X$ and $p \cdot X \subseteq X$, we have $q \cdot p \cdot r = q \cdot p \cdot p \cdot r \in X \cdot X \subseteq X$. Thus it remains to prove $q \cdot p \cdot r \in M$. Given $f \in \mathcal{H}$, we have that $\widetilde{f}(q \cdot p \cdot r) = \widetilde{f}(q) \cdot \widetilde{f}(p) \cdot \widetilde{f}(r) = p \cdot \widetilde{f}(p) \cdot p$ and $\widetilde{f}(p) \cdot p = p$ so that $\widetilde{f}(q \cdot p \cdot r) = p$. Thus $q \cdot p \cdot r \in M$, and $(M \cap X) \cdot p$ is a subsemigroup of δS. Note that it is automatically a subsemigroup of X, since $X \cdot p \subseteq X$.

Since $(M \cap X) \cdot p$ is compact, pick an idempotent $q \in (M \cap X) \cdot p$ and notice that $q \cdot p = q$.

Let $r = p \cdot q$. Then $r \in p \cdot X \subseteq X$ and $r \cdot p = p \cdot r = r$. Also, $r \cdot r = p \cdot q \cdot p \cdot q = p \cdot q \cdot q = p \cdot q = r$. Thus r is an idempotent in δS and $r \leq p$. Finally, let $f \in \mathcal{H}$. Then $\widetilde{f}(r) = \widetilde{f}(p \cdot q) = \widetilde{f}(p) \cdot \widetilde{f}(q) = \widetilde{f}(p) \cdot p = p$. $\square$

4.4 Theorem. *Let S be a layered partial semigroup with 2 layers and let S_0, S_1, and S_2 be as in Definition 3.1. Let $\mathcal{F}$ be a finite nonempty set of partial semigroup homomorphisms from S to $S_0 \cup S_1$ with the property that for each $f \in \mathcal{F}$, either $f_{|S_1} = \iota_{S_1}$ or $f[S_1] = \{e\}$. Let $\mathcal{G}_2 = \{\iota_S\}$, and let $\mathcal{G}_1 = \mathcal{F}$. Statements (2) and (3) are equivalent and are implied by statement (1). If S_2 is countable, then all three statements are equivalent.*

(1) Whenever $r \in \mathbb{N}$, $S \subseteq \bigcup_{i=1}^r C_i$, and $J_1 \in \mathcal{P}_f(S_1)$, there exist a function $\gamma : \{1,2\} \to \{1,2,\ldots,r\}$ and a sequence $\langle w_n \rangle_{n=1}^\infty$ in S_2 such that

(a) $\prod_{n \in F} g_n(w_n)$ is defined for each $F \in \mathcal{P}_f(\mathbb{N})$ and each choice of $g_n \in \bigcup_{i=1}^2 \mathcal{G}_i$;

(b) $\{\prod_{n \in F} g_n(w_n) : F \in \mathcal{P}_f(\mathbb{N})$ and for each $n \in F$, $g_n \in \mathcal{G}_1\} \subseteq C_{\gamma(1)} \cap \bigcap_{x \in J_1} \varphi_{S_1}(x)$; and

(c) for each $J_2 \in \mathcal{P}_f(S_2)$, there exists $m \in \mathbb{N}$ such that

$$\{\textstyle\prod_{n \in F} g_n(w_n) : \ F \in \mathcal{P}_f(\mathbb{N}), \min F \geq m, g_n \in \mathcal{G}_1 \cup \mathcal{G}_2 \text{ for each } n \in F, \text{ and there exists } n \in F \text{ such that } g_n \in \mathcal{G}_2\} \subseteq C_{\gamma(2)} \cap \bigcap_{x \in J_2} \varphi_{S_2}(x)\,.$$

(2) Whenever $r \in \mathbb{N}$, $J_1 \in \mathcal{P}_f(S_1)$, and $\bigcap_{x \in J_1} \varphi_{S_1}(x) \subseteq \bigcup_{i=1}^r A_i$, there exists $i \in \{1,2,\ldots,r\}$ such that for every $J_2 \in \mathcal{P}_f(S_2)$

$$\textstyle\bigcap_{y \in J_2} \varphi_{S_2}(y) \cap \bigcap_{f \in \mathcal{F}} f^{-1}[A_i] \neq \emptyset\,.$$

(3) There exist idempotents p_1 in δS_1 and p_2 in δS_2 such that $p_2 \leq p_1$ and $\widetilde{f}(p_2) = p_1$ for each $f \in \mathcal{F}$.

Proof. To see that (1) implies (2), let $r \in \mathbb{N}$ and $J_1 \in \mathcal{P}_f(S_1)$ be given and assume that $\bigcap_{x \in J_1} \varphi_{S_1}(x) \subseteq \bigcup_{i=1}^r A_i$. Let $A_{r+1} = S \setminus \bigcup_{i=1}^r A_i$. Pick a function $\gamma : \{1,2\} \to \{1,2,\ldots,r+1\}$ and a sequence $\langle w_n \rangle_{n=1}^\infty$ as guaranteed by (1) and the fact that $S \subseteq \bigcup_{i=1}^{r+1} A_i$. Let $i = \gamma(1)$ and pick $g \in \mathcal{F}$. Since $g(w_1) \in A_{\gamma(1)} \cap \bigcap_{x \in J_1} \varphi_{S_1}(x)$, we have that $i \neq r+1$. Now let $J_2 \in \mathcal{P}_f(S_2)$ and pick $m \in \mathbb{N}$ as guaranteed by (1)(c). Then $w_m \in \bigcap_{y \in J_2} \varphi_{S_2}(y) \cap \bigcap_{f \in \mathcal{F}} f^{-1}[A_i]$.

To see that (2) implies (3), let

$$\mathcal{R} = \{A \subseteq S_1 : \text{ there exists } J_2 \in \mathcal{P}_f(S_2) \text{ with } \textstyle\bigcap_{y \in J_2} \varphi_{S_2}(y) \cap \bigcap_{f \in \mathcal{F}} f^{-1}[A] = \emptyset\}$$

and let $\mathcal{A} = \{S_1 \backslash A : A \in \mathcal{R}\} \cup \{\varphi_{S_1}(x) : x \in S_1\}$. We claim that $\mathcal{A}$ has the finite intersection property. To see this, let $\{A_1, A_2, \ldots, A_r\} \subseteq \mathcal{R}$ and let $J_1 \in \mathcal{P}_f(S_1)$. If we had $\bigcap_{i=1}^r (S_1 \backslash A_i) \cap \bigcap_{x \in J_1} \varphi_{S_1}(x) = \emptyset$ we would have $\bigcap_{x \in J_1} \varphi_{S_1}(x) \subseteq \bigcup_{i=1}^r A_i$ so that, by (2), there would be some $i \in \{1, 2, \ldots, r\}$ with $A_i \notin \mathcal{R}$.

Since $\mathcal{A}$ has the finite intersection property, there is some $p \in \beta S_1$ such that $\mathcal{A} \subseteq p$. Since $\{\varphi_{S_1}(x) : x \in S_1\} \subseteq \mathcal{A}$, one has that any such p is in δS_1. Let $X = \{p \in \delta S_1 : p \cap \mathcal{R} = \emptyset\}$. We have just seen that $X \neq \emptyset$. Further X is trivially compact.

We claim that X is a subsemigroup of δS_1. To see this, let $p, q \in X$ and let $A \in p \cdot q$. We need to show that $A \notin \mathcal{R}$. To this end, let $J_2 \in \mathcal{P}_f(S_2)$. We need to show that $\bigcap_{y \in J_2} \varphi_{S_2}(y) \cap \bigcap_{f \in \mathcal{F}} f^{-1}[A] \neq \emptyset$. Let $B = \{x \in S_1 : x^{-1}A \in q\}$. Then $B \in p$ so $B \notin \mathcal{R}$ so pick $a \in \bigcap_{y \in J_2} \varphi_{S_2}(y) \cap \bigcap_{f \in \mathcal{F}} f^{-1}[B]$. Let $C = \bigcap_{f \in \mathcal{F}} f(a)^{-1}A$. Then $C \in q$ so $C \notin \mathcal{R}$ so pick $b \in \bigcap_{y \in J_2} \varphi_{S_2}(y \cdot a) \cap \bigcap_{f \in \mathcal{F}} f^{-1}[C]$. Then $a \cdot b \in \bigcap_{y \in J_2} \varphi_{S_2}(y) \cap \bigcap_{f \in \mathcal{F}} f^{-1}[A]$.

Since X is a compact right topological semigroup, pick an idempotent $p_1 \in X$. Further, given $f \in \mathcal{F}$, either $f_{|S_1} = \iota_{S_1}$ or $f[S_1] = S_0$ so that either $\widetilde{f}(p_1) = p_1$ or $\widetilde{f}(p_1) = e$. In either case, $\widetilde{f}(p_1) \cdot p_1 = p_1$.

Since Lemma 2.16 implies that δS_2 is an ideal of δS, by Lemma 4.3 it suffices to show that there exists $q \in \delta S_2$ such that $\widetilde{f}(q) = p_1$ for every $f \in \mathcal{F}$. For this, it suffices to show that

$$\mathcal{B} = \{f^{-1}[A] : A \in p_1 \text{ and } f \in \mathcal{F}\} \cup \{\varphi_{S_2}(y) : y \in S_2\}$$

has the finite intersection property. For this, it in turn suffices to let $A \in p_1$, let $J \in \mathcal{P}_f(S_2)$, and show that $\bigcap_{y \in J} \varphi_{S_2}(y) \cap \bigcap_{f \in \mathcal{F}} f^{-1}[A] \neq \emptyset$. But this is precisely the assertion that $p_1 \cap \mathcal{R} = \emptyset$.

To see that (3) implies (2), let $r \in \mathbb{N}$, $J_1 \in \mathcal{P}_f(S_1)$, and $\bigcap_{x \in J_1} \varphi_{S_1}(x) \subseteq \bigcup_{i=1}^r A_i$. Pick $i \in \{1, 2, \ldots, r\}$ such that $A_i \in p_1$ and let $J_2 \in \mathcal{P}_f(S_2)$. Then for each $f \in \mathcal{F}$, $f^{-1}[A_i] \in p_2$ and $p_2 \in \delta S_2$ so

$$\bigcap_{y \in J_2} \varphi_{S_2}(y) \cap \bigcap_{f \in \mathcal{F}} f^{-1}[A_i] \in p_2 \,.$$

Finally, assume that S_2 is countable. We show that (3) implies (1). Enumerate S_2 as $\langle x_n \rangle_{n=1}^\infty$ (with repetition in the somewhat boring event that S_2 is finite) and for each $m \in \mathbb{N}$, let $B_{2,m} = \bigcap_{i=1}^m \varphi_{S_2}(x_i)$. (Since $p_2 \in \delta S_2$, we have each $B_{2,m} \in p_2$.) Let $J_1 \in \mathcal{P}_f(S_1)$ and for each $m \in \mathbb{N}$, let $B_{1,m} = \bigcap_{x \in J_1} \varphi_{S_1}(x)$.

Let $r \in \mathbb{N}$ and let $S \subseteq \bigcup_{i=1}^r C_i$. Pick a sequence $\langle w_n \rangle_{n=1}^\infty$ in S_2 and $\gamma : \{1, 2\} \to \{1, 2, \ldots, r\}$ as guaranteed by Lemma 3.12. To see that conclusion (1)(c) holds, let

$J_2 \in \mathcal{P}_f(S_2)$ be given and pick $m \in \mathbb{N}$ such that $J_2 \subseteq \{x_1, x_2, \ldots, x_m\}$. Then $B_{2,m} \subseteq \bigcap_{x \in J_2} \varphi_{S_2}(x)$. □

The requirement in Theorem 4.4 (as well as in Theorem 4.5 below) that all maps in $\mathcal{F}$ are either equal to the identity on S_1 or send S_1 into $\{e\}$ may seem unnatural, but some form of this requirement is necessary in order to have (1) (see Theorem 6.4). By Theorem 3.13, (1) is true if at most one $f \in \mathcal{F}$ sends S_1 into $\{e\}$. It is thus natural to ask whether we can draw the same conclusion if we have more than one such map? By Theorem 4.2, not always. See also Question 6.6 and the remarks following it.

While Theorem 4.4 may seem a bit technical, we have a considerably simpler situation in the event that S is a semigroup. Note that statement (2) resembles the statement of the Hales–Jewett Theorem (Corollary 3.15) and is apparently much weaker than statement (1).

4.5 Theorem. *Let S be a layered semigroup with 2 layers and let S_0, S_1, and S_2 be as in Definition 3.1. Let $\mathcal{F}$ be a finite nonempty set of homomorphisms from S to $S_0 \cup S_1$ with the property that for each $f \in \mathcal{F}$, either $f_{|S_1} = \iota_{S_1}$ or $f[S_1] = \{e\}$. Let $\mathcal{G}_2 = \{\iota_S\}$, and let $\mathcal{G}_1 = \mathcal{F}$. The following statements are equivalent.*

(1) Whenever $r \in \mathbb{N}$ and $S \subseteq \bigcup_{i=1}^{r} C_i$ there exist a function $\gamma : \{1,2\} \to \{1,2,\ldots,r\}$ and a sequence $\langle w_n \rangle_{n=1}^{\infty}$ in S_2 such that

(a) $\{\prod_{n\in F} g_n(w_n) : F \in \mathcal{P}_f(\mathbb{N})$ and for each $n \in F$, $g_n \in \mathcal{G}_1\} \subseteq C_{\gamma(1)}$ and

(b) $\{\prod_{n\in F} g_n(w_n) : F \in \mathcal{P}_f(\mathbb{N}), g_n \in \mathcal{G}_1 \cup \mathcal{G}_2$ for each $n \in F$, and there exists $n \in F$ such that $g_n \in \mathcal{G}_2\} \subseteq C_{\gamma(2)}$.

(2) Whenever $r \in \mathbb{N}$ and $S_1 \subseteq \bigcup_{i=1}^{r} A_i$, there exists $i \in \{1,2,\ldots,r\}$ such that $\bigcap_{f\in\mathcal{F}} f^{-1}[A_i] \neq \emptyset$.

(3) There exist idempotents p_1 in βS_1 and p_2 in βS_2 such that $p_2 \leq p_1$ and $\widetilde{f}(p_2) = p_1$ for each $f \in \mathcal{F}$.

Proof. That (2) and (3) are equivalent and implied by (1) follows from Theorem 4.4. That (3) implies (1) follows from Lemma 3.12, taking each $B_{2,m} = S_2$ and each $B_{1,m} = S_1$. □

The above result can be used to prove Theorem 1.4. Then all $f \in \mathcal{F}$ are such that their restriction to S_1 is equal to the identity, and the requirement (2) can be proved by using the Hales–Jewett Theorem.

Let us state a variant of Gowers' theorem for semigroups in which the layers are not being fixed in advance. Note that the requirement imposed on S, namely that there is an identity and $ab = e$ if and only if $a = b = e$, is true in many of the cases interesting from the point of view of Ramsey theory.

4.6 Theorem. *Assume S is a semigroup with identity e, and that $ab = e$ implies $a = b = e$ for all $a, b \in S$. Assume further that $\sigma: S \to S$ is a homomorphism such that for some $k \in \mathbb{N}$ we have $\sigma^k(x) = e$ for all $x \in S$, yet $\sigma^{k-1}(a) \neq e$ for some $a \in S$. Then for every partition $S = \bigcup_{j=1}^{r} C_j$ there exist a sequence $\langle y_n \rangle_{n=1}^{\infty}$ and $\gamma: \{1, 2, \ldots, k\} \to \{1, 2, \ldots, r\}$ such that*

(i) *$\sigma^{k-1}(y_n) \neq e$, for all n, and*

(ii) *for every $F \in \mathcal{P}_f(\mathbb{N})$ and every $g: F \to \{0, 1, \ldots, k-1\}$ we have*

$$\prod_{i \in F} \sigma^{g(i)}(y_i) \in C_{\gamma(k-m)},$$

where $m = \min\{g(i) : i \in F\}$.

Proof. For $i \in \{0, 1, \ldots, k\}$, let $T_i = \{a \in S : \sigma^i(a) = e\}$, let $S_0 = T_0$, and for $i \in \{1, 2, \ldots, k\}$, let $S_i = T_i \backslash T_{i-1}$. Each T_n is clearly a subsemigroup of S, and by the assumption $T_k = S$ and $S_k \neq \emptyset$. Since $S_k \neq \emptyset$ and $\sigma[S_n] = S_{n-1}$ for each $n \in \{1, 2, \ldots, k\}$, we have that each $S_n \neq \emptyset$. We claim that S_n is an ideal of T_n for all n. Let $a \in S_n$ and $b \in T_n$, and pick m such that $ab \in S_m$. Note that since T_n is a semigroup, $m \leq n$. Then $\sigma^m(a)\sigma^m(b) = e$, and by the assumption on S we have $\sigma^m(a) = e$. Therefore $m = n$ and $ab \in S_n$. The proof that $ba \in S_n$ is analogous.

For each $n \in \{2, e, \ldots, k\}$, let $\mathcal{F}_n = \{\sigma_{|T_n}\}$. For each $n \in \{1, 2, \ldots, k\}$, if $\mathcal{G}_n$ is as in Lemma 3.12, then $\mathcal{G}_n = \{\widetilde{\sigma}^{k-n}\}$.

Let $X_k = \beta S_k$ and $X_{n-1} = \widetilde{\sigma}[X_n]$ for $n \in \{2, 3, \ldots, k\}$, where $\widetilde{\sigma} : \beta S \to \beta S$ is the continuous extension of σ. Since σ is a homomorphism, so is $\widetilde{\sigma}$ by [9, Corollary 4.22], and thus each X_n is a compact semigroup. Note that for every $l \in \{1, 2, \ldots, k\}$, since $\sigma^{l-1}(a) \neq e$ for all $a \in S_l$ and $\sigma^l[S_l] = \{e\}$, we have

$$\beta S_l = \{p \in \beta S : \widetilde{\sigma}^l(p) = e \text{ and } \widetilde{\sigma}^{l-1}(p) \neq e\},$$
$$X_l = \{\widetilde{\sigma}^{k-l}(p) : p \in \beta S \text{ and } \widetilde{\sigma}^{k-1}(p) \neq e\}.$$

Notice in particular that each $X_l \subseteq \beta S_l$. We shall find idempotents $p_1, p_2, \ldots, p_k$ in $X_1, X_2, \ldots, X_k$ respectively such that

(1) if $i, j \in \{1, 2, \ldots, k\}$ and $i \leq j$, then $p_j \leq p_i$ and

(2) if $i \in \{2, 3, \ldots, k\}$, then $\widetilde{\sigma}(p_i) = p_{i-1}$.

Pick an arbitrary idempotent p_1 in X_1. Let $l \in \{1, 2, \ldots, k-1\}$ and assume that $p_1 \in X_1, p_2 \in X_2, \ldots, p_l \in X_l$ have been found satisfying statements (1) and (2).

We need to check the assumptions of Lemma 4.3 are satisfied, with $p = p_l$, $X = X_{l+1}$ and $\mathcal{H} = \{\sigma_{|T_{l+1}}\}$. First note that $\widetilde{\sigma}(p_1) = e$ and, if $l > 1$, $\widetilde{\sigma}(p_l) = p_{l-1}$ and $p_l \leq p_{l-1}$ so that, in any event, $\widetilde{\sigma}(p_l) \cdot p_l = p_l$.

We now check that $X_{l+1} \cdot p_l \subseteq X_{l+1}$ and $p_l \cdot X_{l+1} \subseteq X_{l+1}$, by proving that if $q \in X_{l+1}$ and $r \in X_l$, then $q \cdot r$ and $r \cdot q$ are in X_{l+1}. Let us prove this for $l = k-1$. By the characterization of βS_l above we have $\widetilde{\sigma}^{k-1}(q \cdot r) = \widetilde{\sigma}^{k-1}(q) \cdot \widetilde{\sigma}^{k-1}(r) = \widetilde{\sigma}^{k-1}(q) \cdot e = \widetilde{\sigma}^{k-1}(q) \neq e$, while $\widetilde{\sigma}^{k}(q \cdot r) = e$. Thus $q \cdot r \in \beta S_k = X_k$. Similarly, $r \cdot q \in X_k$. Now consider the case when $l < k-1$, and pick $q \in X_{l+1}$ and $r \in X_l$. Such a q is of the form $\widetilde{\sigma}^{k-(l+1)}(q_0)$ for some $q_0 \in X_k = \beta S_k$, while r is of the form $\widetilde{\sigma}^{k-l}(r_0)$, for $r_0 \in \beta S_k$. Let $r_1 = \widetilde{\sigma}(r_0)$; then $r_1 \in X_{k-1}$, and $q_0 \cdot r_1 \in \beta S_k$. Since $\widetilde{\sigma}^{k-(l-1)}(q_0 \cdot r_1) = q \cdot r$, we have $q \cdot r \in X_{l+1}$. This proves that $X_{l+1} \cdot X_l \subseteq X_{l+1}$. The proof that $X_l \cdot X_{l+1} \subseteq X_{l+1}$ is identical.

Since $p_l \in X_l$ and $\widetilde{\sigma}[X_{l+1}] = X_l$, there is $q \in X_{l+1}$ such that $\widetilde{\sigma}(q) = p_l$. Thus Lemma 4.3 implies that there is an idempotent $p_{l+1} \in X_{l+1}$ such that $p_{l+1} \leq p_l$ and $\widetilde{\sigma}(p_{l+1}) = p_l$.

This describes the construction of $p_1, p_2, \ldots, p_k$. An application of Lemma 3.12 to this k-tuple of idempotents concludes the proof. □

5. Central Sets in Layered Partial Semigroups

In this section we derive a common generalization of the noncommutative Central Sets Theorem ([2, Theorem 2.8], or see [9, Theorem 14.15]) and Theorem 3.13. We also present as an application an extension of the Hales–Jewett Theorem.

5.1 Definition. Let S be an adequate partial semigroup and let $\langle y_n \rangle_{n=1}^{\infty}$ be a sequence in S. Then $\langle y_n \rangle_{n=1}^{\infty}$ is *adequate* if and only if $\prod_{n \in F} y_n$ is defined for each $F \in \mathcal{P}_f(\mathbb{N})$ and for every $K \in \mathcal{P}_f(S)$, there exists $m \in \mathbb{N}$ such that $FP(\langle y_n \rangle_{n=m}^{\infty}) \subseteq \bigcap_{x \in K} \varphi(x)$.

The noncommutative Central Sets Theorem, which we shall be generalizing, is itself a generalization of the commutative Central Sets Theorem (Theorem 1.6). The extension to arbitrary semigroups requires the introduction of additional notation.

5.2 Definition. Let $m \in \mathbb{N}$. Then

$$\begin{aligned} \mathcal{I}_m \quad = \quad & \{(H_1, H_2, \ldots, H_m) \in \mathcal{P}_f(\mathbb{N})^m : \text{if } m > 1 \text{ and} \\ & t \in \{1, 2, \ldots, m-1\}, \text{ then } \max H_t < \min H_{t+1}\}\,. \end{aligned}$$

The basic idea behind the proof of Theorem 5.4 (which was also the basic idea behind the proof of the noncommutative Central Sets Theorem) is an elaboration of an idea of H. Furstenberg and Y. Katznelson [4] which they developed in the context of enveloping semigroups. The following lemma supplies the technical details that are required.

5.3 Lemma. *Let S be an adequate partial semigroup and for each $l \in \mathbb{N}$, let $\langle y_{l,n}\rangle_{n=1}^{\infty}$ be an adequate sequence in S. For $m \in \mathbb{N}$, $\vec{a} \in S^{m+1}$, $\vec{H} \in \mathcal{I}_m$, and $t \in \mathbb{N}$, let $w(\vec{a},\vec{H},t) = \left(\prod_{i=1}^{m}(a_i \cdot \prod_{n\in H_i} y_{t,n})\right) \cdot a_{m+1}$. For $K \in \mathcal{P}_f(S)$ and $i,\alpha \in \mathbb{N}$, let*

$$I_{K,i,\alpha} = \{\vec{x} \in \times_{t=1}^{\infty} S : \text{ there exist } m \in \mathbb{N}\,,\ \vec{a} \in S^{m+1},\text{ and } \vec{H} \in \mathcal{I}_m \text{ such that } \min H_1 \geq i \text{ and for all } t \in \{1,2,\ldots,\alpha\}\,,\ x_t = w(\vec{a},\vec{H},t) \text{ and } x_t \in \textstyle\bigcap_{y\in K} \varphi(y)\} \text{ and let}$$

$$E_{K,i,\alpha} = I_{K,i,\alpha} \cup \{\vec{x} \in \times_{t=1}^{\infty} S : \text{ there exists } a \in \textstyle\bigcap_{y\in K} \varphi(y) \text{ such that for all } t \in \{1,2,\ldots,\alpha\}\,,\ x_t = a\}\,.$$

Let $Y = \times_{t=1}^{\infty}\delta S$, let $Z = \times_{t=1}^{\infty}\beta S$, let

$$E = \bigcap_{K\in\mathcal{P}_f(S)} \bigcap_{i=1}^{\infty} \bigcap_{\alpha=1}^{\infty} c\ell_Z(E_{K,i,\alpha}) \text{ and let}$$

$$I = \bigcap_{K\in\mathcal{P}_f(S)} \bigcap_{i=1}^{\infty} \bigcap_{\alpha=1}^{\infty} c\ell_Z(I_{K,i,\alpha})\,.$$

Then E is a subsemigroup of Y and I is an ideal of E. Further, for any $p \in K(\delta S)$, $\overline{p} = (p,p,p,\ldots) \in E \cap K(Y) = K(E) \subseteq I$.

Proof. We show first that $E \subseteq Y$. To this end, let $\vec{p} \in E$. We need to show that for each $t \in \mathbb{N}$, $p_t \in \delta S$. So let $t \in \mathbb{N}$ and $y \in S$ be given. We need to show that $\varphi(y) \in p_t$. So suppose instead that $S\backslash\varphi(y) \in p_t$. Then $B = \{\vec{q} \in Z : q_t \in \overline{S\backslash\varphi(y)}\}$ is a neighborhood of $\vec{p}$ so pick $\vec{x} \in B \cap E_{\{y\},1,t}$. Then $x_t \in \varphi(y)$, a contradiction.

Next we show that $I \neq \emptyset$ for which it suffices to show that each $I_{K,i,\alpha} \neq \emptyset$ (because if $K \subseteq F$, $i \leq j$, and $\alpha \leq \delta$, then $I_{F,j,\delta} \subseteq I_{K,i,\alpha}$). So let $K \in \mathcal{P}_f(S)$ and $i,\alpha \in \mathbb{N}$ be given. Pick $a_1 \in \bigcap_{y\in K} \varphi(y)$. For each $t \in \{1,2,\ldots,\alpha\}$, pick $m_t \in \mathbb{N}$ such that $FP(\langle y_{t,n}\rangle_{n=m_t}^{\infty}) \subseteq \bigcap_{y\in K} \varphi(y \cdot a_1)$. Let $r = \max\{i\} \cup \big\{m_t : t \in \{1,2,\ldots,\alpha\}\big\}$ and let $H_1 = \{r\}$. Pick $a_2 \in \bigcap_{t=1}^{\alpha} \bigcap_{y\in K} \varphi(y \cdot a_1 \cdot y_{t,r})$. Let $\vec{a} = \langle a_1,a_2\rangle$ and let $\vec{H} = \langle H_1\rangle$. Let $x_t = a_1 \cdot y_{t,r} \cdot a_2$ if $t \in \{1,2,\ldots,\alpha\}$ and $x_t = a_1$ if $t > \alpha$. Then $\vec{x} \in I_{K,i,\alpha}$.

Now we show that E is a subsemigroup of Y and I is an ideal of E. To this end, let $\vec{p},\vec{q} \in E$. We show that $\vec{p}\cdot\vec{q} \in E$ and, if either $\vec{p} \in I$ or $\vec{q} \in I$, then $\vec{p}\cdot\vec{q} \in I$.

Let $K \in \mathcal{P}_f(S)$ and $i,\alpha \in \mathbb{N}$ be given. We show that that $\vec{p}\cdot\vec{q} \in c\ell_Z E_{K,i,\alpha}$ and, if either $\vec{p} \in I$ or $\vec{q} \in I$, then $\vec{p}\cdot\vec{q} \in c\ell_Z I_{K,i,\alpha}$. To this end, let a neighborhood U of $\vec{p}\cdot\vec{q}$ in Z be given. Pick $\gamma \geq \alpha$ in $\mathbb{N}$ and for each $t \in \{1,2,\ldots,\gamma\}$ pick $A_t \subseteq S$ such that

$$\vec{p}\cdot\vec{q} \in \bigcap_{t=1}^{\gamma} {\pi_t}^{-1}[\,\overline{A_t}\,] \subseteq U\,.$$

Then for each $t \in \{1, 2, \ldots, \gamma\}$, $p_t \cdot q_t \in \overline{A_t}$ so $B_t = \{x \in S : x^{-1}A_t \in q_t\} \in p_t$. Thus $\bigcap_{t=1}^{\gamma} {\pi_t}^{-1}[\,\overline{B_t}\,]$ is a neighborhood of $\vec{p}$ in Z so pick

$$\vec{x} \in E_{K,i,\alpha} \cap \bigcap_{t=1}^{\gamma} {\pi_t}^{-1}[\,\overline{B_t}\,]$$

with $\vec{x} \in I_{K,i,\alpha}$ if $\vec{p} \in I$. Then we have that for each $t \in \{1, 2, \ldots, \alpha\}$, $x_t \in \bigcap_{y \in K} \varphi(y)$.

If $\vec{x} \in I_{K,i,\alpha}$, pick $m \in \mathbb{N}$, $\vec{a} \in S^{m+1}$, and $\vec{H} \in \mathcal{I}_m$ such that $\min H \geq i$ and for each $t \in \{1, 2, \ldots, \alpha\}$, $x_t = w(\vec{a}, \vec{H}, t)$. If $\vec{x} \in I_{K,i,\alpha}$, let $j = \max H_m + 1$. Otherwise let $j = i$. In either case let $F = \big\{y \cdot x_t : y \in K \text{ and } t \in \{1, 2, \ldots, \alpha\}\big\}$.

Now, for each $t \in \{1, 2, \ldots, \alpha\}$, we have $x_t \in B_t$, so

$${x_t}^{-1}A_t = \{z \in \varphi(x_t) : x_t \cdot z \in A_t\} \in q_t\,.$$

Thus $\bigcap_{t=1}^{\gamma} {\pi_t}^{-1}[\,\overline{{x_t}^{-1}A_t}\,]$ is a neighborhood of $\vec{q}$ in Z so pick

$$\vec{z} \in E_{F,j,\alpha} \cap \bigcap_{t=1}^{\gamma} {\pi_t}^{-1}[\,\overline{{x_t}^{-1}A_t}\,]$$

with $\vec{z} \in I_{F,j,\alpha}$ if $\vec{q} \in I$. Then we have that for each $t \in \{1, 2, \ldots, \alpha\}$, $z_t \in \bigcap_{y \in F} \varphi(y)$.

If $\vec{z} \in I_{F,j,\alpha}$, pick $n \in \mathbb{N}$, $\vec{b} \in S^{n+1}$, and $\vec{G} \in \mathcal{I}_n$ such that $\min G \geq j$ and for each $t \in \{1, 2, \ldots, \alpha\}$, $z_t = w(\vec{b}, \vec{G}, t)$. Then directly we have that $\vec{x} \cdot \vec{z} \in \bigcap_{t=1}^{\gamma} {\pi_t}^{-1}[\,\overline{A_t}\,] \subseteq U$ so we need only show that $\vec{x} \cdot \vec{z} \in E_{K,i,\alpha}$ with $\vec{x} \cdot \vec{z} \in I_{K,i,\alpha}$ if $\vec{p} \in I$ or $\vec{q} \in I$. First let $t \in \{1, 2, \ldots, \alpha\}$ and let $y \in K$. Then $x_t \in \varphi(y)$ and $y \cdot x_t \in F$ so $z_t \in \varphi(y \cdot x_t)$ and hence $x_t \cdot z_t \in \varphi(y)$. We now consider four possibilities:

(1) $\vec{x} \notin I_{K,i,\alpha}$ and $\vec{z} \notin I_{F,j,\alpha}$.

Then pick $a \in \bigcap_{y \in K} \varphi(y)$ such that for all $t \in \{1, 2, \ldots, \alpha\}$, $x_t = a$ and pick $b \in \bigcap_{y \in F} \varphi(y)$ such that for all $t \in \{1, 2, \ldots, \alpha\}$, $z_t = b$. Then for all $t \in \{1, 2, \ldots, \alpha\}$, $x_t \cdot z_t = a \cdot b$ so $\vec{x} \cdot \vec{z} \in E_{K,i,\alpha}$.

(2) $\vec{x} \in I_{K,i,\alpha}$ and $\vec{z} \notin I_{F,j,\alpha}$.

Then pick $b \in \bigcap_{y \in F} \varphi(y)$ such that for all $t \in \{1, 2, \ldots, \alpha\}$, $z_t = b$ and let $\vec{c} = \langle a_1, a_2, \ldots, a_m, a_{m+1} \cdot b\rangle$. Then for $t \in \{1, 2, \ldots, \alpha\}$, $x_t \cdot z_t = w(\vec{c}, \vec{H}, t)$ so that $\vec{x} \cdot \vec{z} \in I_{K,i,\alpha}$.

(3) $\vec{x} \notin I_{K,i,\alpha}$ and $\vec{z} \in I_{F,j,\alpha}$.

Then pick $a \in \bigcap_{y \in K} \varphi(y)$ such that for all $t \in \{1, 2, \ldots, \alpha\}$, $x_t = a$ and let $\vec{c} = \langle a \cdot b_1, b_2, \ldots, b_{n+1}\rangle$. Then for $t \in \{1, 2, \ldots, \alpha\}$, $x_t \cdot z_t = w(\vec{c}, \vec{G}, t)$ so that $\vec{x} \cdot \vec{z} \in I_{K,i,\alpha}$.

(4) $\vec{x} \in I_{K,i,\alpha}$ and $\vec{z} \in I_{F,j,\alpha}$.

Then let $\vec{c} = \langle a_1, a_2, \ldots, a_m, a_{m+1} \cdot b_1, b_2, \ldots, b_{n+1}\rangle$ and let $\vec{J} = \langle H_1, H_2, \ldots, H_m, G_1, G_2, \ldots, G_n\rangle$. Then $\vec{J} \in \mathcal{I}_{m+n}$ and for all $t \in \{1, 2, \ldots, \alpha\}$, $x_t \cdot z_t = w(\vec{c}, \vec{J}, t)$ so that $\vec{x} \cdot \vec{z} \in I_{K,i,\alpha}$.

To complete the proof of the lemma, we need to show that $E \cap K(Y) = K(E) \subseteq I$ and for any $p \in K(\delta S)$, $\overline{p} \in E \cap K(Y)$. For this, it suffices to let $p \in K(\delta S)$ and show that $\overline{p} \in E$. For then $\overline{p} \in E \cap \times_{t=1}^{\infty} K(\delta S)$ and $\times_{t=1}^{\infty} K(\delta S) = K(Y)$ by [9, Theorem 2.23] so that $\overline{p} \in E \cap K(Y)$. Since $E \cap K(Y) \neq \emptyset$ we have by [9, Theorem 1.65] that $K(E) = E \cap K(Y)$ and since I is an ideal of E, $K(E) \subseteq I$.

So let U be a neighborhood of $\overline{p}$ in Z and let $K \in \mathcal{P}_f(S)$ and $i, \alpha \in \mathbb{N}$. We need to show that $U \cap E_{K,i,\alpha} \neq \emptyset$. Pick $\gamma \geq \alpha$ in $\mathbb{N}$ and for each $t \in \{1, 2, \ldots, \gamma\}$ pick $A_t \subseteq S$ such that $\overline{p} \in \bigcap_{t=1}^{\gamma} {\pi_t}^{-1}[\,\overline{A_t}\,] \subseteq U$. Let $A = \bigcap_{t=1}^{\gamma} A_t$. Then $A \in p$ and $\bigcap_{y \in K} \varphi(y) \in p$ so pick $a \in A \cap \bigcap_{y \in K} \varphi(y)$. Then $\overline{a} \in E_{K,i,\alpha} \cap \bigcap_{t=1}^{\gamma} {\pi_t}^{-1}[\,\overline{A_t}\,]$. □

5.4 Theorem. *Let S be a layered partial semigroup with k layers and let $\langle \mathcal{F}_n \rangle_{n=2}^{k}$ be a layered action on S. Let $S_0, S_1, \ldots, S_k$ be as in Definition* 3.1 *and let $\mathcal{G}_1, \mathcal{G}_2, \ldots, \mathcal{G}_k$ be as in Definition* 3.9.

For each $l \in \mathbb{N}$, let $\langle y_{l,n} \rangle_{n=1}^{\infty}$ be an adequate sequence in S_k. For $m \in \mathbb{N}$, $\vec{a} \in {S_k}^{m+1}$, $\vec{H} \in \mathcal{I}_m$, and $l \in \mathbb{N}$, let

$$w(\vec{a}, \vec{H}, l) = \big(\textstyle\prod_{i=1}^{m} (a_i \cdot \prod_{t \in H_i} y_{l,t}) \big) \cdot a_{m+1} \,.$$

Let $r \in \mathbb{N}$ and let $S = \bigcup_{i=1}^{r} C_i$. Then there exist $\gamma : \{1, 2, \ldots, k\} \to \{1, 2, \ldots, r\}$ and sequences $\langle m(n) \rangle_{n=1}^{\infty}$, $\langle \vec{a_n} \rangle_{n=1}^{\infty}$, and $\langle \vec{H_n} \rangle_{n=1}^{\infty}$ such that

- *(a) for each $j \in \{1, 2, \ldots, k\}$, $C_{\gamma(j)} \cap S_j$ is central in S_j;*
- *(b) for each $n \in \mathbb{N}$, $m(n) \in \mathbb{N}$, $\vec{a_n} \in {S_k}^{m(n)+1}$, and $\vec{H_n} \in \mathcal{I}_{m(n)}$;*
- *(c) for each $n \in \mathbb{N}$, $\max H_{n,m(n)} < \min H_{n+1,1}$; and*
- *(d) for every $f : \mathbb{N} \to \mathbb{N}$ such that $f(n) \leq n$ for each $n \in \mathbb{N}$ and for every $j \in \{1, 2, \ldots, k\}$,*

$$\{\textstyle\prod_{n \in F} g_n\big(w\big(\vec{a_n}, \vec{H_n}, f(n)\big)\big) : \; F \in \mathcal{P}_f(\mathbb{N})\,,\ g_n \in \bigcup_{i=1}^{j} \mathcal{G}_i \text{ for each } n \in F\,, \text{ and there exists } n \in F \text{ such that } g_n \in \mathcal{G}_j\} \subseteq C_{\gamma(j)}\,.$$

Further, $\gamma(1)$ can be any $i \in \{1, 2, \ldots, r\}$ such that $C_i \cap S_1$ is central in S_1.

Proof. Choose $\gamma(1) \in \{1, 2, \ldots, r\}$ such that $C_{\gamma(1)} \cap S_1$ is central in S_1. Choose a minimal idempotent p in δS_1 such that $C_{\gamma(1)} \cap S_1 \in p$ and choose $p_1, p_2, \ldots, p_k$ with $p_1 = p$ as guaranteed by Theorem 3.8. For each $i \in \{2, 3, \ldots, k\}$ choose $\gamma(i) \in \{1, 2, \ldots, r\}$ such that $C_{\gamma(i)} \cap S_i \in p_i$.

For each $K \in \mathcal{P}_f(S_k)$ and each $i, \alpha \in \mathbb{N}$, let $I_{K,i,\alpha}$ and $E_{K,i,\alpha}$ be as in Lemma 5.3 applied to the semigroup S_k. Also let Z, Y, E, and I be as in Lemma 5.3. We inductively construct sequences $\langle m(n) \rangle_{n=1}^{\infty}$, $\langle \vec{a_n} \rangle_{n=1}^{\infty}$, $\langle \vec{H_n} \rangle_{n=1}^{\infty}$, and for each $i \in \{1, 2, \ldots, k\}$ a sequence $\langle A_{i,n} \rangle_{n=1}^{\infty}$ in p_i, such that

(1) for each n, $m(n) \in \mathbb{N}$, $\vec{a_n} \in {S_k}^{m(n)+1}$, $\vec{H_n} \in \mathcal{I}_{m(n)}$, and if $n > 1$, then $\min H_{n,1} > \max H_{n-1,m(n-1)}$;

(2) for $j \in \{1, 2, \ldots, k\}$, $n \in \mathbb{N}$, $g \in \mathcal{G}_j$, and $l \in \{1, 2, \ldots, n\}$, $g\big(w(\vec{a_n}, \vec{H_n}, l)\big) \in A_{j,n}$;

(3) for $i, j \in \{1, 2, \ldots, k\}$, $n \in \mathbb{N}$, $g \in \mathcal{G}_j$, and $l \in \{1, 2, \ldots, n\}$, $g\big(w(\vec{a_n}, \vec{H_n}, l)\big)^{-1} A_{\max\{i,j\},n} \in p_i$;

(4) for $i \in \{1, 2, \ldots, k\}$ and $n \in \mathbb{N}$,

$$\begin{array}{rcl} A_{i,n+1} & = & A_{i,n} \cap \bigcap_{j=1}^{i} \bigcap_{g \in \mathcal{G}_j} \bigcap_{l=1}^{n} \left(g\big(w(\vec{a_n}, \vec{H_n}, l)\big)^{-1} A_{i,n} \right) \\ & & \cap \bigcap_{j=i}^{k} \bigcap_{g \in \mathcal{G}_j} \bigcap_{l=1}^{n} \left(g\big(w(\vec{a_n}, \vec{H_n}, l)\big)^{-1} A_{j,n} \right); \text{ and} \end{array}$$

(5) if $n \in \mathbb{N}$, $\emptyset \neq F \subseteq \{1, 2, \ldots, n\}$, $b = \min F$, $f : F \to \mathbb{N}$, $f(u) \leq u$ for every $u \in F$, $g : F \to \bigcup_{t=1}^{k} \mathcal{G}_t$, and $i = \max\{t : g[F] \cap \mathcal{G}_t \neq \emptyset\}$, then $\prod_{u \in F} g(u)\big(w\big(\vec{a_u}, \vec{H_u}, f(u)\big)\big) \in A_{i,b}$.

For $i \in \{1, 2, \ldots, k\}$, let $A_{i,1} = C_{\gamma(i)} \cap S_i$. Let $n \in \mathbb{N}$ and assume that for each $i \in \{1, 2, \ldots, k\}$, we have $A_{i,n} \in p_i$. By Lemma 3.11(c), we have for any $i, j \in \{1, 2, \ldots, k\}$ and any $g \in \mathcal{G}_j$, $\{w \in S_k : g(w)^{-1} A_{\max\{i,j\},n} \in p_i\} \in p_k$. By Lemma 3.11($b$), for any $j \in \{1, 2, \ldots, k\}$ and any $g \in \mathcal{G}_j$, $g^{-1}[A_{j,n}] \in p_k$. Let

$$\begin{array}{rcl} B & = & \bigcap_{i=1}^{k} \bigcap_{j=1}^{k} \bigcap_{g \in \mathcal{G}_j} \{w \in S_k : g(w)^{-1} A_{\max\{i,j\},n} \in p_i\} \\ & & \cap \bigcap_{j=1}^{k} \bigcap_{g \in \mathcal{G}_j} g^{-1}[A_{j,n}]\,. \end{array}$$

Then $B \in p_k$. If $n = 1$, let $t = 1$. Otherwise $m(n-1)$ and $\overrightarrow{H_{n-1}}$ have been chosen and we let $t = \max H_{n-1,m(n-1)} + 1$. If $n = 1$, let K be any member of $\mathcal{P}_f(S_k)$. Otherwise let

$$\begin{array}{rl} K = \{\prod_{u \in F} g_u\big(w\big(\vec{a_u}, \vec{H_u}, f(u)\big)\big) : & \emptyset \neq F \subseteq \{1, 2, \ldots, n-1\}\,,\ f(u) \in \{1, 2, \ldots, u\} \\ & \text{for each } u \in F\,,\ g_u \in \bigcup_{i=1}^{k} \mathcal{G}_i \text{ for each } u \in F\,, \\ & \text{and there exists } u \in F \text{ such that } g_u \in \mathcal{G}_k\}\,. \end{array}$$

By hypothesis (5), $K \subseteq S_k$. Since each $\mathcal{G}_i$ is finite by Lemma 3.11(a), we have that K is finite.

Let $\overline{p} = (p_k, p_k, p_k, \ldots)$. By Lemma 5.3, $\overline{p} \in I$. Let $D = \bigcap_{i=1}^{n} \pi_i^{-1}[\,\overline{B}\,]$. Then D is a neighborhood of $\overline{p}$ in Z so pick $\vec{x} \in D \cap I_{K,t,n}$. Pick $m(n) \in \mathbb{N}$, $\vec{a_n} \in {S_k}^{m(n)+1}$, and $\vec{H_n} \in \mathcal{I}_{m(n)}$ such that $\min H_{n,1} \geq t$ and for all $s \in \{1, 2, \ldots, n\}$, $x_s = w(\vec{a_n}, \vec{H_n}, s)$ and $x_s \in \bigcap_{y \in K} \varphi_{S_k}(y)$. Notice in particular that for each $s \in \{1, 2, \ldots, n\}$, $w(\vec{a_n}, \vec{H_n}, s) \in B$.

Now for each $i \in \{1, 2, \ldots, k\}$ let $A_{i,n+1}$ be as required by hypothesis (4). Then hypotheses (1), (2), and (3) hold directly, and $A_{i,n+1} \in p_i$ by hypothesis (3).

To complete the construction as well as the proof of the theorem, we verify hypothesis (5). (The conclusion of the theorem follows because each $A_{i,b} \subseteq A_{i,1} \subseteq C_{\gamma(i)}$.) So let $\emptyset \neq F \subseteq \{1, 2, \ldots, n\}$, let $b = \min F$, let $f : F \to \mathbb{N}$ such that $f(u) \leq u$ for every $u \in F$, let $g : F \to \bigcup_{t=1}^{k} \mathcal{G}_t$, and let $i = \max\{t : g[F] \cap \mathcal{G}_t \neq \emptyset\}$. We need to show that $\prod_{u \in F} g(u)\big(w\big(\vec{a_u}, \vec{H_u}, f(u)\big)\big) \in A_{i,b}$, which we do by induction on $|F|$.

If $F = \{b\}$, by hypotheses (2) and (4), $g(b)\big(w\big(\vec{a_b}, \vec{H_b}, f(b)\big)\big) \in A_{i,n} \subseteq A_{i,b}$. So assume that $|F| > 1$, let $G = F \backslash \{b\}$, let $c = \min G$, and let $s = \max\{v : g[G] \cap \mathcal{G}_v \neq \emptyset\}$. We have by induction that

$$\prod_{u \in G} g(u)\big(w\big(\vec{a_u}, \vec{H_u}, f(u)\big)\big) \in A_{s,c} \subseteq A_{s,b+1}\,.$$

Pick $j \in \{1, 2, \ldots, k\}$ such that $g(b) \in \mathcal{G}_j$ and note that $i = \max\{j, s\}$.

Case 1: $i = s$ (so $s \geq j$). Then

$$\prod_{u \in G} g(u)\big(w\big(\vec{a_u}, \vec{H_u}, f(u)\big)\big) \in A_{s,b+1} \subseteq \big(g(b)\big(w\big(\vec{a_b}, \vec{H_b}, f(b)\big)\big)\big)^{-1} A_{s,b}$$

and so $\prod_{u \in F} g(u)\big(w\big(\vec{a_u}, \vec{H_u}, f(u)\big)\big) \in A_{s,b} = A_{i,b}$.

Case 2: $i = j$ (so $s \leq j$). Then

$$\prod_{u \in G} g(u)\big(w\big(\vec{a_u}, \vec{H_u}, f(u)\big)\big) \in A_{s,b+1} \subseteq \big(g(b)\big(w\big(\vec{a_b}, \vec{H_b}, f(b)\big)\big)\big)^{-1} A_{j,b}$$

and so $\prod_{u \in F} g(u)\big(w\big(\vec{a_u}, \vec{H_u}, f(u)\big)\big) \in A_{j,b} = A_{i,b}$. □

In the event that the partial semigroup S is commutative (which means that whenever either $x \cdot y$ or $y \cdot x$ are defined, they both are defined and are equal), the statement of Theorem 5.4 becomes considerably simpler.

5.5 Corollary. *Let S be a commutative layered partial semigroup with k layers and let $\langle \mathcal{F}_n \rangle_{n=2}^{k}$ be a layered action on S. Let $S_0, S_1, \ldots, S_k$ be as in Definition* 3.1 *and let $\mathcal{G}_1, \mathcal{G}_2, \ldots, \mathcal{G}_k$ be as in Definition* 3.9.

For each $l \in \mathbb{N}$, let $\langle y_{l,n} \rangle_{n=1}^{\infty}$ be an adequate sequence in S_k. Let $r \in \mathbb{N}$ and let $S = \bigcup_{i=1}^{r} C_i$. Then there exist $\gamma : \{1, 2, \ldots, k\} \to \{1, 2, \ldots, r\}$ and sequences $\langle b_n \rangle_{n=1}^{\infty}$ and $\langle G_n \rangle_{n=1}^{\infty}$ such that

(a) for each $j \in \{1, 2, \ldots, k\}$, $C_{\gamma(j)} \cap S_j$ is central in S_j;

(b) for each $n \in \mathbb{N}$, $b_n \in S_k$, and $G_n \in \mathcal{P}_f(S_k)$;

(c) for each $n \in \mathbb{N}$, $\max G_n < \min G_{n+1}$; and

(d) for every $f : \mathbb{N} \to \mathbb{N}$ such that $f(n) \leq n$ for each $n \in \mathbb{N}$ and for every $j \in \{1, 2, \ldots, k\}$,

$$\{\prod_{n\in F} g_n\big(b_n\cdot\prod_{t\in G_n} y_{f(n),t}\big) :\ F\in\mathcal{P}_f(\mathbb{N})\,,\ g_n\in\bigcup_{i=1}^{j}\mathcal{G}_i \text{ for each } n\in F\,, \text{ and there exists } n\in F \text{ such that } g_n\in\mathcal{G}_j\}\subseteq C_{\gamma(j)}\,.$$

Further, $\gamma(1)$ can be any $i\in\{1,2,\ldots,r\}$ such that $C_i\cap S_1$ is central in S_1.

Proof. Choose $\gamma:\{1,2,\ldots,k\}\to\{1,2,\ldots,r\}$ and sequences $\langle m(n)\rangle_{n=1}^{\infty}$, $\langle\vec{a_n}\rangle_{n=1}^{\infty}$, and $\langle\vec{H_n}\rangle_{n=1}^{\infty}$ as guaranteed by Theorem 5.4. For each n, let $b_n=\prod_{i=1}^{m(n)+1} a_{n,i}$ and let $G_n=\bigcup_{i=1}^{m(n)} H_{n,i}$. Then given $F\in\mathcal{P}_f(\mathbb{N})$, $f:F\to\mathbb{N}$ with $f(n)\le n$ for each n, and $g:F\to\bigcup_{i=1}^{k}\mathcal{G}_i$, we have

$$\prod_{n\in F} g(n)\big(b_n\cdot\prod_{t\in G_n} y_{f(n),t}\big)=\prod_{n\in F} g(n)\big(w\big(\vec{a_n},\vec{H_n},f(n)\big)\big)\,.$$

□

6. Remarks and questions

Let us take a look at results of this paper from a somewhat different angle. If $\langle x_n\rangle_{n=1}^{\infty}$ is a sequence in a partial semigroup S and $\mathcal{H}$ is a set of partial semigroup homomorphisms from S to itself, let $\langle\!\langle\, \langle x_n\rangle_{n=1}^{\infty}\,\rangle\!\rangle_{\mathcal{H}}$ be the 'closure' of $\langle x_n\rangle_{n=1}^{\infty}$ with respect to $\mathcal{H}$ and the semigroup operation. Namely, if $\mathrm{cl}(\mathcal{H})$ is the set consisting of all compositions of maps from $\mathcal{H}\cup\{\iota\}$, let

$$\langle\!\langle\, \langle x_n\rangle_{n=1}^{\infty}\,\rangle\!\rangle_{\mathcal{H}}=\big\{\prod_{i\in F} g_i(x_i) : F\in\mathcal{P}_f(\mathbb{N}), g_i\in\mathrm{cl}(\mathcal{H}) \text{ for all } i\in F \text{ and } \prod_{i\in F} g_i(x_i) \text{ is defined}\big\}\,.$$

6.1 Problem. *For each $k\in\mathbb{N}$, describe the class of triples $(S,S',\mathcal{H})$ such that S is a partial semigroup, S' is an ideal of S, and $\mathcal{H}$ is a set of partial semigroup homomorphisms from S to itself such that for every $r\in\mathbb{N}$ and every partition $S=\bigcup_{j=1}^{r} C_j$ there is a sequence $\langle x_n\rangle_{n=1}^{\infty}$ included in S' such that the set*

$$\{j\in\{1,2,\ldots,r\} : \langle\!\langle\, \langle x_n\rangle_{n=1}^{\infty}\,\rangle\!\rangle_{\mathcal{H}}\cap C_j\neq\emptyset\}$$

has at most $k+1$ elements.

The requirement that $\langle x_n\rangle_{n=1}^{\infty}$ is included in S' is there to assure that all x_n's are 'large' in some prescribed sense, as this is usually required in applications. Let us show that we already have a substantial class of such triples $(S,S',\mathcal{H})$.

6.2 Theorem. *If S is a layered semigroup with k layers, $\langle\mathcal{F}_n\rangle_{n=2}^{k}$ is a layered action on S, $\langle\mathcal{G}_n\rangle_{n=1}^{k}$ is as in Definition 3.9, and $\mathcal{H}=\bigcup_{n=1}^{k}\mathcal{G}_n$, then for all $r\in\mathbb{N}$ and every partition $S=\bigcup_{j=1}^{r} C_j$ there is a sequence $\langle x_n\rangle_{n=1}^{\infty}$ included in the top layer such that the set $\{j\in\{1,2,\ldots,r\} : \langle\!\langle\, \langle x_n\rangle_{n=1}^{\infty}\,\rangle\!\rangle_{\mathcal{H}}\cap C_j\neq\emptyset\}$ has at most $k+1$ elements.*

Proof. This is an immediate consequence of Theorem 3.13 and the fact from Lemma 3.10 that $\mathrm{cl}(\mathcal{H}) \subseteq \mathcal{H} \cup \{\overline{e}\}$. (The set $\{j \in \{1,2,\ldots,r\} : \langle\!\langle \langle x_n\rangle_{n=1}^\infty \rangle\!\rangle_{\mathcal{H}} \cap C_j \neq \emptyset\}$ might have $k+1$ elements rather than just k because it is possible, indeed likely, that $e \in \langle\!\langle \langle x_n\rangle_{n=1}^\infty \rangle\!\rangle_{\mathcal{H}}$.) □

6.3 Conjecture. *If S is a free semigroup with identity e and S' is an ideal of S, then for every set $\mathcal{H}$ of homomorphisms from S to itself and every $l \in \mathbb{N} \setminus \{1\}$, the following are equivalent:*

(1) For all $r \in \mathbb{N}$ and every partition $S = \bigcup_{j=1}^r C_j$ there is a sequence $\langle x_n\rangle_{n=1}^\infty$ included in S' such that the set $\{j \in \{1,2,\ldots,r\} : \langle\!\langle \langle x_n\rangle_{n=1}^\infty \rangle\!\rangle_{\mathcal{H}} \cap C_j \neq \emptyset\}$ has at most $l+1$ elements.

(2) There exist a partial subsemigroup R of S and $k \in \{2,3,\ldots,l\}$ such that

(a) R is a layered partial semigroup with k layers, $S_1, S_2, \ldots, S_k$;

(b) $S_k \subseteq S'$; and

(c) if for each $n \in \{1,2,\ldots,k\}$,

$$\mathcal{G}_n = \{f_{|R} : f \in \mathcal{H}\,,\ f[R] \subseteq \textstyle\bigcup_{i=0}^n S_i\,,\ \text{and } f[R] \setminus \bigcup_{i=0}^{n-1} S_i \neq \emptyset\}\,,$$

then the conclusion of Theorem 3.13 is satisfied.

It is trivial that (2) implies (1) for arbitrary semigroups.

The following result is related to Conjecture 6.3.

6.4 Theorem. *Let S be a layered semigroup with two layers, let S_0, S_1, and S_2 be as in Defintion 3.1, and let f be a homomorphism from S to $S_0 \cup S_1$. The following statements are equivalent.*

(1) For every $r \in \mathbb{N}$ and every partition $S = \bigcup_{i=1}^r C_i$, there exist a sequence $\langle x_n\rangle_{n=1}^\infty$ in S_2 and $\gamma(1), \gamma(2) \in \{1,2,\ldots,r\}$ such that

$$\langle\!\langle \langle x_n\rangle_{n=1}^\infty \rangle\!\rangle_{\{f\}} \cap S_1 \subseteq C_{\gamma(1)} \text{ and}$$

$$\langle\!\langle \langle x_n\rangle_{n=1}^\infty \rangle\!\rangle_{\{f\}} \cap S_2 \subseteq C_{\gamma(2)}\,.$$

(2) There is a sequence $\langle x_n\rangle_{n=1}^\infty$ in S_2 such that either

(a) $f^2(x_n) = e$ for all n or

(b) $f^2(x_n) = f(x_n)$ for all n.

(3) There exist idempotents $p_1 \in \beta S_1 \cup \{e\}$ and $p_2 \in \beta S_2$ such that $p_2 \leq p_1$, $\widetilde{f}(p_2) = p_1$, and either $\widetilde{f}(p_1) = p_1$ or $\widetilde{f}(p_1) = e$.

(4) Either there exists $y \in S_2$ such that $f(y) = e$ or there is a subsemigroup R of S which is a layered semigroup with two layers (where $R_0 = \{e\}$, $R_1 = R \cap S_1$, and

$R_2 = R \cap S_2$ are as given by Definition 3.1) such that, with $\mathcal{F}_2 = \{f_{|R}\}$, $\langle \mathcal{F}_n \rangle_{n=2}^{2}$ is a layered action on R.

Proof. (1) implies (2). Let g be a function from S to S such that $g(x) = f(x)$ if $f(x) \neq x$ and $g(x) \in S\backslash\{x\}$ if $f(x) = x$. Then g has no fixed points so pick by Katetŏv's Theorem [9, Lemma 3.33] sets C_0, C_1, and C_2 such that $S = C_0 \cup C_1 \cup C_2$ and $C_i \cap g[C_i] = \emptyset$ for each $i \in \{0,1,2\}$. Pick $i \in \{0,1,2\}$ and a sequence $\langle x_n \rangle_{n=1}^{\infty}$ in S_2 such that $\langle\!\langle \langle x_n \rangle_{n=1}^{\infty} \rangle\!\rangle_{\{f\}} \cap S_1 \subseteq C_i$. If for infinitely many n's, $f^2(x_n) = e$ then conclusion 2(a) holds. Thus we may assume that for all n, $f^2(x_n) \neq e$ (and in particular, since f is a homomorphism, $f(x_n) \neq e$). Consequently, for all $n \in \mathbb{N}$, we have that $f(x_n), f^2(x_n) \in C_i$. We claim that for all n, $f^2(x_n) = f(x_n)$. So suppose instead that we have some n with $f^2(x_n) \neq f(x_n)$. Then $g\big(f(x_n)\big) = f^2(x_n) \in C_i \cap g[C_i]$, a contradiction.

(2) implies (3). For $m \in \mathbb{N}$, let

$$W_m = \{\textstyle\prod_{n\in F} f^{\alpha_n}(x_n) : \begin{array}{l} F \in \mathcal{P}_f(\mathbb{N})\,,\ \min F \geq m\,,\ \{\alpha_n : n \in F\} \subseteq \{0,1\}\,, \\ \text{and some } \alpha_n = 0\} \end{array}$$

and let $V_m = \{\prod_{n\in F} f(x_n) : F \in \mathcal{P}_f(\mathbb{N}) \text{ and } \min F \geq m\}$.

Let $X = \bigcap_{m=1}^{\infty} c\ell_{\beta S}(W_m)$. Since S_2 is an ideal of S, we have that $W_1 \subseteq S_2$ and so $X \subseteq \beta S_2$. We claim first that X is a subsemigroup of βS_2. To see this, let $p, q \in X$ and let $m \in \mathbb{N}$. To see that $W_m \in p \cdot q$, we show that $W_m \subseteq \{y \in S : y^{-1}W_m \in q\}$. To this end let $y \in W_m$ and pick $F \in \mathcal{P}_f(\mathbb{N})$ and $\{\alpha_n : n \in F\} \subseteq \{0,1\}$ such that $\min F \geq m$, some $\alpha_n = 0$, and $y = \prod_{n\in F} f^{\alpha_n}(x_n)$. Let $l = \max F + 1$. Then $W_l \subseteq y^{-1}W_n$.

Thus X is a compact right topological semigroup so pick an idempotent $q \in X$. Now, by [9, Corollary 4.22], $\widetilde{f} : \beta S \to \beta S$ is a homomorphism. Let $p_1 = \widetilde{f}(q)$. Then p_1 is an idempotent. Further $p_1 \in \beta S_1 \cup \{e\}$. If $p_1 = e$, let $p_2 = q$ and we are done. So assume that $p_1 \in \beta S_1$.

We claim that for each $m \in \mathbb{N}$, $V_m \in p_1$, for which it suffices that $f[W_m] \subseteq V_m$. So let $m \in \mathbb{N}$ and let $y \in W_m$. Pick $F \in \mathcal{P}_f(\mathbb{N})$ and $\{\alpha_n : n \in F\} \subseteq \{0,1\}$ such that $\min F \geq m$, some $\alpha_n = 0$, and $y = \prod_{n\in F} f^{\alpha_n}(x_n)$. If for each n, $f^2(x_n) = e$, then $f(y) = \prod_{n\in G} f(x_n)$, where $G = \{n \in F : \alpha_n = 0\}$. If for each n, $f^2(x_n) = f(x_n)$, then $f(y) = \prod_{n\in F} f(x_n)$.

If for each n, $f^2(x_n) = e$, then $f[V_1] = \{e\}$ so that $\widetilde{f}(p_1) = e$. If for each n, $f^2(x_n) = f(x_n)$, then f is the identity on V_1 so that $\widetilde{f}(p_1) = p_1$. In either case we have that $\widetilde{f}(p_1) \cdot p_1 = p_1$.

Now we claim that $p_1 \cdot X \subseteq X$ and $X \cdot p_1 \subseteq X$. To this end, let $r \in X$ and let $m \in \mathbb{N}$.

To see that $W_m \in p_1 \cdot r$ one checks as in the proof that X is a semigroup, that for each $y \in V_m$ there is some $l \in \mathbb{N}$ such that $W_l \subseteq y^{-1}W_m$ so that $V_m \subseteq \{y \in S : y^{-1}W_m \in r\}$. To see that $W_m \in r \cdot p_1$ one checks that for each $y \in W_m$ there is some $l \in \mathbb{N}$ such that $V_l \subseteq y^{-1}W_m$ so that $W_m \subseteq \{y \in S : y^{-1}W_m \in p_1\}$.

Now by Lemma 4.3(b) (with $\mathcal{H} = \{f\}$) pick $p_2 \in X$ with $p_2 \leq p_1$ and $\widetilde{f}(p_2) = p_1$.

(3) implies (1). Assume first that $p_1 = e$. Then since $\widetilde{f}(p_2) = e$, there is some $y \in S_2$ such that $f(y) = e$. Then for all $n \in \mathbb{N}$, $f(y^n) = e$. Pick by the Finite Products Theorem [9, Corollary 5.9] some sequence $\langle x_n \rangle_{n=1}^{\infty}$ in $\{y^n : n \in \mathbb{N}\}$ and $\gamma(2) \in \{1, 2, \ldots, r\}$ such that $\{\prod_{n \in F} x_n : F \in \mathcal{P}_f(\mathbb{N})\} \subseteq C_{\gamma(2)}$. Then $\langle\!\langle \langle x_n \rangle_{n=1}^{\infty} \rangle\!\rangle_{\{f\}} = \{e\} \cup \{\prod_{n \in F} x_n : F \in \mathcal{P}_f(\mathbb{N})\}$.

Now assume that $p_1 \in \beta S_1$. If $\widetilde{f}(p_1) = p_1$, then by [9, Theorem 3.35] $\{x \in S_1 : f(x) = x\} \in p_1$ so $\{x \in S_2 : f^2(x) = f(x)\} \in p_2$. If $\widetilde{f}(p_1) = e$, then $\{x \in S_1 : f(x) = e\} \in p_1$ so $\{x \in S_2 : f^2(x) = e\} \in p_2$. Thus, in either case, $\{x \in S_2 : f^2(x) = f(x)$ or $f^2(x) = e\} \in p_2$. For each $m \in \mathbb{N}$ let $B_{2,m} = \{x \in S_2 : f^2(x) = f(x)$ or $f^2(x) = e\}$ and let $B_{1,m} = S_1$

Let $\mathcal{F}_2 = \{f\}$, $\mathcal{G}_2 = \{\iota_S\}$, and $\mathcal{G}_1 = \{f\}$. Let $r \in \mathbb{N}$, let $S = \bigcup_{i=1}^{r} C_i$, and pick $\gamma(1), \gamma(2) \in \{1, 2, \ldots, r\}$ and a sequence $\langle x_n \rangle_{n=1}^{\infty}$ in S_2 as guaranteed by Lemma 3.12. To complete the proof, we need to show that

$$\langle\!\langle \langle x_n \rangle_{n=1}^{\infty} \rangle\!\rangle_{\{f\}} \cap S_2 \subseteq \{\textstyle\prod_{n \in F} g_n(x_n) : \begin{array}{l} F \in \mathcal{P}_f(\mathbb{N}), \{g_n : n \in F\} \subseteq \{\iota_S, f\} \\ \text{and } g_n = \iota_S \text{ for some } n\} \end{array}$$

and $\langle\!\langle \langle x_n \rangle_{n=1}^{\infty} \rangle\!\rangle_{\{f\}} \cap S_1 \subseteq \{\prod_{n \in F} f(x_n) : F \in \mathcal{P}_f(\mathbb{N})\}$.

For the first of these inclusions, let $y \in \langle\!\langle \langle x_n \rangle_{n=1}^{\infty} \rangle\!\rangle_{\{f\}} \cap S_2$. Pick $F \in \mathcal{P}_f(\mathbb{N})$ and $\{\alpha_n : n \in F\} \subseteq \omega$ such that $y = \prod_{n \in F} f^{\alpha_n}(x_n)$. Let $G = \{n \in F : \alpha_n = 0$ or $f^{\alpha_n}(x_n) = f(x_n)\}$. Since each $x_n \in B_{2,1}$, we have that, if $n \in F \backslash G$, then $f^{\alpha_n}(x_n) = e$, so $y = \prod_{n \in G} f^{\delta_n}(x_n)$ where $\delta_n = \min\{\alpha_n, 1\}$. Further, since $y \in S_2$, some $\delta_n = 0$.

For the second inclusion, let $y \in \langle\!\langle \langle x_n \rangle_{n=1}^{\infty} \rangle\!\rangle_{\{f\}} \cap S_1$. Pick $F \in \mathcal{P}_f(\mathbb{N})$ and $\{\alpha_n : n \in F\} \subseteq \omega$ such that $y = \prod_{n \in F} f^{\alpha_n}(x_n)$. Since S_2 is an ideal of S we have that each $\alpha_n \geq 1$. Let $G = \{n \in F : f^{\alpha_n}(x_n) = f(x_n)\}$. Since each $x_n \in B_{2,1}$, we have if $n \in F \backslash G$, then $f^{\alpha_n}(x_n) = e$. Further, $y \in S_1$ so $G \neq \emptyset$. Thus $y = \prod_{n \in G} f(x_n)$.

(2) implies (4). If for some n, $f(x_n) = e$, the first alternative of (4) holds, so assume that for each n, $f(x_n) \neq e$. Let R be the subsemigroup of S generated by

$\{e\} \cup \{x_n : n \in \mathbb{N}\} \cup \{f(x_n) : n \in \mathbb{N}\}$, let $R_1 = R \cap S_1$, and let $R_2 = R \cap S_2$. Then

$$\begin{aligned} R &= \{\textstyle\prod_{i=1}^m f^{\alpha_i}(x_{n_i}) : && m \in \mathbb{N}, \langle n_i \rangle_{i=1}^m \text{ is a sequence in } \mathbb{N}\,, \\ & && \text{and } \langle \alpha_i \rangle_{i=1}^m \text{ is a sequence in } \{0,1\}\} \cup \{e\}\,, \\ R_2 &= \{\textstyle\prod_{i=1}^m f^{\alpha_i}(x_{n_i}) : && m \in \mathbb{N}, \langle n_i \rangle_{i=1}^m \text{ is a sequence in } \mathbb{N}\,, \\ & && \langle \alpha_i \rangle_{i=1}^m \text{ is a sequence in } \{0,1\}, \text{ and some } \alpha_i = 0\}\,, \text{ and} \\ R_1 &= \{\textstyle\prod_{i=1}^m f(x_{n_i}) : && m \in \mathbb{N} \text{ and } \langle n_i \rangle_{i=1}^m \text{ is a sequence in } \mathbb{N}\}\,. \end{aligned}$$

Using these characterizations it is easy to see that if for all n, $f^2(x_n) = e$, then $f_{|R}$ is a shift on R with $f[R_0 \cup R_1] = \{e\}$, and if for all n, $f^2(x_n) = f(x_n)$, then the restriction of f to $R_0 \cup R_1$ is the identity.

(4) implies (1). We showed in the proof that (3) implies (1) that the existence of some $y \in S_2$ such that $f(y) = e$ implies (1). So assume that we have a layered semigroup as in (4). Let $\mathcal{G}_2 = \{\iota_R\}$ and let $\mathcal{G}_1 = \{f_{|R}\}$. Pick $\gamma(1)$, $\gamma(2)$, and a sequence $\langle x_n \rangle_{n=1}^\infty$ in R_2 as guaranteed by Theorem 3.13. Since $\langle \mathcal{F}_n \rangle_{n=2}^2$ is a layered action on R, we have that ${f_{|R}}^2 = f_{|R}$ or ${f_{|R}}^2 = \overline{e}$ and thus

$$\langle\!\langle \langle x_n \rangle_{n=1}^\infty \rangle\!\rangle_{\{f\}} \cap S_2 = \{\textstyle\prod_{n \in F} g_n(x_n) : \; F \in \mathcal{P}_f(\mathbb{N})\,, \{g_n : n \in F\} \subseteq \{\iota_R, f_{|R}\} \text{ and } g_n = \iota_R \text{ for some } n\}$$

and $\langle\!\langle \langle x_n \rangle_{n=1}^\infty \rangle\!\rangle_{\{f\}} \cap S_1 = \{\prod_{n \in F} f(x_n) : F \in \mathcal{P}_f(\mathbb{N})\}$. □

Notice that the option in statement (4) that some $y \in S_2$ has $f(y) = e$ cannot be simply omitted. To see this, let S be any layered semigroup with two layers and let $f = \overline{e}$. Let R be any subsemigroup of S which has $R_2 = R \cap S_2 \neq \emptyset$ and $R_1 = R \cap S_1 \neq \emptyset$ (as is required if these are to be layers of R). Then $f[R_2]$ is not central* in R_1 and $f_{|R_0 \cup R_1}$ is not the identity. The same example shows that the possibility that $p_1 = e$ in statement (3) cannot be eliminated.

Notice also that the requirement in statement (3) that $\widetilde{f}(p_1) = p_1$ or $\widetilde{f}(p_1) = e$ cannot be simply omitted. To see this, let $S_2 = W(\{a,b,c\}) \backslash W(\{a,b\})$, $S_1 = W(\{a,b\}) \backslash \{e\}$, and $S_0 = \{e\}$. Let $f : S \to S$ be the homomorphism with $f(c) = b$, $f(b) = a$, and $f(a) = e$. Then, letting $C_0 = W(\{a\})$, $C_1 = W(\{a,b\}) \backslash W(\{a\})$, and $C_2 = S_2$, if $\langle x_n \rangle_{n=1}^\infty$ is any sequence in S_2, then $f(x_1) \in C_1$ and $f^2(x_2) \in C_0$, while both are in $\langle\!\langle \langle x_n \rangle_{n=1}^\infty \rangle\!\rangle_{\{f\}} \cap S_1$. On the other hand, taking p_1 to be a minimal idempotent in βC_1 one produces $p_2 \in \beta S_2$ with $p_2 \leq p_1$ and $\widetilde{f}(p_2) = p_1$ as in the proof of Theorem 3.8.

It would of course be interesting to see whether Theorem 3.13 can be strengthened by relaxing the definition of action. Theorem 6.4 suggests that the only possible way to do so is by allowing more than one shift at each transition between layers (i.e., by allowing (1)(b) of Definition 3.3 to hold for more than one member of $\mathcal{F}_n$, perhaps with some other restrictions). To support this claim, let us note that the above proof shows that (1) implies (2) in a more general case when f is replaced by an arbitrary finite set of homomorphisms.

6.5 Question. *Let S be a layered partial semigroup and let $\mathcal{H}$ be a set of partial homomorphisms satisfying the conclusion of Theorem* 3.13*. Is there necessarily a partial subsemigroup R of S such that $R \cap S$ is a layered partial semigroup with layers $R \cap S_n$ ($n \leq k$) and the restriction of $\mathcal{H}$ to R is included in $\bigcup_{n=1}^{k} \mathcal{G}_n$, where $\mathcal{G}_n$ are obtained from some action on R as in Definition* 3.9*?*

A positive answer to Question 6.5 would suggest that Theorem 3.13 is rather optimal. In Question 6.6 below we state a more modest variant of Question 6.5.

As pointed out earlier, the fact that we are unable to deal with more than one shift at a time (see Theorem 4.2) is a bit annoying. It is curious that we were unable to find a nontrivial example of a layered partial semigroup with two layers and two different shifts to which the conclusion of Theorem 3.13 applies.

6.6 Question. *Assume S is a layered semigroup with layers S_0, S_1 and S_2. Are the following equivalent for every set $\mathcal{H}$ of shifts of S into $S_0 \cup S_1$?*

(1) The conclusion of Theorem 3.13 *holds.*

(2) There is a subsemigroup R of S closed under all $f \in \mathcal{H}$ and a shift g on R such that $R \cap S_1$ and $R \cap S_2$ are nonempty and for every $f \in \mathcal{H}$ we have either $f_{|R} = g_{|R}$ or $f[R] = \{e\}$.

By Theorem 3.13, (2) implies (1). Thus a positive answer to Problem 6.6 would be a sort of a converse to Theorem 3.13 in case $k = 2$, suggesting that at least in this case this theorem is optimal. We were able to give a positive answer to Question 6.6 in case when $S = W(\Sigma \cup \{a\})$, $S_2 = W(\Sigma \cup \{a\}) \setminus W(\{a\})$, $S_1 = W(\{a\}) \setminus \{e\}$ for some finite alphabet Σ. This result suggests that, at least in the case of free semigroups, Question 6.6 has a positive solution. Its proof will appear elsewhere.

Let us finish with a variation of Problem 6.1.

6.7 Problem. *Describe the class of triples $(S, S', \mathcal{H})$ such that S is a partial semigroup, S' is an ideal of S, and $\mathcal{H}$ is a set of partial semigroup homomorphisms from S to itself such that for every $r \in \mathbb{N}$ and every partition $S = \bigcup_{j=1}^{r} C_j$ there is a sequence $\langle x_n \rangle_{n=1}^{\infty}$ included in S' and $j \in \{1, 2, \ldots, r\}$ such that $\langle\!\langle \langle x_n \rangle_{n=1}^{\infty} \rangle\!\rangle_{\mathcal{H}} \cap S' \subseteq C_j$.*

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